

CLUSTER ALGEBRAS AND QUIVER REPRESENTATIONS

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ABSTRACT. These notes contain an extended abstract and references for a minicourse given at the Summer School ‘Microlocal and Geometric Methods in Representation Theory’ (Schloss Reinsburg, July 2006) of the International Research Training Group Metz–Paderborn.

1. HISTORY

Cluster algebras were invented by Fomin-Zelevinsky in 2000, *cf.* [23], with the aim to find a combinatorial approach to the objects constructed by Lusztig in two series of articles: the first one on canonical bases in quantum groups, *cf.* [45], ..., the second one on total positivity in algebraic groups, *cf.* [46], Despite great progress in the last few years, we are still far from these initial goals. However, in the meantime, cluster algebras themselves have turned into a thriving subject, thanks notably to their many links to other fields. These include Poisson geometry [35] [36], Teichmüller spaces [21] [20], integrable systems [26] [53], combinatorics and combinatorial polyhedra [17] [16] [44], and, last not least, the representation theory of quivers and finite-dimensional algebras [3] [12] [14] [34] [48] [49] [8] Here, the idea is that of categorification: One tries to interpret cluster algebras as combinatorial (perhaps K -theoretic) invariants associated with categories of representations. The lift from combinatorial objects to categories provides rich additional structure, which sometimes allows one to prove statements which seem out of reach of the purely combinatorial methods.

We refer to [55] [54] for introductions to cluster algebras and to [25] for a survey of the first three years of their six-year-long history.

2. REMINDER ON FINITE ROOT SYSTEMS

We recall the definition and the classification of finite (reduced, irreducible) root systems from [51]: one proceeds from a choice of positive roots to construct successively the simple roots, the Cartan matrix and the Dynkin diagram.

3. AN INFORMAL INTRODUCTION TO CLUSTER-FINITE CLUSTER ALGEBRAS

We present a brief description of cluster algebras [23] and an informal statement of the classification theorem of cluster-finite cluster algebras [24]. We then construct the cluster algebra (without coefficients) associated to a Dynkin diagram in three examples: A_2 , A_3 and G_2 . On these examples, we observe the main properties proved in [23] and [24]: The Laurent phenomenon, the periodicity and the bijection between non-trivial cluster variables and positive roots given by comparing the

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exponents of the denominator with the coefficients in the expansion of a positive root as a linear combination of simple roots. We make extensive use of the *knitting algorithm*, whose invention in representation theory goes back to the late seventies, cf. e.g. [29].

4. GEOMETRIC CLUSTER ALGEBRAS

4.1. Geometric cluster algebras without coefficients. A quiver is an oriented graph. We define a seed to be a pair (R, u) formed by a quiver R with vertices $1, \dots, n$, without loops and without 2-cycles, and a free generating set $u = \{u_1, \dots, u_n\}$ of the field of fractions of the field of polynomials with integer coefficients in n indeterminates $x_1, \dots, x_n$. We define mutation of quivers and mutation of seeds. Then, for a given quiver Q without loops or 2-cycles, we define the clusters, the cluster variables and the cluster algebra as in [7] and the seed exchange graph as in section 4 of [22]. We point out the link between quiver mutation and matrix mutation. We illustrate these constructions on the examples A_2 , A_3 and the Kronecker quiver. We note that the knitting algorithm amounts to performing mutations only at sources and at sinks. Therefore, the knitting algorithm does not yield all clusters (except in rank 2) and does not yield all cluster variables (except for a Dynkin quiver). We illustrate this on a quiver of type $A_{2,1}$. We state Fomin-Zelevinsky's criterion [24] for a quiver Q to yield only finitely many cluster variables: This is the case if and only if the quiver is mutation-equivalent to a Dynkin quiver, whose underlying graph Δ is then determined uniquely and called the cluster type of Q . Good criteria for a quiver to be cluster-finite are now known: in terms of quadratic forms [5] or in terms of forbidden minimal cluster-infinite subquivers [52], closely linked to minimal representation-infinite algebras [11]. In general, mutation equivalence is an extremely complex relation. We illustrate this using the quiver mutation application [42]. No normal form for quivers under mutation equivalence is known. For example, it is unknown when a cyclic quiver with three vertices linked respectively by r , s and t arrows is mutation-equivalent to a quiver without oriented cycles.

