Quiver mutation and combinatorial DT-invariants

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Context

quiver

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quiver mutation

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cluster algebras
Fomin-Zelevinsky

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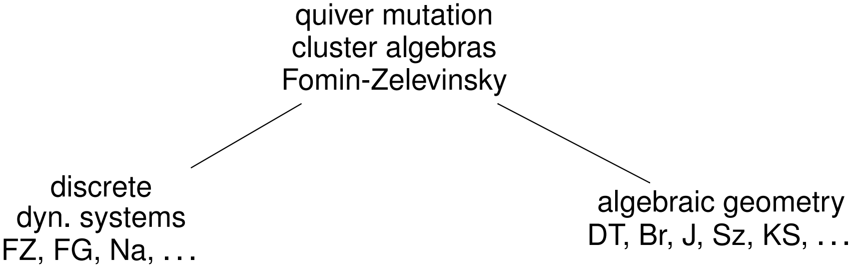

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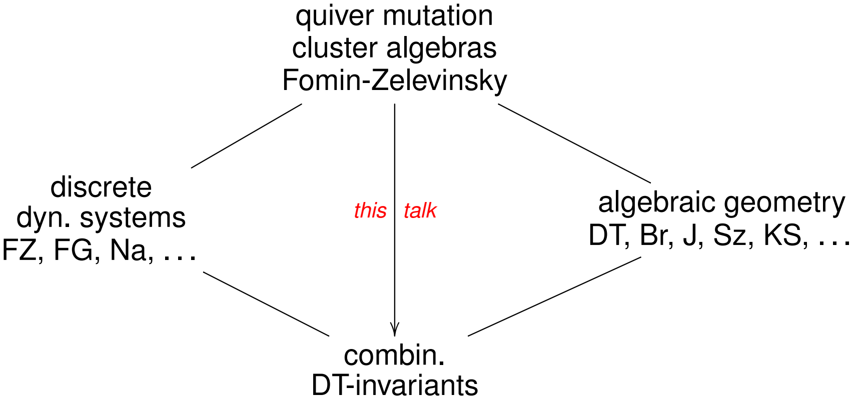

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Plan

1. Quiver mutation
2. Combinatorial DT-invariants

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Quivers: example and terminology

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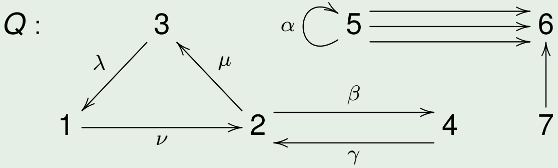

We have $Q_0 = \{1, 2, 3, 4, 5, 6, 7\}$, $Q_1 = \{\alpha, \beta, \ldots\}$.
α is a *loop*, (β, γ) is a 2-*cycle*, (λ, μ, ν) is a 3-*cycle*.

Definition

A vertex *i* is a *source* of Q if no arrows stop at *i*.
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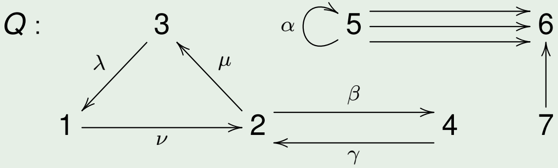

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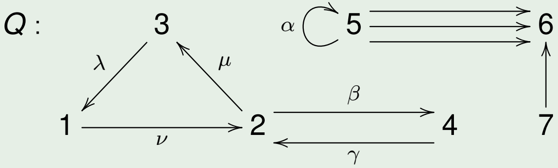

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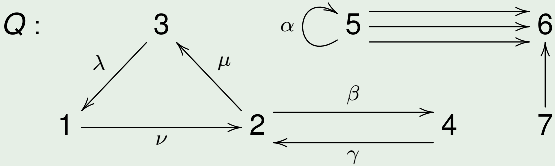

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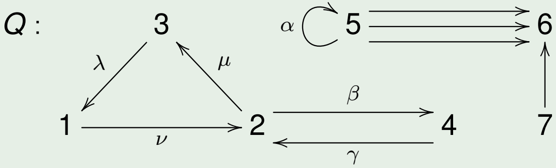

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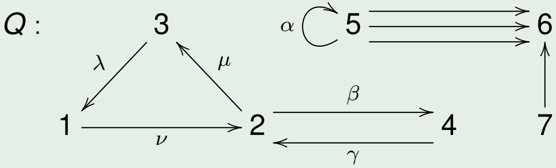

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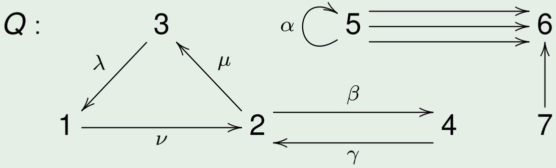

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Definition of quiver mutation

Let Q be a quiver without loops or 2-cycles.

