CORRECTIONS TO ‘ON TRIANGULATED ORBIT CATEGORIES’

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1. Description of the triangulated hull

The description of the triangulated hull of the orbit category given in section 7 of [3] is probably not correct in general. One obtains a correct description by replacing the quotient $\mathcal{D}^b(B)/\operatorname{per}(B)$ by its full subcategory generated by the image of A (considered as a B-module via the projection $B \to A$). The error occurs in the last three lines of the proof of theorem 5.1 of [3]: It is true that each object of $\mathcal{D}^b(B)$ is an extension of two objects which lie in the image of $\mathcal{D}(\operatorname{mod} A)$ but it is not clear (and most probably not true) that these objects can be chosen to have bounded homology.

At least if k is algebraically closed, there is nevertheless a description of the triangulated hull of the orbit category of the form

$$\mathcal{D}_{fd}(B')/\operatorname{per}(B')$$

for a suitable dg algebra B', where $\mathcal{D}_{fd}(B')$ denotes the full subcategory of the derived category formed by all dg modules whose homology is of finite total dimension. One obtains B' as follows: Let $A^{\#}$ be the Koszul dual dg algebra of A (in the sense of [2]). Our hypotheses imply that $A^{\#}$ has its homology of finite total dimension and that it is derived Morita equivalent to A. Let Y be the dg $A^{\#}$-bimodule corresponding to the dg A-bimodule X. Then we can take $B' = A \oplus Y[-1]$ with the multiplication of the trivial extension.

2. On the Calabi-Yau property for higher cluster categories

I thank Alex Dugas for his message of February 26, 2009, where he points out that the proof of the Calabi-Yau property for higher cluster categories which is implicit in section 8.4 of [3] is incomplete. The following sections are meant to fill in the gap.

Moreover, as pointed out by Alex Dugas, in characteristic different from 2, it is not true that τ is isomorphic to the identity functor of the category of finitely generated projective modules over the algebra $\Lambda(L_n)$. This was erroneously claimed at the end of section 7.4 of [3].

2.1. **Functors induced in orbit categories, after Asashiba** [1]**.** Let k be a field, $\mathcal{C}$ a k-linear category, $F : \mathcal{C} \to \mathcal{C}$ an automorphism and $\mathcal{C}/F$ the *orbit category*: its objects are the same as those of $\mathcal{C}$ and, if X and Y are two objects, the space of morphisms from X to Y is

$$\bigoplus_{p\in\mathbb{Z}} \mathcal{C}(X, F^p Y).$$

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The composition of a morphism $f : Y \to F^q Z$ with a morphism $g : X \to F^p Y$ is given by $(F^p f) \circ g$. Let $\pi : \mathcal{C} \to \mathcal{C}/F$ be the projection functor. It is endowed with a canonical isomorphism of functors $\phi : \pi \circ F \to \pi$ given by $\pi X = \mathbf{1}_{FX}$ for each object X of $\mathcal{C}$.

Let $\mathcal{C}'$ be another k-linear category. A (left) *F-invariant functor* from $\mathcal{C}$ to $\mathcal{C}'$ is given by a pair (H, η), where $H : \mathcal{C} \to \mathcal{C}'$ is a k-linear functor and $\eta : HF \to H$ a functor isomorphism. A *morphism* of F-invariant functors $(H, \eta) \to (H', \eta')$ is given by a morphism of functors $\alpha : H \to H'$ such that the square

$$\begin{array}{ccc} HF & \xrightarrow{\eta} & H \\ {\scriptstyle \alpha F}\downarrow & & \downarrow{\scriptstyle \alpha} \\ H'F & \xrightarrow[\eta']{} & H' \end{array}$$