4.2. Geometric cluster algebras with coefficients. Starting from a bipartite quiver without loops or 2-cycles (or equivalently, a rectangular integer matrix with antisymmetric principal part), we construct the associated geometric cluster algebra with coefficients as in [7]. Following [23] we show that such algebras occur in geometry as coordinate algebras of varieties: $SL(2)$ (cluster type A_1), the cone on the Grassmannian of planes in $(n+3)$ -space (cluster type A_n) and, following [35], the big cell of the Grassmannian of k -dimensional subspaces in n -space (possible cluster types A_n , D_4 , D_6 , E_8 or cluster-infinite). Here, the resulting cluster structure is *compatible* with the Sklyanin Poisson bracket on the Grassmannian, i.e. the variables in each cluster form a log-canonical local coordinate system. More generally, as shown in [35], for each bipartite quiver whose associated matrix is of full rank, the cluster variety (obtained by glueing tori corresponding to the clusters) admits compatible Poisson structures. The field of rational functions central for this bracket is generated by certain monomials in the coefficients. If the matrix associated with the bipartite quiver is not of full rank, there is no compatible Poisson bracket in general, but instead, one can find compatible 2-forms. This is studied in detail in [36], in view of possible applications to Teichmüller spaces.

Finally, we exhibit the cluster algebra structure of the subgroup of unipotent upper triangular matrices in $SL(n)$. On this example, we point out the link with total positivity [46] and with Lusztig's dual semicanonical basis [47]. More generally, the coordinate algebra of a maximal unipotent subgroup in any simple connected simply connected complex algebraic group has a canonical cluster algebra structure. This cluster algebra and its representation-theoretic pendant, the category of representations of the corresponding preprojective algebra, have been thoroughly studied by Geiss-Leclerc-Schröer in [34] [33] [31] [30] [32].

We point out that many other examples of geometric cluster algebras with coefficients, in particular coordinate algebras of double Bruhat cells, are contained in [6].

5. QUIVER REPRESENTATIONS AND GABRIEL'S THEOREM

Several books on the representation theory of quivers and finite-dimensional algebras are now available: The books [2] and [1] are written for the novice, whereas [50] and [28] are research monographs.

We introduce the notion of representation of a (finite) quiver and illustrate it on examples. We introduce morphisms of representations, subrepresentations, direct sums, simple representations and indecomposable representations. We state the decomposition theorem (also known as Azumaya-Krull-Remak-Schmidt theorem). We introduce the Tits form and state Gabriel's theorem [27] for representation-finite quivers. We indicate on one example how to treat the case of non-simply laced Dynkin diagrams and refer to [18] [19] for the corresponding representation theory of species. We sketch the definition of a tame (resp. wild) algebra and state the extension of Gabriel's theorem to tame quivers due to Nazarova, Gelfand-Ponomarev, Donovan-Freislich and Dlab-Ringel.

We give a first qualitative link between representation-theory and cluster algebras: For a quiver Q without oriented cycles, let μ be the supremum of the multiplicities of the arrows of the quivers in the mutation class of Q . Then Q is representation-finite iff $\mu = 1$, it is tame iff $\mu = 2$ and it is wild iff $\mu = \infty$. Moreover, as shown in [10], the mutation-class of Q is finite iff Q is representation-finite or tame (if this seems confusing, consider the case of the Kronecker quiver).

Using [42] we illustrate the trichotomy finite-tame-wild on the example of the three quivers whose representations correspond essentially to configurations of 3, 4 or 5 subspaces in a vector space.

6. CLUSTER VARIABLES FROM QUIVER REPRESENTATIONS

A theorem by Fomin-Zelevinsky [24] states that there is a canonical bijection between positive roots of a Dynkin quiver Q and the non trivial cluster variables of the corresponding cluster algebra. By combining this with Gabriel's theorem, we obtain a bijection $V \mapsto X_V$ from the set of isoclasses of indecomposable representations of Q to the set of non-trivial cluster variables. Using [42] we illustrate this bijection on the examples A_2 , A_3 , D_4 and point out the link between the number of terms in the numerator of X_V and the 'number' of subrepresentations of V . This motivates the Caldero-Chapoton formula [12]: If V is a representation of Q of

dimension vector d , then

$$CC(V) = \frac{1}{x_1^{d_1} x_2^{d_2} \dots x_n^{d_n}} \sum_{0 \leq e \leq d} \chi(\text{Gr}_e(V)) \prod_{i=1}^n x_i^{\sum_{j \rightarrow i} e_j + \sum_{i \rightarrow j} (d_j - e_j)}.$$

Caldero-Chapoton prove in [12] that if V is an indecomposable representation of a Dynkin quiver, then $CC(V)$ is a cluster variable and equals X_V . We illustrate the theorem on examples for A_2 and D_4 .

More generally, it is now known that for an arbitrary quiver without oriented cycles, the Caldero-Chapoton map yields a bijection from the set of isoclasses of rigid indecomposable representations (*i.e.* those with $\text{Ext}^1(V, V) = 0$) to the set of non-trivial cluster variables. This was proved in [15] using the Caldero-Chapoton formula as well as results from [9] [4] [3] [48]. A slightly different proof was given in the appendix to [9]. A new and original proof for extended Dynkin quivers is due to A. Hubery [39], who establishes a link with Green's theorem on the comultiplication in the Hall algebra.

