Comm. in Alg. **19** (1991), 379–417.

DERIVED CATEGORIES AND UNIVERSAL PROBLEMS

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Introduction

In this paper we search for a universal property of the (bounded positive) derived category of an exact category. We thereby hope to obtain a better understanding of the category of S-functors [10] starting from the derived category. Such S-functors play an essential rôle in the study of hearts of t-structures [2] [1], in J. Rickard's 'Morita theory for derived categories' [13] and in D. Happel's description of the derived category of a finite-dimensional algebra [7].

We briefly outline the contents of the paper. Let $\mathcal{A}$ be an exact category and $\mathcal{DA}$ the bounded positive derived category (cf. section 1). We start with what we consider the most natural approach, namely the question whether the canonical ∂-functor $\mathcal{A} \to \mathcal{DA}$ is universal among the ∂-functors from $\mathcal{A}$ to suspended categories $\mathcal{S}$. This, however, is not the case. We analyse the situation in section 1. Our conclusion is that the concepts of suspended category and S-functor alone do not provide rich enough a framework for an adequate treatment of the question. As a supplement, we propose 'towers' of suspended (resp. exact) categories: A tower $\mathcal{T}$ consists of a sequence

$$\mathcal{T}_0, \mathcal{T}_1, \ \dots \ , \mathcal{T}_n, \dots \ , \ n \in \mathbf{N}$$

of categories of the respective type and of a multitude of compatible functors joining them. A tower $\mathcal{A}^\wedge$ with base $\mathcal{A}^\wedge_0 = \mathcal{A}$ is associated with the exact category $\mathcal{A}$. The derived categories $\mathcal{D}(\mathcal{A}^\wedge_n)$ form a tower of suspended categories $\mathcal{D}\mathcal{A}^\wedge$ and the tower of ∂-functors $\mathcal{A}^\wedge \to \mathcal{D}\mathcal{A}^\wedge$ is universal (Theorem 2.6). Passing from the tower $\mathcal{D}\mathcal{A}^\wedge$ to its base $\mathcal{D}\mathcal{A}^\wedge_0 = \mathcal{D}\mathcal{A}$ we obtain new information on the problem of extending a ∂-functor $\mathcal{A} \to \mathcal{S}$ to an S-functor $\mathcal{D}\mathcal{A} \to \mathcal{S}$. For example, if Λ and Γ are derived equivalent rings [13], then the S-equivalence $\mathcal{H}_b P_\Gamma \to \mathcal{H}_b P_\Lambda$ constructed in [loc. cit.] is unique among the S-functors which map Γ to the tilting complex T and occur as the base of a tower of S-functors. These 'basic' S-functors apparently form a very large class. For example, all left derived functors are basic and all left S-adjoint functors of basic S-functors are basic (2.8). In fact, we do not know any example of a non-basic S-functor. Neither do we know an example of a suspended category which does not occur as the base of a tower of suspended categories.

The essential ingredient of the proof of the universal property of $\mathcal{D}\mathcal{A}^\wedge$ is the description of the derived category as a localisation of a category of presheaves. We refer to section 3 for details.

The description of $\mathcal{D}\mathcal{A}$ by presheaves also leads to a characterization of the 'construction $\mathcal{D}$', i.e. the 'hyperfunctor' (= 2-functor) which assigns $\mathcal{D}\mathcal{B}$ to $\mathcal{B}$ for each exact category $\mathcal{B}$ (section 4). In fact, $\mathcal{D}$ is universal among the constructions $\mathcal{D}'$ which map $J : \mathcal{I}\mathcal{B} \to \mathcal{M}\mathcal{B}$ to an S-equivalence and

$$0 \to \mathcal{B} \xrightarrow{I} \mathcal{I}\mathcal{B} \xrightarrow{P} \mathcal{B} \to 0$$

to an exact sequence of suspended categories, for each $\mathcal{B}$. Here $\mathcal{M}\mathcal{B}$ is the category of morphisms of $\mathcal{B}$, J is the inclusion of the subcategory $\mathcal{I}\mathcal{B}$ of inflations of $\mathcal{B}$, I maps B to $0 \to B$ and P maps $i : A \to B$ to A.

1. Extending ∂-functors

By an additive, exact [8] [12] or suspended category we shall always mean a svelte (=equivalent to a small) category of the respective type. We shall also assume that idempotents split in the exact categories we consider. Following [4] we use the words 'inflation', 'deflation' and 'conflation' instead of 'admissible monomorphism', 'admissible epimorphism' and 'admissible short exact sequence‘ [12], respectively. If $\mathcal{B}$ is an exact category with enough injectives, we denote by $\underline{\mathcal{B}}$ the residue class category of $\mathcal{B}$ modulo the ideal of morphisms factoring through an injective. The exact structure on $\mathcal{B}$ yields a structure of suspended category on $\underline{\mathcal{B}}$ [10].

Let $\mathcal{A}$ be an exact category and $\mathcal{D}_b\mathcal{A}$ the *bounded derived category* of $\mathcal{A}$: It is the localisation [15] of the *homotopy category* $\mathcal{H}_b\mathcal{A}$ *of bounded chain complexes*

$$\ldots \to K_{n+1} \xrightarrow{d} K_n \xrightarrow{d} K_{n-1} \to \ldots\,,\ dd = 0\,,\ K_n = 0\ \ \forall\, n >> 0 \text{ and } \forall\, n << 0\,,$$

at the subcategory of acyclic complexes, i.e. complexes A admitting conflations

$$Z_{n+1} \xrightarrow{j_n} A_n \xrightarrow{q_n} Z_{n-1}$$

such that $d_n = j_{n-1}q_n$, $\forall\, n$ (we always suppress the zeroes at the ends of a conflation). The *positive derived category* $\mathcal{DA}$ is the full subcategory of $\mathcal{D}_b\mathcal{A}$ consisting of the *positive* complexes K, i.e. $K_n = 0, \forall\, n < 0$. Note that our use of the notation $\mathcal{DA}$ *does not agree* with the usual conventions. We shall see in 5.1 that $\mathcal{DA}$ identifies with a localisation of the *positive homotopy category* $\mathcal{H}^b_{0]}\mathcal{A}$, the full subcategory of $\mathcal{HA}$ consisting of the positive complexes. Clearly $\mathcal{DA}$ is a suspended category in the sense of [10]. We shall work with $\mathcal{DA}$ rather than with $\mathcal{D}_b\mathcal{A}$ since this is technically simpler and at the same time leads to slightly more general results. In fact, it is usually easy to pass from $\mathcal{DA}$ to $\mathcal{D}_b\mathcal{A}$ using the fact that the latter category identifies with the smallest triangulated category containing $\mathcal{DA}$ as a suspended subcategory (cf. [10, 2.1]).

By sending $A \in \mathcal{A}$ to the complex K with $K_0 = A$ and $K_n = 0$ for all $n \neq 0$, we obtain an additive functor *can* from $\mathcal{A}$ to $\mathcal{DA}$. Moreover, for each conflation

$$\varepsilon :\ A \xrightarrow{i} B \xrightarrow{d} C$$

of $\mathcal{A}$, there is a unique [2, 1.1.10] connecting morphism $\partial\varepsilon : C \to SA$ such that the sequence

$$A \xrightarrow{i} B \xrightarrow{d} C \xrightarrow{\partial\varepsilon} SA$$

is a triangle of $\mathcal{DA}$ (we omit *can* from the notation whenever the context makes it clear that we are speaking of complexes rather than of objects of $\mathcal{A}$). Thus, $(can\,, \partial)$ is a ∂-functor in the sense of the following definition: If $\mathcal{S}$ is a suspended category, a ∂-*functor* from $\mathcal{A}$ to $\mathcal{S}$ consists of an additive functor $D : \mathcal{A} \to \mathcal{S}$ and a natural transformation δ which assigns a morphism $\delta\varepsilon : DC \to SDA$ to each conflation ε in such a way that

$$DA \xrightarrow{Di} DB \xrightarrow{Dd} DC \xrightarrow{\delta\varepsilon} SDA$$

is a triangle of $\mathcal{S}$. A *morphism of ∂-functors* $(D, \delta) \to (D', \delta')$ is given by a morphism of functors $\mu : D \to D'$ such that the square

$$\begin{array}{ccc} DC & \xrightarrow{\delta\varepsilon} & SDA \\ \mu C \downarrow & & \downarrow S\mu A \\ D'C & \xrightarrow{\delta'\varepsilon} & SD'A \end{array}$$

commutes for each conflation ε. It is clear how to compose two such morphisms. We see that the ∂-functors from $\mathcal{A}$ to $\mathcal{S}$ form a category $\Delta(\mathcal{A}, \mathcal{S})$. Each S-functor $(F, \varphi) : \mathcal{DA} \to \mathcal{S}$ yields a ∂-functor $(F\,can\,, \delta)$ whose second component assigns $\delta\varepsilon = (\varphi\,can\,A)(F\partial\varepsilon)$ to the conflation ε. Also, if ν is a morphism of S-functors from $\mathcal{DA}$ to $\mathcal{S}$, $can\,\nu$ is a morphism of the associated ∂-functors. Clearly these assignments define a functor

$$can^* : \mathcal{S}usp\,(\mathcal{DA}, \mathcal{S}) \to \Delta\,(\mathcal{A}, \mathcal{S}),$$

where the left hand side is the category of S-functors from $\mathcal{DA}$ to $\mathcal{S}$. It is reasonable to ask the

Question: Is can^* an equivalence ?

An affirmative answer would entitle us to call $\mathcal{DA}$ the solution of a universal problem. The answer, however, is no. The following simple example shows that even if each conflation of $\mathcal{A}$ splits, there are ∂-functors $\mathcal{A} \to \mathcal{S}$ which do not 'extend' to S-functors $\mathcal{DA} \to \mathcal{S}$.

Example. Let $\mathcal{B}$ be an exact category,

$$A \xrightarrow{i} B \xrightarrow{d} C$$

a non-split conflation of $\mathcal{B}$ and $\partial : C \to SA$ the connecting morphism in $\mathcal{DB}$. Let $\mathcal{A}$ be the full additive subcategory of $\mathcal{DB}$ consisting of the objects $\coprod_{i\in I} X_i$, where I is finite and each X_i belongs to $\{A, B, C, SA\}$. We equip $\mathcal{A}$ with the split conflations. Suppose there is an S-functor F from $\mathcal{DA} = \mathcal{H}^b_{0]}\,\mathcal{A}$ to $\mathcal{DB}$ whose restriction to $\mathcal{A}$ is isomorphic to the inclusion $\mathcal{A} \subset \mathcal{DB}$. Then it is not hard to see that

$$F(\,\ldots \to 0 \to A \xrightarrow{i} B \xrightarrow{d} C) = 0.$$

Since the morphism $\partial : C \to SA$ of $\mathcal{A}$ factors through

$$\ldots \to 0 \to A \xrightarrow{i} B \xrightarrow{d} C$$

in $\mathcal{DA}$, this implies $F\partial = 0$, a contradiction.

For a general suspended category $\mathcal{S}$, we know nothing about the image of can^*. The problem only becomes tractable when we make more specific assumptions about $\mathcal{S}$: If $\mathcal{S}$ is the stable category of an exact category with enough injectives then a ∂-functor D is isomorphic to $F can$ for some $F : \mathcal{DA} \to \mathcal{S}$ if it satisfies

Condition 1: For each $n > 0$ and all $A, B \in \mathcal{A}$ we have $\mathcal{S}\,(S^n DA, DB) = 0$.

This is an easy consequence of [10, 3.2]. We shall also reprove it in this paper. The condition is not necessary. For example, if $\mathcal{A}$ has enough injectives, one can define (cf. [14] or example 2.6) an S-functor $F : \mathcal{DA} \to \underline{\mathcal{A}}$ which extends the canonical projection $\mathcal{A} \to \underline{\mathcal{A}}$. However, the latter does not satisfy condition 1 unless $S : \underline{\mathcal{A}} \to \underline{\mathcal{A}}$ is zero, i.e. the injective dimension of $\mathcal{A}$ is 1.

Now let us consider two S-functors $F, F' : \mathcal{DA} \to \mathcal{S}$ and a morphism μ from $D = F can$ to $D' = F' can$. Even if $\mathcal{S}$ is the stable category of an exact category with enough injectives and D and D' satisfy condition 1, we do not know how to extend μ to a morphism of S-functors $\nu : F \to F'$. We illustrate the state of our ignorance by posing the

Problem: If $\mathcal{B}$ is an additive category and $F : \mathcal{H}^b_{0]}\,\mathcal{B} \to \mathcal{H}^b_{0]}\,\mathcal{B}$ is an S-functor whose restriction to $\mathcal{B}$ is isomorphic to $can : \mathcal{B} \to \mathcal{H}^b_{0]}\,\mathcal{B}$, is it true that F is isomorphic to the identity of $\mathcal{H}^b_{0]}\,\mathcal{B}$?

We remark that it would be enough to produce any $\nu : F \to \mathbf{1}$ such that $can\,\nu$ is an isomorphism since such a ν would automatically be invertible, as an easy induction argument shows. We should also mention the following positive result: If $D \cong D'$ and D satisfies the above condition, then there is an isomorphism $FK \xrightarrow{\sim} F'K$ for each $K \in \mathcal{DA}$. It is not hard to prove this for general $\mathcal{S}$ by induction on the maximal n with $K_n \neq 0$. While this is certainly useful in the applications, it does not solve the problem we are considering since there seems to be no way to ensure the functoriality of the isomorphisms $FK \xrightarrow{\sim} F'K$.

Let us return to the problem of extending $\mu : D \to D'$ to $\nu : F \to F'$. Suppose that $\mathcal{S}$ is the stable category of an exact category with enough injectives, that D and D' satisfy condition 1 and that F and F' were obtained from D and D' by the method of [10, 3.2]. Then one can prove that ν exists if D and D' satisfy

Condition 2: For each $n > 0$ and all $A, B \in \mathcal{A}$ we have $\mathcal{S}(S^n DA, D'B) = 0$.

Let us summarize our findings in vague terms: Extending a ∂-functor is possible if $\mathcal{S}$ is of a special type and condition 1 is satisfied; extending a morphism between the restrictions of two S-functors is possible if $\mathcal{S}$ is of a special type, the S-functors are of a special type and their restrictions satisfy conditions 1 and 2.

Thus suspended categories and S-functors 'of a special type' behave better than general suspended categories and S-functors. For example, the above problem becomes trivial if F is assumed to be 'of a special type'. In the following section, we shall introduce the class of *basic* S-functors (resp. suspended categories), which contains all those of 'special type'. The class of basic S-functors is closed under left adjoints and contains all left derived functors. The good properties of S-functors 'of a special type' are shared by all basic S-functors. They derive from the good properties of towers of S-functors.

2. Towers

2.1 For each $n \in \mathbf{N}$ let $\mathcal{P}_n$ be the partially ordered set

$$\{0 < 1\}^n = \{x = (x_1, \ldots, x_n) : x_i \in \{0, 1\}\}$$

and let $\mathcal{P}_0 = \{*\}$. Let $\mathcal{P}$ be the category whose objects are the $\mathcal{P}_n$, $n \in \mathbf{N}$ and whose morphisms are all possible compositions of the maps

$$\begin{aligned} p^j_\varepsilon : \mathcal{P}_n \to \mathcal{P}_{n+1}\,, \qquad & (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{j-1}, \varepsilon, x_j, \ldots, x_n)\,, \\ q^j : \mathcal{P}_{n+1} \to \mathcal{P}_n\,, \qquad & (x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1})\,, \end{aligned}$$

where $n \in \mathbf{N}$ and $1 \le j \le n + 1$. More explicitly, an order preserving map $a : \mathcal{P}_m \to \mathcal{P}_n$ is a morphism of $\mathcal{P}$ iff for all $1 \le i < j \le n$ we have

$$\pi_i\, a = \pi_u \text{ and } \pi_j\, a = \pi_v \Longrightarrow u < v\,,$$

where $\pi_i\, x = x_i$ for $x \in \mathcal{P}_n$. The category $\mathcal{P}$ is the 'cubical category' implicit in the definition of cubical homology (see e.g. [3]). As for the simplicial category one proves that any relation between its generators is a consequence of

$$\begin{array}{ll} p_\varepsilon^j\, p_\eta^k = p_\eta^{k+1}\, p_\varepsilon^j & j \le k \\ q^j\, q^k = q^k\, q^{j+1} & j \ge k \end{array} \qquad q^j\, p_\varepsilon^k = \begin{cases} p_\varepsilon^{k-1}\, q^j & j < k \\ 1 & j = k \\ p_\varepsilon^k\, q^{j+1} & j > k. \end{cases}$$

We define an order relation on the morphisms from $\mathcal{P}_l$ to $\mathcal{P}_m$ by $b \le c \Leftrightarrow b(x) \le c(x)\ ,\ \forall x$. Note that $b \le c$ implies $ab \le ac$ and $be \le ce$ for all morphisms $a : \mathcal{P}_m \to \mathcal{P}_n$ and $e : \mathcal{P}_k \to \mathcal{P}_l$.

Remarks. a) The 'juxtaposition functor'

$$\sqcup : \mathcal{P} \times \mathcal{P} \to \mathcal{P}\ ,\ (\mathcal{P}_n, \mathcal{P}_m) \mapsto \mathcal{P}_{n+m}$$

makes $\mathcal{P}$ into a strictly monoidal category with neutral object $\mathcal{P}_0$ (cf. [11]). If $(\mathcal{C}, \sqcup, e)$ is another strictly monoidal category and if $F : \{\mathcal{P}_0, \mathcal{P}_1\} \to \mathcal{C}$ is a functor defined on the full subcategory $\{\mathcal{P}_0, \mathcal{P}_1\}$ and sending $\mathcal{P}_0$ to e, there is a unique extension $G : \mathcal{P} \to \mathcal{C}$ of F which commutes with $\sqcup$.

b) Taking the inequalities $b \le c$ as 2-morphisms $b \to c$ we can view $\mathcal{P}$ as a 2-category [5, V.1]. Its 2-morphisms are generated by the

$$\varphi^j : p_0^j\, q^j \to 1\ ,\ \psi^j : 1 \to p_1^j\, q^j$$

subject to the relations

$$q^j\, \varphi^j = 1\ ,\ \varphi^j\, p_0^j = 1\ ,\ q^j \psi^j = 1\ ,\ \psi^j\, p_1^j = 1.$$

The functor $\sqcup$ is indeed a 2-functor and $\mathcal{P}$ is a strictly monoidal 2-category. The analogue of a) holds with respect to the full 2-subcategory whose non-identical 1-morphisms are

$$\begin{array}{c} \mathcal{P}_0 \\ p_0^1 \downarrow \quad q^1 \uparrow \quad \downarrow p_1^1 \\ \mathcal{P}_1 \end{array} \quad .$$

For example, if $\mathcal{A}dd$ is the category of additive categories, there is a unique 2-functor from $\mathcal{P}$ to $\mathcal{F}un\,(\mathcal{A}dd, \mathcal{A}dd)$ (cf. A.1) which carries $\sqcup$ to the composition of 2-functors and sends $\mathcal{P}_0$ to the identity, $\mathcal{P}_1$ to $\mathcal{M}$ (cf. example 2.2 a), p_0^1, q^1, p_1^1 to P_0^1, Q_0^1, P_1^1 and φ^1, ψ^1 to the adjunction morphisms.