commutes. In this way, we obtain the *category* $\mathsf{inv}_F(\mathcal{C}, \mathcal{C}')$ of F-invariant functors. In particular, (π, ϕ) is an F-invariant functor and if we compose an arbitrary functor $K : \mathcal{C}/F \to \mathcal{C}'$ with π, it naturally becomes F-invariant. Moreover, an arbitrary morphism of functors $\alpha : K \to K'$ from $\mathcal{C}/F$ to $\mathcal{C}'$ yields an F-invariant morphism $\alpha\pi : K\pi \to K'\pi$. We thus obtain a functor

$$(\pi, \phi)^* : \mathsf{fun}_k(\mathcal{C}/F, \mathcal{C}') \to \mathsf{inv}_F(\mathcal{C}, \mathcal{C}')$$

and it is not hard to check that this functor is an *isomorphism of categories*. For example, if (H, η) is an F-invariant functor, it *induces* a k-linear functor $\overline{H}$ which takes a morphism $f : X \to FY$ to the composition

$$HX \xrightarrow{Hf} HFY \xrightarrow{\alpha FY} FHY.$$

An *F-equivariant functor* is a pair (H, α) formed by a k-linear functor $H : \mathcal{C} \to \mathcal{C}$ and an isomorphism of functors $\eta : HF \to HF$. A *morphism* of F-equivariant functors is defined in the natural way. The *composition* of F-equivariant functors (H, α) and (H', α') is defined as the functor HH' endowed with the composed isomorphism $(\alpha' H)(H'\alpha)$. If (H, α) is an F-equivariant functor, then $\pi H : \mathcal{C} \to \mathcal{C}/F$ becomes an F-invariant functor in a natural way and we obtain in fact functors

$$\mathsf{equ}_F(\mathcal{C}, \mathcal{C}) \to \mathsf{inv}_F(\mathcal{C}, \mathcal{C}/F) = \mathsf{fun}_k(\mathcal{C}/F, \mathcal{C}/F).$$

The composed functor takes an F-equivariant functor (H, α) to the functor $\overline{H}$ *induced* by (H, α). It takes a morphism $f : X \to FY$ to the composition

$$HX \xrightarrow{Hf} HFY \xrightarrow{\alpha FY} FHY.$$

For example, the functor F itself can be made into the F-equivariant functor $(F, \mathbf{1}_{F^2})$ and this F-equivariant functor induces a functor isomorphic to the identity functor in the orbit category. On the other hand, the functor $(F, -\mathbf{1}_{F^2})$ will not induce a functor isomorphic to the identity functor in general.

The composed functor

$$\mathsf{equ}_F(\mathcal{C}, \mathcal{C}) \to \mathsf{inv}_F(\mathcal{C}, \mathcal{C}/F) = \mathsf{fun}_k(\mathcal{C}/F, \mathcal{C}/F).$$

takes compositions of F-equivariant functors to the compositions of the k-linear functors which they induce.

For an automorphism $\varepsilon : F \to F$, let us *denote by* $\Delta(\varepsilon)$ the functor induced in $\mathcal{C}/F$ by the F-equivariant functor $(\mathbf{1}, \varepsilon)$. Then, the functor induced by $(F, -\mathbf{1}_{F^2})$

is isomorphic to $\Delta(-\mathbf{1}_F)$, since $(F, -\mathbf{1}_{F^2})$ is the composition of $(F, \mathbf{1}_{F^2})$ with $(\mathbf{1}, -\mathbf{1}_F)$.

2.2. **Self commutation morphisms.** Keep the hypotheses of the preceding section. Let F_1 and F_2 be two k-linear functors endowed with commutation morphisms

$$\phi_{ij} : F_i F_j \to F_j F_i \; , \; 1 \le i, j \le 2.$$

We assume that ϕ_{ii} is a scalar multiple of the identity morphism of $F_i F_i$, say $\phi_{ii} = \varepsilon_i \mathbf{1}_{F_i F_i}$, and that ϕ_{ij} is the inverse of ϕ_{ji} for all i, j. Let $F = F_1 F_2$. Then the ϕ_{ij} yield a natural autocommutation morphism $FF \to FF$, namely the composition

$$F_1F_2F_1F_2 \xrightarrow{F_1\phi_{21}F_2} F_1F_1F_2F_2 \xrightarrow{\phi_{11}*\phi_{22}} F_1F_1F_2F_2 \xrightarrow{F_1\phi_{12}F_2} F_1F_2F_1F_2.$$

Now since the ϕ_{ii} are multiples of the identity morphisms and ϕ_{12} is the inverse of ϕ_{21}, this composition is equal to $\varepsilon_1\varepsilon_2\mathbf{1}_{FF}$.