Definition (Fomin-Zelevinsky)

Let $j \in Q_0$. The *mutation* $\mu_j(Q)$ is the quiver obtained from Q as follows

1) for each subquiver $i \xrightarrow{\alpha} j \xrightarrow{\beta} k$, add a new arrow $i \xrightarrow{[\alpha\beta]} k$;
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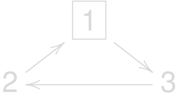

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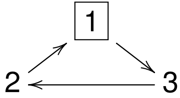

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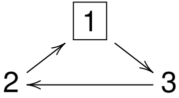

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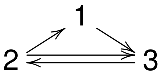

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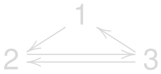

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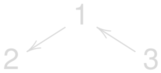

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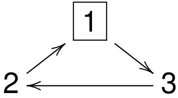

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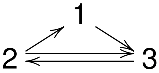

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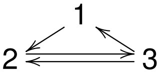

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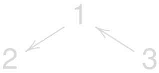

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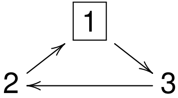

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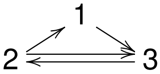

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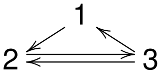

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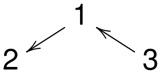
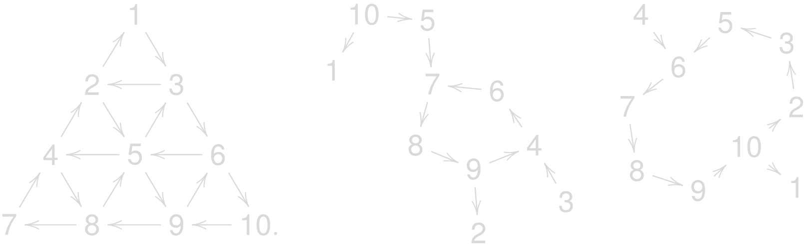


More complicated examples: Google ‘quiver mutation’!



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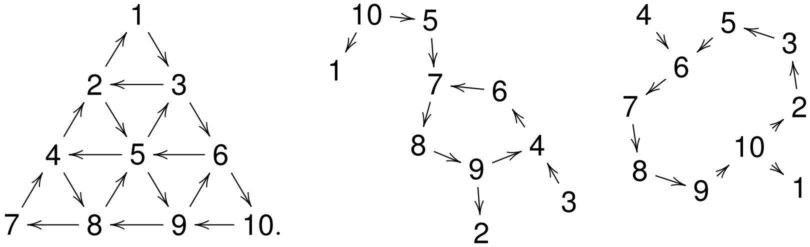
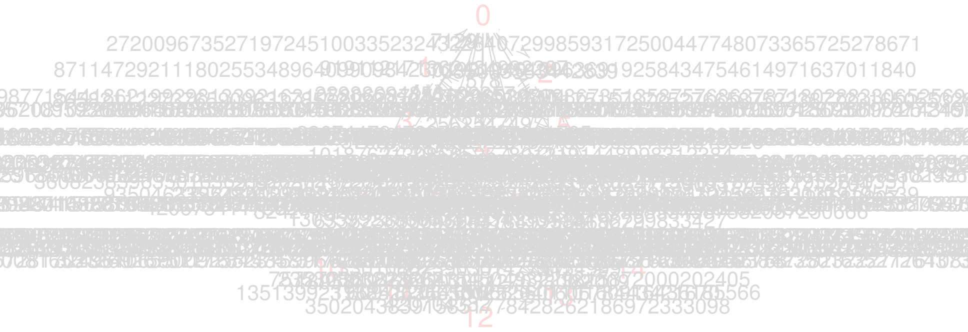


