# ON DIFFERENTIAL GRADED CATEGORIES

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Abstract. Differential graded categories enhance our understanding of triangulated categories appearing in algebra and geometry. In this survey, we review their foundations and report on recent work by Drinfeld, Dugger-Shipley, ..., Toën and Toën-Vaquié.

## 1. Introduction

1.1. **Triangulated categories and dg categories.** Derived categories were invented by Grothendieck-Verdier in the early sixties in order to formulate Grothendieck's duality theory for schemes, *cf.* [65]. Today, they have become an important tool in many branches of algebraic geometry, in algebraic analysis, non commutative algebraic geometry, representation theory, mathematical physics ... . In an attempt to axiomatize the properties of derived categories, Grothendieck-Verdier introduced the notion of a triangulated category. For a long time, triangulated categories were considered too poor to allow the development of more than a rudimentary theory. This vision has changed in recent years [106] [107], but the fact remains that many important constructions of derived categories cannot be performed with triangulated categories. Notably, tensor products and functor categories formed from triangulated categories are no longer triangulated. One approach to overcome these problems has been the theory of derivators initiated by Grothendieck [56] at the beginning of the nineties. Another, perhaps less formidable one is the theory of differential graded categories (=dg categories), together with its cousin, the theory of $A_\infty$-categories.

Dg categories already appear in [81]. In the seventies, they found applications [121] [35] in the representation theory of finite-dimensional algebras. The idea to use dg categories to 'enhance' triangulated categories goes back at least to Bondal-Kapranov [22], who were motivated by the study of exceptional collections of coherent sheaves on projective varieties.

The synthesis of Koszul duality [9] [10] with Morita theory for derived categories [115] was the aim of the study of the unbounded derived category of a dg category in [73].

It is now well-established that invariants like $K$-theory, Hochschild (co-)homology and cyclic homology associated with a ring or a variety 'only depend' on its derived category. However, in most cases, the derived category (even with its triangulated structure) is not enough to compute the invariant, and the datum of a triangle equivalence between derived categories is not enough to construct an isomorphism between invariants (*cf.* Dugger-Shipley's [132] results in section 3.9). Differential graded categories provide the necessary structure to fill this gap. This idea was applied to $K$-theory by Thomason-Trobaugh [143] and to cyclic homology in [75] [77].

The most useful operation which *can be performed* on triangulated categories is the passage to a Verdier quotient. It was therefore important to lift this operation to the world of differential graded categories. This was done implicitly in [77] but explicitly, by Drinfeld, in [34].

*Date*: February 26, 2006.

1991 *Mathematics Subject Classification.* 18E30, 16D90.

*Key words and phrases.* Homological algebra, derived category, homotopy category, derived functor, $K$-theory, Hochschild cohomology, Morita theory, non commutative algebraic geometry.

In a certain sense, differential graded categories and differential graded functors contain too much information and the main problem in working with them consists in 'discarding what is irrelevant'. It now appears clearly that the best tool for doing this are Quillen model categories [112]: They provide a homotopy theoretic framework which allows simple, yet precise statements and rigorous but readable proofs. Building on the techniques of [34], a suitable model structure on the category of small differential graded categories was constructed in [137]. Starting from this structure, Toën has given a new approach to Morita theory for dg categories [146]. In their joint work [147], Toën and Vaquié have applied this to the construction of moduli stacks of objects in dg categories, and notably in categories of perfect complexes arising in geometry and representation theory.

Thanks to [146], [82] and to recent work by Tamarkin [140], we are perhaps getting closer to answering Drinfeld's question [34]: *What do DG categories form?*

1.2. **Contents.** After introducing notations and basic definitions in section 2 we review the derived category of a dg category in section 3. This is the first opportunity to practice the language of model categories. We present the structure theorems for algebraic triangulated categories which are compactly generated or, more generally, well-generated. We conclude with a survey of recent important work by Dugger and Shipley on topological Morita equivalence for dg categories. In section 4, we present the homotopy categories of dg categories and of 'triangulated' dg categories following Toën's work [146]. The most important points are the description of the mapping spaces of the homotopy category via quasi-functors (Theorem 4.3), the closed monoidal structure (Theorem 4.5) and the characterization of dg categories of finite type (Theorem 4.12). We conclude with a summary of the applications to moduli problems. In the final section 5, we present the most important invariance results for $K$-theory, Hochschild (co-)homology and cyclic homology. The derived Hall algebra presented in section 5.6 is a new invariant due to Toën [144]. Its further development might lead to significant applications in representation theory.