It follows that if we make F_1 and F_2 into F-equivariant functors using the ϕ_{ij}, then the functor induced in $\mathcal{C}/F$ by their composition is isomorphic to the functor induced by $(\mathbf{1}, \varepsilon_1\varepsilon_2\mathbf{1}_F)$, *i.e.* to $\Delta(\varepsilon_1\varepsilon_2\mathbf{1}_F)$. In other words, if $\overline{F}_i$ is the functor induced by the natural F-equivariant functor associated with F_i, then $\overline{F}_1\overline{F}_2$ is isomorphic to $\Delta(\varepsilon_1\varepsilon_2\mathbf{1}_F)$ and the inverse of $\overline{F}_1$ is isomorphic to the composition $\overline{F_2}\,\Delta(\varepsilon_1\varepsilon_2\mathbf{1}_F)$.

2.3. **Serre functors.** Keep the hypotheses of section 2.1. Assume moreover that $\mathcal{C}$ is Hom-finite and admits a Serre functor $S : \mathcal{C} \to \mathcal{C}$ endowed with trace maps

$$t_X : \mathcal{C}(X, SX) \to k.$$

Define a morphism $\sigma_F : FS \to SF$ by requiring that we have

$$t_{FX}((\sigma_F X) \circ Ff) = t_X(f)$$

for all morphisms $f : X \to SX$. Now define trace maps on $\mathcal{C}/F$ by requiring that $t_{\pi X}$ vanishes on all morphisms $X \to F^p SX$ for $p \neq 0$ and coincides with t_X on the morphisms $X \to SX$.

Lemma 2.1. a) *The F-equivariant functor (S, σ_F^{-1}) induces the Serre functor of $\mathcal{C}/F$ and the $t_{\pi X}$ are trace maps.*

b) *We have $\sigma_S = \mathbf{1}_{S^2}$ and, if $F = F_1F_2$ for two automorphisms F_1 and F_2, we have*

$$\sigma_F = (\sigma_{F_1}F_2)(F_1\sigma_{F_2}).$$

2.4. **The triangulated case and the Calabi-Yau property.** Keep the hypotheses of section 2.1. Assume moreover that $\mathcal{C}$ is endowed with the structure of a triangulated category with suspension functor Σ and that $F : \mathcal{C} \to \mathcal{C}$ is a triangle functor. Thus, F is endowed with an isomorphism of functors $\alpha : F\Sigma \to \Sigma F$ such that, for each triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X,$$

the sequence $(Fu, Fv, (\alpha X)(Fw))$ is a triangle. Thus, the pair (Σ, α^{-1}) is an F-equivariant functor. By definition, the *suspension functor* of the orbit category $\mathcal{C}/F$ is induced by this F-equivariant functor. In particular, we have a canonical isomorphism $\pi\Sigma \xrightarrow{\sim} \Sigma\pi$.

Now consider the case where $F = \Sigma$. Thus, we have to make Σ into a triangle functor. The canonical way to do this is to take $(\Sigma, -\mathbf{1}_{\Sigma^2})$. This pair is always a triangle functor, due to the fact that if (u, v, w) is a triangle, then so is $(\Sigma u, \Sigma v, -\Sigma w)$. On the other hand, the pair $(\Sigma, \mathbf{1}_\Sigma)$ *is not, in general, a triangle functor.* So we consider for F the triangle functor $(\Sigma, -\mathbf{1}_{\Sigma^2})$. Then, according to the above definition, the suspension functor of $\mathcal{C}/F = \mathcal{C}/\Sigma$ is induced by the Σ-equivariant functor $(\Sigma, -\mathbf{1}_{\Sigma^2})$ and, contrary to what one might have expected, this functor *is not, in general, isomorphic to the identity functor* but to the functor $\Delta(-\mathbf{1}_\Sigma)$.