2.2 A *tower $\mathcal{T}$ of additive categories* is given by the following data

- a sequence of additive categories $\mathcal{T}_n$, $n \in \mathbf{N}$,

- an additive functor $\mathcal{T}a : \mathcal{T}_n \to \mathcal{T}_m$ for each morphism $a : \mathcal{P}_m \to \mathcal{P}_n$,
- a morphism of functors $\mathcal{T}(c, b) : \mathcal{T}c \to \mathcal{T}b$ for each pair of morphisms $b \leq c$ from $\mathcal{P}_l$ to $\mathcal{P}_m$.

These data are subject to the following conditions

- $\mathcal{T}1 = 1$ and $\mathcal{T}b\mathcal{T}a = \mathcal{T}ab$,
- $\mathcal{T}(b, b) = 1$ and $\mathcal{T}(c, b)\,\mathcal{T}(d, c) = \mathcal{T}(d, b)$,
- $\mathcal{T}(c, b)\mathcal{T}a = \mathcal{T}(ac, ab)$ and $\mathcal{T}e\mathcal{T}(c, b) = \mathcal{T}(ce, be)$,

whenever a is a morphism $\mathcal{P}_m \to \mathcal{P}_n$, $b \leq c \leq d$ are morphisms $\mathcal{P}_l \to \mathcal{P}_m$ and e is a morphism $\mathcal{P}_k \to \mathcal{P}_l$. Note that the first condition implies that $\mathcal{T}$ gives rise to a functor from $\mathcal{P}^{op}$ to the category of additive categories and that the second condition implies that $\mathcal{T}$ gives rise to a functor from $\mathcal{P}\,(\mathcal{P}_l, \mathcal{P}_m)^{op}$ to the category of additive functors from $\mathcal{T}_m$ to $\mathcal{T}_l$. Here (as always) we regard an ordered set S as a category with objects $s \in S$ and a morphism $s \to t$ for each pair $s \leq t$. This definition can be summed up by saying that $\mathcal{T}$ is a 2-functor from $\mathcal{P}^{OP}$ to $\mathcal{A}dd$ (cf. remark 2.1 b) and A.1), where $\mathcal{P}^{OP}$ denotes the 2-category $\mathcal{P}$ with reversed 1- and 2-morphisms.

In a completely analogous fashion, one defines *towers of exact categories* and *towers of suspended categories*.

For any tower $\mathcal{T}$, we denote $\mathcal{T}p^j_\varepsilon$ by P^j_ε and $\mathcal{T}q^j$ by Q^j_0.

Examples. a) Let $\mathcal{B}$ be an additive category and $\mathcal{B}^\wedge_m$ the category of contravariant functors from $\mathcal{P}_m$ to $\mathcal{B}$. Thus $\mathcal{B}^\wedge_m$ consists of the m-dimensional commutative hypercubes in $\mathcal{B}$. If $X \in \mathcal{B}^\wedge_m$ and $b : \mathcal{P}_l \to \mathcal{P}_m$ is a morphism, we define $(\mathcal{B}^\wedge bX)(x) = Xb(x)$. If c is another morphism $\mathcal{P}_l \to \mathcal{P}_m$ and $b \leq c$ then

$$\mathcal{B}^\wedge(c, b) : \mathcal{B}^\wedge c \to \mathcal{B}^\wedge b$$

is furnished by the morphisms

$$Xc(x) \to Xb(x)\;,\; x \in \mathcal{P}_l.$$

Clearly $\mathcal{B}^\wedge$ is a tower of additive categories. Note that $B^\wedge_0$ is isomorphic to $\mathcal{B}$ and that $\mathcal{B}^\wedge_n$ is isomorphic to

$$\mathcal{M}^n\mathcal{B} = \mathcal{M}\mathcal{M}\ldots\mathcal{M}\mathcal{B}$$

where $\mathcal{MB}$ is the category of morphisms of $\mathcal{B}$. More precisely, we have an isomorphism

$$M^j : \mathcal{B}^\wedge_{n+1} \to \mathcal{MB}^\wedge_n \,,\; X \mapsto (P^j_1 X \to P^j_0 X)$$

for each $1 \leq j \leq n+1$. For later reference, we record the functors

$$Q^j_{-1} : \mathcal{B}^\wedge_n \to \mathcal{P}^\wedge_{n+1} \text{ and } Q^j_1 : \mathcal{P}^\wedge_n \to \mathcal{P}^\wedge_{n+1} \,,$$

which we define by their compositions with M^j:

$$M^j Q^j_{-1} X = (0 \to X) \,, \quad M^j Q^j_1 X = (X \to 0).$$

Observe that we have a chain of adjoint functors

$$Q^j_{-1} \dashv P^j_0 \dashv Q^j_0 \dashv P^j_1 \dashv Q^j_1.$$

b) Let $\mathcal{A}$ be an exact category. We convert $\mathcal{A}^\wedge$ into a tower of exact categories by endowing $\mathcal{A}^\wedge_n$ with the pairs whose evaluation at each $x \in \mathcal{P}_n$ is a conflation of $\mathcal{A}$. Now we inductively define exact subcategories $\mathcal{I}^n\mathcal{A}$ of $\mathcal{A}^\wedge$: $\mathcal{I}^0\mathcal{A} = \mathcal{A}^\wedge_0 = \mathcal{A}$ and $\mathcal{I}^n\mathcal{A}$ consists of the $X \in \mathcal{A}^\wedge_n$ such that the morphism

$$P^1_1 X \to P^1_0 X$$

is an inflation of $\mathcal{I}^{n-1}\mathcal{A}$, $n > 0$. Using the snake lemma it is easy to verify that this is equivalent to requiring that

$$P^j_1 X \to P^j_0 X$$

be an inflation for each j. It is easy to see that $\mathcal{A}^\wedge a X$ lies in $\mathcal{I}^m\mathcal{A}$ if X lies in $\mathcal{I}^n\mathcal{A}$ and $a : \mathcal{P}_m \to \mathcal{P}_n$ is a morphism. Thus the $\mathcal{I}^n\mathcal{A}$ yield an exact 'subtower' $\mathcal{I}^*\mathcal{A}$ of $\mathcal{A}^\wedge$.

c) If $\mathcal{A}$ is an exact category, then clearly the $\mathcal{DA}^\wedge_n$ form a tower $\mathcal{DA}^\wedge$ of suspended categories. If moreover $\mathcal{A}$ has enough injectives, then $\mathcal{I}^n\mathcal{A}$ has enough injectives (namely the objects with injective components) and the $\underline{\mathcal{I}^n\mathcal{A}}$ form a tower $\underline{\mathcal{I}^*\mathcal{A}}$ of suspended categories: This is due to the fact that the passage from exact to stable categories is compatible with composition of functors and with morphisms of functors.

2.3 Now let $\mathcal{S}$ and $\mathcal{T}$ be towers of additive categories. A *tower of additive functors* $F : \mathcal{S} \to \mathcal{T}$ consists of

- a sequence of additive functors $F_n : \mathcal{S}_n \to \mathcal{T}_n$, $n \in \mathbf{N}$,
- an isomorphism $Fa : \mathcal{T}a F_n \xrightarrow{\sim} F_m \mathcal{S}a$ for each $a : \mathcal{P}_m \to \mathcal{P}_n$.

$$\begin{tikzcd} \mathcal{S}_n \arrow[r, "\mathcal{S}a"] \arrow[d, "F_n"'] & \mathcal{S}_m \arrow[d, "F_m"] \\ \mathcal{T}_n \arrow[r, "\mathcal{T}a"'] \arrow[ur, phantom, "Fa"] & \mathcal{T}_m \end{tikzcd}$$

We require that the Fa be compatible with compositions in the sense that $Fab = (Fb\mathcal{S}a)(\mathcal{T}bFa)$ and that they be compatible with the morphisms $\mathcal{S}(c,b)$ and $\mathcal{T}(c,b)$ in the sense that $F_l\,\mathcal{S}(c,b)\cdot Fc = Fb\,\mathcal{T}(c,b)\cdot F_m$. Note that F *does not give rise* to a morphism between the functors $\mathcal{P}_m \mapsto \mathcal{S}_m$ and $\mathcal{P}_m \mapsto \mathcal{T}_m$ from $\mathcal{P}$ to the category of additive categories unless $Fa = 1$ for all morphisms a. We may view F as a 1-morphism of $\mathcal{F}un\,(\mathcal{P}^{OP}, \mathcal{A}dd)$. It is clear how to define *towers of exact functors* and *towers of S-functors*.

We *compose* two towers of functors $F : \mathcal{S} \to \mathcal{T}$ and $G : \mathcal{R} \to \mathcal{S}$ by setting $(FG)_n = F_n G_n$ and $FGa = (F_m Ga)(Fa G_n)$.

Examples. a) Each additive functor $F : \mathcal{B} \to \mathcal{C}$ yields a tower of additive functors $F^\wedge$ from $\mathcal{B}^\wedge$ to $\mathcal{C}^\wedge$: $F_n^\wedge$ is the induced functor on the category of n-dimensional hypercubes and $F^\wedge a$ is the identity for each morphism a.

b) For each $x \in \mathcal{P}_n$ let $a_x : \mathcal{P}_0 \to \mathcal{P}_n$ be the morphism with $a_x(*) = x$. If $\mathcal{T}$ is a tower of additive categories, we define $\Phi : \mathcal{T} \to \mathcal{T}_0^\wedge$ by

$$\Phi_n : \mathcal{T}_n \to (\mathcal{T}_0^\wedge)_n \,,\ X \mapsto (x \mapsto (\mathcal{T}a_x)X)$$

and by $\Phi a = 1$ for each morphism a.

c) Let $\mathcal{S}$ be a tower of suspended categories and let $\mathcal{S}|$ be the tower of additive categories obtained from $\mathcal{S}$ by forgetting the suspended structure. The suspension functors $S_n : \mathcal{S}_n \to \mathcal{S}_n$ yield a tower of additive functors $S : \mathcal{S}| \to \mathcal{S}|$ where $(Sa)^{-1}$ is the commutation isomorphism $(\mathcal{S}a)S_n \to S_n(\mathcal{S}a)$ for each morphism a of $\mathcal{P}$.

2.4 Let $\mathcal{E}$ be a tower of exact categories and $\mathcal{T}$ a tower of suspended categories. A *tower of ∂-functors* $D : \mathcal{E} \to \mathcal{T}$ consists of

- a sequence of ∂-functors $D_n : \mathcal{E}_n \to \mathcal{T}_n$, $n \in \mathbf{N}$

- an isomorphism of ∂-functors $Fa : \mathcal{T}aD_n \to D_m\mathcal{E}a$ for each $a : \mathcal{P}_m \to \mathcal{P}_n$.

As in 2.3 we require $Dab = (Db\mathcal{E}a)(\mathcal{T}bDa)$ and $Db{\cdot}\mathcal{T}(c,b)D_m = D_n\mathcal{E}(c,b){\cdot}Dc$. The *composition* of a tower of ∂-functors $D : \mathcal{E} \to \mathcal{T}$ with a tower of S-functors $F : \mathcal{T} \to \mathcal{T}$ is defined in analogy with 2.3.

Examples. In the situation of 2.2, Example b), we have an obvious tower of ∂-functors $D : \mathcal{A}^\wedge \to \mathcal{D}\mathcal{A}^\wedge$ with $Da = 1$ for each morphism a. If $\mathcal{A}$ has enough injectives we also have a ∂-functor $D : \mathcal{I}^*\mathcal{A} \to \underline{\mathcal{I}^*\mathcal{A}}$ and again $Da = 1$ for each morphism a. In this case, there is also a less obvious tower $D : \mathcal{A}^\wedge \to \underline{\mathcal{I}^*\mathcal{A}}$: D_0 is the canonical ∂-functor $\mathcal{A} \to \underline{\mathcal{A}}$, the underlying additive functor of $D_1 : \mathcal{M}\mathcal{A} \to \underline{\mathcal{I}^1\mathcal{A}}$ sends

$$f : X_1 \to X_0 \text{ to } \begin{bmatrix} f \\ i \end{bmatrix} : X_1 \to X_0 \oplus I,$$

where $i : X_1 \to I$ is an inflation into an injective. The connecting morphism of D_1 is obtained by applying this construction to the category $\mathcal{E}\mathcal{A}$ of conflations of $\mathcal{A}$: It provides us with an additive functor

$$\mathcal{E}\mathcal{M}\mathcal{A} \xrightarrow{\sim} \mathcal{M}\mathcal{E}\mathcal{A} \to \underline{\mathcal{I}^1\mathcal{E}\mathcal{A}}\,,$$

which we compose with the canonical functor from $\underline{\mathcal{I}^1\mathcal{E}\mathcal{A}} \xrightarrow{\sim} \underline{\mathcal{E}\mathcal{I}^1\mathcal{A}}$ to the category of triangles of $\underline{\mathcal{I}^1\mathcal{A}}$. The construction of the higher D_n is similar. One can give a more rigorous treatment of this tower using 7.1 and A.2.

2.5 If F and G are towers of functors from $\mathcal{S}$ to $\mathcal{T}$, a *morphism* $\mu : F \to G$ is given by a sequence $\mu_n : F_n \to G_n$ of morphisms of functors such that $(\mu_m\mathcal{T}a)Fa = Ga\,(\mathcal{T}a\mu_n)$ for each morphism $a : \mathcal{P}_m \to \mathcal{P}_n$. Thus μ is a 2-morphism of $\mathcal{F}un\,(\mathcal{P}^{OP}, \mathcal{A}dd)$. It is clear how to *compose* such morphisms. The towers of additive functors from $\mathcal{S}$ to $\mathcal{T}$ and their morphisms are easily seen to form a *category* $Hom_{add}(\mathcal{S}, \mathcal{T})$. *Morphisms of towers of S-functors* and of ∂-functors are defined similarly. If $\mathcal{S}$ and $\mathcal{T}$ are towers of suspended categories, we denote the corresponding *category of towers of S-functors* by $Hom_S(\mathcal{S}, \mathcal{T})$. If $\mathcal{E}$ is a tower of exact categories, we denote the *category of towers of ∂-functors* by $Hom_\partial(\mathcal{E}, \mathcal{T})$.

Examples. a) If $\mu : F \to G$ is a morphism of additive functors from $\mathcal{B}$ to $\mathcal{C}$, there is an obvious morphism $\mu^\wedge : F^\wedge \to G^\wedge$ of towers.

b) We continue 2.3, Example b). Let $\mathcal{A}$ be an additive category and $F : \mathcal{S} \to \mathcal{A}^\wedge$ a tower of additive functors. There is an isomorphism $\varphi F : F \to F_0^\wedge\Phi$, which is obtained as follows: For each $X \in \mathcal{S}_n$ and each $x \in \mathcal{P}_n$ we have $(F_n X)(x) = (\mathcal{A}_0^\wedge a_x)(F_n X)$ by definition; we set

$$(\varphi F)_n X(x) = (F a_x)X : (\mathcal{A}_0^\wedge a_x)(F_n X) \xrightarrow{\sim} F_0 \mathcal{S} a_x X.$$

It is straightforward to verify that φ is a well-defined morphism. Moreover if $\mu : F \to F'$ is a morphism of towers, we have $(\mu^\wedge \Phi)(\varphi F) = (\varphi F')\mu$. This shows that $G \mapsto G^\wedge \Phi$ is a quasiinverse for the functor

$$Hom_{add}(\mathcal{S}, \mathcal{A}^\wedge) \to \mathcal{A}dd\,(\mathcal{S}_0, \mathcal{A})\ ,\ F \mapsto F_0\ ,$$

where $\mathcal{A}dd\,(\mathcal{S}_0, \mathcal{A})$ is the category of additive functors $\mathcal{S}_0 \to \mathcal{A}$. Observe that by the construction of φ, this functor induces an isomorphism of the full subcategory of the F with $Fa = 1$, $\forall\, a$ onto $\mathcal{A}dd\,(\mathcal{S}_0, \mathcal{A})$.

2.6 Let $\mathcal{T}$ be an *epivalent* tower of suspended categories, i.e. for each n and for each $1 \leq j \leq n+1$ the functor

$$M^j : \mathcal{T}_{n+1} \to \mathcal{M}\mathcal{T}_n\ ,\ X \mapsto (P_1^j X \to P_0^j X)$$

is an epivalence (i.e. M^j is full and essentially surjective and a morphism f is invertible iff $M^j f$ is). If the $\mathcal{T}_n$ are triangulated categories, this means that $\mathcal{T}_{n+1}$ is a recollement of two copies of $\mathcal{T}_n$ ($P_0^j = i^!$ and $P_1^j = j^*$ in the notations of [2, 1.4.3]) as we shall see in 6.1. Now let $\mathcal{A}$ be an exact category.

Theorem. *The canonical functor $Hom_S(\mathcal{D}\mathcal{A}^\wedge, \mathcal{T}) \to Hom_\partial(\mathcal{A}^\wedge, \mathcal{T})$ is an equivalence.*

We shall prove this in section 9.

Examples. With the notations of 2.2, Example c), $\mathcal{D}\mathcal{A}^\wedge$ and $\underline{\mathcal{I}^*\mathcal{A}}$ are epivalent towers, as we shall see in 6.1. The towers $\mathcal{A}^\wedge \to \mathcal{D}\mathcal{A}^\wedge$ and $\mathcal{A}^\wedge \to \underline{\mathcal{I}^*\mathcal{A}}$ of 2.4 yield the identical tower $\mathcal{D}\mathcal{A}^\wedge \to \mathcal{D}\mathcal{A}^\wedge$ and a tower $\mathcal{D}\mathcal{A}^\wedge \to \underline{\mathcal{I}^*\mathcal{A}}$, respectively. The base $\mathcal{D}\mathcal{A} \to \underline{\mathcal{A}}$ of the latter extends the canonical projection $\mathcal{A} \to \underline{\mathcal{A}}$.

