INTRODUCTION TO ABELIAN AND DERIVED CATEGORIES

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Abstract. This is an account of three 1-hour lectures given at the Instructional Conference on Representation Theory of Algebraic Groups and Related Finite Groups, Isaac Newton Institute, Cambridge, 6–11 January 1997.

In section 1, we define abelian categories following Grothendieck [12]. We then characterize module categories among abelian categories. Finally we sketch a proof of Mitchell's full embedding theorem [25]: each small abelian category embeds fully and exactly into a module category.

We come to our main topic in section 2, where we define the derived category of an abelian category following Verdier [33] and the total right derived functor of an additive functor following Deligne [6].

We treat the basics of triangulated categories including K_0-groups and the example of perfect complexes over a ring in section 3.

Section 4 is devoted to Rickard's Morita theory for derived categories [29]. We give his characterization of derived equivalences, list the most important invariants under derived equivalence, and conclude by stating the simplest version of Broué's conjecture [2].

1. Abelian categories

1.1. **Definition and basic properties.** A $\mathbf{Z}$-*category* is a category $\mathcal{C}$ whose morphism sets $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ are abelian groups such that all composition maps

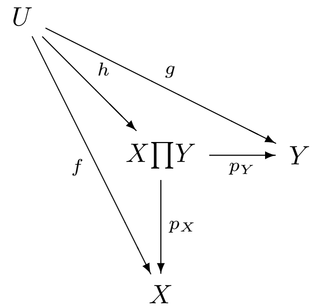
$$\mathrm{Hom}_{\mathcal{C}}(Y,Z)\times \mathrm{Hom}_{\mathcal{C}}(X,Y)\to \mathrm{Hom}_{\mathcal{C}}(X,Z)$$

are bilinear. For example, if R is a ring (associative, with 1) and $\mathcal{C}$ is the category having exactly one object, whose endomorphism set is R, then $\mathcal{C}$ is a $\mathbf{Z}$-category. A general $\mathbf{Z}$-category should be thought of as a 'ring with several objects' [25].

An *additive category* is a $\mathbf{Z}$-category $\mathcal{A}$ which has a zero object 0 (i.e. we have $\mathrm{Hom}_{\mathcal{A}}(0,X) = 0 = \mathrm{Hom}_{\mathcal{A}}(X,0)$ for all X) and such that all pairs of objects $X,Y\in\mathcal{C}$, admit a *product* in $\mathcal{C}$, i.e. an object $X\prod Y$ endowed with morphisms $p_X : X\prod Y\to X$ and $p_Y : X\prod Y \to Y$ such that the map

$$\mathrm{Hom}_{\mathcal{C}}(U,X\textstyle\prod Y)\to \mathrm{Hom}_{\mathcal{C}}(U,X)\times \mathrm{Hom}_{\mathcal{C}}(U,Y)\ ,\ h\mapsto (p_X\,h, p_Y\,h)$$

is bijective. In other words, the pair of maps (p_X, p_Y) is universal among all pairs of morphisms (f, g) from an object U to X respectively Y.



Universal properties of this type are most conveniently expressed in the language of *representable functors*: Recall that a contravariant functor F defined on a category $\mathcal{C}$ with values in the category of sets is *representable* if there is an object $Z \in \mathcal{C}$ and an isomorphism of functors

$$\varphi : \mathrm{Hom}_{\mathcal{C}}(?, Z) \xrightarrow{\sim} F.$$

Note that this determines the object Z uniquely up to canonical isomorphism. For example, the product $X \prod Y$ represents the product functor

$$\mathrm{Hom}_{\mathcal{C}}(?, X) \times \mathrm{Hom}_{\mathcal{C}}(?, Y).$$

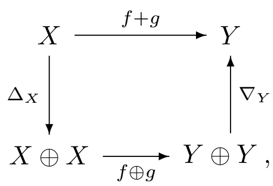
Dually, a covariant functor $G : \mathcal{C} \to \mathrm{Sets}$ is *corepresentable* if it is isomorphic to $\mathrm{Hom}_{\mathcal{C}}(Z, ?)$ for some $Z \in \mathcal{C}$.

Accordingly the *coproduct* $X \coprod Y$ is defined to corepresent the functor

$$\mathrm{Hom}_{\mathcal{C}}(X, ?) \times \mathrm{Hom}_{\mathcal{C}}(Y, ?)$$

(if this functor is corepresentable). We leave it to the reader as an exercise to check that in an additive category, the coproduct of any pair of objects exists and is canonically isomorphic to their product. We will henceforth write $X \oplus Y$ for both.

Note that in an additive category, the group law on $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is determined by the underlying category of $\mathcal{C}$. Indeed, for $f, g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, we have the following commutative diagram



where by definition the composition of the diagonal morphism Δ_X with both of the canonical projections $X \oplus X \to X$ is the identity of X and the codiagonal morphism ∇_Y is defined dually.

If R is a ring, the *category* $\mathrm{Mod}\, R$ of (right) R-modules is an additive category. So are its full *subcategories* $\mathrm{Free}\, R$ *and* $\mathrm{mod}\, R$ whose objects are the free, and the finitely presented R-modules, respectively.

Now let $\mathcal{A}$ be an additive category and $f : A \to B$ a morphism of $\mathcal{A}$. By definition, the *kernel* $\ker f$ represents the functor

$$\ker(f_* : \mathrm{Hom}_{\mathcal{A}}(?, A) \to \mathrm{Hom}_{\mathcal{A}}(?, B)).$$

This means that the kernel of f is defined only if this functor is representable, and in this case, the isomorphism from $\mathrm{Hom}_{\mathcal{A}}(?, \ker f)$ to the kernel functor corresponds to a morphism $i : \ker f \to A$ such that $f\,i = 0$ and i is universal with respect to this property. Dually, the *cokernel* $\mathrm{cok}\, f$ corepresents the functor

$$\ker(f^* : \mathrm{Hom}_{\mathcal{A}}(B, ?) \to \mathrm{Hom}_{\mathcal{A}}(A, ?)).$$

(note that this is the kernel and *not* the cokernel of a morphism between functors). Finally, one defines the *image* $\mathrm{im}\, f = \ker(B \to \mathrm{cok}\, f)$ and the *coimage* $\mathrm{coim}\, f = \mathrm{cok}(\ker f \to A)$. Now suppose that these four objects are well-defined for f. It is then easy to see that there is a unique morphism $\overline{f}$ making the following diagram commutative

$$\begin{array}{ccccccc} & A & & \xrightarrow{\quad f \quad} & & B & \\ \nearrow & & \searrow & & \nearrow & & \searrow \\ \ker f & & & \mathrm{coim}\, f \xrightarrow[\overline{f}]{} \mathrm{im}\, f & & & \mathrm{cok}\, f \end{array}$$

By definition [4] [12], *an abelian category* is an additive category $\mathcal{A}$ such that each morphism of $\mathcal{A}$ admits a kernel and a cokernel and that the canonical morphism $\overline{f}$ is invertible for each morphism f.

