Quiver mutation and derived equivalence

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quiver mutation = elementary operation on quivers discovered
  - in mathematics: cluster algebras (Fomin-Zelevinsky, 2000)
  - in physics: Seiberg duality (Vafa, Berenstein-Douglas, ...)

Aim: Categorify quiver mutation using recent work by
  - Derksen-Weyman-Zelevinsky
  - Ginzburg
Plan

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Quiver mutation and derived equivalence
A quiver is an oriented graph

**Definition**

A *quiver* $Q$ is an oriented graph: It is given by

- a set $Q_0$ (the set of vertices)
- a set $Q_1$ (the set of arrows)
- two maps
  - $s : Q_1 \to Q_0$ (taking an arrow to its source)
  - $t : Q_1 \to Q_0$ (taking an arrow to its target).

**Remark**

A quiver is a ‘category without composition’.
A quiver can have loops, cycles, several components.

Example

The quiver $\tilde{A}_3 : 1 \leftarrow \alpha \rightarrow 2 \leftarrow \beta \rightarrow 3$ is an orientation of the Dynkin diagram $A_3 : 1 \rightarrow 2 \rightarrow 3$.

Example

$Q :$

$Q_0 = \{1, 2, 3, 4, 5, 6\}, \quad Q_1 = \{\alpha, \beta, \ldots\}$.

$\alpha$ is a loop, $(\beta, \gamma)$ is a 2-cycle, $(\lambda, \mu, \nu)$ is a 3-cycle.
Definition of quiver mutation

Let $Q$ be a quiver without loops or 2-cycles.

Definition (Fomin-Zelevinsky)

Let $i \in Q_0$. The mutation $\mu_i(Q)$ is the quiver obtained from $Q$ as follows

1) for each subquiver $j \xrightarrow{b} i \xrightarrow{a} l$, add a new arrow $j \xrightarrow{[ab]} l$;

2) reverse all arrows incident with $i$;

3) remove the arrows in a maximal set of pairwise disjoint 2-cycles (e.g. $\bullet \xrightarrow{} \bullet \xrightarrow{} \bullet$ yields $\bullet \xrightarrow{} \bullet$, ‘2-reduction’).
Examples of quiver mutation

A simple example:

1) 

2) 

3)
More complicated examples: Google ‘quiver mutation’!

Aim: Categorify these combinatorics!
## Special case: source mutation

### Notation
- $k$ a field,
- $kQ$ the path algebra: $\bigoplus_{\text{path}} kp$,
- $e_i$ the lazy path attached to a vertex $i$,
- $P_i = e_i kQ$ the projective right module associated with $e_i$,
- $\text{Mod}(kQ)$ the category of all right $kQ$-modules,
- $\mathcal{D}(kQ)$ the derived category of $\text{Mod}(kQ)$.

### Theorem (Bernstein-Gelfand-Ponomarev 1973 + Happel 1986)

Let $Q' = \mu_i(Q)$, where $i$ is a source. There is an equivalence (reflection functor) $R : \mathcal{D}(kQ') \xrightarrow{\sim} \mathcal{D}(kQ)$ such that $P'_j \mapsto P_j$ for $j \neq i$ and $P'_i \mapsto (P_i \mapsto \bigoplus_{i \to j} P_j)$. 

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General case: Much more complicated

What if $i$ is neither source nor sink? An easy counter-example

\[Q : 1 \rightarrow 2 \rightarrow 3 \quad Q' = \mu_1(Q) : 1 \rightarrow 2 \rightarrow 3\]

Easy:
\[\mathcal{D}(kQ) \text{ and } \mathcal{D}(kQ') \text{ are very far from being equivalent.}\]