2.7 A suspended category is *basic* if it occurs as the base of an epivalent tower of suspended categories; an S-functor between two basic suspended categories is *basic* if it occurs as the base of a tower of S-functors; a morphism between two basic S-functors ... . Finally, if $\mathcal{A}$ is an exact category and $\mathcal{T}$ a tower of suspended categories, a ∂-functor from $\mathcal{A}$ to $\mathcal{T}_0$ is *basic* if it occurs as the base of a tower of ∂-functors from $\mathcal{A}^\wedge$ to $\mathcal{T}$ and similarly for morphisms. We point out that 'basic' always refers to a *fixed choice* of the respective towers.

Theorem 2.6 is to be conceived of as a means for studying basic S-functors, ∂-functors ... . Thus it is crucial to know how large the classes of these basic entities are. Empirically, we have found that they are quite large (see 2.8). Indeed we do not know of an example of a non-basic entity. On the other hand, we are not able to prove that, for example, each suspended category is basic, or that a tower over a given base is 'unique' if it exists.

The following theorem shows that certain ∂-functors are necessarily basic and that all morphisms between certain pairs of ∂-functors are basic. We use the above notations.

Theorem.

a) *If $D : \mathcal{A} \to \mathcal{T}_0$ is a ∂-functor with $\mathcal{T}_0(S^n DA, DB) = 0$ for all $n > 0$, $A, B \in \mathcal{A}$, there is a tower of ∂-functors $D^+ : \mathcal{A}^\wedge \to \mathcal{T}$ with $\mathcal{D}_0^+ \xrightarrow{\sim} D$. The tower D^+ is unique up to unique isomorphism.*

b) *If D, D' are towers of ∂-functors $\mathcal{A}^\wedge \to \mathcal{T}$ with $\mathcal{T}_0(S^n D_0 A, D'_0 B) = 0$ for all $n > 0$, $A, B \in \mathcal{A}$, the map*

$$\mathrm{Hom}\,(D, D') \to \mathrm{Hom}\,(D_0, D'_0)\ ,\ \mu \mapsto \mu_0$$

is bijective.

Combined with 2.6 this immediately yields the

Corollary.

a) *If $D : \mathcal{A} \to \mathcal{T}_0$ is a ∂-functor with $\mathcal{T}_0(S^n DA, DB) = 0$ for all $n > 0$, $A, B \in \mathcal{A}$, there is a basic S-functor $F : \mathcal{D}\mathcal{A} \to \mathcal{T}_0$ extending D. It is unique up to a unique basic isomorphism.*

b) *If F and F' are two basic S-functors $\mathcal{D}\mathcal{A} \to \mathcal{T}_0$ with $\mathcal{T}_0(S^n FA, F'A) = 0$ for all $n > 0$, $A, B \in \mathcal{A}$ and μ is a morphism between the restrictions of F and F' to $\mathcal{A}$, there is a unique basic morphism of S-functors $\nu : F \to F'$ extending μ.*

2.8 The following theorem and its corollary account for the all-pervasiveness of basic S-functors. Suppose that $\mathcal{S}$ and $\mathcal{T}$ are basic suspended categories.

Theorem. *If $L : \mathcal{T} \to \mathcal{S}$ is left S-adjoint to a basic S-functor, it is basic.*

Note that the dual statement holds for basic co-suspended categories and that both hold for basic triangulated categories (we entrust the reader with the definition of these concepts). In view of the desription of the morphisms of $\mathcal{P}$, the theorem follows from A.2 and 6.5.

Now assume $\mathcal{A}$ is an exact category with enough projectives. The localisation functor $\mathcal{H}_+\mathcal{A} \to \mathcal{D}_+\mathcal{A}$ is of course basic and has the projective resolution functor (dual to [9, 4.1]) as a left adjoint. By the theorem, projective resolution is a basic S-functor. Hence we have the

Corollary. *If $F : \mathcal{A} \to \mathcal{B}$ is an additive functor, the left derived functor $\underline{L}F : \mathcal{D}_+\mathcal{A} \to \mathcal{D}_+\mathcal{B}$ is a basic S-functor.*

3. Presheaves and the derived category

Let $\mathcal{A}$ be an exact category and $\mathbf{N}_1$ the set of natural numbers $n \geq 1$. We endow $\mathbf{N}_1$ with the topology of the cofinite sets. A *rough presheaf* $\mathcal{F}$ on $\mathbf{N}_1$ with values in $\mathcal{A}^{op}$ consists of objects $\mathcal{F}(U) \in \mathcal{A}^{op}$ for each open subset U of $\mathbf{N}_1$ and of morphisms $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ of $\mathcal{A}^{op}$ for each inclusion $U \subset V$ of open sets of $\mathbf{N}_1$. We require that

- $\rho_{UU} = 1$ for each open $U \subset \mathbf{N}_1$,
- $\rho_{UW} = \rho_{VW}\rho_{UV}$ whenever $U \subset V \subset W$ are open in $\mathbf{N}_1$ and

- there is an open $M \in \mathbf{N}_1$ such that $\mathcal{F}(U)$ vanishes if U does not contain M.

In particular we have $\mathcal{F}(\emptyset) = 0$. We endow the *category* $\mathcal{RA}$ of rough presheaves with the exact structure consisting of the pairs

$$\mathcal{F}' \xrightarrow{i} \mathcal{F} \xrightarrow{d} \mathcal{F}''$$

such that

$$\mathcal{F}'(U) \xrightarrow{iU} \mathcal{F}(U) \xrightarrow{dU} \mathcal{F}''(U)$$

is a conflation of $\mathcal{A}^{op}$ for each open $U \subset \mathbf{N}_1$. By definition the class Σ consists of those morphisms $s : \mathcal{F} \to \mathcal{F}''$ which fit into a conflation

$$\mathcal{F}' \xrightarrow{i} \mathcal{F} \xrightarrow{s} \mathcal{F}'' ,$$

where $\mathcal{F}'$ admits a $j \in \mathbf{N}_1$ such that the restriction

$$\mathcal{F}'(U) \to \mathcal{F}'(U \cap U_j)$$

is invertible for all open U, where $U_j = \mathbf{N}_1 - j$.

The open sets U_j, $j \in \mathbf{N}_1$ clearly form an open covering $\mathcal{U}$ of $\mathbf{N}_1$. If $\mathcal{F}$ is a rough presheaf, the associated Čech-complex

$$C\mathcal{F} = C(\mathcal{U}, \mathcal{F})$$

has the components $C^0\mathcal{F} = \mathcal{F}(\mathbf{N}_1)$ and

$$C^p\mathcal{F} = \prod_{i_1 < \ldots < i_p} \mathcal{F}(U_{i_1} \cap \ldots \cap U_{i_p}) \ , \ p > 0$$

and the differential given by the matrix of the

$$(-1)^k \rho : \mathcal{F}(U_{i_1} \cap \ldots U_{i_{k-1}} \cap U_{i_{k+1}} \cap \ldots \cap U_{i_{p+1}}) \to \mathcal{F}(U_{i_1} \cap \ldots \cap U_{i_{p+1}}).$$

Observe that there are only finitely many non-zero terms in the definition of $C^p\mathcal{F}$ and that $C\mathcal{F}$ is a positive differential complex over $\mathcal{A}$.

Theorem. *The functor C induces an equivalence $(\mathcal{RA})[\Sigma^{-1}] \to \mathcal{DA}$.*

This is an 'abstract' localisation [5, I, 1.1]. We shall give the proof in 5.4.

4. The universal property of the construction $\mathcal{D}$

In this section, we view the assignment $\mathcal{A} \mapsto \mathcal{D}\mathcal{A}$ as a 2-functor (cf. A.1) from the 2-category of exact categories $\mathcal{E}x$ to the 2-category of suspended categories $\mathcal{S}usp$. This simply means that $\mathcal{D}$ is defined on exact categories, exact functors and morphisms of exact functors and is compatible with the various composition functors. The canonical ∂-functors $\mathcal{A} \to \mathcal{D}\mathcal{A}$, $\mathcal{A} \in \mathcal{E}x$ combine into a 2-∂-functor *can* in the sense of the following definition: If $\mathcal{F} : \mathcal{E}x \to \mathcal{S}usp$ is a 2-functor, a *2-∂-functor* D to $\mathcal{F}$ consists of

- a ∂-functor $D\mathcal{A} : \mathcal{A} \to \mathcal{F}\mathcal{A}$ for each exact category $\mathcal{A}$
- an isomorphism of ∂-functors $DF : (\mathcal{F}F)(D\mathcal{A}) \to (D\mathcal{B})F$ for each exact functor $F : \mathcal{A} \to \mathcal{B}$.

These data are required to be compatible with compositions of exact functors and with morphisms of exact functors in the already familiar fashion:

$$D(GF) = (DG)F \cdot (\mathcal{F}G)(DF) \text{ and } (DB)\mu \cdot DF = DG \cdot \mathcal{F}\mu D\mathcal{A}$$

whenever we have exact functors

$$\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$$

or a morphism $\mu : F \to G$ of exact functors from $\mathcal{A}$ to $\mathcal{B}$. A *morphism* $\alpha : D \to D'$ of 2-∂-functors assigns a morphism of ∂-functors $\alpha\mathcal{A} : D\mathcal{A} \to D'\mathcal{A}$ to each exact $\mathcal{A}$. The $\alpha\mathcal{A}$ are required to satisfy

$$D'F \cdot \mathcal{F}F\alpha\mathcal{A} = \alpha\mathcal{B}F \cdot DF$$

for each exact functor $F : \mathcal{A} \to \mathcal{B}$. The 2-$\partial$-functors to a given $\mathcal{F}$ form a *category* $Hom_\partial(\mathcal{F})$. Similarly, the 1-morphisms $\mathcal{D} \to \mathcal{F}$ of $\mathcal{F}un\,(\mathcal{E}x, \mathcal{S}usp)$ form a *category* $Hom_S\,(\mathcal{D}, \mathcal{F})$. The canonical 2-$\partial$-functor induces a functor

$$can^* : Hom_S(\mathcal{D}, \mathcal{F}) \to Hom_\partial(\mathcal{F}).$$

We shall give a sufficient condition for can^* to be an equivalence. For each $\mathcal{A} \in \mathcal{E}x$ let $\mathcal{M}\mathcal{A}$ be the category of morphisms of $\mathcal{A}$, $J\mathcal{A} : \mathcal{I}\mathcal{A} \to \mathcal{M}\mathcal{A}$ the inclusion of the subcategory of $\mathcal{M}\mathcal{A}$ consisting of the inflations of $\mathcal{A}$ and

$$0 \to \mathcal{A} \xrightarrow{Q\mathcal{A}} \mathcal{I}\mathcal{A} \xrightarrow{P\mathcal{A}} \mathcal{A} \to 0$$

the exact sequence of exact categories with $(Q\mathcal{A})B = (0 \to B)$ and $(P\mathcal{A})(A \to B) = A$. We shall see that $\mathcal{D}J\mathcal{A}$ is an equivalence (10.1) and that $\operatorname{Im}\mathcal{D}Q\mathcal{A} = \operatorname{Ker}\mathcal{D}P\mathcal{A}$ (Example 6.1 c).

Theorem. *The functor can^* is an equivalence if $\mathcal{F}J\mathcal{A}$ is an equivalence and $\operatorname{Im}\mathcal{F}Q\mathcal{A} = \operatorname{Ker}\mathcal{F}P\mathcal{A}$ for each exact $\mathcal{A}$.*

We shall prove this in section 10.

5. Proof of Theorem 3

5.1 Proposition.

a) *$\mathcal{D}\mathcal{A}$ is isomorphic to the localisation of $\mathcal{H}_{0]}^{b}\mathcal{A}$ (1.1) at the class of morphisms $\overline{s}$ which fit into a triangle*

$$X \xrightarrow{\overline{s}} Y \to A \to SX$$

of $\mathcal{H}_{0]}^{b}\mathcal{A}$ with acyclic A.

b) *$\mathcal{D}\mathcal{A}$ is ismorphic to the localisation of $\mathcal{C}_{0]}^{b}\mathcal{A}$ (the category of bounded positive complexes) at the class of morphisms d which fit into a sequence*

$$A \xrightarrow{i} X \xrightarrow{d} X'$$

of $\mathcal{C}_{0]}^{b}\mathcal{A}$ such that (i_n, d_n) is a conflation of $\mathcal{A}$ for all $n \in \mathbf{N}$ and A is of the form

$$\ldots \to 0 \to A_j \xrightarrow{\sim} A_{j-1} \to 0 \to \ldots$$

for some $j \geq 1$.

Proof. a) As in the classical case [15, I, §2, no. 1] one sees that the class of morphisms $\overline{s}$ admits a calculus of left fractions [5, I, 2.3]. Now suppose X is a positive complex and X' appears in a triangle

$$B \to X \xrightarrow{\overline{t}} X' \to SB$$

of $\mathcal{H}_b\mathcal{A}$ with an acyclic B. Realising X' as a mapping cone we see that $\overline{t}$ factors as $\overline{rs}$, where $\overline{s}$ is as in the assertion and $\overline{r} : X'_{\geq 0} \to X'$ induces an isomorphism

$$\mathcal{H}_b\mathcal{A}\,(K, X'_{\geq 0}) \xrightarrow{\sim} \mathcal{H}_b\mathcal{A}\,(K, X')$$

for each positive complex K. Using left fractions to compute the morphism groups $\mathcal{D}_b\mathcal{A}\,(K,X)$ we infer

$$\mathcal{D}\mathcal{A}\,(K,X) \xrightarrow{\sim} \mathcal{D}_b\mathcal{A}\,(K,X).$$

b) Let us first assume that each conflation of $\mathcal{A}$ splits. We have to show that $\mathcal{H}^b_{0]}\,\mathcal{A}$ identifies with the localisation of $\mathcal{C}^b_{0]}\,\mathcal{A}$ at the class Θ of morphisms d. Obviously the functor $(\mathcal{C}^b_{0]}\,\mathcal{A})[\Theta^{-1}] \to \mathcal{H}^b_{0]}\,\mathcal{A}$ is full and bijective on objects. Since the $d \in \Theta$ admit right inverses in $\mathcal{C}^b_{0]}\,\mathcal{A}$, the localisation functor $\mathcal{C}^b_{0]}\,\mathcal{A} \to (\mathcal{C}^b_{0]}\,\mathcal{A})[\Theta^{-1}]$ is full as well and it only remains to be shown that two morphisms of complexes $f, g : X \to Y$ have the same image in $(\mathcal{C}^b_{0]}\,\mathcal{A})[\Theta^{-1}]$ if they are homotopic. Let $f - g = k i_X$, where $i_X : X \to IX$ has the components

$$\begin{bmatrix} 1 \\ d_n \end{bmatrix} : X_n \to X_n \oplus X_{n+1} = (IX)_n \text{ and } d^{IX} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Since $[1_X\ 0] : X \oplus IX \to X$ is in Θ, the morphisms $[1_X\ i]^t$ and $[1_X\ 0]^t : X \to X \oplus IX$ have the same image in $(\mathcal{C}^b_{0]}\,\mathcal{A})[\Theta^{-1}]$. Hence the same is true of $f = [g\ k][1_X\ i]^t$ and $g = [g\ k][1_X\ 0]^t$.

If $\mathcal{A}$ is a general exact category, we conclude from the above special case that $(\mathcal{C}^b_{0]}\,\mathcal{A})[\Theta^{-1}]$ identifies with the localisation of $\mathcal{H}^b_{0]}\,\mathcal{A}$ at the image of Θ. We have to show that the saturation of the image contains each morphism $\overline{s}$ as in a). Since the mapping cone over s is acyclic, there is a commutative diagram

$$\begin{tikzcd}
 & X_{n+1} \arrow[r, "d_{n+1}"] \arrow[d] & X_n \arrow[r, "d_n"] \arrow[d] & X_{n-1} \arrow[d, "q'_n"] & & X_1 \arrow[r, "d_1"] \arrow[d] & X_0 \arrow[d, "q'_1"] \\
\cdots & Z_{n+1} \arrow[ur] \arrow[d] & Z_n \arrow[ur, "j'_n"] \arrow[d, "j''_n"'] & Z_{n-1} \arrow[d] & \cdots & Z_1 \arrow[ur, "j'_1"] \arrow[d, "j''_1"'] & Z_0 \arrow[d] \\
 & Y_{n+1} \arrow[r, "d_{n+1}"'] \arrow[ur] & Y_n \arrow[r, "d_n"'] \arrow[ur, "q''_n"] & Y_{n-1} & & Y_1 \arrow[r, "d_1"'] \arrow[ur, "q''_1"] & Y_0
\end{tikzcd}$$

where $s_n = j''_n\, q'_{n+1}$, $j'_n\, q''_{n+1} = 0$ and

$$Z_n \xrightarrow{j_n} X_{n-1} \oplus Y_n \xrightarrow{q_n} Z_{n-1}\ ,\ j_n = [-j'_n\ j''_n]^t\ ,\ q_n = [q'_n\ q''_n]\ ,$$

is a conflation $\forall\, n > 0$. We see that $s = t^0 \cdot t^1 \cdot t^2 \ldots$, where

$$\begin{array}{rcl}
(t_0^0, t_1^0, t_2^0, \ldots) & = & (j_0'', 1_{Y_1}, 1_{Y_2}, \ldots) \\
(t_0^1, t_1^1, t_2^1, \ldots) & = & (q_1', j_1'', 1_{Y_2}, \ldots) \\
& \vdots & \\
(t_0^n, t_1^n, t_2^n, \ldots) & = & (1_{X_0}, 1_{X_1}, \ldots, 1_{X_{n-2}}, q_n', j_n'', 1_{Y_{n+1}}, \ldots) \\
& \vdots &
\end{array}$$

We may therefore assume that there is an $n \geq 1$ such that $s_i = 1_{X_i}$ for $i > n$ and $i < n-1$ and that the sequence $([d_n\ \ -s_n]^t, [s_{n-1}\ d_n])$ is a conflation. The sequence of complexes

$$\begin{array}{ccccccc}
\ldots 0 & \rightarrow & X_n & \xrightarrow{1} & X_n & \rightarrow & 0 \ldots \\
\downarrow & & u_n \downarrow & & \downarrow u_{n-1} & & \downarrow \\
\ldots X_{n+1} & \rightarrow & X_n \oplus Y_n & \xrightarrow{d_n \oplus 1} & X_{n-1} \oplus Y_n & \rightarrow & X_{n-2} \ldots \\
\downarrow & & v_n \downarrow & & \downarrow v_{n-1} & & \downarrow \\
\ldots Y_{n+1} & \rightarrow & Y_n & \longrightarrow & Y_{n-1} & \rightarrow & Y_{n-2} \ldots
\end{array}$$

where $u_n = [1\ \ -s_n]^t$, $v_n = [s_n\ 1]$, $u_{n-1} = [d_n\ \ -s_n]^t$, $v_{n-1} = [s_{n-1}\ d_n]$, shows that such an $\overline{s}$ becomes invertible in $(\mathcal{C}^b_{0]}\,\mathcal{A})[\Theta^{-1}]$.