Towards green quiver mutation



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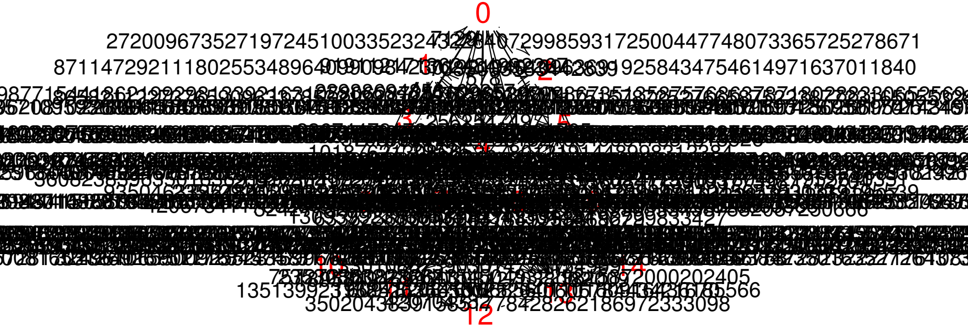

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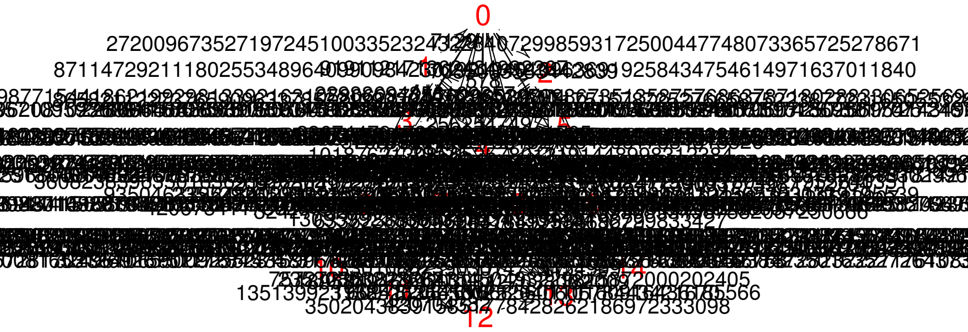

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Green quiver mutation: the framed quiver

Let Q be a quiver without loops or 2-cycles, $Q_0 = \{1, \ldots, n\}$.

Definition

The framed quiver $\widetilde{Q}$ is obtained from Q by adding, for each vertex i, a new vertex i' and a new arrow $i \to i'$.

Example

$$\begin{tikzcd} Q : 1 \arrow[r] & 2 \end{tikzcd} \qquad \begin{tikzcd} \widetilde{Q} : 1 \arrow[r] \arrow[d] & 2 \arrow[d] \\ 1' & 2' \end{tikzcd}$$

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The vertices i' are frozen vertices, i.e. we never mutate at them.

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Green and red vertices

Suppose that we have transformed $\widetilde{Q}$ into $\widetilde{Q}'$ by a finite sequence of mutations (at non frozen vertices).

Definition

A vertex i of Q is green in $\widetilde{Q}'$ if there are no arrows $j' \to i$ in $\widetilde{Q}'$. It is red if there are no arrows $i \to j'$ in $\widetilde{Q}'$.

Green and red vertices in $\widetilde{Q}'$

Theorem (Derksen-Weyman-Zelevinsky 2010)

Each vertex of Q is either green or red in $\widetilde{Q}'$ but not both.

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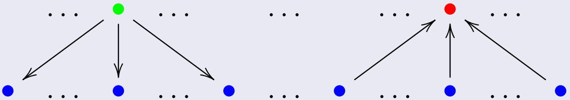

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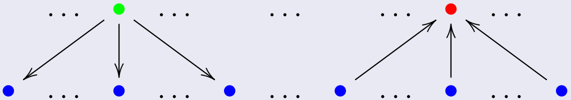

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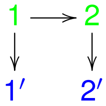
Green sequences

Definition

A sequence $\underline{i} = (i_1, \ldots, i_N)$ is green if, for each $1 \leq t \leq N$, the vertex i_t is green in the partially mutated quiver

$$\mu_{i_{t-1}} \ldots \mu_{i_2} \mu_{i_1} (\widetilde{Q}) =: \widetilde{Q}(\underline{i}, t).$$