This definition implies in particular that in an abelian category a morphism f is invertible iff it is both, a monomorphism (i.e. $\ker f = 0$) and an epimorphism (i.e. $\mathrm{cok}\, f = 0$).

Clearly, if R is a ring, the category $\mathrm{Mod}\, R$ is abelian. If X is a topological space, the category $\mathrm{Sh}\, X$ of sheaves of abelian groups on X (cf. [10] [16]) is abelian as well. One of the principal aims of Grothendieck's study [12] of abelian categories was to develop a unified homology theory for these two classes of examples.

It may be helpful to point out two non-examples: If the ring R is not semi-simple, the category $\mathrm{Proj}\, R$ of projective modules over R is not abelian since in this case there exist morphisms between projective R-modules which do not admit a cokernel in $\mathrm{Proj}\, R$. But there are also examples of non-abelian categories where each morphism does admit a kernel and a cokernel: This holds for the category of filtered abelian groups

$$A = \bigcup_{n \in \mathbf{N}} A_n.$$

Indeed, if $0 \neq A$, the canonical morphism from A to the filtered group $A(1)$ defined by

$$A(1)_p = A_{p+1}$$

is monomorphic and epimorphic but not invertible.

A functor between abelian categories is *left exact* if it preserves kernels, *right exact* if it preserves cokernels, and *exact* if it is both right and left exact. Recall that a functor $F : \mathcal{A} \to \mathcal{B}$ is said to be *fully faithful* if it induces bijections

$$\mathrm{Hom}_{\mathcal{A}}(A, B) \to \mathrm{Hom}_{\mathcal{B}}(FA, FB)$$

for all objects $A, B \in \mathcal{A}$.

Theorem (Mitchell [24]**)** *Let $\mathcal{A}_0$ be an abelian category whose objects form a set (i.e. a* small *category). Then there is a ring R and a fully faithful exact functor*

$$F : \mathcal{A}_0 \to \mathrm{Mod}\, R.$$

This theorem, known as the 'full embedding theorem', allows us to deal with objects of an abelian category 'as if they were modules'. More precisely, any theorem about modules involving only a finite diagram and such notions as exactness, existence or vanishing of morphisms … holds true in any abelian category $\mathcal{A}$ (to deduce this from the theorem, construct a full small abelian subcategory $\mathcal{A}_0 \subset \mathcal{A}$ containing all the objects involved).

However, it is important to note that not all theorems about module categories carry over to arbitrary abelian categories. For example, the product of an arbitrary set-indexed family of exact sequences of modules is exact; but the analogous statement for sheaves is false, in general. This is not in contradiction with the full embedding theorem, since the functor $\mathcal{A}_0 \to \operatorname{Mod} R$ obtained may not commute with infinite products.

1.2. **Characterization of module categories and Morita equivalence.** Let R be a ring and $\mathcal{A} = \operatorname{Mod} R$ the category of R-modules. Then it is easy to check that $\mathcal{A}$ has the following properties

- It is *cocomplete*, i.e. for each set-indexed family $(M_i)_{i\in I}$ of objects of $\mathcal{A}$, there exists the coproduct $\bigoplus_{i\in I} M_i$ (which corepresents $\prod_{i\in I} \operatorname{Hom}_{\mathcal{A}}(M_i, ?)$).
- It has a *generator* $P = R$ (the free R-module of rank 1), i.e. for each $M \in \mathcal{A}$, there is an epimorphism $\bigoplus_I P \to M$ for some set I.
- The generator P is *projective*, i.e. the functor $\operatorname{Hom}_{\mathcal{A}}(P, ?) : \mathcal{A} \to \operatorname{Mod} \mathbf{Z}$ is exact.
- The generator P is *compact*, i.e. the functor $\operatorname{Hom}_{\mathcal{A}}(P, ?) : \mathcal{A} \to \operatorname{Mod} \mathbf{Z}$ commutes with arbitrary set-indexed coproducts.

This proves the necessity of the condition of the following

Theorem [7] [8] *Let $\mathcal{A}$ be an arbitrary abelian category and R a ring. Then $\mathcal{A}$ is equivalent to* $\operatorname{Mod} R$ *if and only if $\mathcal{A}$ is cocomplete and has a compact projective generator P with* $\operatorname{Hom}_{\mathcal{A}}(P, P) \cong R$.

To prove the sufficiency, one shows that the functor $F : \operatorname{Hom}_{\mathcal{A}}(P, ?) : \mathcal{A} \to \operatorname{Mod} R$ is an equivalence. In particular, we can take $\mathcal{A}$ to be a module category as well:

Corollary (Morita) *Let R and S be two rings. Then the following conditions are equivalent:*

i) *There is an equivalence of categories $F : \operatorname{Mod} R \to \operatorname{Mod} S$.*
ii) *There is an R-S-bimodule X such that the functor $? \otimes_R X : \operatorname{Mod} R \to \operatorname{Mod} S$ is an equivalence.*
iii) *There is a finitely generated projective S-module P such that P generates* $\operatorname{Mod} S$ *and R is isomorphic to* $\operatorname{Hom}_S(P, P)$.

The equivalence between i) and iii) follows from the theorem, once it is shown that a projective S-module P is compact iff it is finitely generated. This is left to the reader as an easy exercise. Clearly ii) implies i). To prove that iii) implies ii), one notes that P has a structure of R-S-bimodule and puts $X = P$. Then it is not hard to verify that ii) holds.

By definition, R is *Morita equivalent* to S if the conditions of the corollary hold. In the best known example, R is the ring of $n \times n$ matrices over S and P is S^n (realized as a set of row vectors on which R acts from the right).

