On green sequences

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Slightly extended beamer version of a blackboard talk

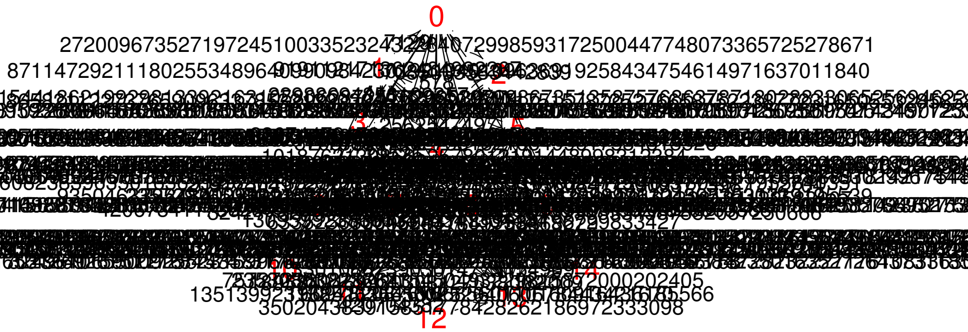
Plan

1. Combinatorics

2. Representation theory

If you do not know about quiver mutation, google it!

In general, it is terribly complicated:



But we are interested in its environmentally friendly variant:
Green quiver mutation.

Green quiver mutation: the framed quiver

Let Q be a quiver without loops or 2-cycles.

Definition

The framed quiver $\widehat{Q}$ is obtained from Q by adding, for each vertex i, a new vertex i' and a new arrow $i \to i'$.

Example

$$Q: \; 1 \longrightarrow 2 \qquad\qquad \widehat{Q}: \; \begin{array}{ccc} 1 & \longrightarrow & 2 \\ \downarrow & & \downarrow \\ 1' & & 2' \end{array}$$

Definition

The vertices i' are frozen vertices, i.e. we never mutate at them.

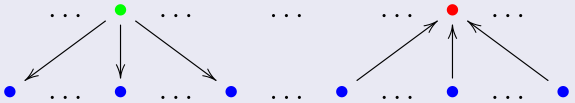
Green vertices, green sequences

Suppose that we have transformed $\widehat{Q}$ into $\widehat{Q}'$ by a finite sequence of mutations (at non frozen vertices).

Definition

A non frozen vertex i is green in $\widehat{Q}'$ if there are no arrows $j' \to i$ in $\widehat{Q}'$. It is red if there are no arrows $i \to j'$ in $\widehat{Q}'$.

Green and red vertices in $\widehat{Q}'$



Theorem (Derksen-Weyman-Zelevinsky 2010)

Each non frozen vertex of $\widehat{Q}'$ is either green or red.

c-vectors and green sequences

Definition

The c-vector α_i associated with a vertex i of $\widehat{Q}'$ has the components

$$|\{i \to j'\}| - |\{j' \to i\}| \ , \ j \in Q_0.$$

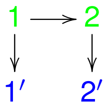
Definition ('10)

A sequence $\underline{i} = (i_1, \ldots, i_N)$ is green if, for each $1 \leq t \leq N$, the vertex i_t is green in the partially mutated quiver

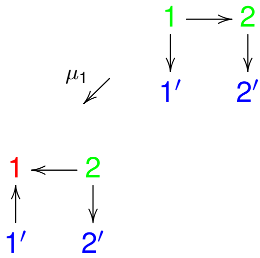
$$\mu_{i_{t-1}} \ldots \mu_{i_2} \mu_{i_1}(\widehat{Q}).$$

It is maximal green if all non frozen vertices of $\mu_{\underline{i}}(\widehat{Q})$ are red.