Green quiver mutation: example



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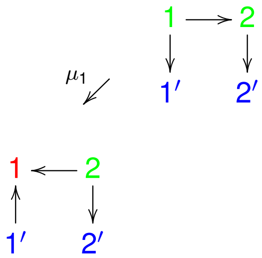

$$\begin{tikzcd}
{} & {} & {\color{green}1} \arrow[r] \arrow[d] & {\color{green}2} \arrow[d] \\
{} & {} \arrow[ld, "\mu_1"'] & {\color{blue}1'} & {\color{blue}2'} \\
{\color{red}1} & {\color{green}2} \arrow[l] \arrow[d] & & \\
{\color{blue}1'} \arrow[u] & {\color{blue}2'} & &
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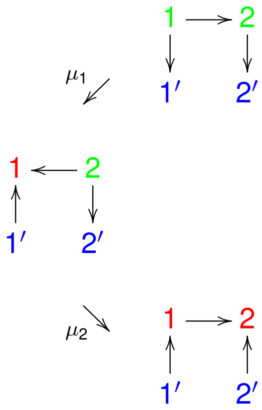

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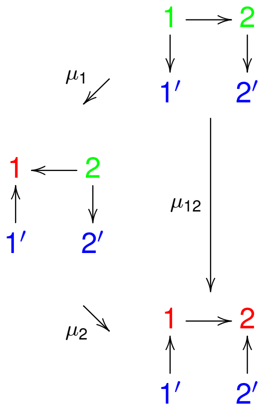

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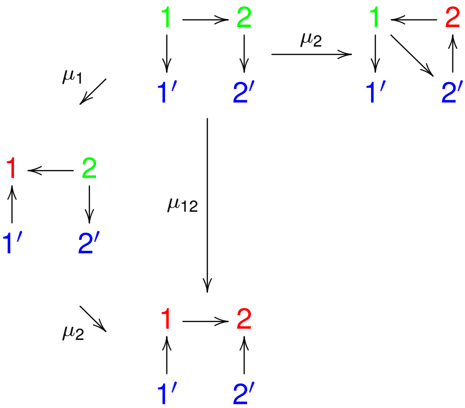

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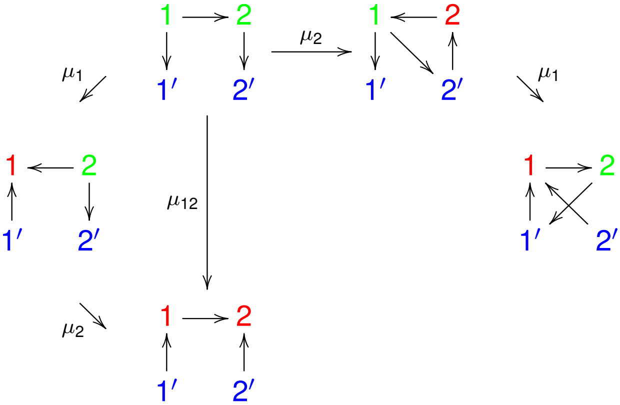

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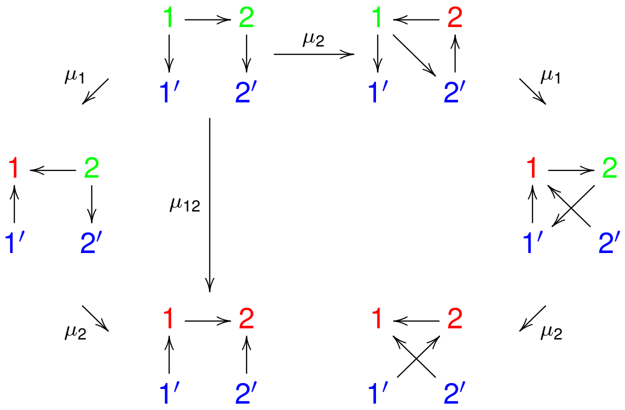

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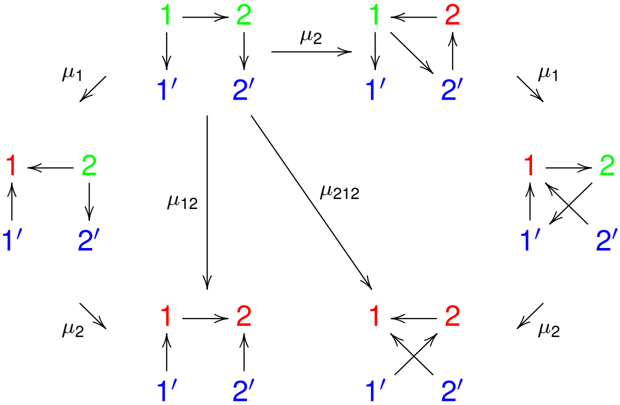