1.3. **On the proof of the full embedding theorem.** The following sketch of the proof of the full embedding theorem is to give the reader an idea of some more advanced techniques of the theory of abelian categories. We follow Freyd [7].

The proof rests on the following

Theorem (Mitchell) *Let $\mathcal{A}$ be a cocomplete abelian category with a projective generator P. Then each small full abelian subcategory $\mathcal{A}_0 \subset \mathcal{A}$ admits a fully faithful exact functor $F : \mathcal{A}_0 \hookrightarrow \operatorname{Mod} R$ for some ring R.*

Note that the generator P is *not* supposed to be compact. For the proof, one chooses Q to be a large sum of copies of P; so large indeed that for each object A of $\mathcal{A}_0$ there exists an epimorphism $Q \to A$. Since P is a generator and $\mathcal{A}_0$ is small, this is possible. Now one takes $R = \operatorname{Hom}_{\mathcal{A}}(Q, Q)$ and checks that the restriction F of $\operatorname{Hom}_{\mathcal{A}}(Q, ?)$ to $\mathcal{A}_0$ is fully faithful (cf. [7, Theorem 4.44]).

This proof is still of the same level of difficulty as the proofs of the preceding section. Now, however, we will need some deeper results: As a first trial at 'embedding' $\mathcal{A}_0$, consider the Yoneda embedding

$$Y : \mathcal{A}_0^{op} \to \operatorname{Fun}(\mathcal{A}_0, \operatorname{Mod} \mathbf{Z}) \ , \ A \mapsto \operatorname{Hom}_{\mathcal{A}_0}(A, ?).$$

Here, $\operatorname{Fun}(\mathcal{A}_0, \operatorname{Mod} \mathbf{Z})$ denotes the category of additive functors from $\mathcal{A}_0$ to $\operatorname{Mod} \mathbf{Z}$ (note that this is indeed a category since $\mathcal{A}_0$ is small). Recall that $\mathcal{A}_0$ should be thought of as a 'ring with several objects' and accordingly, $\operatorname{Fun}(\mathcal{A}_0, \operatorname{Mod} \mathbf{Z})$ is viewed as the category of modules over this 'multi-ring'. From this viewpoint, we have already got quite close to our aim of embedding $\mathcal{A}_0$ in a module category. However, the Yoneda functor is *not exact* (only left exact). To remedy this, we observe that the $\operatorname{Hom}_{\mathcal{A}_0}(A, ?)$ are not arbitrary functors : they are *left exact*. We therefore restrict the domain of the Yoneda functor to the category $\operatorname{Lex} = \operatorname{Lex}(\mathcal{A}_0, \operatorname{Mod} \mathbf{Z})$ of left exact functors $\mathcal{A}_0 \to \operatorname{Mod} \mathbf{Z}$

$$Y : \mathcal{A}_0^{op} \to \operatorname{Lex}.$$

The crucial point of the proof is to show that the category of left exact functors is abelian [8]. It is then not hard to see that it is also cocomplete, has a generator (to wit, the direct sum of the functors $\operatorname{Hom}_{\mathcal{A}}(A, ?)$, $A \in \mathcal{A}_0$), and has exact filtered direct limits. In other words, it is a *Grothendieck category* (Grothendieck invented, but did not name, Grothendieck categories in [12]; cf. [28] for a comprehensive account of the subject). Now as a Grothendieck category, the category Lex is also complete and has an injective cogenerator. So we have embedded $\mathcal{A}_0^{op}$ in a complete abelian category with an injective cogenerator. Looking at this through a mirror we see that we have embedded $\mathcal{A}_0$ in Lex^{op}, a cocomplete abelian category with a projective generator. Now we obtain the required embedding $\mathcal{A}_0 \to \operatorname{Mod} R$ from the theorem above.

2. Derived categories and derived functors

Derived categories are a 'formalism for hyperhomology' [34]. Used at first only by the circle around Grothendieck they have now become wide-spread in a number of subjects beyond algebraic geometry, and have found their way into graduate text books [35], [17], [22], [16]. We refer to L. Illusie's account [15] for a brief history of the origins of derived categories.

In order to illustrate the relation between the language of classical homological algebra and that of derived categories, let us consider the example of the Lyndon-Hochschild-Serre spectral sequence: Recall that if G is a group, H a normal subgroup, and A a G-module, then this sequence reads as follows

$$E_2^{pq} = H^p(G/H, H^q(H,A)) \Rightarrow H^{p+q}(G,A). \tag{1}$$

The corresponding statement in the language of derived categories is

$$\mathbf{R}\,\mathrm{Fix}_{G/H} \circ \mathbf{R}\,\mathrm{Fix}_H = \mathbf{R}\,\mathrm{Fix}_G\ , \tag{2}$$

where the equality denotes a canonical isomorphism between functors defined on the derived category $\mathcal{D}^+ \operatorname{Mod} \mathbf{Z}G$ with values in $\mathcal{D}^+ \operatorname{Mod} \mathbf{Z}$ and $\mathbf{R}\,\mathrm{Fix}_G$ the total right derived functor of the fixed point functor $\mathrm{Fix}_G : \operatorname{Mod} \mathbf{Z}G \to \operatorname{Mod} \mathbf{Z}$ defined by

$$\mathrm{Fix}_G\, M = \{m \in M \mid gm = m\ ,\ \forall g \in G\}.$$

Of course, the composition formula (2) is based on the observation that $\mathrm{Fix}_{G/H} \circ \mathrm{Fix}_H = \mathrm{Fix}_G$. It is stronger than (1) in the sense that (1) can be derived from (2) by standard techniques [34]. The precise meaning of (2) will become clear below. To link the two formulas, we have to evaluate $\mathbf{R}\,\mathrm{Fix}_G$ at the module A. This is done by applying the functor Fix to an injective resolution

$$I^{\cdot} : 0 \to I^0 \to I^1 \to \dots$$

of A. By definition, $\mathbf{R}\,\mathrm{Fix}_G\, A$ is the complex thus obtained. The link between (1) and (2) is then the formula

$$H^n \mathbf{R}\,\mathrm{Fix}_G\, A = \mathbf{R}^n\,\mathrm{Fix}_G\, A = H^n(G,A)\ ,$$

where $\mathbf{R}^n\,\mathrm{Fix}_G$ is the n-th right derived functor of Fix_G in the sense of Cartan-Eilenberg [5].

