Quiver mutation and derived equivalence

Bernhard Keller and Dong Yang

U.F.R. de Mathématiques et Institut de Mathématiques
Université Paris Diderot – Paris 7

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History in a nutshell

quiver mutation = elementary operation on quivers discovered

- in mathematics: cluster algebras (Fomin-Zelevinsky, 2000)
- in physics: Seiberg duality (Vafa, Berenstein-Douglas, ...)

Aim: Categorify quiver mutation using recent work by

- Derksen-Weyman-Zelevinsky
- Ginzburg

Plan

A quiver is an oriented graph

Definition

A *quiver* Q is an oriented graph: It is given by

- a set Q_0 (the set of vertices)
- a set Q_1 (the set of arrows)
- two maps
  - $s : Q_1 \to Q_0$ (taking an arrow to its source)
  - $t : Q_1 \to Q_0$ (taking an arrow to its target).

Remark

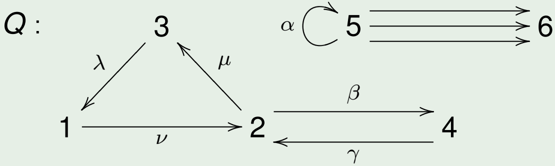
A quiver is a 'category without composition'.

A quiver can have loops, cycles, several components.

Example

The quiver $\vec{A}_3 : \; 1 \xleftarrow{\alpha} 2 \xleftarrow{\beta} 3$ is an orientation of the Dynkin diagram $A_3 : \; 1 \text{ --- } 2 \text{ --- } 3$.

Example



We have $Q_0 = \{1, 2, 3, 4, 5, 6\}$, $Q_1 = \{\alpha, \beta, \ldots\}$.
α is a *loop*, (β, γ) is a 2-*cycle*, (λ, μ, ν) is a 3-*cycle*.

Definition of quiver mutation

Let Q be a quiver without loops or 2-cycles.

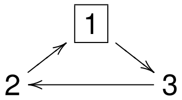
Definition (Fomin-Zelevinsky)

Let $i \in Q_0$. The *mutation* $\mu_i(Q)$ is the quiver obtained from Q as follows

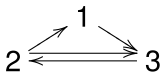
1) for each subquiver $j \xrightarrow{b} i \xrightarrow{a} l$, add a new arrow $j \xrightarrow{[ab]} l$;
2) reverse all arrows incident with i;
3) remove the arrows in a maximal set of pairwise disjoint 2-cycles (e.g. $\bullet \rightleftarrows\!\!\!\rightarrow \bullet$ yields $\bullet \longrightarrow \bullet$, '2-reduction').

Examples of quiver mutation

A simple example:



1)



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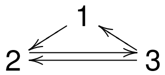

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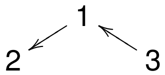
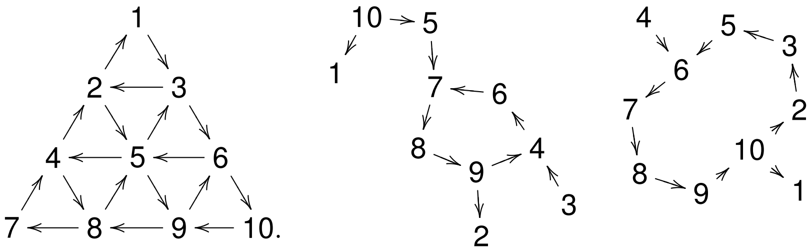


More complicated examples: Google ‘quiver mutation’!



Aim: Categorify these combinatorics!

Special case: source mutation

Notation

- k a field,
- kQ the path algebra: $\bigoplus_{p \text{ path}} kp$,
- e_i the lazy path attached to a vertex i,
- $P_i = e_i kQ$ the projective right module associated with e_i,
- $\text{Mod}(kQ)$ the category of all right kQ-modules,
- $\mathcal{D}(kQ)$ the derived category of $\text{Mod}(kQ)$.

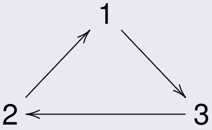
Theorem (Bernstein-Gelfand-Ponomarev 1973 + Happel 1986)

Let $Q' = \mu_i(Q)$, where i is a source. There is an equivalence (=reflection functor) $R : \mathcal{D}(kQ') \xrightarrow{\sim} \mathcal{D}(kQ)$ such that $P'_j \mapsto P_j$ for $j \neq i$ and $P'_i \mapsto (P_i \to \bigoplus_{i \to j} P_j)$.