5.2 Lemma. *The composition*

$$\mathcal{RA} \xrightarrow{C} \mathcal{C}^b_{0]}\,\mathcal{A} \rightarrow \mathcal{DA}$$

makes the $s \in \Sigma$ invertible.

Proof. The image of the conflation

$$\mathcal{F}' \xrightarrow{i} \mathcal{F} \xrightarrow{s} \mathcal{F}''$$

of section 3 fits into a triangle of $\mathcal{DA}$. It is therefore enough to show that $C\mathcal{F}'$ is acyclic. Indeed, $C\mathcal{F}'$ is split acyclic, as it is not hard to verify.

5.3 Let $K \in \mathcal{C}^b_{0]}\,\mathcal{A}$. Using the notations of section 3 we define a rough presheaf RK on $\mathbf{N}_1$:

- $(RK)(U) = 0$ unless $U = U_1 \cap \ldots \cap U_n$ for some $n \in \mathbf{N}$ in which case $(RK)(U) = K_n$,
- $\rho_{UV} = 0$ unless $U = U_1 \cap \ldots \cap U_n$, $V = U \cap U_{n+1}$ for some $n \in \mathbf{N}$ in which case $\rho_{UV} = d_{n+1}$.

Lemma. *The composition*

$$\mathcal{C}^b_{0]}\,\mathcal{A} \xrightarrow{R} \mathcal{RA} \to (\mathcal{RA})[\Sigma^{-1}]$$

induces a functor $\mathcal{DA} \to (\mathcal{RA})[\Sigma^{-1}]$.

Proof. If

$$A \xrightarrow{i} X \xrightarrow{d} X'$$

is a sequence as in 5.1 b), the pair (Ri, Rd) is a conflation of $\mathcal{RA}$ and

$$(RA)(U) \to (RA)(U \cap U_j)$$

is clearly invertible for each open $U \subset \mathbf{N}_1$. Hence $Rd \in \Sigma$.

5.4 We prove theorem 3. According to 5.2 and 5.3, C and R induce functors

$$(\mathcal{RA})[\Sigma^{-1}] \underset{\leftarrow}{\overset{\rightarrow}{}} \mathcal{DA}\,,$$

which we also denote by C and R. It is clear that CR is isomorphic to the identity. We shall produce an isomorphism between RC and the identity of $(\mathcal{RA})[\Sigma^{-1}]$. More precisely, we shall first construct an isomorphism

$$1 \xrightarrow{\Psi} G$$

in $(\mathcal{RA})[\Sigma^{-1}]$, where G is a functor $\mathcal{RA} \to \mathcal{RA}$ such that Gs becomes invertible in $(\mathcal{RA})[\Sigma^{-1}]$ for all $s \in \Sigma$ and that $(G\mathcal{F})(U) \neq 0$ only if U is of the form $U_1 \cap \ldots \cap U_n$ for some $n \in \mathbf{N}$. It is then clear that we have an isomorphism $RCG \xrightarrow{\sim} G$ of functors $\mathcal{RA} \to \mathcal{RA}$, hence an isomorphism

$$RC \xrightarrow{RC\Psi} RCG \xrightarrow{\sim} G \xleftarrow{\Psi} 1$$

of functors $(\mathcal{RA})[\Sigma^{-1}] \to (\mathcal{RA})[\Sigma^{-1}]$. We first have to introduce some notation: If $M \subset \mathbf{N}_1$ is a finite subset and $V \subset \mathbf{N}_1$ its complement then, since $\mathbf{N}_1 = M \coprod V$, a rough presheaf $\mathcal{F}$ on $\mathbf{N}_1$ is given by the presheaf $W \mapsto \mathcal{F}_W$ on the discrete space M whose value at $W \subset M$ is the rough presheaf

$$U \mapsto \mathcal{F}_W(U) = \mathcal{F}(W \cup U)\ ,\ U \subset V$$

on V. Thus if $M = \{m, n\}$, we may describe $\mathcal{F}$ by the commutative square

$$\begin{array}{ccc} \mathcal{F}_{\{m,n\}} & \xrightarrow{a} & \mathcal{F}_{\{m\}} \\ b \downarrow & & \downarrow c \\ \mathcal{F}_{\{n\}} & \xrightarrow{d} & \mathcal{F}_{\emptyset}\ , \end{array}$$

where $a, \ldots, d$ are the restriction morphisms. We now define a presheaf $\mathcal{G} = F'_{n,m}\mathcal{F}$ by the square

$$\begin{array}{ccc} \mathcal{G}_{\{m,n\}} & \to & \mathcal{G}_{\{m\}} \\ \downarrow & & \downarrow \\ \mathcal{G}_{\{n\}} & \to & \mathcal{G}_{\emptyset} \end{array} \quad := \quad \begin{array}{rcl} \mathcal{F}_{\{m,n\}} & \xrightarrow{a} & \mathcal{F}_{\{m\}} \\ {[b\ \ -a]^t} \downarrow & & \downarrow [c\ \ -1]^t \\ \mathcal{F}_{\{n\}} \oplus \mathcal{F}_{\{m\}} & \xrightarrow{d \oplus 1} & \mathcal{F}_{\emptyset} \oplus \mathcal{F}_{\{m\}} \end{array}$$

and a presheaf $\mathcal{H} = F_{n,m}\mathcal{F}$ by the square

$$\begin{array}{ccc} \mathcal{H}_{\{m,n\}} & \to & \mathcal{H}_{\{m\}} \\ \downarrow & & \downarrow \\ \mathcal{H}_{\{n\}} & \to & \mathcal{H}_{\emptyset} \end{array} \quad := \quad \begin{array}{rcl} \mathcal{F}_{\{m,n\}} & \longrightarrow & 0 \\ {[b\ \ -a]^t} \downarrow & & \downarrow \\ \mathcal{F}_{\{n\}} \oplus \mathcal{F}_{\{m\}} & \xrightarrow{[d\ c]} & \mathcal{F}_{\emptyset} \end{array} .$$

We have a morphism $\alpha : F'_{n,m}\mathcal{F} \to \mathcal{F}$ whose components are the obvious projections and a morphism $\beta : F'_{n,m}\mathcal{F} \to F_{n,m}\mathcal{F}$ whose components are obvious except for

$$[1\ c] : \mathcal{F}_{\emptyset} \oplus \mathcal{F}_{\{m\}} \to \mathcal{F}_{\emptyset}$$

in the lower right corner. Clearly α and β are deflations,

$$\operatorname{Ker} \alpha(U) \to \operatorname{Ker} \alpha(U \cap U_m)$$

is invertible $\forall\, U$ and

$$\operatorname{Ker} \beta(U) \to \operatorname{Ker} \beta(U \cap U_n)$$

is invertible $\forall\, U$. Hence α, β lie in Σ. This implies that $F_{n,m}s$ is invertible in $\mathcal{RA}[\Sigma^{-1}]$ for all $s \in \Sigma$ and that we have an isomorphism

$$1 \to F_{n,m}$$

of functors $\mathcal{RA}[\Sigma^{-1}] \to \mathcal{RA}[\Sigma^{-1}]$. Observe that if $\mathcal{F}$ has *width*

$$w = \min \{p : C^q \mathcal{F} = 0\ \forall\, q > p\}$$

then $F_{n,m}\mathcal{F}$ has width w and that $F_{n,m}\mathcal{F} \xrightarrow{\sim} \mathcal{F}$ in $\mathcal{RA}$ if $n > w$. Moreover if $(F_{n,m}\mathcal{F})(U) \neq 0$, then $U \cap \{n, m\} \neq \{m\}$ and we have $\mathcal{F}(U) \neq 0$ or $\mathcal{F}(U \cup \{m\} - \{n\}) \neq 0$. Using this it is not hard to verify that $\mu(F_{n,m}\mathcal{F}) \geq \mu(\mathcal{F})$ if $(n, m) \leq \mu(\mathcal{F})$ and that $\mu(F_{n,m}\mathcal{F}) > (n, m)$ if $(n, m) = \mu(\mathcal{F})$, where $\mu(\mathcal{F}) \leq \infty$ is minimal (with respect to the lexicographic ordering) among the pairs (k, l), $k > l$ such that there is an open $U \subset \mathbf{N}_1$ with $U \cap \{k, l\} = l$ and $\mathcal{F}(U) \neq 0$. Now we define functors $G_n : \mathcal{RA} \to \mathcal{RA}$, $n \in \mathbf{N}_1$ by

$$G_1 = 1\ ,\ G_2 = F_{2,1}\ ,\ \ldots\ ,\ G_n = F_{n,n-1}\, F_{n,n-2} \ldots F_{n,1}\, G_{n-1}.$$

It is clear that $\mu(G_n\mathcal{F}) \geq (n, n-1)$ and that $G_n\mathcal{F} \xrightarrow{\sim} G_{n+1}\mathcal{F}$ in $\mathcal{RA}$ if n is greater than the width of $\mathcal{F}$. The functor G defined by

$$G\mathcal{F} = \varinjlim G_n\mathcal{F}\ ,\ n > w$$

has the required properties.

6. Epivalence and Recollement

6.1 Let $\mathcal{S}_0$ and $\mathcal{S}_1$ be suspended categories and $Q_0 : \mathcal{S}_0 \to \mathcal{S}_1$ a fully faithful S-functor admitting a left S-adjoint P_0 and a right S-adjoint P_1. The functor

$$M : \mathcal{S}_1 \to \mathcal{MS}_0\ ,\ X \mapsto (MX : P_1X \to P_0X)$$

from $\mathcal{S}_1$ to the category of morphisms of $\mathcal{S}_0$ is defined by requiring that Q_0MX equal the composition of the adjunction morphisms

$$Q_0P_1X \to X \to Q_0P_0X.$$

We consider the conditions

(E) The functor M is an epivalence (I, 5.2).

(R) There are chains of S-adjoint functors

$$P_{-1} \dashv Q_{-1} \dashv P_0 \text{ and } Q_0 \dashv P_1 \dashv Q_1\ ,$$

Q_{-1} and Q_1 are fully faithful and $\operatorname{Ker} P_1 = \operatorname{Im} Q_{-1}$.

If the suspension functors of $\mathcal{S}_0$ and $\mathcal{S}_1$ are equivalences, condition (R) is equivalent to the recollement setup of [2, 1.4.3] with $P_0 = i^!$ and $P_1 = j^*$, as it is not hard to verify.

Examples. a) If $\mathcal{T}$ is an epivalent tower the S-functor $Q_0^j : \mathcal{T}_n \to \mathcal{T}_{n+1}$ satisfies (E).

b) If $\mathcal{A}$ is an exact category with enough injectives, then $\mathcal{IA}$, the full subcategory of $\mathcal{MA}$ consisting of the inflations of $\mathcal{A}$, is exact with enough injectives (I, 5.1). The functor

$$Q_0 : \underline{\mathcal{A}} \to \underline{\mathcal{IA}}\ ,\ A \mapsto (A \xrightarrow{1} A)$$

has the left adjoint

$$P_0 : \underline{\mathcal{IA}} \to \underline{\mathcal{A}}\ ,\ (A_1 \xrightarrow{i} A_0) \mapsto A_0$$

and the right adjoint

$$P_1 : \underline{\mathcal{IA}} \to \underline{\mathcal{A}}\,,\ (A_1 \xrightarrow{i} A_0) \mapsto A_1.$$

The functor M is simply given by

$$\underline{\mathcal{IA}} \to \mathcal{M}\underline{\mathcal{A}}\,,\ (A_1 \xrightarrow{i} A_0) \mapsto (A_1 \xrightarrow{\bar{i}} A_0).$$

Condition (E) holds (compare [9, 5.2]). We infer that the tower $\mathcal{T} = \underline{\mathcal{I}^*\mathcal{A}}$ is epivalent.

c) Let $\mathcal{A}$ be an exact category. The functor

$$\mathcal{A} \to \mathcal{MA}\,,\ A \mapsto (A \xrightarrow{1} A)$$

has two successive left adjoints given by

$$(A_1 \xrightarrow{f} A_0) \mapsto A_0\,,\ A \mapsto (0 \to A)$$

and two successive right adjoints given by

$$(A_1 \xrightarrow{f} A_0) \mapsto A_1\,,\ A \mapsto (A \to 0).$$

These functors induce a chain of S-adjoint functors

$$Q_{-1} \dashv P_0 \dashv Q_0 \dashv P_1 \dashv Q_1$$

between $\mathcal{DA}$ and $\mathcal{DMA}$, Q_0 being induced by $A \mapsto (A \xrightarrow{1} A)$. It is clear that $1 \xrightarrow{\sim} P_0 Q_{-1}$, $1 \xrightarrow{\sim} P_1 Q_0$ and $P_1 Q_1 \xrightarrow{\sim} 1$, which means that Q_{-1}, Q_0 and Q_1 are fully faithful. Moreover for each complex K, there is a triangle

$$Q_{-1}P_0K \to K \to Q_1P_1K \to SQ_{-1}P_0K\,,$$

which shows that $\operatorname{Ker} P_1 = \operatorname{Im} Q_{-1}$ since the suspension functor of $\mathcal{DMA}$ is fully faithful. In order to construct the missing left adjoint P_{-1}, we use that $\mathcal{DMA} \xleftarrow{\sim} \mathcal{DIA}$ (10.1). P_{-1} is 'induced' by

$$\mathrm{Cok} : \mathcal{IA} \to \mathcal{A}\,,\ (A_1 \xrightarrow{i} A_0) \mapsto \mathrm{Cok}\, i.$$

We have thus shown that $Q_0 : \mathcal{DA} \to \mathcal{DMA}$ satisfies (R). By the following lemma Q_0 also satisfies (E). We infer that $\mathcal{T} = \mathcal{DA}^\wedge$ is epivalent.

Lemma.

a) Suppose (E) holds. Then (R) holds and the functors Q_{-1}, Q_1 and Q_0 induce S-equivalences from $\mathcal{S}_0$ to the full subcategories of $\mathcal{S}_1$ consisting of the X with $P_1X = 0$, $P_0X = 0$ and with invertible MX, respectively. Moreover for each $X \in \mathcal{S}_1$ there is an exact sequence

$$\mathcal{S}_0\,(SP_1X, P_0Y) \to \mathcal{S}_1\,(X, Y) \xrightarrow{M} \mathcal{M}\mathcal{S}_0\,(MX, MY) \to 0\,,$$

which is functorial in $Y \in \mathcal{S}_1$.

b) Suppose (R) holds. For each $X \in \mathcal{S}_1$ there is a unique morphism ζX such that the S-sequence

$$Q_0P_1X \to X \to Q_{-1}P_{-1}X \xrightarrow{\zeta X} SQ_0P_1X$$

is a triangle. There is a triangle

$$P_1X \xrightarrow{MX} P_0X \to P_{-1}X \to SP_1X$$

functorial in $X \in \mathcal{S}_1$. There is a canonical isomorphism $\eta : P_{-1}Q_1 \xrightarrow{\sim} S$. If the suspension functor of $\mathcal{S}_1$ is fully faithful, then (E) holds.

Proof. a) *Construction of Q_{-1}:* Let $Y \in \mathcal{S}_0$. We choose $X \in \mathcal{S}_1$ such that there is an isomorphism

$$\begin{array}{ccc} 0 & \to & Y \\ \downarrow & & \downarrow f_0 \\ P_1X & \to & P_0X. \end{array}$$

Since M is an epivalence, f_0 yields a surjection

$$\mathcal{S}_1\,(X, U) \to \mathcal{S}_0\,(Y, P_0U)\,,\ g \mapsto (P_0g)f_0.$$

Suppose that g is mapped to 0. We form a triangle

$$X \xrightarrow{g} U \xrightarrow{h} V \to SX$$

in $\mathcal{S}_1$. Since $P_0g = 0$, P_0h admits a retraction. Since $P_0X = 0$, P_1h is invertible. Hence Mh admits a retraction. Since M is an epivalence, this implies that h admits a retraction, so $g = 0$. Using 6.7 we complete Q_{-1} to a left S-adjoint. By the construction, we have $P_1Q_{-1} = 0$ and $1 \xrightarrow{\sim} P_0Q_{-1}$. If $P_1U = 0$, then the construction shows that the image of $U \to Q_{-1}P_0U$ under M is invertible. Since M detects isomorphisms, $U \xrightarrow{\sim} Q_{-1}P_0U$, so $\operatorname{Ker} P_1 = \operatorname{Im} Q_{-1}$. The *construction of* Q_1 is similar. It shows that Q_1 is an equivalence of $\mathcal{S}_0$ onto the full subcategory of $\mathcal{S}_1$ consisting of the X with $P_0X = 0$. – Since Q_0 is fully faithful, MQ_0X is invertible for all $Y \in \mathcal{S}_0$. Conversely, if MX is invertible, then the image of $Q_0P_1X \to X$ under M is invertible.