2.1. **Definition of derived categories.** Let $\mathcal{A}$ be an abelian category (for example, the category $\operatorname{Mod} R$ of modules over a ring R). We denote by $\mathcal{CA}$ *the category of differential complexes*

$$A^\bullet = (\dots \to A^n \xrightarrow{d_A^n} A^{n+1} \to \dots)\ ,\ A^n \in \mathcal{A}\ ,\ n \in \mathbf{Z}\ ,\ d^2 = 0.$$

Recall that a morphism of complexes $f : A^\bullet \to B^\bullet$ is *null-homotopic* if $f^n = d_B\, h^n + h^{n+1} d_A$ for all $n \in \mathbf{Z}$ for some family of morphisms $h^n : A^n \to B^{n-1}$. Clearly, any composition gfe is null-homotopic if f is null-homotopic. The *homotopy category* $\mathcal{HA}$ has the same objects as $\mathcal{CA}$. Its morphisms from $A^\bullet$ to $B^\bullet$ are the classes of morphisms of complexes $f : A^\bullet \to B^\bullet$ modulo the null-homotopic morphisms.

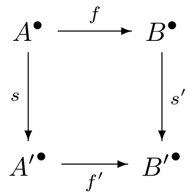
Note that the homology functor $H^n : \mathcal{CA} \to \mathcal{A}$ induces a well-defined functor $\mathcal{HA} \to \mathcal{A}$. We define a *quasi-isomorphism* to be a morphism $s : A^\bullet \to A'^\bullet$ of $\mathcal{HA}$ such that the induced morphisms $H^n s : H^n A^\bullet \to H^n A'^\bullet$ are invertible for all $n \in \mathbf{Z}$. We denote by Σ the class of all quasi-isomorphisms. Our aim is to define the derived category $\mathcal{DA}$ as the 'localization' of $\mathcal{HA}$ at the class Σ. Now by construction, $\mathcal{HA}$ is a $\mathbf{Z}$-category (even an additive category), and should be viewed as a 'ring with several objects'. The following lemma states that the analogues of the Ore conditions in the localization theory of rings hold for the class Σ (the assumption that the elements to be made invertible be non-zero divisors is weakened into condition c).

Lemma 1

a) *Identities are quasi-isomorphisms and compositions of quasi-isomorphisms are quasi-isomorphisms.*
b) *Each diagram*

$$A'^{\bullet} \xleftarrow{s} A^{\bullet} \xrightarrow{f} B^{\bullet} \quad (\textit{resp. } A'^{\bullet} \xrightarrow{f'} B'^{\bullet} \xleftarrow{s'} B^{\bullet})$$

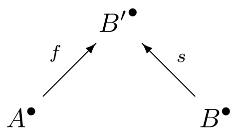
of $\mathcal{HA}$, where s (resp. s') is a quasi-isomorphism, may be embedded into a square



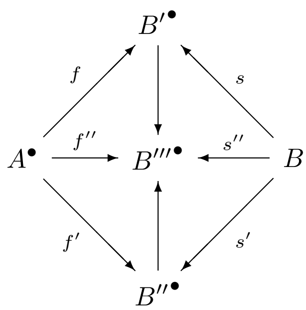
which commutes in $\mathcal{HA}$.
c) *Let f be a morphism of $\mathcal{HA}$. Then there is a quasi-isomorphism s such that $sf = 0$ in $\mathcal{HA}$ if and only if there is a quasi-isomorphism t such that $ft = 0$ in $\mathcal{HA}$.*

The lemma is proved for example in [17, 1.6.7]. Clearly condition a) would also be true for the pre-image of Σ in the category of complexes. However, for b) and c) to hold, it is essential to pass to the homotopy category. Historically [15], this observation was the main reason for inserting the homotopy category between the category of complexes and the derived category (the latter can also be defined directly as an 'abstract localization' [9] of the category of complexes at the pre-image of Σ).

Now we define [33] the *derived category* $\mathcal{DA}$ to be the localization of the homotopy category at the class of quasi-isomorphisms. This means that the derived category has the same objects as the homotopy category and that morphisms in the derived category from $A^{\bullet}$ to $B^{\bullet}$ are given by 'left fractions' "$s^{-1} \circ f$", i.e. equivalence classes of diagrams



where s is a quasi-isomorphism and a pair (f, s) is equivalent to (f', s') iff there is a commutative diagram of $\mathcal{HA}$



where s'' is a quasi-isomorphism. Composition is defined by

$$“t^{-1}g” \circ “s^{-1}f” = “(s't)^{-1} \circ g'f” ,$$

where $s' \in \Sigma$ and g' are constructed using condition b) as in the following commutative diagram of $\mathcal{HA}$

$$\begin{array}{ccccc} & & C''^{\bullet} & & \\ & {\scriptstyle g'}\nearrow & & \nwarrow{\scriptstyle s'} & \\ & B'^{\bullet} & & C'^{\bullet} & \\ {\scriptstyle f}\nearrow & & \nwarrow{\scriptstyle s} \quad {\scriptstyle g}\nearrow & & \nwarrow{\scriptstyle t} \\ A^{\bullet} & & B^{\bullet} & & C^{\bullet}. \end{array}$$

One can then check that composition is associative and admits the obvious morphisms as identities.

Using ‘right fractions’ instead of left fractions we would have obtained an isomorphic category (use lemma 1 b). We have a canonical functor $\mathcal{HA} \to \mathcal{DA}$ sending a morphism $f : A^{\bullet} \to B^{\bullet}$ to the fraction “$\mathbf{1}_B^{-1} f$”. This functor makes all quasi-isomorphisms invertible and is universal among functors with this property. The following lemma yields a more concrete description of some morphisms of the derived category. In part c) we use the following notation: An object $A \in \mathcal{A}$ is identified with the complex $\ldots \to 0 \to A \to 0 \to \ldots$ having A in degree 0. If $K^{\bullet}$ is an arbitrary complex, we denote by $K^{\bullet}[n]$ the complex with components $K^{\bullet}[n]^p = K^{n+p}$ and differential $d_{K[n]} = (-1)^n d_K$.