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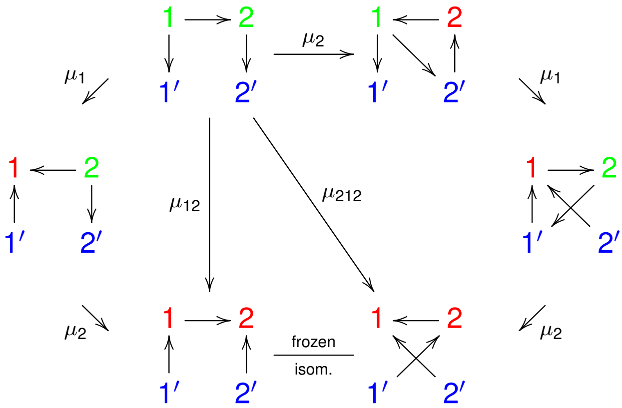

Reddening sequences

Theorem 1

If $\underline{i}$ and $\underline{i}'$ are maximal green sequences, then there is a frozen isomorphism

$$\mu_{\underline{i}}(\widetilde{Q}) \xrightarrow{\sim} \mu_{\underline{i}'}(\widetilde{Q}).$$

Remarks

- More generally, this holds for reddening sequences, i.e. arbitrary sequences $\underline{i} = (i_1, \ldots, i_N)$ such that all vertices of Q are red in the final quiver

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- Not all quivers admit reddening sequences:

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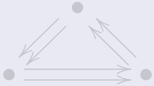

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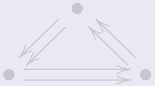

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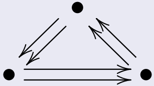


The exchange quiver

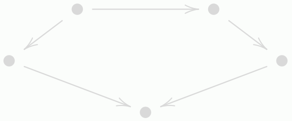
Definition

The exchange quiver $\mathcal{E}_Q$ has

- vertices: frozen isomorphism classes $\mu_{\underline{i}}(\widetilde{Q})$, where $\underline{i}$ is any sequence of vertices,
- arrows: $\widetilde{Q}' \to \mu_j(\widetilde{Q}')$ if j is green in $\widetilde{Q}'$.

Example

$Q : 1 \to 2$ yields $\mathcal{E}_Q$:



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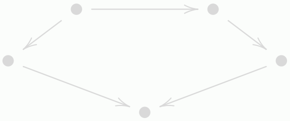

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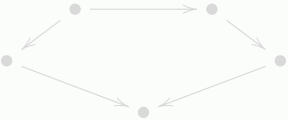

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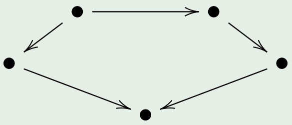

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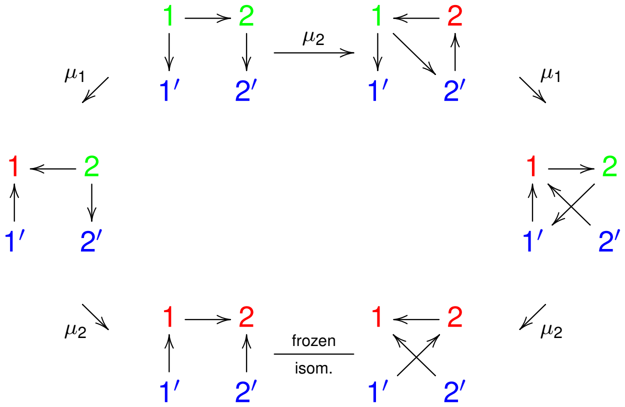

Key properties of the exchange quiver

Example (Reminder)

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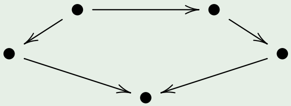

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- green seq. = formal comp. of arrows in $\mathcal{E}_Q$ = path in $\mathcal{E}_Q$,
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Theorem (Nagao 2010)

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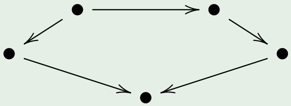

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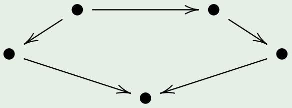

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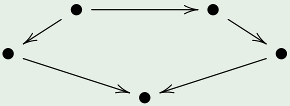

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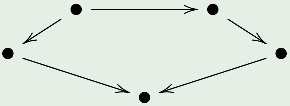

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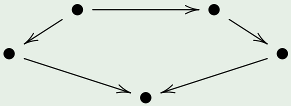