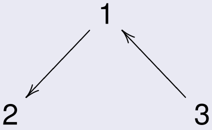
General case: Much more complicated

What if i is neither source nor sink? An easy counter-example

Q :



$Q' = \mu_1(Q)$:



Easy:

$\mathcal{D}(kQ)$ and $\mathcal{D}(kQ')$ are very far from being equivalent.

Remedy? Hint from physics:

Study quivers with potentials!

Completed path algebras, potentials

Notation

- k a field, Q a finite quiver without loops or 2-cycles
- $\widehat{kQ}$ = completed path algebra = $\prod_{p \text{ path}} kp$,
- $HH_0 = \widehat{kQ}/[\widehat{kQ}, \widehat{kQ}] = \{\text{infinite lin. comb. of cycles of } Q\}$,
- each $a \in Q_1$ yields the *cyclic derivative* $\partial_a : HH_0 \to \widehat{kQ}$ such that

$$\text{path } p \mapsto \sum_{p=uav} vu.$$

Definition

A *potential* on Q is an element $W \in HH_0$ not involving cycles of length 0.

Mutation of quivers with potential

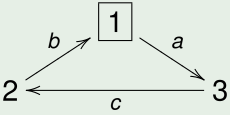
Theorem (Derksen-Weyman-Zelevinsky)

The mutation operation $Q \mapsto \mu_i(Q)$ admits a good extension to quivers with potentials

$$(Q, W) \mapsto \mu_i(Q, W) = (Q', W'),$$

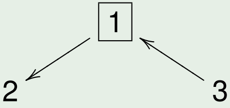
i.e. $\mu_i(Q)$ is isomorphic to the quiver Q' at least if W is generic *(and to the 2-reduction of Q' if W is arbitrary).*

Example



$W = abc$

$\Leftrightarrow$



$W' = 0$

Ginzburg's dg algebra

(Q, W) a quiver with potential (Q may have loops and 2-cycles)

Definition (Ginzburg)

$\widetilde{Q}$ = quiver with $\widetilde{Q}_0 = Q_0$ and

- the arrows of Q in degree 0,
- a new arrow $a^* : j \to i$ of degree -1 for each $a : i \to j$ of Q,
- a loop $t_i : i \to i$ of degree -2 for each vertex i of Q.

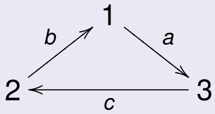
$\Gamma = \Gamma(Q, W) = \widehat{k\widetilde{Q}}$ endowed with the unique d of degree 1 such that

- $d(a) = 0$ for each arrow a of Q,
- $d(a^*) = \partial_a W$ for each arrow a of Q,
- $d(t_i) = e_i(\sum_{a \in Q_1} [a, a^*])e_i$ for each vertex i of Q.

Example of Ginzburg's dg algebra

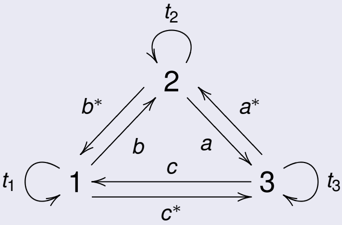
Quiver with potential

Q :



, $W = abc$,

Ginzburg dg algebra



, $d(a^*) = bc, \; d(t_1) = cc^* - b^*b, \; \ldots$

$H^0\Gamma$ and the derived category of Γ

Remark

Γ is an enhancement of

$$H^0\Gamma = \widehat{kQ}/(\partial_a W | a \in Q_1) = \textit{Jacobi algebra} \text{ of } (Q, W).$$

Definition

- $\mathcal{D}\Gamma$ =derived category of Γ
  (objects: differential graded Γ-modules),
- $\operatorname{per}\Gamma$ =perfect derived category = closure of Γ_Γ under
  shifts, extensions, direct summands,
- $\mathcal{D}^b\Gamma$ = bounded derived category
  $= \{M \in \mathcal{D}\Gamma \mid \dim H^* M < \infty\}$.