Lemma 2

a) *The category $\mathcal{DA}$ is additive and the canonical functors $\mathcal{CA} \to \mathcal{HA} \to \mathcal{DA}$ are additive.*

b) *If the complex $I^{\bullet}$ is left bounded (i.e. $I^n = 0$ for all $n \ll 0$) and has injective components, then the canonical morphism*

$$\mathrm{Hom}_{\mathcal{HA}}(A^{\bullet}, I^{\bullet}) \to \mathrm{Hom}_{\mathcal{DA}}(A^{\bullet}, I^{\bullet})$$

is invertible for all complexes $A^{\bullet}$. Dually, the canonical morphism

$$\mathrm{Hom}_{\mathcal{HA}}(P^{\bullet}, B^{\bullet}) \to \mathrm{Hom}_{\mathcal{DA}}(P^{\bullet}, B^{\bullet})$$

is invertible if $P^{\bullet}$ is right bounded with projective components and $B^{\bullet}$ is any complex.

c) *For all $A, B \in \mathcal{A}$, there is a canonical isomorphism*

$$\partial : \mathrm{Ext}^n_{\mathcal{A}}(A, B) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{DA}}(A, B[n]).$$

The calculus of fractions yields part a) of the lemma (cf. [9]). Part b) follows from [14, I, Lemma 4.5]. Part c) is in [14, I, §6].

Let us prove c) in the case where $\mathcal{A}$ has enough injectives (i.e. each object admits a monomorphism into an injective). In this case, the object B admits an injective resolution, i.e. a quasi-isomorphism $s : B \to I^{\bullet}$ of the form

$$\begin{array}{ccccccccccc} \ldots & \to & 0 & \to & B & \to & 0 & \to & 0 & \to & \ldots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \ldots & \to & 0 & \to & I^0 & \to & I^1 & \to & I^2 & \to & \ldots \end{array}$$

where the I^p are injective. Then, since s becomes invertible in $\mathcal{DA}$, it induces an isomorphism

$$\mathrm{Hom}_{\mathcal{DA}}(A, B[n]) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{DA}}(A, I^\bullet[n]).$$

By part b) of the lemma, we have the isomorphism

$$\mathrm{Hom}_{\mathcal{DA}}(A, I^\bullet[n]) \xleftarrow{\sim} \mathrm{Hom}_{\mathcal{HA}}(A, I^\bullet[n]).$$

Finally, the last group is exactly the n-th homology of the complex $\mathrm{Hom}_{\mathcal{A}}(A, I^\bullet)$, which identifies with $\mathrm{Ext}^n_{\mathcal{A}}(A, B)$ by (the most common) definition.

In two very special cases, we can directly describe the derived category in terms of the module category (cf. [17, Exercise I.18]): First suppose that $\mathcal{A} = \mathrm{Mod}\, k$, where k is a field (or more generally, suppose that $\mathcal{A}$ is semi-simple, i.e. $\mathrm{Ext}^1_{\mathcal{A}}(A, B) = 0$ for all $A, B \in \mathcal{A}$). Then the functor $A^\bullet \mapsto H^* A^\bullet$ establishes an equivalence between $\mathcal{DA}$ and the category of $\mathbf{Z}$-graded k-vector spaces. In the second case, suppose that $\mathcal{A}$ is hereditary (i.e. $\mathrm{Ext}^2_{\mathcal{A}}(A, B) = 0$ for all $A, B \in \mathcal{A}$). Then each object $A^\bullet$ of $\mathcal{DA}$ is quasi-isomorphic to the sum of the $(H^n A^\bullet)[-n]$, $n \in \mathbf{Z}$. Morphisms from $A^\bullet$ to $B^\bullet$ are then in bijection with the families (f_n, ε_n), $n \in \mathbf{Z}$, of morphisms $f_n : H^n A^\bullet \to H^n B^\bullet$ and extensions $\varepsilon_n \in \mathrm{Ext}^1_{\mathcal{A}}(H^n A^\bullet, H^{n-1} B^\bullet)$.

2.2. **Definition of derived functors.** The difficulty in finding a general definition of derived functors is to establish a framework which allows one to derive in full generality as many as possible of the pleasant properties found in the examples. This seems to be best achieved by Deligne's definition [6], which we will give in this section (compare with Grothendieck-Verdier's definition in [33]).

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $F : \mathcal{A} \to \mathcal{B}$ an additive functor (for example, the fixed point functor $\mathrm{Fix}_G : \mathrm{Mod}\, \mathbf{Z}G \to \mathrm{Mod}\, \mathbf{Z}$ from the introduction of this section). Then F clearly induces a functor $\mathcal{CA} \to \mathcal{CB}$ (obtained by applying F componentwise) and a functor $\mathcal{HA} \to \mathcal{HB}$. By abuse of notation, both will be denoted by F as well. We are looking for a functor $? : \mathcal{DA} \to \mathcal{DB}$ so as to make the following square commutative

$$\begin{array}{ccc} \mathcal{HA} & \xrightarrow{F} & \mathcal{HB} \\ \downarrow & & \downarrow \\ \mathcal{DA} & \xrightarrow{?} & \mathcal{DB} \end{array}$$

However, if F is not exact, it will not transform quasi-isomorphisms into quasi-isomorphisms and the functor in question cannot exist. What we will define then is a functor $\mathbf{R}F$ called the 'total right derived functor', which will be a 'right approximation' to an induced functor. More precisely, for a given $A^\bullet \in \mathcal{DA}$, we will not define $\mathbf{R}F(A^\bullet)$ directly but only the functor

$$(\mathbf{r}F)(?, A^\bullet) : (\mathcal{DB})^{op} \to \mathrm{Mod}\, \mathbf{Z}$$

which, if representable, will be represented by $\mathbf{R}F(A^\bullet)$. For $X^\bullet \in \mathcal{DB}$, we define $(\mathbf{r}F)(X^\bullet, A^\bullet)$ to be the set of 'left F-fractions', i.e. equivalence classes of diagrams

$$\begin{array}{ccccc} & FA'^\bullet & & A'^\bullet & \\ {\scriptstyle f}\nearrow & & & & \nwarrow{\scriptstyle s} \\ X^\bullet & & & & A^\bullet \end{array}$$

where f is a morphism of $\mathcal{DB}$ and s a quasi-isomorphism of $\mathcal{HA}$. Equivalence is defined in complete analogy with section 2.1. We say that $\mathbf{R}F$ is *defined at* $A^\bullet \in \mathcal{DA}$ if the functor $(\mathbf{r}F)(?, A^\bullet)$ is representable and if this is the case, then the value $\mathbf{R}FA^\bullet$ is defined by the isomorphism

$$\mathrm{Hom}_{\mathcal{DB}}(?, (\mathbf{R}F)(A^\bullet)) \xrightarrow{\sim} (\mathbf{r}F)(?, A^\bullet).$$

The link between this definition and more classical constructions is established by the

Proposition *Suppose that $\mathcal{A}$ has enough injectives and $A^\bullet$ is left bounded. Then $\mathbf{R}F$ is defined at $A^\bullet$ and we have*

$$\mathbf{R}FA^\bullet = FI^\bullet$$

where $A^\bullet \to I^\bullet$ is a quasi-isomorphism with a left bounded complex with injective components.