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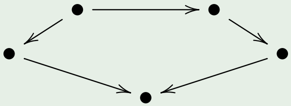

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The 3-chain

$$\begin{tikzcd} Q : & & 1 \arrow[dl] & \\ & 2 & & 3 \arrow[ul] \end{tikzcd}$$

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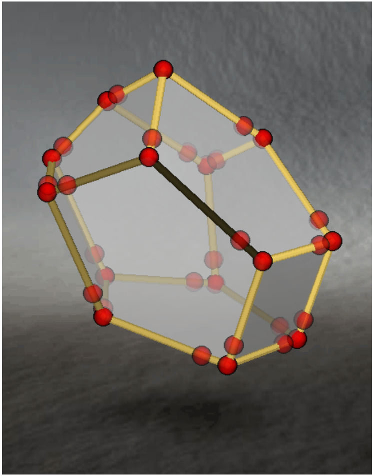

The 3-cycle

$$Q: \begin{tikzcd} & 1 \arrow[dr] & \\ 2 \arrow[ur] & & 3 \arrow[ll] \end{tikzcd}$$

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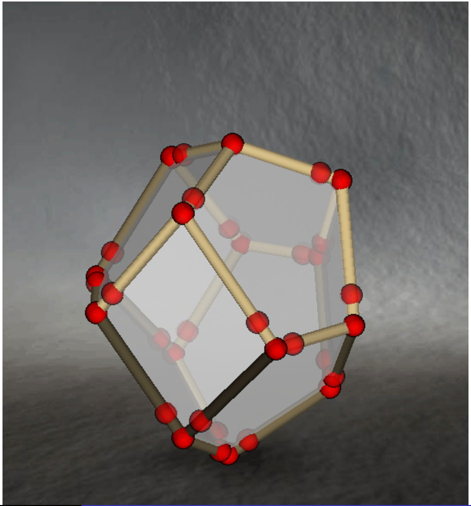

The 3-cycle with double arrow

$$Q : \begin{tikzcd} & 1 \arrow[dr] & \\ 2 \arrow[ur] & & 3 \arrow[ll, shift left] \arrow[ll, shift right] \end{tikzcd}$$

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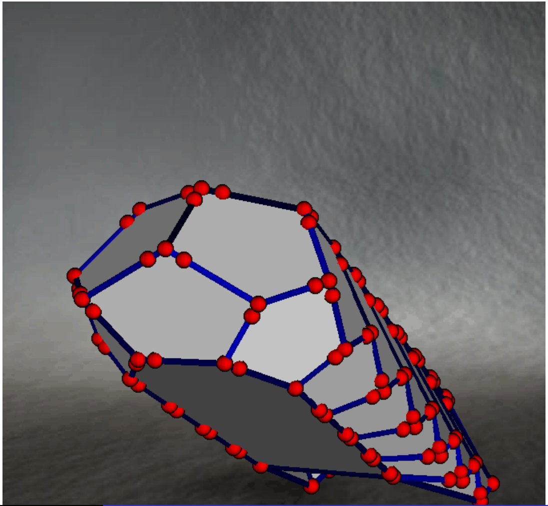

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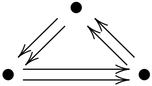

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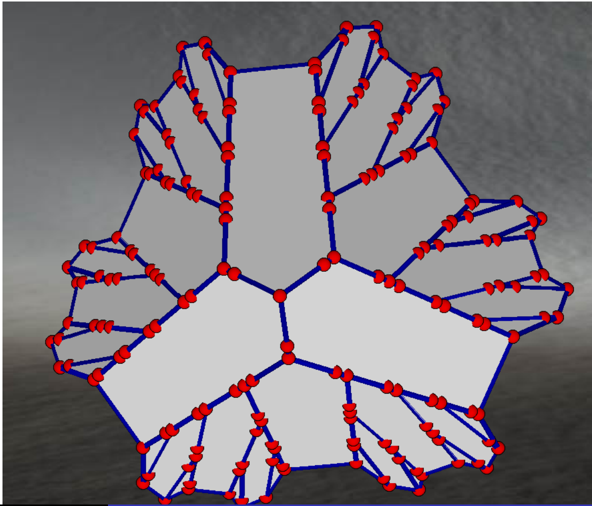

Plan

1. Quiver mutation
2. Combinatorial DT-invariants

The quantum dilogarithm and the quiver $\vec{A}_1$

Aim

Define the combinatorial DT-invariant $\mathbb{E}_Q$ as a formal power series intrinsically associated with Q, whenever Q admits a reddening sequence.