Γ is smooth and 3-Calabi Yau

Remarks

- Γ is *homologically smooth*, *i.e.*

$$\Gamma \in \operatorname{per}(\Gamma^e)\,, \quad \text{where } \Gamma^e = \Gamma \widehat{\otimes} \Gamma^{op}.$$

- Therefore, we have $\mathcal{D}^b\Gamma \subset \operatorname{per}\Gamma$.
- Γ is 3-*Calabi Yau (as a bimodule)*, *i.e.*

$$\operatorname{RHom}_{\Gamma^e}(\Gamma, \Gamma^e) \xrightarrow{\sim} \Gamma[-3] \text{ in } \mathcal{D}(\Gamma^e).$$

- Therefore, $\mathcal{D}^b\Gamma$ is 3-*CY as a triang. cat.*, *i.e.*

$$D\operatorname{Hom}(L, M) = \operatorname{Hom}(M, L[3])$$

for all L, M in $\mathcal{D}^b\Gamma$, where $D = \operatorname{Hom}_k(?, k)$.

The main theorem

Q w/o loops or 2-cycles, i a vertex of Q, $\Gamma' = \Gamma(\mu_i(Q, W))$.

Theorem (-Yang)

There is a canonical equivalence $\mathcal{D}\Gamma' \xrightarrow{\sim} \mathcal{D}\Gamma$ *taking* P'_j *to* P_j *for* $j \neq i$ *and* P'_i *to* $\mathrm{cone}(P_i \to \bigoplus_{i\to j} P_j)$. *It induces equivalences in* per *and* $\mathcal{D}^b$.

Remarks

- This improves on a result by Jorge Vitória (0709.3939v2), cf. also Mukhopadhyay-Ray, Berenstein-Douglas, ...
- The canonical t-structure on $\mathcal{D}^b\Gamma'$ yields a new t-structure on $\mathcal{D}^b\Gamma$. If W is generic, we get lots of new t-structures on $\mathcal{D}^b\Gamma$ ...

Links to cluster theory

$$
\begin{array}{ccccccccc}
 & & \text{NC 3-CY variety (KS)} & & & & \text{generalized cluster cat.: 2-CY} & & \\
 & & \rightsquigarrow & & & & \rightsquigarrow & & \\
0 & \longrightarrow & \mathcal{D}^b\Gamma & \longrightarrow & \operatorname{per}\Gamma & \longrightarrow & \mathcal{C}_\Gamma & \longrightarrow & 0 \\
 & & \Big\downarrow \text{KS (soon)} & & & & \Big\downarrow (*) & & \\
 & & \text{coeff. alg. (FZ)} & & & & \text{cluster alg. (FZ)} & & \\
 & & \mathcal{X}\text{-space (FG)} & & & & \mathcal{A}\text{-space (FG)} & &
\end{array}
$$

KS=Kontsevich-Soibelman, FZ=Fomin-Zelevinsky,
FG=Fock-Goncharov

Link to cluster algebras, application

Illustration of the link $(*)$ from $\mathcal{C}_\Gamma$ to the cluster algebra:

Theorem (-Caldero)

Suppose Q does not have any oriented cycles. Then

$$\left\{ \begin{array}{l} \textit{rigid indecomp.} \\ \textit{objects of } \mathcal{C}_\Gamma \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{l} \textit{cluster variables} \\ \textit{of the cl. alg. } \mathcal{A}_Q \end{array} \right\}.$$

Application of these techniques:

Theorem (K)

The periodicity conjecture (Al. Zamolodchikov, 1991) for pairs of Dynkin diagrams (Kuniba-Nakanishi, 1992) is true.

Summary

- Quiver mutation is derived equivalence of Ginzburg algebras.
- The periodicity conjecture is true.
- Google ‘quiver mutation’!

The periodicity conjecture

Notation

- Δ and Δ' two Dynkin diagrams with vertex sets J, J',
- h, h' their Coxeter numbers, C, C' their Cartan matrices,
- $Y_{i,i',t}$ variables where $i \in J$, $i' \in J'$, $t \in \mathbb{Z}$.

Y-system associated with (Δ, Δ')

$$Y_{i,i',t-1} Y_{i,i',t+1} = \frac{\prod_{j \neq i} (1 + Y_{j,i',t})^{-c_{ij}}}{\prod_{j' \neq i'} (1 + Y_{i,j',t}^{-1})^{-c'_{i'j'}}}.$$

Periodicity conjecture (Al. Zamolodchikov, Kuniba-Nakanishi)

All solutions to this system are periodic of period dividing $2(h + h')$.