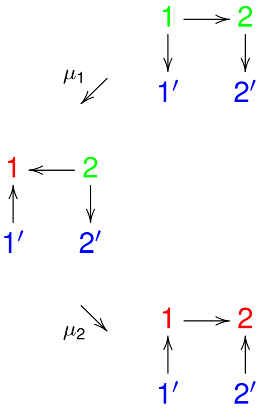
Green quiver mutation: an example



Green quiver mutation: an example



Green quiver mutation: an example



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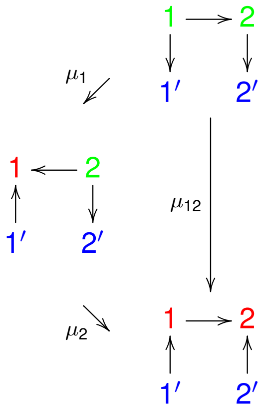

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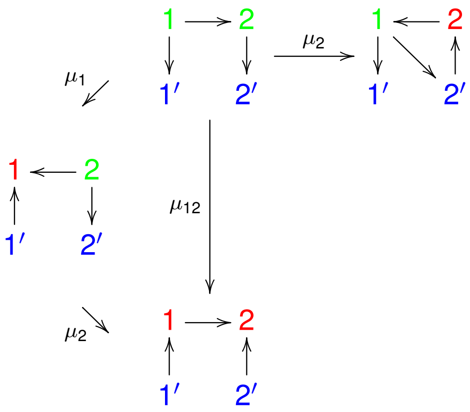

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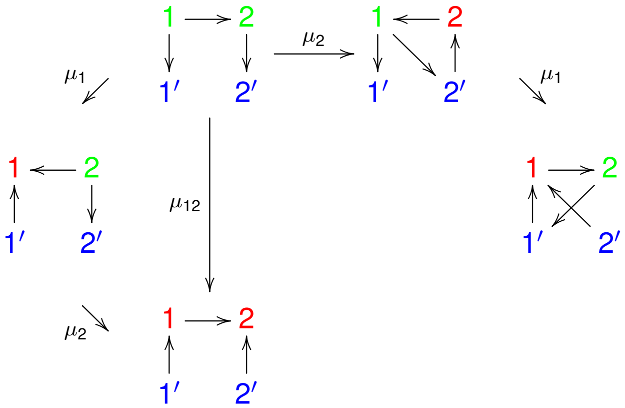

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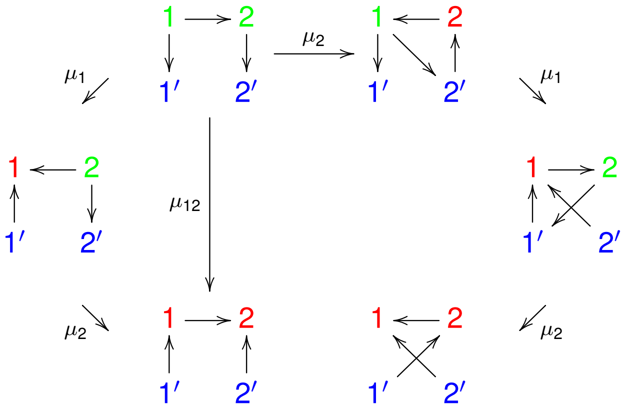

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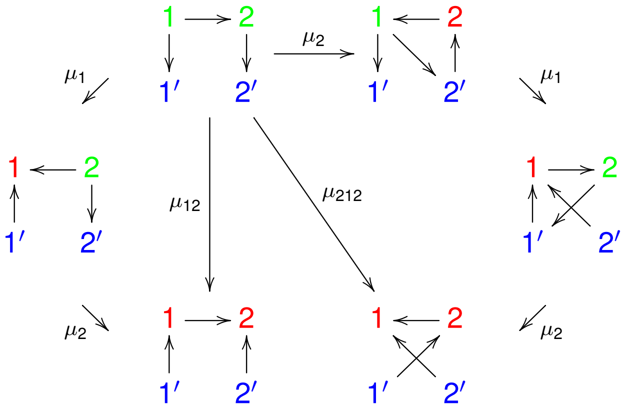

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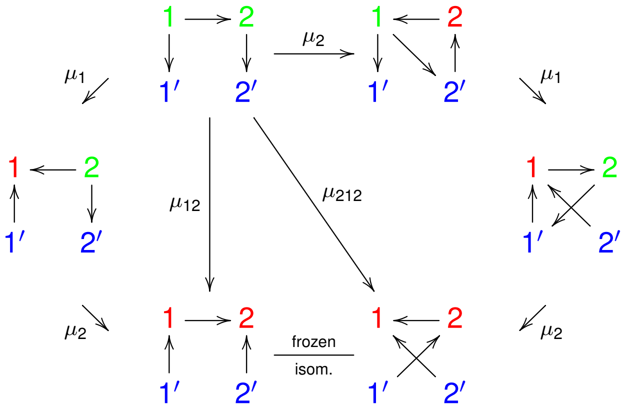


Abundance and applications of max. green sequences

Fact

Large classes of quivers arising in Lie theory and higher Teichmüller theory do have maximal green sequences.

