

THE ABELIAN HULL OF AN EXACT CATEGORY

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ABSTRACT. Using classical ideas we show how to construct a universal exact functor from an exact category to an abelian category.

1. THE CONSTRUCTION AND ITS PROPERTIES

1.1. A 2-universal functor. Let \mathcal{A} be a small exact category in the sense of Quillen [3] (the ‘obscure axiom’ is redundant as shown in [5, App. A]). Following ideas of Freyd [4] and Gabriel [2], we will construct an exact functor $\mathcal{A} \rightarrow \mathcal{U}\mathcal{A}$ to an abelian category such that for each abelian category \mathcal{B} , the induced functor between categories of exact functors

$$\mathcal{F}un_{ex}(\mathcal{U}\mathcal{A}, \mathcal{B}) \rightarrow \mathcal{F}un_{ex}(\mathcal{A}, \mathcal{B})$$

is an equivalence. We say that the functor $\mathcal{A} \rightarrow \mathcal{U}\mathcal{A}$ is *2-universal among exact functors from \mathcal{A} to abelian categories and exact functors*.

1.2. A word of warning. Suppose that \mathcal{A} is abelian and endowed with the class of all exact sequences. Then the functor $\mathcal{A} \rightarrow \mathcal{U}\mathcal{A}$ is an equivalence. However, if \mathcal{A} is abelian but endowed with a smaller class of exact sequences, the functor $\mathcal{A} \rightarrow \mathcal{U}\mathcal{A}$ will not be an equivalence in general.

1.3. The construction. We construct $\mathcal{U}\mathcal{A}$ in three steps. The first two steps are due to Freyd [4] and were further analyzed by Beligiannis [1]. They only use the additive structure of \mathcal{A} .

Step 1. Let $\mathbf{fp}(\mathcal{A}^{op})$ be the category of finitely presented covariant functors from \mathcal{A} to the category of abelian groups. Thus each object of $\mathbf{fp}(\mathcal{A}^{op})$ is a functor $X : \mathcal{A} \rightarrow \mathbf{Ab}$ admitting a presentation

$$\mathcal{A}(A_1, ?) \rightarrow \mathcal{A}(A_0, ?) \rightarrow X \rightarrow 0$$

for some morphism $A_0 \rightarrow A_1$ of \mathcal{A} . The category $\mathbf{fp}(\mathcal{A}^{op})$ is an (additive) category with cokernels. Hence $\mathbf{fp}(\mathcal{A}^{op})^{op}$ is an additive category with kernels. The Yoneda functor

$$F_1 : \mathcal{A} \rightarrow \mathbf{fp}(\mathcal{A}^{op})^{op}, A \mapsto \mathbf{Hom}(A, ?)$$

is 2-universal among additive functors from \mathcal{A} to additive categories with kernels and kernel preserving functors.

Step 2. If \mathcal{B} is an additive category with kernels, then, by a classical theorem [4] which goes back to Cartan, the category $\mathbf{fp}(\mathcal{B})$ is abelian and the Yoneda functor $\mathcal{B} \rightarrow \mathbf{fp}(\mathcal{B})$ is 2-universal among kernel preserving functors from \mathcal{B} to abelian categories and exact functors. It follows that the category

$$\mathcal{U}_0\mathcal{A} := \mathbf{fp}(\mathbf{fp}(\mathcal{A}^{op})^{op})$$

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is abelian and that the canonical functor $F : \mathcal{A} \rightarrow \mathcal{U}_0\mathcal{A}$ is 2-universal among additive functors from \mathcal{A} to abelian categories and exact functors.

Step 3. We will now use the exact structure of \mathcal{A} . It is easy to check that the functor

$$\mathcal{A} \rightarrow \mathcal{U}_0\mathcal{A}$$

is left exact. We force it to become an exact functor: Let \mathcal{N} be the smallest épaisse subcategory of $\mathcal{U}_0\mathcal{A}$ containing all cokernels

$$\text{cok}(FB \xrightarrow{Fd} FC)$$

where $d : B \rightarrow C$ runs through the deflations (=admissible epimorphisms) of \mathcal{A} . If we set

$$\mathcal{U}\mathcal{A} := (\mathcal{U}_0\mathcal{A})/\mathcal{N},$$

the canonical functor $\mathcal{A} \rightarrow \mathcal{U}\mathcal{A}$ clearly becomes exact. It is also clear that an exact functor $G : \mathcal{U}_0\mathcal{A} \rightarrow \mathcal{B}$ to an abelian category \mathcal{B} vanishes on \mathcal{N} iff its composition with $\mathcal{A} \rightarrow \mathcal{U}_0\mathcal{A}$ is exact. Combined with the 2-universal property of localization at \mathcal{N} , this shows that $\mathcal{A} \rightarrow \mathcal{U}\mathcal{A}$ is 2-universal among exact functors from \mathcal{A} to abelian categories with exact functors.

1.4. Properties of the universal functor. By construction, the universal functor $\mathcal{A} \rightarrow \mathcal{U}\mathcal{A}$ is exact. We claim that it is also fully faithful and that it detects exactness.

Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an additive fully faithful functor into an abelian category. We claim that its (exact) extension to $\mathcal{U}_0\mathcal{A}$ is still fully faithful. Indeed, by the 2-universal property of $\mathcal{A} \rightarrow \text{fp}(\mathcal{A}^{op})^{op}$, the functor $F : \mathcal{A} \rightarrow \mathcal{B}$ extends to a left exact functor $\text{fp}(\mathcal{A}^{op})^{op} \rightarrow \mathcal{B}$ unique up to isomorphism. This extension is faithful on \mathcal{A} , hence on the category of injectives of $\text{fp}(\mathcal{A}^{op})^{op}$ and hence on $\text{fp}(\mathcal{A}^{op})^{op}$. By the 2-universal property of $\text{fp}(\mathcal{A}^{op})^{op} \rightarrow \mathcal{U}_0\mathcal{A}$, there is a further extension $\mathcal{U}_0\mathcal{A} \rightarrow \mathcal{B}$. This extension is exact. Since it is faithful on the projectives, it is faithful on $\mathcal{U}_0\mathcal{A}$. Thus we obtain the fully faithful exact functor $\mathcal{U}_0\mathcal{A} \rightarrow \mathcal{B}$ extending F .

In particular, the exact extension $\mathcal{U}_0\mathcal{A} \rightarrow \text{Mod } \mathcal{A}$ of the Yoneda functor $\mathcal{A} \rightarrow \text{Mod } \mathcal{A}$ is fully faithful.

Let $\mathcal{E} \subset \text{Mod } \mathcal{A}$ be the subcategory of effaceable functors (cf. [5, App. A]). It is shown in [loc.cit.] that this is a localizing subcategory, that the quotient category is equivalent to the category of left exact functors on \mathcal{A} and that \mathcal{A} embeds in the quotient category in such a way that a sequence is a conflation in \mathcal{A} iff its image is exact in the quotient category, i.e. the functor $\mathcal{A} \rightarrow (\text{Mod } \mathcal{A})/\mathcal{E}$ detects exactness. As always, the quotient category is equivalent to the full subcategory of $\text{Mod } \mathcal{A}$ formed by the closed objects. An object of $\text{Mod } \mathcal{A}$ is closed with respect to \mathcal{E} iff it is a left exact functor.

It is easy to check that the image of \mathcal{N} under $\mathcal{U}_0\mathcal{A} \rightarrow \text{Mod } \mathcal{A}$ is contained in the subcategory \mathcal{E} of effaceable functors (but it is not clear whether objects of $\mathcal{U}_0\mathcal{A}$ with effaceable image always belong to \mathcal{N}). Now all left exact functors are closed with respect to \mathcal{E} in $\text{Mod } \mathcal{A}$. By the full faithfulness of $\mathcal{U}_0\mathcal{A} \rightarrow \text{Mod } \mathcal{A}$, we conclude that all objects FA , $A \in \mathcal{A}$, are closed in $\mathcal{U}_0\mathcal{A}$ with respect to \mathcal{N} . Thus the functor $\mathcal{A} \rightarrow \mathcal{U}\mathcal{A}$ is fully faithful. By construction, it is also exact. Its composition with $\mathcal{U}\mathcal{A} \rightarrow (\text{Mod } \mathcal{A})/\mathcal{E}$ is exact and detects exactness. Thus the functor $\mathcal{A} \rightarrow \mathcal{U}\mathcal{A}$ also detects exactness.

1.5. **Warning.** It is not clear whether the functor $\mathcal{U}\mathcal{A} \rightarrow (\text{Mod } \mathcal{A})/\mathcal{E}$ is faithful and it is not clear whether objects of $\mathcal{U}_0\mathcal{A}$ with effaceable image in $\text{Mod } \mathcal{A}$ belong to \mathcal{N} . Both are true, however, if \mathcal{A} has weak kernels.

2. APPENDIX : A DIAGRAMMATIC DESCRIPTION OF $\mathcal{U}_0\mathcal{A}$

The category $\mathcal{U}_0\mathcal{A}$ can be described more concretely: Let $\mathcal{U}_1\mathcal{A}$ be the category whose objects are the commutative squares

$$S : \begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ C & \xrightarrow{g} & D \end{array}$$

in \mathcal{A} . A *morphism* from the square S to the square S' is given by four morphisms $A \rightarrow A'$, \dots , $D \rightarrow D'$ such that in the resulting cube with front face S and back face S' , all faces except the top face commute and the top face commutes up to a morphism factoring through the morphism $A \rightarrow C$. Then the functor mapping the square S to

$$\text{cok}(\ker(f)) \rightarrow \ker(g)$$

induces an equivalence from $\mathcal{U}_1\mathcal{A}$ to $\mathcal{U}_0\mathcal{A}$.

Now suppose that \mathcal{A} has weak kernels. Then the above description can be simplified: Let \mathcal{U}_2 be the category whose objects are the complexes

$$A \rightarrow B \rightarrow C$$

over \mathcal{A} and whose morphisms are the morphisms of complexes modulo those that factor through sums of complexes of the following types:

$$A \rightarrow 0 \rightarrow 0, \quad A \xrightarrow{1} A \rightarrow 0, \quad 0 \rightarrow A \xrightarrow{1} A, \quad 0 \rightarrow 0 \rightarrow A.$$

Then the homology functor yields an equivalence from $\mathcal{U}_2\mathcal{A}$ onto $\mathcal{U}\mathcal{A}$.

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