Under the hypotheses of the proposition, the quasi-isomorphism $A^\bullet \to I^\bullet$ always exists [17, 1.7.7]. Viewed in the homotopy category $\mathcal{HA}$ it is functorial in $A^\bullet$ since it is in fact the universal morphism from $A^\bullet$ to a left bounded complex with injective components. For example, if $A^\bullet$ is concentrated in degree 0, i.e. $A^\bullet = A$ for some $A \in \mathcal{A}$, then $I^\bullet$ may be chosen to be an injective resolution of A and we find that

$$H^n\mathbf{R}FA = (\mathbf{R}^nF)(A)\ ,$$

the n-th right derived functor of F in the sense of Cartan-Eilenberg [5].

We suggest it to the reader as an exercise to prove the identity

$$\mathbf{R}\,\mathrm{Fix}_{G/H} \circ \mathbf{R}\,\mathrm{Fix}_H = \mathbf{R}\,\mathrm{Fix}_G$$

of the introduction of this subsection, where all derived functors are defined on the full subcategory of left bounded complexes $\mathcal{D}^+ \operatorname{Mod} \mathbf{Z}G$ of $\mathcal{D} \operatorname{Mod} \mathbf{Z}G$.

3. Triangulated categories

3.1. **Definition and examples.** Let $\mathcal{A}$ be an abelian category (for example, the category $\operatorname{Mod} R$ of modules over a ring R). One can show that the derived category $\mathcal{DA}$ is abelian only if all short exact sequences of $\mathcal{A}$ split. This deficiency is partly compensated by the so-called triangulated structure of $\mathcal{DA}$, which we are about to define. In this section, to ease the notation, we will write X instead of $X^\bullet$ when speaking of the 'complex X'. Most of the material of this section first appears in [33].

A *standard triangle* of $\mathcal{DA}$ is a sequence

$$X \xrightarrow{Qi} Y \xrightarrow{Qp} Z \xrightarrow{\partial\varepsilon} X[1]\ ,$$

where $Q : \mathcal{CA} \to \mathcal{DA}$ is the canonical functor,

$$\varepsilon : 0 \to X \xrightarrow{i} Y \xrightarrow{p} Z \to 0$$

a short exact sequence of complexes, and $\partial\varepsilon$ a certain morphism of $\mathcal{DA}$, functorial in ε, and which lifts the connecting morphism $H^\bullet Z \to H^{\bullet+1}X$ of the long exact homology sequence associated with ε. More precisely, $\partial\varepsilon$ is the fraction $''s^{-1} \circ j''$ where j is the inclusion of the subcomplex Z into the complex $X'[1]$ with components $Z^n \oplus Y^{n+1}$ and differential

$$\begin{bmatrix} d_Z & p \\ 0 & -d_Y \end{bmatrix},$$

and $s : X[1] \to X'[1]$ is the morphism $\begin{bmatrix} 0 \\ i \end{bmatrix}$.

A *triangle* of $\mathcal{DA}$ is a sequence (u', v', w') of $\mathcal{DA}$ *isomorphic* to a standard triangle, i.e. such that we have a commutative diagram

$$\begin{array}{ccccccc} X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \\ x\downarrow & & \downarrow & & \downarrow & & \downarrow x[1] \\ X & \to & Y & \to & Z & \to & X[1]\,, \end{array}$$

where the vertical arrows are isomorphisms of $\mathcal{DA}$ and the bottom row is a standard triangle.

Lemma 3

T1 *For each object X, the sequence*

$$0 \to X \xrightarrow{\mathbf{1}} X \to 0[1]$$

is a triangle.

T2 *If (u, v, w) is a triangle, then so is $(v, w, -u[1])$.*

T3 *If (u, v, w) and (u', v', w') are triangles and x, y morphisms such that $yu = u'x$, then there is a morphism z such that $zv = v'y$ and $(x[1])w = w'z$.*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ x\downarrow & & y\downarrow & & z\downarrow & & \downarrow x[1] \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1]. \end{array}$$

T4 *For each pair of morphisms*

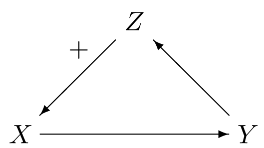
$$X \xrightarrow{u} Y \xrightarrow{v} Z$$

there is a commutative diagram

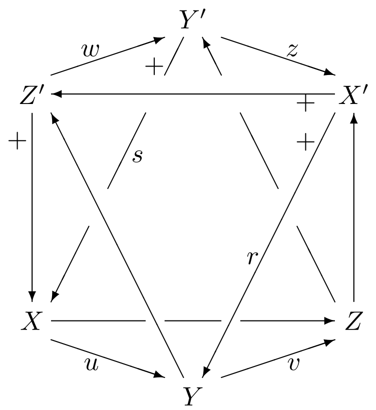
$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{x} & Z' & \to & X[1] \\ \| & & v\downarrow & & \downarrow w & & \| \\ X & \to & Z & \xrightarrow{y} & Y' & \xrightarrow{s} & X[1] \\ & & \downarrow & & \downarrow t & & \downarrow u[1] \\ & & X' & \xrightarrow{\mathbf{1}} & X' & \xrightarrow{r} & Y[1] \\ & & r\downarrow & & \downarrow & & \\ & & Y[1] & \xrightarrow{x[1]} & Z'[1]\,, & & \end{array}$$

where the first two rows and the two central columns are triangles.