Applications

- Donaldson-Thomas invariants
- Twist automorphisms
- Fock–Goncharov duality conjectures

Application: Donaldson-Thomas invariants

Theorem (K '10)

If $\underline{i} = (i_1, \ldots, i_N)$ is maximal green, then

$$DT_Q = \mathbb{E}(y^{\alpha_1}) \cdots \mathbb{E}(y^{\alpha_N}) \text{ in } \widehat{\mathbb{A}}_Q.$$

- DT_Q = refined Donaldson–Thomas invariant of Kontsevich–Soibelman ('08 and '10), extremely hard to compute in general
- $\mathbb{E}$ = quantum dilogarithm series,
- $\alpha_1, \ldots, \alpha_N$ = sequence of c-vectors associated with $\underline{i}$.

Independently discovered by Gaiotto–Moore–Neitzke ('09).

For $Q: 1 \longrightarrow 2$, the two max. green sequences give

$$\mathbb{E}(y_1)\mathbb{E}(y_2) = DT_Q = \mathbb{E}(y_2)\mathbb{E}(q^{-1/2}y_1 y_2)\mathbb{E}(y_1)$$

where $y_1 y_2 = q y_2 y_1$. This is the pentagon identity.

Application: Twist automorphisms

If Q has a maximal green sequence $\underline{i}$, then the cluster algebra $\mathcal{A}_Q$ has a well-defined twist automorphism which is easily computed from $\underline{i}$.

The twist automorphism corresponds to the suspension of the generalized cluster category (Geiss–Leclerc–Schröer).

Appears in work of Berenstein–Fomin–Zelevinsky ('96), Geiss–Leclerc–Schröer ('10), Marsh–Scott ('13), Rietsch–L. Williams ('17), Cautis–H. Williams ('18).

Application: Fock–Goncharov duality conjecture

Theorem (Gross–Hacking–Keel–Kontsevich '14)

If Q has a maximal green sequence and $\mathcal{A}_Q = \mathcal{U}_Q$, then the Fock–Goncharov duality conjecture holds for Q so we have

$$\mathcal{X}_{Q^L}(\mathbb{Z}^{trop}) \xrightarrow{\sim} \{\textit{can. basis of } \mathcal{A}_Q\} \ ,$$

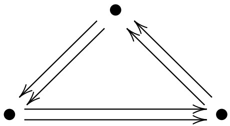
where Q^L is the Langlands dual (valued) quiver.

Remark

The key tool in the proof are the scattering diagrams from mirror symmetry (Kontsevich–Soibelman '06, Gross–Siebert '11).

Non examples

- The quiver $Q_{2,2,2}$



  does not admit a maximal green sequence (Brüstle–Dupont–Pérotin '12).
- Nor does $Q_{a,b,c}$ for $a, b, c \geq 2$ (Muller '15).
  Remarkably $Q_{2,2,3}$ is mutation-acyclic!
- Nor do the quivers associated with triangulations of once-punctured closed surfaces of genus ≥ 1 (Ladkani '13).
- Nor does X_7 (Seven '14).

Mutation-finite quivers

Theorem (Mills '16)

If Q is a mutation-finite quiver, it has a maximal green sequence except if it comes from a once-punctured closed surface of genus ≥ 1 or is in the mutation class of X_7.

The proof uses the combinatorics of marked surfaces and their laminations well as the

Rotation Lemma (Brüstle–Hermes–Igusa–Todorov '15)

If $\underline{i} = (i_1, \ldots, i_N)$ is maximal green for Q, then

$$(i_2, \ldots, i_N, k)$$

is maximal green for $\mu_{i_1}(Q)$, where $i_1' \to k$ in $\mu_{\underline{i}}(\widehat{Q})$.

Muller's theorem

Theorem (Muller '15)

If Q has a maximal green sequence, then each full subquiver $Q' \subseteq Q$ has a maximal green sequence.

Remark

Muller uses scattering diagrams. But there is also a proof via representation theory, as we will see.