Construction of P_{-1}*:* Let $X \in \mathcal{S}_1$. We form a triangle

$$Q_0P_1X \to X \to U \to SQ_0P_1X$$

over the adjunction morphism. The associated long exact Hom-sequence shows that $\operatorname{Hom}(X, Q_{-1}Y) \xleftarrow{\sim} \operatorname{Hom}(U, Q_{-1}Y)$ for each $Y \in \mathcal{S}_0$ since $P_1Q_{-1} = 0$. Applying P_1 to the triangle we see that $P_1U = 0$ hence $U \in \operatorname{Im} Q_{-1}$. Since Q_{-1} is fully faithful, we can conclude that P_{-1} exists as an additive functor. Using 6.7 we turn it into a left S-adjoint. Now let $X \in \mathcal{S}_1$, $a = -Q_{-1}MX$ and let $b : Q_{-1}P_1X \to Q_0P_1X$ be the morphism such that P_0b is the composition

$$P_0Q_{-1}P_1X \xleftarrow{\sim} P_1X \xrightarrow{\sim} P_0Q_0P_1X.$$

We form a triangle

$$Q_{-1}P_1X \xrightarrow{[a\ b]^t} Q_{-1}P_0X \oplus Q_0P_1X \xrightarrow{c} U \to SQ_{-1}P_1X.$$

Let

$$e : Q_{-1}P_0X \oplus Q_0P_1X \to X$$

have the adjunction morphisms as components. Then $e[a\ b]^t = 0$, so $e = fc$ for some f. Since the images of the triangle under P_0 and P_1 are split exact sequences, Mf is invertible. So f is invertible and we have a triangle

$$Q_{-1}P_1X \to Q_{-1}P_0X \oplus Q_0P_1X \to X \to SQ_{-1}P_1X.$$

Applying $\mathcal{S}_1(?, Y)$ to this triangle we get the sequence of the assertion.

b) As in [2, 1.4.3] one sees that the 'morphism of degree 1' is unique if it exists. To derive the first triangle, we form a triangle

$$Q_0P_1X \to X \to U \to SQ_0P_1X$$

over the adjunction morphism. Applying P_1 to the triangle we see that $P_1U = 0$ hence $U \in \operatorname{Im} Q_{-1}$. Since the triangle also shows that

$$\operatorname{Hom}(X, Q_{-1}Y) \xleftarrow{\sim} \operatorname{Hom}(U, Q_{-1}Y)\,,\ \forall Y \in \mathcal{S}_0$$

this implies that U is canonically isomorphic to $Q_{-1}P_{-1}X$. We obtain the second triangle by applying P_0 to the first. If we apply the second triangle to $X = Q_1 Y$ we get an isomorphism $P_{-1}Q_1Y \to SP_1Q_1Y$ since $P_0Q_1Y = 0$ by adjunction. The required isomorphism is the composition

$$P_{-1}Q_1Y \to SP_1Q_1Y \to SY.$$

Now suppose the suspension functor of $\mathcal{S}_1$ is fully faithful. Let $X \in \mathcal{S}_1$. We form a triangle

$$X \to Q_1P_1X \to U \to SX$$

in $\mathcal{S}_1$. Applying P_1 we see that $P_1U = 0$, hence $U \in \operatorname{Im} Q_{-1}$. Moreover

$$\operatorname{Hom}(Q_{-1}Y, U) \xrightarrow{\sim} \operatorname{Hom}(Q_{-1}Y, SX)\,,\ \forall\, Y \in \mathcal{S}_0.$$

So U is canonically isomorphic to $Q_{-1}P_0SX$ and we have a triangle

$$Q_{-1}P_0X \to X \to Q_1P_1X \to SQ_{-1}P_0X.$$

Now it is obvious that a morphism $f : X \to X'$ is invertible iff P_0f and P_1f are invertible, i.e. M detects isomorphisms. In fact, this was all we needed besides condition (R) to derive the exact sequence of a). Hence M is an epivalence.

6.2 Suppose $Q_0 : \mathcal{S} \to \mathcal{T}$ satisfies condition (E) and $X, Y \in \mathcal{S}_1$.

Lemma.

a) The map

$$\mathcal{S}_1(X, Y) \to \mathcal{M}\mathcal{S}_0(MX, MY)$$

is bijective if $\mathcal{S}_0(SP_1X, P_0Y) = 0$. *We have*

$$\mathcal{S}_1(S^kX, Y) = 0\,,\ \forall\, k > 0$$

if $\mathcal{S}_0(S^kP_1X, P_1Y) = \mathcal{S}_0(S^kP_0X, P_0Y) = \mathcal{S}_0(S^kP_1X, P_0Y) = 0$ *for each* $k > 0$.

b) If

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

are morphisms of $\mathcal{S}_1$ *such that* $gf = 0$, *that there are triangles*

$$P_1X \xrightarrow{P_1f} P_1Y \xrightarrow{P_1g} P_1Z \to SP_1X\,,\ P_0X \xrightarrow{P_0f} P_0Y \xrightarrow{P_0g} P_0Z \to SP_0X$$

in $\mathcal{S}_0$ *and that* $\mathcal{S}_0\,(SP_1X, P_1Z) = \mathcal{S}_0\,(SP_0X, P_0Z) = 0$*, then there is a triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \to SX$$

in $\mathcal{S}_1$.

Proof. a) is an immediate consequence of 6.1 a). b) Let

$$X \xrightarrow{f} Y \xrightarrow{h} U \xrightarrow{k} SX$$

be a triangle in $\mathcal{S}_1$. Its 'images' under P_1 and P_0 are isomorphic to the given triangles. Hence there are isomorphisms $i_1 : P_1U \to P_1Z$ and $i_0 : P_0U \to P_0Z$ with $i_1\,P_1h = P_1g$ and $i_0\,P_0h = P_0g$, respectively. Since $\mathcal{S}_0\,(SP_1X, P_1Z) = 0 = \mathcal{S}_0\,(SP_0X, P_0Z)$, i_0 and i_1 are uniquely determined by these equalities. Now $gf = 0$, so $g = jh$ for some $j : U \to Z$. We have $P_0j = i_0$ and $P_1j = i_1$. Since M is an epivalence, j is invertible. Hence (f, g, kj^{-1}) is a triangle.

6.3 We prepare for the proof of the redundancy of the connecting morphisms (section 7). Let

$$\begin{array}{ccc} \mathcal{S}_0 & \xrightarrow{F_0} & \mathcal{S}'_0 \\ Q_0 \downarrow & & \downarrow Q_0 \\ \mathcal{S}_1 & \xrightarrow{F_1} & \mathcal{S}'_1 \end{array}$$

be a diagram of suspended categories and S-functors and let

$$\mu : F_1Q_0 \xrightarrow{\sim} Q_0F_0$$

be an isomorphism of S-functors. We assume that both functors Q_0 satisfy condition (R) of 6.1, that the morphisms

$$P_0F_1 \to F_0P_0\,,\ P_1F_1 \leftarrow F_0P_1$$

associated (A.4) with μ and μ^{-1} are invertible and that the morphisms

$$F_1Q_{-1} \leftarrow Q_{-1}F_0\,,\ F_1Q_1 \to Q_1F_0$$

associated with the inverses of the above morphisms are also invertible. We now infer from the first triangle of 6.1 b) that the associated morphism

$$P_{-1}F_1 \to F_0P_{-1}$$

is also invertible.

Remark. If the functors Q_0 satisfy condition (E) and the morphisms

$$P_0F_1 \to F_0P_0\,,\; P_1F_1 \leftarrow F_0P_1$$

are invertible, the morphisms

$$F_1Q_{-1} \leftarrow Q_{-1}F_0\,,\; F_1Q_1 \to Q_1F_0$$

are automatically invertible (apply M to these morphisms and use 6.1 a).

Let $\varphi_0 : SF_0 \to F_0S$ be the commutation isomorphism. A straightforward verification shows that the diagram

$$\begin{array}{ccccc} F_0S & & \xrightarrow{\varphi_0} & & SF_0 \\ F_0\eta \downarrow & & & & \downarrow \eta F_0 \\ F_0P_{-1}Q_1 & \xrightarrow{\sim} & P_{-1}F_1Q_1 & \xrightarrow{\sim} & P_{-1}Q_1F_0 \end{array} \qquad (*)$$

is commutative. This means that φ_0 is uniquely determined by the η and by the underlying additive functors of the S-functors at hand. We make this more precise. Suppose that we are given *additive* functors

$$F_0 : \mathcal{S}_0 \to \mathcal{S}_0' \text{ and } F_1 : \mathcal{S}_1 \to \mathcal{S}_1'$$

and an isomorphism of functors

$$\mu : Q_0F_0 \xrightarrow{\sim} F_1Q_0.$$

We assume that

a) the associated morphisms $P_0F_1 \to F_0P_0$, $P_1F_0 \to F_0P_1$ are invertible,

b) the morphisms $F_1Q_{-1} \to Q_{-1}F_0$, $F_1Q_1 \to Q_1F_0$ associated to the inverses of the morphisms of a) are invertible and

c) the associated morphism $P_{-1}F_1 \to F_0P_{-1}$ is invertible.

Observe that b) follows from a) if the functors Q_0 satisfy condition (E) (apply M to the morphisms of b). Moreover, assuming a) and b), a sufficient condition for c) is that for each triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$$

of $\mathcal{S}_1$ there is *some* triangle

$$F_1X \xrightarrow{F_1u} F_1Y \xrightarrow{F_1v} F_1Z \xrightarrow{w'} SF_1X$$

of $\mathcal{S}_1'$. This follows from the unicity of the first triangle of 6.1 b).

Under the hypotheses a), b) and c), (F_0, φ_0) *is an* S*-functor if* φ_0 *is defined by diagram* $(*)$ *and* Q_0 *satisfies condition (E).* The proof is a straightforward verification based on 6.1 b). We shall also need to know that (F_0, φ_0) 'functorially depends' on the triple consisting of F_0, F_1, and μ. That is to say that if we are given another triple

$$G_0 : \mathcal{S}_0 \to \mathcal{S}_0' \,,\; G_1 : \mathcal{S}_1 \to \mathcal{S}_1' \,,\; \nu : Q_0G_0 \xrightarrow{\sim} G_1Q_0$$

satisfying the analogues of a), b) and c), and if

$$\alpha_0 : F_0 \to G_0 \text{ and } \alpha_1 : F_1 \to G_1$$

are morphisms of functors such that the diagram

$$\begin{array}{ccc} Q_0F_0 & \xrightarrow{\mu} & F_1Q_0 \\ Q_0\alpha_0 \downarrow & & \downarrow \alpha_1Q_0 \\ Q_0G_0 & \xrightarrow{\nu} & G_1Q_0 \end{array}$$

commutes, then α_0 *gives rise to a morphism of* S*-functors* $(F_0, \varphi_0) \to (G_0, \gamma_0)$. Again we omit the straightforward proof. Finally we have to consider compositions of functors. Suppose that we are given additive functors

$$\mathcal{S}_0 \xrightarrow{F_0} \mathcal{S}_0' \xrightarrow{G_0} \mathcal{S}_0'' \,,\; \mathcal{S}_1 \xrightarrow{F_1} \mathcal{S}_1' \xrightarrow{G_1} \mathcal{S}_1''$$

and commutation isomorphisms

$$\mu : Q_0F_0 \to F_1Q_0 \,,\; \nu : Q_0G_0 \to G_1Q_0$$

yielding invertible associated morphisms. Then the triple consisting of G_0F_0, G_1F_1 and $(G_1\mu)(\nu F_0)$ yields an S-functor (G_0F_0, θ). It is not hard to verify that *this* S*-functor is the composition of the* S*-functors* (G_0, γ_0) *and* (F_0, φ_0) *constructed from* F_0*,* F_1*,* μ *and* G_0*,* G_1*,* ν*,* respectively.

6.4 Let $\mathcal{A}$ be an exact category. We have the functors

$$\begin{array}{ll} Q_{-1} : \mathcal{A} \to \mathcal{MA} \,,\; A \mapsto (0 \to A) \,, & P_0 : \mathcal{MA} \to \mathcal{A} \,,\; (A_1 \to A_0) \mapsto A_0 \,, \\ Q_0 : \mathcal{A} \to \mathcal{MA} \,,\; A \mapsto (A \xrightarrow{1} A) \,, & P_1 : \mathcal{MA} \to \mathcal{A} \,,\; (A_1 \to A_0) \mapsto A_1 \,, \\ Q_1 : \mathcal{A} \to \mathcal{MA} \,,\; A \mapsto (A \to 0). & \end{array}$$

Moreover if $X = (A \xrightarrow{i} B)$ is an inflation, we have a well defined $P_{-1}X = \operatorname{Cok} i$ in $\mathcal{A}$. Now let $Q_0 : \mathcal{S}_0 \to \mathcal{S}_1$ be an S-functor satisfying condition (R), $D_0 : \mathcal{A} \to \mathcal{S}_0$ and $D_1 : \mathcal{MA} \to \mathcal{S}_1$ ∂-functors and $\lambda : Q_0 D_0 \xrightarrow{\sim} D_1 Q_0$ an isomorphism of ∂-functors. As in 6.3, we require that the associated morphisms

$$P_0 D_1 \to D_0 P_1 \ , \ P_1 D_1 \leftarrow D_0 P_1 \ , \ D_1 Q_{-1} \leftarrow Q_{-1} D_0 \ , \ D_1 Q_1 \to Q_1 D_0$$

are invertible. If $Q_0 : \mathcal{S}_0 \to \mathcal{S}_1$ even satisfies condition (E), then we only have to require this for the first pair of morphisms. It follows for the second pair by 6.1 a). If

$$\varepsilon : A \xrightarrow{i} B \xrightarrow{d} C$$

is a conflation of $\mathcal{A}$, we have a conflation

$$\varepsilon^+ : \qquad \begin{array}{ccccc} A_1 & \xrightarrow{1} & A_1 & \to & 0 \\ \| & & \downarrow i & & \downarrow \\ A_1 & \xrightarrow{i} & A_0 & \to & \operatorname{Cok} i \end{array}$$

in $\mathcal{MA}$. Its 'codification' is

$$Q_0 P_1 X \to X \to Q_{-1} P_{-1} X \ ,$$

where $X = (A \xrightarrow{i} B)$. The image of the morphism $Q_0 P_1 X \to X$ under D_1 identifies with $Q_0 P_1 D_1 X \to D_1 X$ by assumption. So the 'image' of the conflation is isomorphic to the first triangle of 6.1 b) applied to $D_1 X$. We obtain an isomorphism

$$Q_{-1} P_{-1} D_1 X \xrightarrow{\sim} D_1 Q_{-1} P_{-1} X \ ,$$

hence an isomorphism $P_{-1} D_1 X \to D_0 P_{-1} X$, which is easily seen to be associated with the above $D_1 Q_{-1} \xleftarrow{\sim} Q_{-1} D_0$. Thus, as for S-functors, the associated morphism

$$P_{-1} D_1 \to D_0 P_{-1}$$

is invertible whenever it is defined. Moreover we have commutative diagrams

$$\begin{array}{ccc} Q_{-1} P_{-1} D_1 X & \xrightarrow{\zeta D_1 X} & S Q_0 P_1 D_1 X \\ \downarrow & & \uparrow \\ D_1 Q_{-1} P_{-1} X & \xrightarrow{\delta_1 \varepsilon^+} & S D_1 Q_0 P_1 X \end{array}$$

and

$$\begin{array}{ccccc} P_0 D_1 Q_{-1} P_{-1} X & \xrightarrow{P_0 \delta_1 \varepsilon^+} & P_0 S D_1 Q_0 P_1 X & \xrightarrow{\sim} & S P_0 D_1 Q_0 P_1 X \\ \downarrow & & & & \downarrow \\ D_0 C & & \xrightarrow{\delta_0 \varepsilon} & & S D_0 A \end{array} \qquad (*)$$

which show how to compute $\delta_0\varepsilon$ from D_1X. Now suppose that $D_0 : \mathcal{A} \to \mathcal{S}_0$ and $D_1 : \mathcal{MA} \to \mathcal{S}_1$ are *additive* functors and $\lambda : D_1Q_0 \to Q_0D_0$ is an isomorphism of functors. We assume that

a) the associated morphisms $P_0D_1 \to D_0P_0$, $P_1D_1 \to D_0P_1$ are invertible,

b) the morphisms $D_1Q_{-1} \to Q_{-1}D_0$, $D_1Q_1 \to Q_1D_0$ are invertible and

c) for each inflation $X = (A \xrightarrow{i} B)$ the morphism $P_{-1}D_1X \to D_0P_{-1}X$ is invertible.

Observe that b) follows from a) if the functor Q_0 satisfies condition (E). Moreover, assuming a) and b), a sufficient condition for c) is that for each conflation

$$X \xrightarrow{j} Y \xrightarrow{q} Z$$

of $\mathcal{MA}$ there is *some* triangle

$$D_1X \xrightarrow{D_1j} D_1Y \xrightarrow{D_1q} D_1Z \to SD_1X.$$

Under the hypotheses a), b) and c), one can verify that the diagrams $(*)$ *define* a 'connecting morphism' δ_0 such that (D_0, δ_0) is a ∂-functor. Moreover this construction transforms 'morphisms of triples D_0, D_1, λ' to morphisms of ∂-functors and is compatible with 'composition with exact functors from the right' and with 'composition with S-functors from the left'. All of these statements are easy to make precise (compare 6.3) and to prove. We omit the details.

6.5 We prepare for the proof of the theorem on left adjoints (2.8). We use the notations and hypotheses of the beginning of 6.3. In addition, we assume that both functors Q_0 satisfy condition (E).

Lemma. *If $F_0 : \mathcal{S}_0 \to \mathcal{S}'_0$ admits a left S-adjoint G_0, then F_1 admits a left S-adjoint G_1. The morphisms*

$$G_1Q_0 \to Q_0G_0\ ,\ G_0P_0 \to P_0G_1\ ,\ G_0P_1 \to P_1G_1$$

associated with

$$Q_0F_0 \to F_1Q_0\ ,\ P_0F_1 \to F_0P_0\ ,\ P_1F_1 \leftarrow F_0P_1$$

are invertible.

Proof. Let $X \in \mathcal{S}'_1$. The functor $\mathcal{S}'_1(X, F_1\,?)$ is represented by $S^k Q_0 G_0 Y$ for $X = S^k Q_0 Y$ and by $S^k Q_{-1} G_0 Y$ for $X = S^k Q_{-1} Y$. Since for any $X \in \mathcal{S}'_1$ we have a triangle (cf. proof of 6.1 a)

$$Q_{-1}P_1X \to Q_{-1}P_0X \oplus Q_0P_1X \to X \to SQ_{-1}P_1X\ ,$$

we conclude from 6.7 that the functor is representable for arbitrary X and that F_1 has a left S-adjoint. Using that F_1 'commutes' with the right adjoints of Q_0, P_0 and P_1 it is easy to see that the associated morphisms are invertible.