Remedy? Hint from physics:
Study quivers with potentials!
Notation

- $k$ a field, $Q$ a finite quiver without loops or 2-cycles
- $\hat{k}Q = \text{completed path algebra} = \prod_{p \text{ path}} kp$,
- $HH_0 = \hat{k}Q/[\hat{k}Q, \hat{k}Q] = \{\text{infinite lin. comb. of cycles of } Q\}$,
- each $a \in Q_1$ yields the cyclic derivative $\partial_a : HH_0 \to \hat{k}Q$ such that
  \[
  \text{path } p \mapsto \sum_{p = uav} vu.
  \]

Definition

A potential on $Q$ is an element $W \in HH_0$ not involving cycles of length 0.
Theorem (Derksen-Weyman-Zelevinsky)

The mutation operation $Q \mapsto \mu_i(Q)$ admits a good extension to quivers with potentials

$$(Q, W) \mapsto \mu_i(Q, W) = (Q', W') ,$$

i.e. $\mu_i(Q)$ is isomorphic to the quiver $Q'$ at least if $W$ is generic (and to the 2-reduction of $Q'$ if $W$ is arbitrary).

Example

\begin{align*}
\begin{array}{c}
1 \\
\downarrow b \\
2 \\
\downarrow c \\
3 \\
\uparrow a
\end{array} & \Leftrightarrow \\
\begin{array}{c}
1 \\
\downarrow b \\
2 \\
\downarrow c \\
3 \\
\uparrow a
\end{array} \\
W = abc & \Rightarrow W' = 0
\end{align*}
Ginzburg’s dg algebra

$(Q, W)$ a quiver with potential ($Q$ may have loops and 2-cycles)

**Definition (Ginzburg)**

$\tilde{Q} = \text{quiver with } \tilde{Q}_0 = Q_0 \text{ and }$

- the arrows of $Q$ in degree 0,
- a new arrow $a^* : j \to i$ of degree $-1$ for each $a : i \to j$ of $Q$,
- a loop $t_i : i \to i$ of degree $-2$ for each vertex $i$ of $Q$.

$\Gamma = \Gamma(Q, W) = \hat{k}\tilde{Q}$ endowed with the unique $d$ of degree 1 such that

- $d(a) = 0$ for each arrow $a$ of $Q$,
- $d(a^*) = \partial_a W$ for each arrow $a$ of $Q$,
- $d(t_i) = e_i(\sum_{a \in Q_1}[a, a^*])e_i$ for each vertex $i$ of $Q$. 

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Example of Ginzburg’s dg algebra

Quiver with potential

\[ Q : \begin{array}{ccc}
    & 1 & \\
    b & & a \\
    \downarrow & \downarrow & \uparrow \\
    2 & c & 3 \\
\end{array} , \ W = abc , \]

Ginzburg dg algebra

\[ \begin{array}{ccc}
    & 2 & \\
    b^* & a^* & \\
    \downarrow & \downarrow & \uparrow \\
    1 & b & 3 \\
\end{array} , \ d(a^*) = bc , \ d(t_1) = cc^* - b^*b , \ldots \]

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Remark

$\Gamma$ is an enhancement of

$$H^0\Gamma = \widehat{kQ}/(\partial_a W | a \in Q_1) = \text{Jacobi algebra of } (Q, W).$$

Definition

- $\mathcal{D}\Gamma =$ derived category of $\Gamma$
  (objects: differential graded $\Gamma$-modules),
- $\text{per } \Gamma =$ perfect derived category = closure of $\Gamma_{\Gamma}$ under shifts, extensions, direct summands,
- $\mathcal{D}^b\Gamma =$ bounded derived category
  $= \{ M \in \mathcal{D}\Gamma \mid \dim H^* M < \infty \}.$
Remarks

- $\Gamma$ is homologically smooth, i.e.

\[ \Gamma \in \text{per}(\Gamma^e), \text{ where } \Gamma^e = \Gamma \hat{\otimes} \Gamma^{op}. \]

- Therefore, we have $\mathcal{D}^b \Gamma \subset \text{per} \Gamma$.