1.3. **Acknowledgments.** I thank Bertrand Toën, Henning Krause and Gonçalo Tabuada for helpful comments on previous versions of this article.

## 2. Definition

2.1. **Notations.** Let $k$ be a commutative ring, for example a field or the ring of integers $\mathbb{Z}$. We will write $\otimes$ for the tensor product over $k$. Recall that a *$k$-algebra* is a $k$-module $A$ endowed with a $k$-linear associative multiplication $A \otimes_k A \to A$ admitting a two-sided unit $1 \in A$. For example, a $\mathbb{Z}$-algebra is just a (possibly non commutative) ring. A *$k$-category* $\mathcal{A}$ is a '$k$-algebra with several objects' in the sense of Mitchell [100]. Thus, it is the datum of a class of objects $\mathrm{obj}(\mathcal{A})$, of a $k$-module $\mathcal{A}(X, Y)$ for all objects $X$, $Y$ of $\mathcal{A}$, and of $k$-linear associative composition maps

$$\mathcal{A}(Y, Z) \otimes \mathcal{A}(X, Y) \to \mathcal{A}(X, Z)\ ,\ (f, g) \mapsto fg$$

admitting units $\mathbf{1}_X \in \mathcal{A}(X, X)$. For example, we can view $k$-algebras as $k$-categories with one object. The category $\mathrm{Mod}\, A$ of right $A$-modules over a $k$-algebra $A$ is an example of a $k$-category with many objects. It is also an example of a *$k$-linear* category, *i.e.* a $k$-category which admits all finite direct sums.

A *graded $k$-module* is a $k$-module $V$ together with a decomposition indexed by the positive and the negative integers:

$$V = \bigoplus_{p \in \mathbb{Z}} V^p.$$

The *shifted module* $V[1]$ is defined by $V[1]^p = V^{p+1}$, $p \in \mathbb{Z}$. A *morphism* $f : V \to V'$ of graded $k$-modules of degree $n$ is a $k$-linear morphism such that $f(V^p) \subset V^{p+n}$ for all $p \in \mathbb{Z}$. The *tensor product* $V \otimes W$ of two graded $k$-modules $V$ and $W$ is the graded $k$-module with components

$$(V \otimes W)^n = \bigoplus_{p+q=n} V^p \otimes W^q \ , \ n \in \mathbb{Z}.$$

The *tensor product* $f \otimes g$ of two maps $f : V \to V'$ and $g : W \to W'$ of graded $k$-modules is defined using the *Koszul sign rule*: We have

$$(f \otimes g)(v \otimes w) = (-1)^{pq} f(v) \otimes g(w)$$

if $g$ is of degree $p$ and $v$ belongs to $V^q$. A *graded $k$-algebra* is a graded $k$-module $A$ endowed with a multiplication morphism $A \otimes A \to A$ which is graded of degree 0, associative and admits a unit $1 \in A^0$. We identify 'ordinary' $k$-algebras with graded $k$-algebras concentrated in degree 0. We write $\mathcal{G}(k)$ for the *category of graded $k$-modules*.

A *differential graded (=dg) $k$-module* is a $\mathbb{Z}$-graded $k$-module $V$ endowed with a *differential* $d_V$, *i.e.* a map $d_V : V \to V$ of degree 1 such that $d_V^2 = 0$. Equivalently, $V$ is a *complex* of $k$-modules. The *shifted dg module* $V[1]$ is the shifted graded module endowed with the differential $-d_V$. The *tensor product* of two dg $k$-modules is the graded module $V \otimes W$ endowed with the differential $d_V \otimes \mathbf{1}_W + \mathbf{1}_V \otimes d_W$.