The Jacobi algebra (Derksen–Weyman–Zel. ’07)

- Q a finite quiver without loops nor 2-cycles
- k an uncountable field
- $\widehat{kQ}$ the completed path algebra
- $W \in HH_0(\widehat{kQ})$ a non degenerate potential
- $A = \mathrm{Jac}(Q, W) =$ Jacobi algebra $= \widehat{kQ}/(\partial_\alpha W \mid \alpha \in Q_1)$
- $\dim A \leq \infty$!
- $\mathrm{mod}\, A = \{k$-finite-dimensional right A-modules$\}$.

Nagao's theorem

Theorem (Nagao '10)

Each green sequence $\underline{i} = (i_1, \ldots, i_N)$ *yields a chain*

$$0 = \mathcal{T}_0 \subset \mathcal{T}_1 \subset \mathcal{T}_2 \subset \ldots \subset \mathcal{T}_N \subseteq \operatorname{mod} A$$

of torsion classes such that

$$\mathcal{T}_{t-1}^{\perp} \cap \mathcal{T}_t = \operatorname{add}(B_t)$$

for a unique brick B_t *with* $\operatorname{End}(B_t) = k$, $\operatorname{Ext}^1(B_t, B_t) = 0$ *and*

$$\underline{\dim}\, B_t = \alpha_t = \textit{c-vector number } t \textit{ associated with } \underline{i}\,,\ 1 \le t \le N.$$

Moreover, $\underline{i}$ *is maximal iff* $\mathcal{T}_N = \operatorname{mod} A$.

Consequences

Corollary (Brüstle–Dupont–Pérotin)

If Q has a maximal green sequence then $\dim A < \infty$.

From Demonet–Iyama–Jasso's work ('15), we get

Theorem

Suppose $\dim A < \infty$. *Then Nagao's map* $(i_t) \mapsto (\mathcal{T}_t)$ *is a bijection from the set of green sequences to the set of paths starting at* 0 *in the quiver* $\mathrm{Hasse}(\mathrm{tors}\, A)$.

Which sequences of bricks occur?

Theorem (Igusa '17, Demonet)

Suppose $\dim A < \infty$. *A sequence of bricks* $B_1, \ldots, B_N$ *is associated with a maximal green sequence for* Q *iff*

$$\mathrm{Hom}(B_i, B_j) = 0 \text{ for all } i < j$$

and the sequence cannot be refined keeping this condition.

Challenge

Prove Mills' theorem using these facts.

Muller's theorem via representation theory

We use

Theorem (Demonet–Iyama–Reading–Reiten–Thomas '17)

Let A be a finite-dimensional algebra and $I \triangleleft A$ a 2-sided ideal. Then the map $\mathcal{T} \mapsto \mathcal{T} \cap \mathrm{mod}(A/I)$ induces a contraction (=functor taking arrows to arrows or identities)

$$\mathrm{Hasse}(\mathrm{tors}\, A) \to \mathrm{Hasse}(\mathrm{tors}\, A/I).$$

Proof of Muller's theorem:

- Q with a max. green sequence, $Q' \subseteq Q$ a full subquiver
- $A = \mathrm{Jac}(Q, W)$, W' the induced potential on Q'
- $A' = \mathrm{Jac}(Q', W') \cong A/AeA$, where $e = \sum_{i \notin Q'_0} e_i$.

The contraction

$$\mathrm{Hasse}(\mathrm{tors}\, A) \to \mathrm{Hasse}(\mathrm{tors}\, A')\ ,\ \mathcal{T} \mapsto \mathcal{T} \cap \mathrm{mod}\, A'$$

clearly maps a finite path from 0 to mod A to a finite path from 0 to mod A'.

A grain of salt

Muller states his theorem for quivers but in fact the statement and the proof go through for valued quivers corresponding to antisymmetrizable matrices (scattering diagrams work in this generality!).

Despite the best efforts of Demonet, Labardini–Zelevinsky, Geiss–Leclerc–Schröer, . . . this is still out of reach of the representation-theoretic methods.

An open problem

Relate the two proofs, i.e. link scattering diagrams to torsion classes.

First step: Bridgeland ('16), who associates a generalized scattering diagram (with values in the motivic Hall algebra) to each quiver with relations.