Definition

The (exponential of the) quantum dilogarithm series is

$$\begin{aligned}\mathbb{E}(y) &= 1 + \frac{q^{1/2}}{q-1} \cdot y + \cdots + \frac{q^{n^2/2} y^n}{(q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})} + \cdots \\ &\in \mathbb{Q}(q^{1/2})[[y]].\end{aligned}$$

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The pentagon identity and the quiver $\vec{A}_2$

Pentagon identity (Faddeev–Kashaev–Volkov 1993)

$$y_1 y_2 = q y_2 y_1 \Longrightarrow \mathbb{E}(y_1)\mathbb{E}(y_2) = \mathbb{E}(y_2)\mathbb{E}(q^{-1/2} y_1 y_2)\mathbb{E}(y_1).$$

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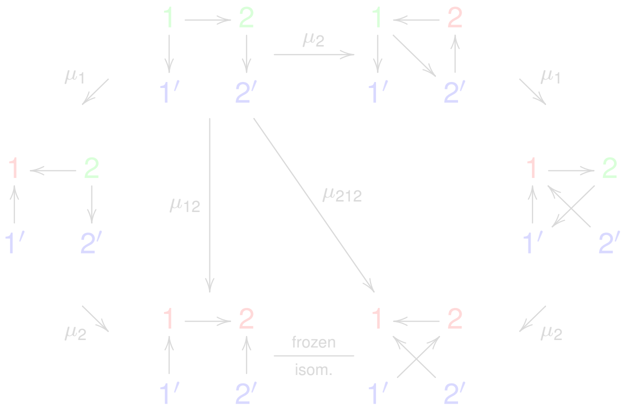

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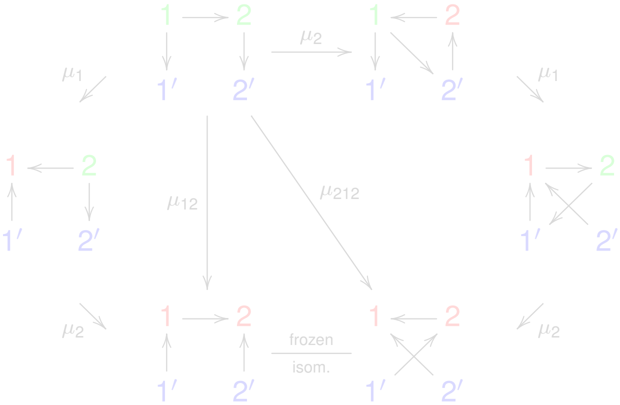

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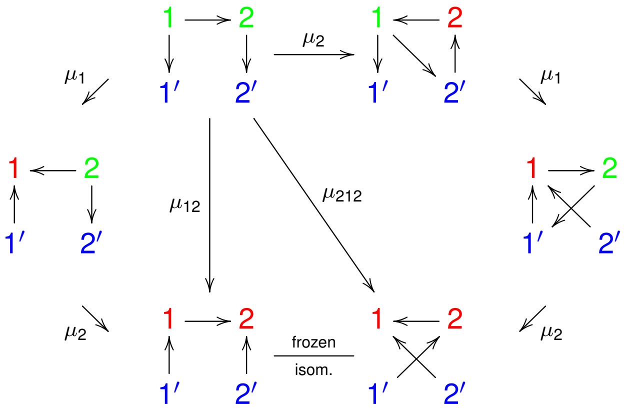

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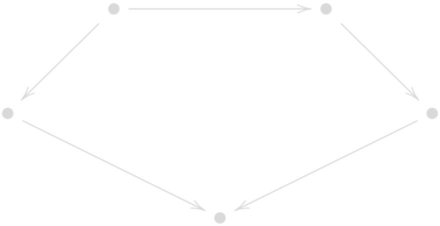

Each walk from the source to the sink of the exchange quiver yields a product expression for the combinatorial DT-invariant.