6.6 We prove the results about S-adjoints that we have been using in this section. Let $\mathcal{S}$ and $\mathcal{T}$ be suspended categories. We study right S-adjoints. Let $(L, \lambda) : \mathcal{S} \to \mathcal{T}$ be an S-functor, R a right adjoint of the underlying additive functor of L and $\Phi : LR \to 1_{\mathcal{T}}$, $\Psi : 1_{\mathcal{S}} \to RL$ compatible adjunction morphisms.

Lemma. *If the composition $\overline{\rho} = (RS\Phi)(R\lambda R)(\Psi SR)$ is invertible and $\rho = \overline{\rho}^{-1}$, then (R, ρ) is a right S-adjoint to (L, λ).*

We omit the proof since it is quite similar to that of lemma 6.7

6.7 We study left S-adjoints. Let $\mathcal{S}$, $\mathcal{T}$ be suspended categories and $(R, \rho) : \mathcal{S} \to \mathcal{T}$ an S-functor. We say that the left adjoint L is *defined* in $X \in \mathcal{T}$ if the functor $\mathcal{T}(X, R\,?)$ is representable, i.e. if there is an object $LX \in \mathcal{S}$ and a morphism $\Psi X : X \to RLX$ which induces a bijection

$$(\Psi X)^* R(LX,\,?) : \mathcal{S}(LX,\,?) \xrightarrow{\sim} \mathcal{T}(X, R\,?)\ ,\ f \mapsto (Rf)(\Psi X).$$

If L is defined in X and in SX, we have a canonical morphism $\lambda X : LSX \to SLX$ defined by

$$(R\lambda)(\Psi SX) = (\rho^{-1}L)(S\Psi X).$$

Let $\mathcal{D}$ be the full subcategory of $\mathcal{T}$ consisting of the objects X satisfying

- L is defined in $S^n X$, $\forall\, n \in \mathbf{N}$ and
- $\lambda S^n X$ is invertible, $\forall\, n$.

By definition we have $S\mathcal{D} \subset \mathcal{D}$. The following lemma shows that $\mathcal{D}$ is a suspended subcategory of $\mathcal{T}$ and that $(L, \lambda) : \mathcal{D} \to \mathcal{S}$ is an S-functor. If

$R\mathcal{S} \subset \mathcal{D}$, then (R, ρ) and (L, λ) yield a pair of S-adjoint functors between $\mathcal{S}$ and $\mathcal{D}$.

Lemma. (cf. [10, 1.5]) *If X and Y lie in $\mathcal{D}$ and*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$$

is a triangle of $\mathcal{T}$, then Z lies in $\mathcal{D}$ and

$$LX \xrightarrow{Lu} LY \xrightarrow{Lv} LZ \xrightarrow{(\lambda X)(Lw)} SLX$$

is a triangle of $\mathcal{S}$.

Proof. *1st step: L is defined in $S^n Z$, $\forall\, n \in \mathbf{N}$.* We form a triangle

$$LX \xrightarrow{Lu} LY \xrightarrow{v'} Z' \xrightarrow{w'} SLX$$

in $\mathcal{S}$. By SP3 [10, 1.1] there is a morphism of triangles

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & SX \\ \Psi X \downarrow & & \Psi Y \downarrow & & f \downarrow & & \downarrow S\Psi X \\ RLX & \xrightarrow{RLu} & RLY & \xrightarrow{Rv'} & RZ' & \xrightarrow{(\rho L)(Rw')} & SRLX. \end{array}$$

It yields a morphism of exact sequences

$$\begin{array}{ccccccccc} \mathcal{T}(X, R?) & \leftarrow & \mathcal{T}(Y, R?) & \leftarrow & \mathcal{T}(Z, R?) & \leftarrow & \mathcal{T}(SX, R?) & \leftarrow & \mathcal{T}(SY, R?) \\ \uparrow \alpha_1 & & \uparrow \alpha_2 & & \uparrow \alpha_3 & & \uparrow \alpha_4 & & \uparrow \alpha_5 \\ \mathcal{S}(LX, ?) & \leftarrow & \mathcal{S}(LY, ?) & \leftarrow & \mathcal{S}(Z', ?) & \leftarrow & \mathcal{S}(SLX, ?) & \leftarrow & \mathcal{S}(SLY, ?), \end{array}$$

where α_1 and α_2 are given by $\Psi^* R(L, ?)$, α_4 and α_5 by

$$(S\Psi)^* (\rho^{-1} L)^* R(SL, ?)$$

and α_3 by $f^* R(Z', ?)$. By assumption, the evaluations of $\Psi^* R(L, ?)$ and

$$(S\Psi)^* (\rho^{-1} L)^* R(SL, ?) = (\Psi S)^* (R\lambda)^* R(SL, ?) = (\Psi S)^* R(LS, ?) \lambda^*$$

at X resp. Y are invertible. By the 5-lemma, α_3 is invertible, so L is defined in X. Since $(Sv, Sw, -Su)$ is a triangle as well, we can use the same argument to conclude that L is defined in SX … .

2nd step: $(Lu, Lv, (\lambda X)(Lw))$ is a triangle. In the above notations, there is an isomorphism $g : LZ \to Z'$ such that $(Rg)(\Psi Z) = f$. We claim that

$$\begin{array}{ccccccc} LX & \xrightarrow{Lu} & LY & \xrightarrow{Lv} & LZ & \xrightarrow{(\lambda X)(Lw)} & SLX \\ \| & & \| & & \downarrow g & & \| \\ LX & \xrightarrow{Lu} & LY & \xrightarrow{v'} & Z' & \xrightarrow{w'} & SLX \end{array}$$

is a morphism of S-sequences, which implies that the first row is a triangle of $\mathcal{T}$. Indeed, $g(Lv) = v'$ follows from

$$(Rg)(RLv)(\Psi Y) = (Rg)(\Psi Z)v = fv = (Rv')(\Psi Y)$$

and $w'g = (\lambda X)(Lw)$ from

$$\begin{aligned}(Rw')(Rg)(\Psi Z) &= (Rw')f = (\rho LX)^{-1}(S\Psi X)w \\ &= (R\lambda X)(\Psi SX)w = (R\lambda X)(RLw)(\Psi Z).\end{aligned}$$

3rd step: Z lies in $\mathcal{D}$. Since λZ occurs in the morphism of triangles

$$\begin{array}{ccccccc} LSX & \xrightarrow{LSu} & LSY & \xrightarrow{LSv} & LSZ & \xrightarrow{-(\lambda SX)(LSw)} & SLSX \\ \downarrow \lambda X & & \downarrow \lambda Y & & \downarrow \lambda Z & & \downarrow S\lambda X \\ SLX & \xrightarrow{SLu} & SLY & \xrightarrow{SLv} & SLZ & \xrightarrow{-(S\lambda X)(SLw)} & SSLX \end{array},$$

λZ is invertible. The same argument shows that $\lambda S^n Z$ is invertible for each $n \in \mathbf{N}$. The assertion now follows from the 1st step.

7. Redundancy of the connecting morphisms

7.1 Let $\mathcal{A}$ be an exact category and $\mathcal{S}$, $\mathcal{T}$ epivalent towers of suspended categories. We denote the underlying towers of additive categories of $\mathcal{A}^\wedge$, $\mathcal{S}$ and $\mathcal{T}$ by $\mathcal{A}^\wedge|$, $\mathcal{S}|$ and $\mathcal{T}|$, respectively.

Lemma.

a) *The forgetful functor*

$$Hom_S(\mathcal{S}, \mathcal{T}) \to Hom_{add}(\mathcal{S}|, \mathcal{T}|)$$

is an isomorphism onto the full subcategory consisting of the towers F such that for each triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$$

of $\mathcal{S}_n$ there is some triangle

$$F_nX \xrightarrow{F_nu} F_nY \xrightarrow{F_nv} F_nZ \xrightarrow{w'} SF_nX$$

of $\mathcal{T}_n$.

b) The forgetful functor

$$Hom_\partial(\mathcal{A}^\wedge, \mathcal{S}) \to Hom_{add}(\mathcal{A}^\wedge|, \mathcal{S}|)$$

is an isomorphism onto the full subcategory consisting of the towers D such that for each conflation

$$A \xrightarrow{i} B \xrightarrow{d} C$$

of $\mathcal{A}_n^\wedge$ there is some triangle

$$D_n A \xrightarrow{D_n i} D_n B \xrightarrow{D_n d} D_n C \to S D_n A$$

of $\mathcal{S}_n$.

Proof. In order to produce an inverse of the forgetful functor, we consider the functor $\Theta : \mathcal{P} \to \mathcal{P}$ which associates $\mathcal{P}_{n+1}$ to $\mathcal{P}_n$ and maps a morphism $a : \mathcal{P}_m \to \mathcal{P}_n$ to

$$\Theta a : \mathcal{P}_{m+1} \to \mathcal{P}_{n+1}\ ,\ (x_1, \ldots, x_{m+1}) \mapsto (x_1, a(x_2, \ldots, x_{m+1})).$$

We have a natural transformation $\varphi : \Theta \to 1_\mathcal{P}$ whose value at $\mathcal{P}_n$ is

$$q_0^1 : \mathcal{P}_{n+1} \to \mathcal{P}_n\ ,\ (x_1, \ldots, x_{n+1}) \mapsto (x_2, \ldots, x_{n+1}).$$

Now Θ is really a 2-functor and Θ is a morphism of 2-functors. So we obtain a morphism of towers

$$\mathcal{S}\varphi : \mathcal{S} \to \mathcal{S}\Theta$$

whose components are the

$$Q_0^1 : \mathcal{S}_n \to \mathcal{S}_{n+1}.$$

Of course we have $\mathcal{S}|\Theta = \mathcal{S}\Theta|$. Let $F : \mathcal{S}| \to \mathcal{T}|$ be a tower as in the assertion. We have a diagram

$$\begin{array}{ccc} \mathcal{S}| & \xrightarrow{F} & \mathcal{T}| \\ \mathcal{S}\varphi \downarrow & & \downarrow \mathcal{T}\varphi \\ \mathcal{S}\Theta| & \xrightarrow{F\Theta} & \mathcal{T}\Theta| \end{array}$$

of towers of additive categories and an isomorphism

$$F\varphi : (\mathcal{T}\varphi)F \to F(\Theta\mathcal{S}\varphi).$$

We apply 6.3 to complete each F_n to an S-functor $\tilde{F}_n$ and to combine the $\tilde{F}_n$ into a tower of S-functors $\tilde{F}$. The proof of b) is completely analogous. We omit the details.

7.2 We use the notations and hypotheses of Theorem 4. We denote the compositions of $\mathcal{D}$ and $\mathcal{F}$ with the forgetful 2-functor $\mathcal{U} : \mathcal{S}usp \to \mathcal{A}dd$ by $\mathcal{D}|$ and $\mathcal{F}|$.

Lemma.

a) *The forgetful functor*

$$Hom_S(\mathcal{D}, \mathcal{F}) \to Hom_{add}(\mathcal{D}|, \mathcal{F}|)$$

is an isomorphism.

b) *The forgetful functor*

$$Hom_\partial(\mathcal{F}) \to Hom_{add}(\mathcal{U}, \mathcal{F}|)$$

is an isomorphism.

Proof. a) We produce an inverse of the forgetful functor. For each exact $\mathcal{A}$ we consider the functor

$$Q : \mathcal{A} \to \mathcal{MA}.$$

If $F : \mathcal{D}| \to \mathcal{F}|$ is given, we have a diagram

$$\begin{array}{rcl} \mathcal{DA} & \overset{F\mathcal{A}}{\to} & \mathcal{FA} \\ \mathcal{D}Q \downarrow & & \downarrow \mathcal{F}Q \\ \mathcal{DMA} & \overset{F\mathcal{MA}}{\to} & \mathcal{FMA} \end{array}$$

and an isomorphism

$$FQ : (\mathcal{FMA})(\mathcal{D}Q) \xrightarrow{\sim} (\mathcal{F}Q)(F\mathcal{A}).$$

By example 6.1 c), $\mathcal{D}Q$ satisfies condition (E) and similarly we see that $\mathcal{F}Q$ satisfies condition (R). It is clear that conditions 6.3 a) and b) are satisfied since the functors P_0, P_1, Q_1 and Q_{-1} are all induced by exact functors between $\mathcal{A}$ and $\mathcal{MA}$ (compare 6.4). Since by assumption $\mathcal{F}$ and $\mathcal{D}$ carry the inclusion $\mathcal{IA} \to \mathcal{MA}$ to an equivalence, the functor P_{-1} is 'induced' by the cokernel functor $\mathcal{IA} \to \mathcal{A}$ (cf. 6.4) and therefore condition 6.3 c) is also satisfied.

So we can complete $F\mathcal{A}$ to an S-functor $\tilde{F}\mathcal{A}$ and combine the $\tilde{F}\mathcal{A}$ into a 1-morphism $\tilde{F} : \mathcal{D} \to \mathcal{F}$. The proof of b) is completely analogous. We omit the details.

8. Proof of Theorem 2.7

8.1 We prove b). It is not hard to see (use [2, 1.1.9]) that under the hypotheses of b), morphisms of ∂-functors from D_0 to D_0' bijectively correspond to morphisms between the underlying additive functors. Therefore, according to 7.1 b), we only have to show that the functor

$$Hom_{add}\,(\mathcal{A}^\wedge|, \mathcal{T}|) \to \mathcal{A}dd\,(\mathcal{A}|, \mathcal{T}_0|)\ ,\ D| \mapsto D|_0\ ,$$

where the $|$ denotes underlying additive categories resp. functors, induces a bijection

$$\mathrm{Hom}\,(D|, D'|) \to \mathrm{Hom}\,(D|_0, D'|_0).$$

From now on we omit the $|$. We factor the above functor as

$$Hom_{add}\,(\mathcal{A}^\wedge, \mathcal{T}) \to Hom_{add}\,(\mathcal{A}^\wedge, \mathcal{T}_0^\wedge) \to Hom\,(\mathcal{A}, \mathcal{T}_0)\ ,$$

where the first functor is induced by $\Phi : \mathcal{T} \to \mathcal{T}_0^\wedge$ (Example 2.3 b). The second functor is an equivalence by Example 2.3 b). So it remains to be shown that Φ induces a bijection

$$\mathrm{Hom}\,(D, D') \to \mathrm{Hom}\,(\Phi D, \Phi D').$$

By A.5 it is enough to check this locally, i.e. we have to show that $\Phi_n : \mathcal{T}_n \to (\mathcal{T}_0^\wedge)_n$ induces a bijection

$$\mathrm{Hom}\,(D_n\mathcal{T}a, D_n'\mathcal{T}a) \to \mathrm{Hom}\,(\Phi_n D_n\mathcal{T}a, \Phi_n D_n'\mathcal{T}a)$$

for each morphism $a : \mathcal{P}_m \to \mathcal{P}_n$. This certainly holds if for all $X, Y \in \mathcal{A}_n^\wedge$, the map

$$\mathrm{Hom}\,(D_nX, D_n'Y) \to \mathrm{Hom}\,(\Phi_n D_nX, \Phi_n D_n'Y)$$

is bijective. This follows from lemma 6.2 a) by induction.

8.2 We prove a). The functor D yields $D^\wedge : \mathcal{A}^\wedge \to \mathcal{T}_0^\wedge$. Let $\mathcal{V}_n \subset (\mathcal{T}_0^\wedge)_n$ be the image of $D_n^\wedge$ and $\mathcal{U}_n \subset \mathcal{T}_n$ the preimage of $\mathcal{V}_n$ under $\Phi_n : \mathcal{T}_n \to (\mathcal{T}_0^\wedge)_n$. Obviously we have a diagram of towers of additive categories

$$\mathcal{A}^\wedge \to \mathcal{V} \leftarrow \mathcal{U} \to \mathcal{T}.$$

We shall show that $\mathcal{U} \to \mathcal{V}$ is an equivalence and that the quasiinverse provides the required D^+ upon composition with $\mathcal{A}^\wedge \to \mathcal{V}$ and $\mathcal{U} \to \mathcal{T}$. By remark A.2 we only have to show that $\mathcal{U}_n \to \mathcal{V}_n$ is an equivalence for each n. This is clear for $n = 0$. Suppose it has been shown up to $n-1$. Moreover suppose that we have shown that $\operatorname{Hom}(S^k X, Y) = 0$, $\forall\, k > 0$, $\forall\, X, Y \in \mathcal{U}_{n-1}$. We have a diagram

$$\begin{array}{ccccccc}
\mathcal{M}\mathcal{A}^\wedge_{n-1} & \longrightarrow & \mathcal{M}\mathcal{V}_{n-1} & \xleftarrow{\sim} & \mathcal{M}\mathcal{U}_{n-1} & \to & \mathcal{M}\mathcal{T}_{n-1} \\
\sim\uparrow & & \uparrow\sim & & \uparrow & & \uparrow M^1 \\
\mathcal{A}^\wedge_n & \xrightarrow{(D^\wedge_0)_n} & \mathcal{V}_n & \leftarrow & \mathcal{U}_n & \to & \mathcal{T}_n
\end{array}$$

which commutes up to isomorphism. By lemma 6.2 a), M^1 induces an equivalence $\mathcal{U}_n \to \mathcal{M}\mathcal{U}_{n-1}$ and $\operatorname{Hom}(S^k X, Y) = 0$, $\forall\, k > 0$, $\forall\, X, Y \in \mathcal{U}_n$. This implies the assertion. The tower of additive functors D^+ thus constructed obviously satisfies $D^+_0 \xrightarrow{\sim} D$. It follows by induction from lemma 6.2 b) that the image of a conflation under D^+_n can be embedded into a triangle. Hence D^+ yields a tower of ∂-functors by 7.1 b).

9. Proof of Theorem 2.6

9.1 We establish the connection between towers and presheaves. Let $\mathcal{A}$ be an exact category. We have a full embedding from $\mathcal{A}^\wedge_m$ to $\mathcal{R}\mathcal{A}$ which with $X \in \mathcal{A}^\wedge_m$ associates the rough presheaf $\mathcal{F}$ such that

- $\mathcal{F}(U) = 0$ if U does not contain $U_1 \cap \ldots \cap U_m$,
- $\mathcal{F}(U) = X(x_1, \ldots, x_m)$ if U contains $U_1 \cap \ldots \cap U_m$ and $x_i = 1$ iff U is contained in U_i.