On the other hand, consider for F the square of the triangle functor $(\Sigma, -\mathbf{1}_{\Sigma^2})$. This square is $(\Sigma^2, \mathbf{1}_{\Sigma^3})$. Now the suspension functor of $\mathcal{C}/F = \mathcal{C}/\Sigma^2$ is induced by $(\Sigma, \mathbf{1}_{\Sigma^3})$ and its square is induced by $(\Sigma^2, \mathbf{1}_{\Sigma^4})$. Thus the square of the suspension functor of $\mathcal{C}/\Sigma^2$ is indeed isomorphic to the identity functor, as one would expect.

Now assume that $\mathcal{C}$ is Hom-finite and admits a Serre functor S. Following Bondal-Kapranov and Van den Bergh, we canonically make S into a triangle functor. Surprisingly enough, in the notations of section 2.3, this canonical enhancement of S into a triangle functor is $(S, -\sigma_\Sigma)$. Notice the sign! Now fix an integer d and consider the triangle functors $F_1 = (S, -\sigma_\Sigma)$ and

$$F_2 = (\Sigma, -\mathbf{1}_{\Sigma^2})^{-d} = (\Sigma^{-d}, (-1)^d \mathbf{1}_{\Sigma^{1-d}}).$$

The structure of triangle functor on F_1 and F_2 yields commutation morphisms ϕ_{12}, ϕ_{21} and ϕ_{22}. Moreover, we endow F_1 with the identical autocommutation morphism ϕ_{11}. Then the ϕ_{ij} yield a commutation morphism between $S = F_1$ and $F = F_1 F_2$. This morphism is not the one defined in section 2.3 but differs from that one by the sign $(-1)^d$ because of the 'twist' in the triangle functor structure of S. Thus the functor induced in $\mathcal{C}/F$ by this equivariant functor is $\Delta((-1)^d) S_{\mathcal{C}/F}$, where $S_{\mathcal{C}/F}$ is the Serre functor. The functor induced by the equivariant enhancement of F_2 obtained from the ϕ_{ij} is the $(-d)$th power of the suspension functor $\Sigma_{\mathcal{C}/F}$. The functor induced by the equivariant enhancement of $F_1 F_2$ is isomorphic, according to section 2.2, to $\Delta(\varepsilon_1 \varepsilon_2 \mathbf{1}_F)$. Now $\varepsilon_1 = 1$ and $\varepsilon_2 = (-1)^d$. So we find that the composition of equivariant functors $F_1 F_2$ induces $\Delta((-1)^d)$ and we have

$$\Delta((-1)^d) S_{\mathcal{C}/F} \Sigma_{\mathcal{C}/F}^{-d} \xrightarrow{\sim} \Delta((-1)^d)$$

as k-linear functors $\mathcal{C}/F \to \mathcal{C}/F$. This implies that

$$S_{\mathcal{C}/F} \xrightarrow{\sim} \Sigma_{\mathcal{C}/F}^d$$

at least as k-linear functors. It should not be hard to upgrade this to an isomorphism of triangle functors.

2.5. **Another proof.** One could give another proof of the Calabi-Yau property of higher cluster categories by adapting the proof that Claire Amiot gives for the ordinary cluster category in her thesis (Corollary 4.4, page 102; the thesis is available at her home page). Amiot shows that the cluster category is triangle equivalent to a subquotient of the derived category of a dg algebra and uses a general theorem on the construction of Serre functors for quotients.

References

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