- $\Gamma$ is 3-Calabi Yau (as a bimodule), i.e.

\[ \text{RHom}_{\Gamma^e}(\Gamma, \Gamma^e) \sim \rightarrow \Gamma[-3] \text{ in } \mathcal{D}(\Gamma^e). \]

- Therefore, $\mathcal{D}^b \Gamma$ is 3-CY as a triang. cat., i.e.

\[ D \text{Hom}(L, M) = \text{Hom}(M, L[3]) \]

for all $L, M$ in $\mathcal{D}^b \Gamma$, where $D = \text{Hom}_k(?, k)$.
The main theorem

$Q$ w/o loops or 2-cycles, $i$ a vertex of $Q$, $\Gamma' = \Gamma(\mu_i(Q,W))$.

Theorem (-Yang)

There is a canonical equivalence $\mathcal{D}\Gamma' \sim \mathcal{D}\Gamma$ taking $P'_j$ to $P_j$ for $j \neq i$ and $P'_i$ to $\text{cone}(P_i \to \bigoplus_{i \to j} P_j)$. It induces equivalences in $\text{per}$ and $\mathcal{D}^b$.

Remarks

- This improves on a result by Jorge Vitória (0709.3939v2), cf. also Mukhopadhyay-Ray, Berenstein-Douglas, . . .
- The canonical $t$-structure on $\mathcal{D}^b\Gamma'$ yields a new $t$-structure on $\mathcal{D}^b\Gamma$. If $W$ is generic, we get lots of new $t$-structures on $\mathcal{D}^b\Gamma$ . . .
Links to cluster theory

NC 3-CY variety (KS)

\[ \begin{array}{c}
0 \rightarrow \mathcal{D}^b \Gamma \rightarrow \text{per} \Gamma \\
\rightarrow \text{KS (soon)}
\end{array} \]

coeff. alg. (FZ)
\[ \chi \text{-space (FG)} \]

KS=Kontsevich-Soibelman, FZ=Fomin-Zelevinsky, FG=Fock-Goncharov

generalized cluster cat.: 2-CY

\[ \begin{array}{c}
\rightarrow \mathcal{C}_\Gamma \\
\rightarrow (\star)
\end{array} \]

cluster alg. (FZ)
\[ \mathcal{A} \text{-space (FG)} \]
Illustration of the link \((\ast)\) from \(C_\Gamma\) to the cluster algebra:

**Theorem (-Caldero)**

Suppose \(Q\) does not have any oriented cycles. Then

\[
\{\text{rigid indecomp. objects of } C_\Gamma\} \overset{\sim}{\longrightarrow} \{\text{cluster variables of the cl. alg. } A_Q\}.
\]

Application of these techniques:

**Theorem (K)**

The periodicity conjecture (Al. Zamolodchikov, 1991) for pairs of Dynkin diagrams (Kuniba-Nakanishi, 1992) is true.
Quiver mutation is derived equivalence of Ginzburg algebras.
The periodicity conjecture is true.
Google ‘quiver mutation’!
The periodicity conjecture

**Notation**
- \( \Delta \) and \( \Delta' \) two Dynkin diagrams with vertex sets \( J, J' \),
- \( h, h' \) their Coxeter numbers, \( C, C' \) their Cartan matrices,
- \( Y_{i,i',t} \) variables where \( i \in J, i' \in J', t \in \mathbb{Z} \).

**Y-system associated with \((\Delta, \Delta')\)**

\[
Y_{i,i',t-1} Y_{i,i',t+1} = \frac{\prod_{j \neq i} (1 + Y_{j,i',t})^{-c_{ij}}}{\prod_{j' \neq i'} (1 + Y_{i,j',t}^{-1})^{-c'_{i'j'}}}.
\]

**Periodicity conjecture (Al. Zamolodchikov, Kuniba-Nakanishi)**

All solutions to this system are periodic of period dividing \( 2(h + h') \).