2.2. **Differential graded categories.** A *differential graded* or *dg category* is a $k$-category $\mathcal{A}$ whose morphism spaces are dg $k$-modules and whose compositions

$$\mathcal{A}(Y, Z) \otimes \mathcal{A}(X, Y) \to \mathcal{A}(X, Z)$$

are morphisms of dg $k$-modules.

For example, dg categories with one object may be identified with *dg algebras*, *i.e.* graded $k$-algebras endowed with a differential $d$ such that the Leibniz rule holds:

$$d(fg) = d(f)g + (-1)^p f\, d(g)$$

for all $f \in A^p$ and all $g$. In particular, each ordinary $k$-algebra yields a dg category with one object. A typical example with several objects is obtained as follows: Let $B$ be a $k$-algebra and $\mathcal{C}(B)$ the category of complexes of right $B$-modules

$$\cdots \longrightarrow M^p \xrightarrow{d_M} M^{p+1} \longrightarrow \ldots \ , \ p \in \mathbb{Z} \ .$$

For two complexes $L, M$ and an integer $n \in \mathbb{Z}$, we define $\mathcal{H}om(L, M)^n$ to be the $k$-module formed by the morphisms $f : L \to M$ of graded objects of degree $n$, *i.e.* the families $f = (f^p)$ of morphisms $f^p : L^p \to M^{p+n}$, $p \in \mathbb{Z}$, of $B$-modules. We define $\mathcal{H}om(L, M)$ to be the graded $k$-module with components $\mathcal{H}om(L, M)^n$ and whose differential is the commutator

$$d(f) = d_M \circ f - (-1)^n f \circ d_L.$$

The *dg category* $\mathcal{C}_{dg}(B)$ has as objects all complexes and its morphisms are defined by

$$\mathcal{C}_{dg}(B)(L, M) = \mathcal{H}om(L, M).$$

The composition is the composition of graded maps.

Let $\mathcal{A}$ be a dg category. The *opposite dg category* $\mathcal{A}^{op}$ has the same objects as $\mathcal{A}$ and its morphisms are defined by

$$\mathcal{A}^{op}(X, Y) = \mathcal{A}(Y, X);$$

the composition of $f \in \mathcal{A}^{op}(Y, X)^p$ with $g \in \mathcal{A}^{op}(Z, Y)^q$ is given by $(-1)^{pq} gf$. The *category* $Z^0(\mathcal{A})$ has the same objects as $\mathcal{A}$ and its morphisms are defined by

$$(Z^0\mathcal{A})(X, Y) = Z^0(\mathcal{A}(X, Y)) \ ,$$

where $Z^0$ is the kernel of $d : \mathcal{A}(X,Y)^0 \to \mathcal{A}(X,Y)^1$. The *category* $H^0(\mathcal{A})$ has the same objects as $\mathcal{A}$ and its morphisms are defined by

$$(H^0(\mathcal{A}))(X,Y) = H^0(\mathcal{A}(X,Y)) ,$$

where $H^0$ denotes the 0th homology of the complex $\mathcal{A}(X,Y)$. For example, if $B$ is a $k$-algebra, we have an isomorphism of categories

$$Z^0(\mathcal{C}_{dg}(B)) = \mathcal{C}(B)$$

and an isomorphism of categories

$$H^0(\mathcal{C}_{dg}(B)) = \mathcal{H}(B) ,$$

where $\mathcal{H}(B)$ denotes the *category of complexes up to homotopy*, *i.e.* the category whose objects are the complexes and whose morphisms are the morphisms of complexes modulo the morphisms $f$ homotopic to zero, *i.e.* such that $f = d(g)$ for some $g \in \mathcal{H}om(L,M)^{-1}$. The *homology category* $H^*(\mathcal{A})$ is the graded category with the same objects as $\mathcal{A}$ and morphisms spaces $H^*\mathcal{A}(X,Y)$.