We define $\mathcal{R}_m\mathcal{A} \subset \mathcal{R}\mathcal{A}$ to be the image of $\mathcal{A}^\wedge_m$. It identifies with the category of presheaves on the discrete set $\{1, \ldots, m\}$. By 'transport of structure' we combine the $\mathcal{R}_m\mathcal{A}$ into a tower $\mathcal{R}_*\mathcal{A}$. Note that the functor

$$Q^{m+1}_{-1} : \mathcal{R}_m\mathcal{A} \to \mathcal{R}_{m+1}\mathcal{A}$$

of example 2.2 a) coincides with the canonical embedding. Thus $\mathcal{R}\mathcal{A}$ is the limit of the direct system

$$\mathcal{R}_0\mathcal{A} \xrightarrow{Q^1_{-1}} \mathcal{R}_1\mathcal{A} \to \ldots \to \mathcal{R}_m\mathcal{A} \xrightarrow{Q^{n+1}_{-1}} \mathcal{R}_{m+1}\mathcal{A} \to \ldots \quad .$$

Now let $n \in \mathbf{N}$. Applying the above to $\mathcal{R}_n\mathcal{A}$ instead of $\mathcal{A}$ we find that the direct limit of

$$\mathcal{R}_0\mathcal{R}_n\mathcal{A} \to \mathcal{R}_1\mathcal{R}_n\mathcal{A} \to \ldots \to \mathcal{R}_m\mathcal{R}_n\mathcal{A} \to \mathcal{R}_{m+1}\mathcal{R}_n\mathcal{A} \to \ldots$$

is $\mathcal{R}\mathcal{R}_n\mathcal{A}$. We shall identify $\mathcal{R}_m\mathcal{R}_n\mathcal{A}$ with $\mathcal{R}_{m+n}\mathcal{A}$ in the canonical fashion: A presheaf $\mathcal{F}$ on

$$\{1, \ldots, m+n\} \xrightarrow{\sim} \{1, \ldots, m\} \coprod \{1, \ldots, n\}$$

is given by the presheaf $W \mapsto \mathcal{F}_W$ on $\{1, \ldots, m\}$ whose value at W is the presheaf

$$U \mapsto \mathcal{F}_W(U) = \mathcal{F}\,(W \cup U).$$

In the following paragraph we shall suppress $\mathcal{A}$ in the symbols $\mathcal{R}\mathcal{A}$, $\mathcal{R}_m\mathcal{A}$, $\mathcal{R}_*\mathcal{A}$, $\mathcal{R}_m\mathcal{R}_n\mathcal{A}$ and $\mathcal{R}\mathcal{R}_n\mathcal{A}$.

9.2 We want to prove theorem 2.6 by constructing a quasiinverse of

$$Hom_S\,(\mathcal{D}\mathcal{R}_*, \mathcal{T}) \to Hom_\partial\,(\mathcal{R}_*, \mathcal{T}).$$

Let D be a tower of ∂-functors $\mathcal{R}_* \to \mathcal{T}$. We first describe the components F_n of the image F of D under the quasiinverse. By example 6.1 a), the functors

$$Q_{-1}^{n+1} : \mathcal{T}_n \to \mathcal{T}_{n+1}$$

admit left adjoints P_{-1}^{n+1}. For $n < l$ we put

$$L_{n,l} = P_{-1}^{n+1} \ldots P_{-1}^{l-1} P_{-1}^{l} : \mathcal{T}_l \to \mathcal{T}_n.$$

We obtain a diagram

$$\begin{array}{ccccc}
\mathcal{R}_0\mathcal{R}_n & \xrightarrow{Q_{-1}^{n+1}} & \mathcal{R}_1\mathcal{R}_n \to \ldots & \mathcal{R}_m\mathcal{R}_n & \longrightarrow \ldots \\
D_n \downarrow & & \downarrow L_{n,n+1}D_{n+1} & \downarrow L_{n,n+m}D_{n+m} & \\
\mathcal{T}_n & \xrightarrow{1} & \mathcal{T}_n \quad \longrightarrow \ldots & \mathcal{T}_n & \longrightarrow \ldots\ ,
\end{array}$$

which commutes up to isomorphism by 6.4. By A.4 and 9.1, we obtain a functor $E_n : \mathcal{R}\mathcal{R}_n \to \mathcal{T}_n$ which 'extends' the $L_{n,n+m}D_{n+m}$. Clearly the image of a conflation under E_n embeds into a triangle. So in order to show that the $s \in \Sigma$ are made invertible by E_n , it is enough to show that $E_n\mathcal{F}'$ vanishes if $\mathcal{F}'$ is a presheaf as in section 3. This is equivalent to showing that $L_{n,l}D_lQ_0^j = 0$

for each $l > n$ and each $1 \leq j \leq l$. But we have $D_n Q_0^j \xrightarrow{\sim} Q_0^j D_n$ since D is a tower, $P_{-1}^k Q_0^j \xrightarrow{\sim} Q_0^j P_{-1}^k$ for $k > j$ (apply 6.3 to the square $Q_0^j Q_0^k \xrightarrow{\sim} Q_0^k Q_0^j$) and $P_{-1}^j Q_0^j = 0$. We conclude that E_n induces an $F_n : \mathcal{DR}_n \to \mathcal{T}_n$. It is clear by the construction that the image of a triangle of $\mathcal{DR}_n$ under F_n can be embedded into a triangle of $\mathcal{T}_n$. So once we have shown that the F_n combine into a tower F of additive functors it will follow from 7.1 a) that this tower gives rise to a unique tower of S-functors.

9.2 We keep the notations and hypotheses of the preceding paragraph. In order to construct F as a tower and to make it clear that F depends on D in a functorial way, we have to add one layer of abstraction.

Let $\Theta : \mathcal{P} \to \mathcal{P}$ be the functor which carries $\mathcal{P}_n$ to $\mathcal{P}_{n+1}$ and maps a morphism $a : \mathcal{P}_m \to \mathcal{P}_n$ to

$$\Theta a : \mathcal{P}_{m+1} \to \mathcal{P}_{n+1}\ ,\ (x_1, \ldots, x_{m+1}) \mapsto (a(x_1, \ldots, x_m), x_{m+1}).$$

We have a natural transformation $\tau : 1_{\mathcal{P}} \to \Theta$ taking the values

$$\tau \mathcal{P}_n = p_0^{n+1} : \mathcal{P}_n \to \mathcal{P}_{n+1}, (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0).$$

Of course, Θ is really a 2-functor and τ a morphism of 2-functors. For each m we obtain a tower of exact functors

$$\mathcal{R}_* \tau \Theta^m : \mathcal{R}_* \Theta^m \leftarrow \mathcal{R}_* \Theta^{m+1}.$$

By example 2.2 a) and A.2, this tower admits a left adjoint, which we denote by $\mathcal{R}_* \sigma \Theta^m$, by abuse of notation. Its components are the

$$Q_{-1}^{n+m+1} : \mathcal{R}_{n+m} \to \mathcal{R}_{n+m+1}\ .$$

We form the direct system $\mathcal{R}_{\mathrm{N}}$:

$$\mathcal{R}_* \xrightarrow{\mathcal{R}_* \sigma} \mathcal{R}_* \Theta \to \ldots \to \mathcal{R}_* \Theta^m \xrightarrow{\mathcal{R}_* \sigma \Theta^m} \ldots\ ,$$

which we view as an object of $\mathcal{F}un\,(\mathbf{N}, \mathcal{F}un\,(\mathcal{P}^{OP}, \mathcal{E}x))$, where $\mathcal{P}^{OP}$ denotes the 2-category $\mathcal{P}$ with reversed 1- and 2-morphisms. By 9.1 and A.4 its direct limit is isomorphic to $\mathcal{RR}_*$, the tower with components $(\mathcal{RR}_*)_n = \mathcal{RR}_n$. Using the notations of A.4 we have two 'universal arrows'

$$\mathcal{R}_{\mathrm{N}} \to \Delta \mathcal{RR}_* \to \Delta \mathcal{DR}_*$$

in $\mathcal{F}un\,(\mathbf{N}, \mathcal{F}un\,(\mathcal{P}^{OP}, \mathcal{E}x))$. We denote their composition by Can. Let us return to $\mathcal{T}$. By remark 6.3 and A.2, the tower

$$\mathcal{T}\tau\Theta^m : \mathcal{T}\Theta^m \leftarrow \mathcal{T}\Theta^{m+1}$$

admits a left adjoint, which we denote by $\mathcal{T}\sigma\Theta^m$. We consider $\sigma^+ : \Delta\mathcal{T} \to \mathcal{T}_{\mathrm{N}}$ given by

$$\begin{array}{ccccccc}
\mathcal{T} & \xrightarrow{1} & \mathcal{T} & \longrightarrow \dots & \mathcal{T} & \longrightarrow \dots \\
\| & & \downarrow & & \downarrow & \\
\mathcal{T} & \xrightarrow{\mathcal{T}\sigma} & \mathcal{T}\Theta & \xrightarrow{\mathcal{T}\sigma\Theta} \dots & \mathcal{T}\Theta^m & \xrightarrow{\mathcal{T}\sigma\Theta^m} \dots .
\end{array}$$

Here the vertical arrows are defined so as to make the squares commutative. By A.2 and 6.3, σ^+ admits a left adjoint $\rho^+ : \mathcal{T}_{\mathrm{N}} \to \Delta\mathcal{T}$

$$\begin{array}{ccccccc}
\mathcal{T} & \xrightarrow{\mathcal{T}\sigma} & \mathcal{T}\Theta & \xrightarrow{\mathcal{T}\sigma\Theta} \dots & \mathcal{T}\Theta^m & \xrightarrow{\mathcal{T}\sigma\Theta^m} \dots \\
\| & & \downarrow \rho_1^+ & & \downarrow \rho_m^+ & \\
\mathcal{T} & \xrightarrow{1} & \mathcal{T} & \longrightarrow \dots & \mathcal{T} & \longrightarrow \dots .
\end{array}$$

Of course, the n-th component of ρ_m^+ is simply

$$(\rho_m^+)_n = L_{n,n+m} : \mathcal{T}_{n+m} \to \mathcal{T}_n.$$

We compose $\rho^+ : \mathcal{T}_{\mathrm{N}} \to \Delta\mathcal{T}$ with $D_{\mathrm{N}} : \mathcal{R}_{\mathrm{N}} \to \mathcal{T}_{\mathrm{N}}$. By A.4 the composition $D_{\mathrm{N}}\rho^+$ gives rise to a tower of additive functors $F : \mathcal{DR}_* \to \mathcal{T}$ which makes the square

$$\begin{array}{ccc}
\mathcal{R}_{\mathrm{N}} & \xrightarrow{Can} & \Delta\mathcal{DR}_* \\
D_{\mathrm{N}} \downarrow & & \downarrow \Delta F \\
\mathcal{T}_{\mathrm{N}} & \xrightarrow{\rho^+} & \Delta\mathcal{T}
\end{array}$$

commutative up to isomorphism. By construction, F depends on D in a functorial manner and $F\,can \xrightarrow{\sim} D$ canonically. There only remains to be constructed a functorial isomorphism $F \xrightarrow{\sim} G$ for the case where $D = G\,can$ for some tower of S-functors $G : \mathcal{DR}_* \to \mathcal{T}$. It is enough to produce an isomorphism between $(\Delta F)\,Can$ and $(\Delta G)\,Can$, i.e. between $\rho^+ D_{\mathrm{N}}$ and $(\Delta G) Can$. We illustrate the situation by the diagram

$$\begin{array}{ccccc}
\mathcal{R}_{\mathrm{N}} & \xrightarrow{can_{\mathrm{N}}} & (\mathcal{DR}_*)_{\mathrm{N}} & \xrightarrow{G_{\mathrm{N}}} & \mathcal{T}_{\mathrm{N}} \\
\| & & \downarrow \rho^+ & & \downarrow \rho^+ \\
\mathcal{R}_{\mathrm{N}} & \xrightarrow{Can} & \Delta\mathcal{DR}_* & \xrightarrow{\Delta G} & \Delta\mathcal{T} .
\end{array}$$

It is clear from A.3 that $D_{\mathrm{N}} = G_{\mathrm{N}}\,can_{\mathrm{N}}$ if $D = G\,can$. Hence $\rho^+ D_{\mathrm{N}} = \rho^+ G_{\mathrm{N}}\,can_{\mathrm{N}}$. We also have $\tau^+ G_{\mathrm{N}} \xrightarrow{\sim} (\Delta G)\tau^+$, canonically, where τ^+ denotes the obvious 1-morphisms $\mathcal{T}_{\mathrm{N}} \to \Delta\mathcal{T}$ and $(\mathcal{DR}_*)_{\mathrm{N}} \to \Delta\mathcal{DR}_*$. By 'twofold association' (A.3 and 6.3), we obtain an isomorphism $\rho^+ G_{\mathrm{N}} \xrightarrow{\sim} (\Delta G)\rho^+$, which by A.3 is functorial in G. So it only remains to be shown that $\rho^+ can_{\mathrm{N}} \xrightarrow{\sim} Can$. We fix a choice of ρ^+ : The

$$\rho_m^+ : \mathcal{DR}_*\Theta^m \to \mathcal{DR}_*$$

are determined by the

$$(\rho_m^+)_n : \mathcal{DR}_{m+n} \to \mathcal{DR}_n$$

which are to be induced by the canonical isomorphisms

$$\mathcal{RR}_{m+n} \to \mathcal{RR}_n$$

provided by the partitions

$$\{1, \ldots, m+n\} \coprod \mathbf{N}_1 \xrightarrow{\sim} \{1, \ldots, n\} \coprod \mathbf{N}_1 \,.$$

It is then clear that $\rho^+ can_{\mathrm{N}} = Can$ but we still have to show that $(\rho_m^+)_n$ is really left adjoint to the composition

$$Q_{-1}^{m+n} \ldots Q_{-1}^{n} : \mathcal{DR}_n \to \mathcal{DR}_{m+n} \,.$$

It is obviously enough to consider the case $m = 1$. After replacing $\mathcal{A}$ by $\mathcal{R}_n\mathcal{A}$ we may also assume that $n = 0$. The assertion then means that

$$Z : \mathcal{RR}_1 \to \mathcal{RA}$$

induces a left adjoint of the functor

$$Q_{-1}^1 : \mathcal{DA} \to \mathcal{DR}_1\mathcal{A} \xrightarrow{\sim} \mathcal{DMA} \,.$$

Now Z induces the mapping cone functor so that the assertion follows from 10.1 b).

10. Proof of Theorem 4

10.1 Lemma.

a) The inclusion $J : \mathcal{IA} \to \mathcal{MA}$ *induces an* S*-equivalence* $\mathcal{DIA} \xrightarrow{\sim} \mathcal{DMA}$.

b) The mapping cone functor $C : \mathcal{DMA} \to \mathcal{DA}$ *is left adjoint to the embedding* $\mathcal{DA} \to \mathcal{DMA}$ *induced by* $A \mapsto (0 \to A)$.

Proof. a) Let $\mathcal{I}_{cs}\mathcal{A}$ and $\mathcal{M}_{cs}\mathcal{A}$ be the categories $\mathcal{IA}$ and $\mathcal{MA}$ endowed with the componentwise split conflations. For each $(f : A_1 \to A_0) \in \mathcal{M}_{cs}\mathcal{A}$ we have a conflation

$$\begin{array}{ccccc} 0 & \longrightarrow & A_1 & \longrightarrow & A_1 \\ \downarrow & & \downarrow i & & \downarrow f \\ A_1 & \xrightarrow{[1\ -f]^t} & A_1 \oplus A_0 & \xrightarrow{[f\ 1]} & A_0 \end{array} \ , \ i = [1\ 0]^t$$

in $\mathcal{M}_{cs}\mathcal{A}$. We see that each object of $\mathcal{M}_{cs}\mathcal{A}$ (hence of $\mathcal{I}_{cs}\mathcal{A}$) admits a projective resolution of lenghth one by objects contained in $\mathcal{U}$, the full subcategory of $\mathcal{I}_{cs}\mathcal{A}$ consisting of the split conflations. Therefore

$$\mathcal{DI}_{cs}\mathcal{A} \xleftarrow{\sim} \mathcal{H}^b_{0]}\mathcal{U} \xrightarrow{\sim} \mathcal{DM}_{cs}\mathcal{A}.$$

It follows from 5.1 a) that $\mathcal{DIA}$ (resp. $\mathcal{DMA}$) identifies with the localisation of $\mathcal{DI}_{cs}\mathcal{A}$ (resp. $\mathcal{DM}_{cs}\mathcal{A}$) at the class of morphisms s which fit into a triangle

$$X \xrightarrow{s} Y \to A \to SX$$

with an $\mathcal{IA}$-acyclic (resp. $\mathcal{MA}$-acyclic) A. The assertion follows because the preimages in $\mathcal{H}^b_{0]}\mathcal{U}$ of the respective classes coincide.

b) It is enough to show the assertion for the restriction $C|\mathcal{DIA}$. Now $Q : \mathcal{DA} \to \mathcal{DIA}$ obviously has the functor Cok $: \mathcal{DIA} \to \mathcal{DA}$ as a left adjoint. Since the mapping cone over the canonical morphism $CX \to \operatorname{Cok} X$, $X \in \mathcal{DIA}$ is acyclic, C and Cok are isomorphic as functors $\mathcal{DIA} \to \mathcal{DA}$.