Property T4 can be given a more symmetric form if we represent a morphism $X \to Y[1]$ by the symbol $X \xrightarrow{+} Y$ and write a triangle in the form



With this notation, the diagram of T4 can be written as an octahedron in which 4 faces represent triangles. The other 4 as well as two of the 3 squares 'containing the center' are commutative.



A *triangulated category* is an additive category $\mathcal{T}$ endowed with an autoequivalence $X \mapsto X[1]$ and a class of sequences (called triangles) of the form

$$X \to Y \to Z \to X[1]$$

which is stable under isomorphisms and satisfies properties T1 through T4.

Note that 'being abelian' is a property of an additive category, whereas 'being triangulated' is the datum of extra structure.

A whole little theory can be deduced from the axioms of triangulated categories. This theory is nevertheless much poorer than that of abelian categories. The main reason for this is the non-uniqueness of the morphism z in axiom T3.

We mention only two consequences of the axioms: a) They are actually self-dual, in the sense that the opposite category $\mathcal{T}^{op}$ also carries a canonical triangulated structure. b) Applying the functor $\mathrm{Hom}_{\mathcal{T}}(U, ?)$ or $\mathrm{Hom}_{\mathcal{T}}(?, V)$ to a triangle yields a long exact sequence of abelian groups. By the 5-lemma, this implies for example that if in axiom T3, two of the three vertical morphisms are invertible, then so is the third.

For later use, we record a number of examples of triangulated categories: If $\mathcal{A}$ is abelian, then not only the derived category $\mathcal{D}\mathcal{A}$ is triangulated but also the homotopy category $\mathcal{H}\mathcal{A}$. Here the triangles are constructed from componentwise split short exact sequences of complexes.

If $\mathcal{T}$ is a triangulated category, a *full triangulated subcategory of* $\mathcal{T}$ is a full subcategory $\mathcal{S} \subset \mathcal{T}$ such that $\mathcal{S}[1] = \mathcal{S}$ and that whenever we have a triangle (X, Y, Z) of $\mathcal{T}$ such that X and Z belong to $\mathcal{T}$ there is an object Y' of $\mathcal{S}$ isomorphic to Y. For example, the full subcategory $\mathcal{H}^b\mathcal{A}$ of bounded complexes (i.e. $X^p = 0$ for all $|p| \gg 0$) of $\mathcal{H}\mathcal{A}$ is a full triangulated subcategory, and so is the full subcategory $\mathcal{D}^b\mathcal{A}$ of bounded complexes of $\mathcal{D}\mathcal{A}$. One can show that this subcategory also identifies with the localization of $\mathcal{H}^b\mathcal{A}$ at the class of quasi-isomorphisms between bounded complexes. Note that the categories $\mathcal{H}\mathcal{A}$ and $\mathcal{H}^b\mathcal{A}$ are in fact defined for any additive category $\mathcal{A}$.

If $\mathcal{T}$ is a triangulated category and $\mathcal{X}$ a class of objects of $\mathcal{T}$, there is a smallest *strictly* (=closed under isomorphism) full triangulated subcategory $\mathrm{Tria}\,(\mathcal{X})$ of $\mathcal{T}$ containing $\mathcal{X}$. It is called the *triangulated subcategory generated by* $\mathcal{X}$. For example, the category $\mathcal{D}^b\mathcal{A}$ is generated by $\mathcal{A}$ (identified with the category of complexes concentrated in degree 0).

If R is a ring, a very important triangulated category is the full subcategory $\operatorname{per} R \subset \mathcal{D} \operatorname{Mod} R$ formed by the *perfect* complexes, i.e. the complexes quasi-isomorphic to bounded complexes with components in $\operatorname{proj} R$, the *category* of finitely generated projective R-modules. The subcategory $\operatorname{per} R$ may be intrinsically characterized [29, 6.3] as the subcategory of *compact objects* of $\mathcal{D} \operatorname{Mod} R$, i.e. objects X whose associated functor $\operatorname{Hom}(X, ?)$ commutes with arbitrary set-indexed coproducts. Note that by lemma 2, the canonical functor

$$\mathcal{H}^b \operatorname{proj} R \to \operatorname{per} R$$

is an equivalence so that the category $\operatorname{per} R$ is relatively accessible to explicit computations.

3.2. **Grothendieck groups.** Then *Grothendieck group* $K_0(\mathcal{T})$ of a triangulated category $\mathcal{T}$ is defined [13] as the quotient of the free abelian group on the isomorphism classes $[X]$ of objects of $\mathcal{T}$ divided by the subgroup generated by the relators

$$[X] - [Y] + [Z]$$

where (X, Y, Z) runs through the triangles of $\mathcal{T}$.

For example, if R is a right coherent ring, then the category $\operatorname{mod} R$ of finitely presented R-modules is abelian and the K_0-group of the triangulated category $\mathcal{D}^b \operatorname{mod} R$ is isomorphic to $G_0 R = K_0(\operatorname{mod} R)$ via the Euler characteristic:

$$[M^\bullet] \mapsto \sum_{i \in \mathbf{Z}} (-1)^i [H^i M^\bullet].$$

If R is any ring, the K_0-group of the triangulated category $\operatorname{per} R$ is isomorphic to $K_0 R$ via the morphism

$$[P^\bullet] \mapsto \sum_{i \in \mathbf{Z}} (-1)^i [P^i] \ , \ P^\bullet \in \mathcal{H}^b \operatorname{proj} R.$$

Note that this shows that any two rings with the 'same' derived category, will have isomorphic K_0-groups. To make this more precise, we need the notion of a triangle equivalence (cf. below)

3.3. **Triangle functors.** Let $\mathcal{S}, \mathcal{T}$ be triangulated categories. A *triangle functor* $\mathcal{S} \to \mathcal{T}$ is a pair (F, φ) formed by an additive functor $F : \mathcal{S} \to \mathcal{T}$ and a functorial isomorphism

$$\varphi X : F(X[1]) \xrightarrow{\sim} (FX)[1] \ ,$$

such that the sequence

$$FX \xrightarrow{Fu} FY \xrightarrow{Fv} FZ \xrightarrow{(\varphi X)\, Fw} (FX)[1]$$

is a triangle of $\mathcal{T}$ for each triangle (u, v, w) of $\mathcal{S}$.