7. SKETCH OF PROOF OF CALDERO-CHAPOTON'S THEOREM

The aim is to show that $CC(V)$ is a cluster variable (showing that it is X_V is then not hard). The idea for this is to show that the map $V \mapsto X_V$ satisfies the initial conditions and the recurrence relation for a suitable version of the knitting algorithm. One first shows that $CC(V) = X_V$ holds if V is an indecomposable projective module. Then, one shows that CC is multiplicative on split short exact sequences and 'almost multiplicative' on almost split short exact sequences [2]. Finally, one uses the knitting algorithm for representations to conclude that $CC(V)$ is a cluster variable for each indecomposable representation V .

8. CLUSTER ALGEBRAS FROM CLUSTER CATEGORIES

8.1. The repetition and the derived category. We formalize the definition of the repetition $\mathbb{Z}Q$ of a quiver Q without oriented cycles. We define the mesh category associated to the repetition and state Happel's theorem [37] [38] according to which the mesh category is isomorphic to the category of indecomposables of the bounded derived category of Q if Q is a Dynkin quiver. We extend the map $V \mapsto X_V$ to the derived category so that it becomes multiplicative on split exact sequences and 'almost multiplicative' on almost split triangles. The resulting map takes the shifted indecomposable projectives to the trivial cluster variables. It is periodic with respect to a certain periodicity automorphism, namely $F = \tau^{-1}S$, where S is the shift functor for complexes. This motivates the definition of the cluster category.

8.2. Clusters from the cluster category. Following [3] we define the cluster category as the orbit category of the bounded derived category under the action of the automorphism $F = \tau^{-1}S$, where S is the shift functor. In type A_n , an isomorphic category was independently defined in [13] using the diagonals of an $(n+3)$ -gon as objects. A theorem of [3] states that for a Dynkin quiver, the isoclasses of indecomposables of the cluster category are in bijection with all the cluster variables (trivial and non trivial) and that under this bijection, the clusters correspond exactly to the *cluster-tilting* subsets of indecomposables, *i.e.* the sets of n indecomposables $T_1, \dots, T_n$ with $\text{Ext}^1(T_i, T_j) = 0$ for all i, j . We illustrate this

on the example A_3 and then state how to obtain the quiver R in a seed (R, u) whose cluster u corresponds to $T_1, \dots, T_n$. It follows that the cluster determines the seed. This statement was extended from Dynkin quivers to arbitrary quivers without oriented cycles by Buan-Marsh-Reiten-Todorov [9], who thus proved a conjecture by Fomin-Zelevinsky.

8.3. Relations from triangles. The derived category is triangulated and the cluster category admits a canonical triangulation for which the projection functor becomes a triangle functor [43]. For objects L, M, N of the cluster category of a Dynkin quiver, denote by $\text{Ext}^1(L, M)$ the set of morphisms from L to M and by $\text{Ext}^1(L, M)_N$ the subset formed by the morphisms $\varepsilon : L \rightarrow M$ such that there is a triangle

$$M \rightarrow N \rightarrow L \rightarrow SM$$

whose third arrow is ε . It is shown in [14] that if $\text{Ext}^1(L, M)$ does not vanish, we have the relation

$$X_L X_M = \sum_N \frac{\chi(\mathbb{P}\text{Ext}^1(L, M)_N) + \chi(\mathbb{P}\text{Ext}^1(M, L)_N)}{\dim \text{Ext}^1(L, M)}.$$

It yields a sufficient set of relations (including those of the form $xx' = 1 + y$), a basis and structure constants if Q is Dynkin. The key ingredients of the proof are the Caldero-Chapoton formula and the fact that the cluster category is 2-Calabi-Yau, *i.e.* we have the functorial isomorphisms

$$D\text{Ext}^1(L, M) \simeq \text{Ext}^1(M, L),$$

where D is the duality over the ground field. The formula has been adapted to the setting of modules over preprojective algebras in [32]. It can be extended to the case where Q does not have oriented cycles if, inspired by [32], we slightly modify our viewpoint and the notations:

$$X_L X_M = \frac{1}{\dim \text{Ext}^1(L, M)} \left(\int_{\varepsilon \in P(L, M)} X_{mt(\varepsilon)} d\chi + \int_{\varepsilon \in P(M, L)} X_{mt(\varepsilon)} d\chi \right)$$

where $mt(\varepsilon)$ denotes the middle term of the extension associated with ε and $P(L, M)$ the projectivization of $\text{Ext}^1(L, M)$. Here, as in [47], the integral $\int_Y f d\chi$ is defined if the integrand is a function on the complex algebraic variety Y all of whose fibers are constructible and such that only finitely many fibers have non zero Euler characteristic. In this case, the value of the integral is the sum of the Euler characteristics of the fibers of f .

8.4. Quivers with oriented cycles. Quivers which are not mutation-equivalent to quivers without oriented cycles are very frequent. The problem of obtaining a representation-theoretic interpretation of the cluster-algebra in this case is being studied in [40] [41]. In very rough terms, the strategy consists in replacing path algebras by ‘Calabi-Yau algebras’.

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