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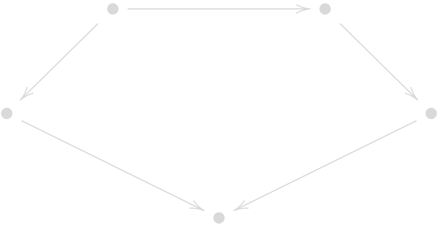

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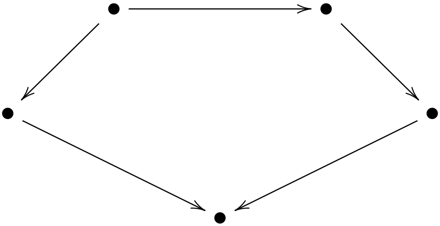

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Key construction

Given any mutation sequence $\underline{i} = (i_1, \ldots, i_N)$, we need to construct a product $\mathbb{E}(\underline{i})$ of quantum dilogarithm series.

This product is taken in the algebra $\widehat{\mathbb{A}}_Q$ constructed as follows: Let $\lambda_Q : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ be bilinear antisymmetric such that

$$\lambda_Q(e_i, e_j) = \#(\text{arrows } i \to j \text{ in } Q) - \#(\text{arrows } j \to i \text{ in } Q).$$

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The combinatorial DT-invariant

Main Theorem

Let $\underline{i}$ and $\underline{i}'$ be mutation sequences.

$$\exists \text{ frozen } \mu_{\underline{i}}(\widetilde{Q}) \cong \mu_{\underline{i}'}(\widetilde{Q}') \quad \Longrightarrow \quad \mathbb{E}(\underline{i}) = \mathbb{E}(\underline{i}').$$

Remark

In particular, if $\underline{i}$ and $\underline{i}'$ are reddening sequences, then by the first theorem, we have $\mathbb{E}(\underline{i}) = \mathbb{E}(\underline{i}')$.

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Let Q be an alternating Dynkin quiver $\vec{\Delta}$, where Δ is an ADE Dynkin diagram, e. g.

$$Q = \vec{A}_5 : \bullet \longleftarrow \circ \longrightarrow \bullet \longleftarrow \circ \longrightarrow \bullet$$

Put

$$\begin{aligned} i_+ &= \text{ sequence of all sources } \circ \\ i_- &= \text{ sequence of all sinks } \bullet . \end{aligned}$$

Then $\underline{i} = i_+ i_-$ is maximal green and so is

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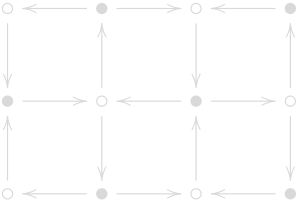
$Q = A_4 \square A_3$:



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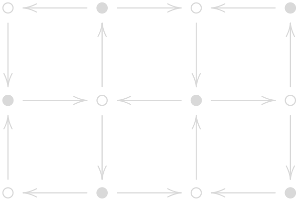

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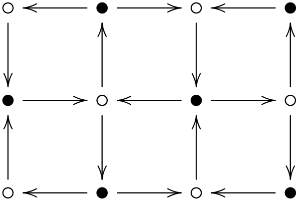

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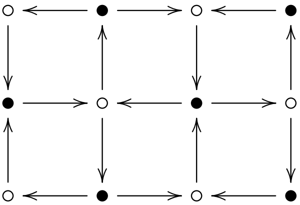

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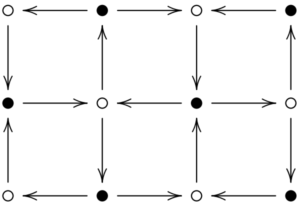

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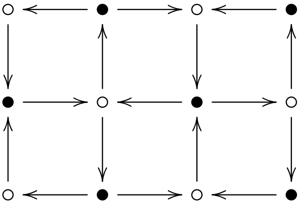

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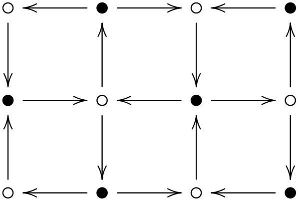

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- ‘Proof’ based on Kontsevich-Soibelman’s theory (preprints from November 2008 and June 2010),
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1. $\mathbb{E}(\underline{i}) = \mathbb{E}(\underline{i}')$ in $\widehat{\mathbb{A}}_Q$ follows from
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4. The equality (3) is equivalent to its specialization at $q^{1/2} = 1$! (by a theorem on quantum cluster algebras due to Berenstein-Zelevinsky)
5. The specialization of $\operatorname{Ad}\mathbb{E}(\underline{i})$ at $q^{1/2} = 1$ can be expressed in terms of Euler characteristics of quiver grassmannians of certain representations of Q (by the 'main theorem of additive categorification').
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- Do these follow from the pentagon ?
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