For example, if $\mathcal{A}$ and $\mathcal{B}$ are abelian categories and $F : \mathcal{A} \to \mathcal{B}$ is an additive functor, one can show [6] that the domain of definition of the right derived functor $\mathbf{R}F$ is a strictly full triangulated subcategory $\mathcal{S}$ of $\mathcal{D}\mathcal{A}$ and that $\mathbf{R}F : \mathcal{S} \to \mathcal{D}\mathcal{B}$ becomes a triangle functor in a canonical way.

A triangle functor (F, φ) is a *triangle equivalence* if the functor F is an equivalence. We leave it to the reader as an exercise to define 'morphisms of triangle functors', and 'quasi-inverse triangle functors', and to show that a triangle functor admits a 'quasi-inverse triangle functor' if and only if it is a triangle equivalence [18].

4. Morita theory for derived categories

The following theorem is the precise analogue of the Morita theorem of section 1.2 in the framework of derived categories.

Let k be a commutative ring. A k-category is a category whose morphism spaces are k-modules such that the composition maps are bilinear (we have already encountered the case $k = \mathbf{Z}$ in section 1.1). A functor between k-categories is *k-linear* if it induces k-linear maps in the morphism spaces.

The following theorem is due to J. Rickard [29] [31]. A direct proof can be found in [21].

Theorem (Rickard) *Let A and B be k-algebras which are flat as modules over k. The following are equivalent*

i) *There is a k-linear triangle equivalence $(F, \varphi) : \mathcal{D}\operatorname{Mod} A \to \mathcal{D}\operatorname{Mod} B$.*

ii) *There is a complex of A-B-modules $X^\bullet$ such that the total left derived functor*

$$\mathbf{L}(? \otimes_A X^\bullet) : \mathcal{D}\operatorname{Mod} A \to \mathcal{D}\operatorname{Mod} B$$

is an equivalence.

iii) *There is a complex T of B-modules such that the following conditions hold*
   a) *T is perfect,*
   b) *T generates $\mathcal{D}\operatorname{Mod} B$ as a triangulated category with infinite direct sums,*
   c) *we have*

$$\operatorname{Hom}_{\mathcal{D}B}(T, T[n]) = 0 \text{ for } n \neq 0 \text{ and } \operatorname{Hom}_{\mathcal{D}B}(T, T) \cong A \ ;$$

Condition b) in iii) means that $\mathcal{D}\operatorname{Mod} B$ coincides with its smallest strictly full triangulated subcategory stable under forming arbitrary (set-indexed) coproducts.

The implication from ii) to i) is clear. To prove the implication from i) to iii), one puts $T = FA$ (where A is regarded as the free right A-module of rank one concentrated in degree 0). Since F is a triangle equivalence, it is then enough to check that the analogues of a), b), and c) hold for the object A of $\mathcal{D}\operatorname{Mod} A$. Properties a) and c) are clear. Checking property b) is non-trivial [21]. The hard part of the proof is the implication from iii) to ii). Indeed, motivated by the proof of the classical Morita theorem we would like to put $X = T$. The problem is that although A acts on T as an object of the derived category, it does not act on the individual components of T, so that T is not a complex of bimodules as required in ii). We refer to [19] for a direct solution of this problem.

Condition b) of iii) may be replaced by the condition that the direct summands of T generate $\operatorname{per} B$ as a triangulated category, which is easier to check in practice.

If the algebras A and B are even projective as modules over k, then the complex $X^\bullet$ may be chosen to be bounded and with components which are projective from both sides. In this case, the tensor product functor $? \otimes_A X^\bullet$ is exact and induces in the derived category a functor isomorphic to its total left derived functor.

By definition [31], the algebra A is *derived equivalent* to B if the conditions of the theorem hold. In this case, T is called a *tilting complex*, X a *two-sided tilting complex* and $\mathbf{L}(? \otimes_A X)$ a *standard equivalence.*

We know that any equivalence between module categories is given by the tensor product with a bimodule. Strangely enough, in the setting of derived categories, it is an open question whether all k-linear triangle equivalences are (isomorphic to) standard equivalences.

One of the main motivations for considering derived categories is the fact that they contain a large amount of information about classical homological invariants. The following theorem illustrates this point.

Theorem *If A is derived equivalent to B, then*

a) *there is a triangle equivalence* $\operatorname{per} A \xrightarrow{\sim} \operatorname{per} B$ *(and conversely, if there is such an equivalence, then A is derived equivalent to B);*
b) *if A and B are right coherent, there is a triangle equivalence* $\mathcal{D}^b \operatorname{mod} A \xrightarrow{\sim} \mathcal{D}^b \operatorname{mod} B$ *(and conversely, if A and B are right coherent and there is such an equivalence, then A is derived equivalent to B);*
c) *there is an isomorphism $K_0 A \xrightarrow{\sim} K_0 B$ and, if A and B are right coherent, an isomorphism $G_0 A \xrightarrow{\sim} G_0 B$;*
d) *the algebras A and B have isomorphic centers, isomorphic Hochschild homology and cohomology and isomorphic cyclic homology.*

The theorem is proved in [29], [31] and, for the case of cyclic homology, in [20].

A large number of derived equivalent (and Morita non equivalent) algebras is provided by Broué's conjecture [2], [3], which, in its simplest form, is the following statement

Conjecture (Broué) *Let k be an algebraically closed field of characteristic p and let G be a finite group with abelian p-Sylow subgroups. Then $B_{pr}(G)$ (the principal block of of kG) is derived equivalent to $B_{pr}(N_G(P))$, where P is a p-Sylow subgroup.*

We refer to [30] for a proof of the conjecture for blocks of group algebras with cyclic p-Sylows.

5. Notes on the references

Chapter I of Kashiwara-Schapira's monograph [17] is a concise and very well-written introduction to derived categories (readers may want to consult [14, Chapter I] or [11] to fill in some details). A modern text on homological algebra including derived categories is Weibel's book [35]. Gelfand-Manin [22] give a comprehensive overview of the same subject.

J. Rickard's paper [29] is the original reference for Morita theory for derived categories. The link with derived equivalences is established in [31]. Reference [21] contains direct proofs of the results of [29] and [31].

The articles [32], [26], and [1] by N. Spaltenstein, A. Neeman and M. Boekstedt contain important advances in the treatment of unbounded complexes. These have lead to an improved understanding [27], [23] of the original applications of derived categories in Grothendieck's duality theory [14].

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