10.2 We construct a quasiinverse of

$$can^* : Hom_S(\mathcal{D}, \mathcal{F}) \to Hom_\partial(\mathcal{F}).$$

We first remark that the functor

$$Q_0 : \mathcal{FA} \to \mathcal{FMA}$$

induced by $A \mapsto (A \to 0)$ satisfies condition (R) of 6.1. To see this, we can use the same argument as in example 6.1 c). Now let $D \in Hom_\partial(\mathcal{F})$ and let $\mathcal{U} : \mathcal{E}x \to \mathcal{A}dd$ be the forgetful 2-functor. We consider $\mathcal{F}$ as a 2-functor $\mathcal{E}x \to \mathcal{A}dd$ and D as a morphism of 2-functors $\mathcal{U} \to \mathcal{F}$. We define the 1-morphism $D_{\mathrm{N}} : \mathcal{U}_{\mathrm{N}} \to \mathcal{F}_{\mathrm{N}}$ to be

$$\begin{array}{ccccccc} \mathcal{U} & \xrightarrow{\mathcal{U}Q} & \mathcal{UM} & \xrightarrow{\mathcal{UM}Q} \dots & \mathcal{UM}^n & \xrightarrow{\mathcal{UM}^nQ} & \dots \\ D \downarrow & & \downarrow & & \downarrow D\mathcal{M}^n & & \\ \mathcal{F} & \xrightarrow{\mathcal{F}Q} & \mathcal{FM} & \xrightarrow{\mathcal{FM}Q} \dots & \mathcal{FM}^n & \xrightarrow{\mathcal{FM}^nQ} & \dots \ , \end{array}$$

where Q is the 1-morphism whose value at $\mathcal{A}$ is the exact embedding

$$\mathcal{A} \to \mathcal{M}\mathcal{A}\,,\ A \mapsto (0 \to A)$$

and the squares are commutative up to canonical isomorphism. From 6.1 it is clear that the 1-morphism $\sigma^+ : \Delta\mathcal{F} \to \mathcal{F}_{\mathrm{N}}$ given by

$$\begin{array}{ccccccc}
\mathcal{F} & \xrightarrow{1} & \mathcal{F} & \xrightarrow{1} \dots & \mathcal{F} & \xrightarrow{\mathcal{F}} \dots & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{F} & \xrightarrow{\mathcal{F}Q} & \mathcal{F}\mathcal{M} & \xrightarrow{\mathcal{F}\mathcal{M}Q} \dots & \mathcal{F}\mathcal{M}^n & \xrightarrow{\mathcal{F}\mathcal{M}^nQ} \dots &
\end{array}$$

(the vertical arrows are defined so as to make the squares commutative) admits a left adjoint $\rho^+ : \mathcal{F}_{\mathrm{N}} \to \Delta\mathcal{F}$. The composition $\rho^+ D_{\mathrm{N}}$ 'factors' through the 'universal arrow' (A.5)

$$\mathcal{U}_{\mathrm{N}} \to \Delta\mathcal{R}\,,$$

where $\mathcal{R}$ assigns the additive category of rough presheaves $\mathcal{R}\mathcal{A}$ to $\mathcal{A} \in \mathcal{E}x$ and the functors

$$\mathcal{M}^n\mathcal{A} \to \mathcal{R}\mathcal{A}$$

are defined in analogy with 9.1. So we have $E : \mathcal{R} \to \mathcal{F}$ such that

$$\mathcal{U}_{\mathrm{N}} \to \Delta\mathcal{R} \xrightarrow{\Delta E} \Delta\mathcal{F}$$

is isomorphic to $\rho^+ D_{\mathrm{N}}$. As in 9.2, one sees that

$$E\mathcal{A} : \mathcal{R}\mathcal{A} \to \mathcal{F}\mathcal{A}$$

makes all the $s \in \Sigma$ invertible. This clearly implies that E 'factors' through

$$\mathcal{R} \to \mathcal{D}$$

giving rise to $F : \mathcal{D} \to \mathcal{F}$. From 7.2 we see that F corresponds to a 1-morphism of $\mathcal{F}un(\mathcal{E}x, \mathcal{S}usp)$. Now one can imitate the end of the proof in 9.3. We omit the details.

Appendix : 2-Functor-Categories

A.1 Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories [5, V. 1]. We set out to define a sub-2-category of the 2-functor-category of [6, I, 2.4] from $\mathcal{C}$ to $\mathcal{D}$.

An *object* of the *2-category* $\mathcal{F}un\,(\mathcal{C}, \mathcal{D})$ is a *2-functor* $X : \mathcal{C} \to \mathcal{D}$, i.e. a map $X : \mathcal{C}_0 \to \mathcal{D}_0$ together with functors

$$X(x, y) : Hom_{\mathcal{C}}\,(x, y) \to Hom_{\mathcal{D}}\,(Xx, Xy)\,,\ f \mapsto Xf\,,$$

which are compatible with units and compositions. A *1-morphism* $F : X \to Y$ assigns a 1-morphism $Fx : Xx \to Xy$ to each object $x \in \mathcal{C}$ and a an invertible 2-morphism $Ff : YfFx \to FyXf$ to each 1-morphism $f : x \to y$

$$\begin{tikzcd}
Xx \arrow[r, "Xf"] \arrow[d, "Fx"'] & Xy \arrow[d, "Fy"] \\
Yx \arrow[r, "Yf"'] \arrow[ur, phantom, "\nearrow Ff"] & Yy
\end{tikzcd}$$

such that $F1_x = 1_{Fx}$, $\forall x \in \mathcal{C}_0$, $Ffg = (FfXg)(YfFg)$ for each pair of composable 1-morphisms f, g of $\mathcal{C}$ and $(FyX\mu)(Ff) = (Fg)(Y\mu Fx)$ for each 2-morphism $\mu : f \to g$ of $\mathcal{C}$. The *composition of two 1-morphisms* F, G is defined by $FGx = FxGx$ for each $x \in \mathcal{C}_0$ and $FGf = (FyGf)(FfGx)$ for each 1-morphism $f : x \to y$. A *2-morphism* $\Theta : F \to G$ assigns a 2-morphism $\Theta x : Fx \to Gx$ to each object x of $\mathcal{C}$ such that the equation $Gf(Yf\Theta x) = (\Theta yXf)Ff$ holds for each 1-morphism $f : x \to y$. The two compositions of 2-morphisms are $(\Theta\Phi)x = \Theta x\Phi x$ and $(\Theta * \Phi)x = \Theta x * \Phi x$.

Example. Let $\mathcal{C}$ be the 2-category with a single object x having only identical 2-morphisms and such that $Hom_{\mathcal{C}}(x, x)$ is a free monoid on $s : x \to x$. Then $\mathcal{F}un\,(\mathcal{C}, \mathcal{C}at)$ is isomorphic to the 2-category whose objects are the pairs $(\mathcal{X}, S)$ of categories $\mathcal{X}$ with an endofunctor $S : \mathcal{X} \to \mathcal{X}$, whose morphisms are the 'S-functors' and whose 2-morphisms are the 'morphisms of S-functors' (compare [10, 1.4]).

A.2 We keep the assumptions and notations of 6.1. An *adjoint pair* in a 2-category consists of 1-morphisms $l : x \to y$, $r : y \to x$ and 2-morphisms $\varphi : lr \to 1_y$, $\psi : 1_x \to rl$ such that $(r\varphi)(\psi r) = 1_r$ and $(\varphi l)(l\psi) = 1_l$. If φ and ψ are invertible, l and r are *quasiinverse equivalences.*

Now let $R : X \to Y$ be a 1-morphism of $\mathcal{F}un\,(\mathcal{C}, \mathcal{D})$ and suppose that for each $x \in \mathcal{C}_0$ we are given an adjoint pair Rx, Lx, Φx, Ψx such that

$$Mf = (\Phi yXfLx)(LyRfLx)(LyYf\Psi x) : LyYf \to XfLx$$

is invertible for each 1-morphism $f : x \to y$ of $\mathcal{C}$.

Proposition. (compare [10, 1.6]) *The assignments*

$$x \mapsto Lx\ ,\ f \mapsto Lf = (Mf)^{-1}$$

define a 1-morphism L of $\mathcal{F}un\,(\mathcal{C}, \mathcal{D})$, the assignments

$$x \mapsto \Phi x \;,\; x \mapsto \Psi x$$

define 2-morphisms Φ, Ψ and R, L, Φ, Ψ is an adjoint pair in $\mathcal{F}un\,(\mathcal{C}, \mathcal{D})$.

Remark. In particular, R has a quasiinverse iff each Rx has a quasiinverse.

Proof. Substituting into the definitions we obtain statements which immediately follow from A.3. We omit the details.

A.3 Let l, r, φ, ψ and l', r', φ', ψ' be adjoint pairs in a 2-category $\mathcal{C}$ which appear in a diagram

$$\begin{array}{ccc} x & \xrightarrow{f} & x' \\ l \downarrow\uparrow r & & l' \downarrow\uparrow r' \\ y & \xrightarrow{g} & y'. \end{array}$$

We call two 2-morphisms $\alpha : fr \to r'g$ and $\beta : l'f \to gl$ *associated* if the following equivalent conditions hold (compare [10, 1.6]) :

$$\begin{array}{ll} \text{i) } \alpha = (r'g\varphi)(r'\beta r)(\psi' fr) & \text{ii) } (r'\beta)(\psi' f) = (\alpha l)(f\psi) \\ \text{iii) } \beta = (\varphi' gl)(l'\alpha l)(l'f\psi) & \text{iv) } (g\varphi)(\beta r) = (\varphi' g)(l'\alpha). \end{array}$$

We can interpret this as follows: i) and iii) define inverse bijections between the 1-morphisms from l to l' and from r to r', where we consider l, l' as 2-functors from $\{0 < 1\}$ (having only identical 2-morphisms) to $\mathcal{C}$ and r, r' as 2-functors from $\{0 < 1\}$ to $\mathcal{C}^{oP}$, the 2-category $\mathcal{C}$ 'with reversed 2-morphisms'. In fact, these bijections are part of an isomorphism of categories

$$Hom\,(l, l') \xrightarrow{\cong} Hom\,(r, r')^{op}.$$

We make this more precise: Let $f_1 : x \to x'$ and $g_1 : y \to y'$ be another pair of morphisms and $\mu : f \to f_1$, $\nu : g \to g_1$ 2-morphisms. If $\alpha : fr \to r'g$, $\beta : l'f \to gl$ and $\alpha_1 : f_1 r \to r' g_1$, $\beta_1 : l' f_1 \to g_1 l$ are associated pairs, then

$$(r'\nu)\alpha = \alpha_1(\mu r) \Longleftrightarrow (\nu l)\beta = \beta_1(l'\mu) \;,$$

i.e. μ, ν define a 2-morphism $(f, g, \alpha) \to (f_1, g_1, \alpha_1)$ iff they define a 2-morphism $(f, g, \beta) \to (f_1, g_1, \beta_1)$.

In 9.2 and A.2 we also need that the isomorphisms

$$Hom\,(l, l') \xrightarrow{\cong} Hom\,(r, r')^{op}$$

are compatible with compositions, i.e. if a diagram

$$\begin{array}{ccccc} x & \xrightarrow{f} & x' & \xrightarrow{h} & x'' \\ l \downarrow\uparrow r & & l' \downarrow\uparrow r' & & l'' \downarrow\uparrow r'' \\ y & \xrightarrow{g} & y' & \xrightarrow{i} & y'' \end{array}$$

is given, where l'', r'', φ'', ψ'' is another adjoint pair, and if $\alpha : fr \to r'g$, $\beta : l'f \to gl$ and $\gamma : hr' \to r''i$, $\delta : l''h \to il'$ are associated pairs, then $(\gamma g)(h\alpha) : (hf)r \to r''(ig)$ and $(i\beta)(\delta f) : l''(hf) \to (ig)l$ are associated.

A.4 We consider the partially ordered set $\mathbf{N}$ as a 2-category having only identical 2-morphisms. Let $\mathcal{E}$ be another 2-category and let $\Delta : \mathcal{E} \to \mathcal{F}un\,(\mathbf{N}, \mathcal{E})$ be the obvious 'diagonal' 2-functor. The *limit of a 2-functor* $X : \mathbf{N} \to \mathcal{E}$ consists of an object $\varinjlim X \in \mathcal{E}$ and a 1-morphism $f : X \to \Delta \varinjlim X$ inducing an equivalence of categories

$$Hom_{\mathcal{E}}\,(\varinjlim X, y) \to Hom_{\mathcal{F}un\,(\mathrm{N},\mathcal{E})}(X, \Delta y)\ ,\ g \mapsto f \cdot \Delta g$$

for each object y of $\mathcal{E}$.

Example. a) Let $\mathcal{A}$ be an exact category. In $\mathcal{E} = \mathcal{A}dd$ we consider the sequence of embeddings (9.1)

$$\mathcal{R}_0 \xrightarrow{Q^1_{-1}} \mathcal{R}_1 \to \ldots \to \mathcal{R}_n \xrightarrow{Q^{n+1}_{-1}} \mathcal{R}_{n+1} \to \ldots\ .$$

For an exact category $\mathcal{B}$, a 1-morphism to $\Delta\mathcal{B}$ is given by a family of functors

$$G_n : \mathcal{R}_n \to \mathcal{B}$$

and of isomorphisms

$$\gamma_n : G_n \xrightarrow{\sim} G_{n+1} Q^{n+1}_{-1}.$$

Using a well-known technique we now exhibit a category $\mathcal{LA}$ and a 1-morphism to $\Delta\mathcal{LA}$ which even induces an *isomorphism*

$$Hom\,(\mathcal{LA}, \mathcal{B}) \xrightarrow{\cong} Hom(\mathcal{R}_0 \to \mathcal{R}_1 \to \ldots\ ,\ \Delta\mathcal{B}).$$

The objects of $\mathcal{LA}$ are the pairs (X, n) of natural numbers n and of objects $X \in \mathcal{R}_n$. The morphisms from (X, n) to (Y, m) bijectively correspond to the elements of $\mathcal{RA}\,(X, Y)$ (we identify X, Y with their images in $\mathcal{RA}$). The functor $can_n : \mathcal{R}_n \to \mathcal{LA}$ associates the pair (X, n) with $X \in \mathcal{R}_n$. The isomorphism $can_n \xrightarrow{\sim} can_{n+1} Q^{n+1}_{-1}$ is produced by the identities of $\mathcal{RA}$.

Obviously $\mathcal{LA}$ is equivalent to $\mathcal{RA}$. Hence $\mathcal{RA}$ is also a limit of the sequence of the $\mathcal{R}_n$. We conclude by theorem 3 that for each additive category $\mathcal{T}_0$ the canonical functor

$$Hom\,(\mathcal{DA}, \mathcal{T}_0) \to Hom\,(\mathcal{R}_0 \to \mathcal{R}_1 \to \dots\,,\ \Delta\mathcal{T}_0)$$

is an equivalence onto the full subcategory of the 'compatible families' (G_n, γ_n) such that G_n makes the $s \in \Sigma$ lying in $\mathcal{R}_n$ invertible.

Now we consider the case where $\mathcal{E} = \mathcal{F}un\,(\mathcal{C}, \mathcal{D})$ for two 2-categories $\mathcal{C}$, $\mathcal{D}$. A 2-functor $X : \mathbf{N} \to \mathcal{F}un\,(\mathcal{C}, \mathcal{D})$ yields a 2-functor $X_c : \mathbf{N} \to \mathcal{D}$ defined by

$$n \mapsto X_c(n) = (Xn)(c)$$

for each $c \in \mathcal{C}$. Let us suppose that for each c there is a *strict limit*, i.e. a limit $\varinjlim X_c$ furnishing *isomorphisms*

$$Hom\,(\varinjlim X_c, y) \xrightarrow{\cong} Hom\,(X_c, \Delta y)\ ,\ \forall\, y \in \mathcal{D}.$$

In this case, the assignment $c \mapsto \varinjlim X_c$ can be completed to a 2-functor $\mathcal{L} : \mathcal{C} \to \mathcal{D}$ in a natural way, and the 1-morphisms $X_c \to \Delta\varinjlim X_c$ can be completed to a 1-morphism $X \to \Delta\mathcal{L}$. A tedious exercise shows that this 1-morphism induces an isomorphism

$$Hom_{\mathcal{F}un\,(\mathcal{C},\mathcal{D})}(\mathcal{L}, Y) \to Hom_{\mathcal{F}un\,(\mathrm{N},\mathcal{F}un\,(\mathcal{C},\mathcal{D}))}(X, \Delta Y)$$

for each $Y \in \mathcal{F}un\,(\mathcal{C}, \mathcal{D})$.

Example. b) We consider the 2-functor $\mathcal{R}_* : \mathcal{P}^{OP} \to \mathcal{A}dd$ and the sequence

$$\mathcal{R}_* \xrightarrow{\mathcal{R}_*\sigma} \mathcal{R}_*\Theta \to \dots \to \mathcal{R}_*\Theta^m \xrightarrow{\mathcal{R}_*\sigma\Theta^m} \dots\ .$$

of 9.2. By definition, its evaluation at $\mathcal{P}_n$ is the sequence

$$\mathcal{R}_n \xrightarrow{Q^{n+1}_{-1}} \mathcal{R}_{n+1} \to \dots \to \mathcal{R}_{n+m} \xrightarrow{Q^{n+m+1}_{-1}} \mathcal{R}_{n+m+1} \to \dots\ .$$

As in example a), we see that this sequence has a strict limit $\mathcal{L}_n$. The $\mathcal{L}_n$ combine into a 2-functor $\mathcal{L} : \mathcal{P}^{OP} \to \mathcal{A}dd$. The canonical morphisms $\mathcal{R}_*\Theta^m \to \mathcal{RR}_*$ induce a 1-morphism $\mathcal{L} \to \mathcal{RR}_*$. Since its components are equivalences, it is an equivalence itself by A.2. Hence $\mathcal{RR}_*$ is a limit of the seqence of the

$\mathcal{R}_*\Theta^m$. In particular, it follows that, for each tower of additive categories $\mathcal{T}$, the functor

$$Hom\,(\mathcal{D}\mathcal{R}_*, \mathcal{T}) \to Hom(\mathcal{R}_* \to \mathcal{R}_*\Theta \to \dots\,,\ \Delta\mathcal{T})$$

is an equivalence onto the full subcategory of the 'compatible families' $G_m : \mathcal{R}_*\Theta^m \to \mathcal{T}$ such that $(G_m)_n : \mathcal{R}_{n+m} \to \mathcal{T}$ makes all the $s \in \Sigma$ (with respect to $\mathcal{R}\mathcal{R}_n$) lying in $\mathcal{R}_{n+m}$ invertible.

A.5 Let $\mathcal{C}$, $\mathcal{D}$ be 2-categories, X, Y, $Z \in \mathcal{F}un\,(\mathcal{C}, \mathcal{D})$, $F, G : X \to Y$ and $H : Y \to Z$ 2-functors. Suppose that, for each 1-morphism $f : x \to y$ of $\mathcal{C}$, Hy induces a bijection

$$Hom_{\mathcal{D}}\,(Xx, Yy)(FyXf, GyXf) \xrightarrow{\sim} Hom_{\mathcal{D}}\,(Xx, Zy)(HFyXf, HGyXf).$$

Lemma. *H induces a bijection of the classes of 2-morphisms*

$$Hom_{\mathcal{F}un\,(\mathcal{C},\mathcal{D})}(X, Y)(F, G) \xrightarrow{\sim} Hom_{\mathcal{F}un\,(\mathcal{C},\mathcal{D})}(X, Z)(HF, HG).$$

We omit the straightforward proof.

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