2.3. **The category of dg categories.** Let $\mathcal{A}$ and $\mathcal{A}'$ be dg categories. A *dg functor* $F : \mathcal{A} \to \mathcal{A}'$ is given by a map $F : \mathrm{obj}(\mathcal{A}) \to \mathrm{obj}(\mathcal{A}')$ and by morphisms of dg $k$-modules

$$F(X,Y) : \mathcal{A}(X,Y) \to \mathcal{A}(FX,FY) ,\ X,Y \in \mathrm{obj}(\mathcal{A}) ,$$

compatible with the composition and the units. The *category of small dg categories* $\mathrm{dgcat}_k$ has the small dg categories as objects and the dg functors as morphisms. Note that it has an initial object, the empty dg category $\emptyset$, and a final object, the dg category with one object whose endomorphism ring is the zero ring. The *tensor product* $\mathcal{A} \otimes \mathcal{B}$ of two dg categories has the class of objects $\mathrm{obj}(\mathcal{A}) \times \mathrm{obj}(\mathcal{B})$ and the morphism spaces

$$(\mathcal{A} \otimes \mathcal{B})((X,Y),(X',Y')) = \mathcal{A}(X,X') \otimes \mathcal{B}(Y,Y')$$

with the natural compositions and units.

For two dg functors $F, G : \mathcal{A} \to \mathcal{B}$, the *complex of graded morphisms* $\mathcal{H}om(F,G)$ has as its $n$th component the module formed by the families of morphisms

$$\phi_X \in \mathcal{B}(FX,GX)^n$$

such that $(Gf)(\phi_X) = (\phi_Y)(Ff)$ for all $f \in \mathcal{A}(X,Y)$, $X, Y \in \mathcal{A}$. The differential is induced by that of $\mathcal{B}(FX,GX)$. The set of *morphisms* $F \to G$ is by definition in bijection with $Z^0\mathcal{H}om(F,G)$.

Endowed with the tensor product, the category $\mathrm{dgcat}_k$ becomes a symmetric tensor category which admits an internal Hom-functor, *i.e.*

$$\mathrm{Hom}(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) = \mathrm{Hom}(\mathcal{A}, \mathcal{H}om(\mathcal{B},\mathcal{C})) ,$$

for $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathrm{dgcat}_k$, where $\mathcal{H}om(\mathcal{B},\mathcal{C})$ has the dg functors as objects and the morphism space $\mathcal{H}om(F,G)$ for two dg functors $F$ and $G$. The unit object is the dg category associated with the $k$-algebra $k$.

A *quasi-equivalence* is a dg functor $F : \mathcal{A} \to \mathcal{A}'$ such that

1) $F(X,Y)$ is a quasi-isomorphism for all objects $X$, $Y$ of $\mathcal{A}$ and
2) the induced functor $H^0(F) : H^0(\mathcal{A}) \to H^0(\mathcal{A}')$ is an equivalence.

Note that neither the tensor product nor the internal Hom-functor respect the quasi-equivalences, a source of technical difficulties.

## 3. The derived category of a dg category

3.1. **Dg modules.** Let $\mathcal{A}$ be a small dg category. A *left dg $\mathcal{A}$-module* is a dg functor

$$L : \mathcal{A} \to \mathcal{C}_{dg}(k)$$

and a *right dg $\mathcal{A}$-module* a dg functor

$$M : \mathcal{A}^{op} \to \mathcal{C}_{dg}(k).$$

Equivalently, a right dg $\mathcal{A}$-module $M$ is given by complexes $M(X)$ of $k$-modules, for each $X \in \mathrm{obj}(\mathcal{A})$, and by morphisms of complexes

$$M(Y) \otimes \mathcal{A}(X, Y) \to M(X)$$

compatible with compositions and units. The *homology* $H^*(M)$ of a dg module $M$ is the induced functor

$$H^*(\mathcal{A}) \to \mathcal{G}(k)\ ,\ X \mapsto H^*(M(X))$$

with values in the category $\mathcal{G}(k)$ of graded $k$-modules (*cf.* 2.1). For each object $X$ of $\mathcal{A}$, we have the right module *represented by* $X$

$$X^\wedge = \mathcal{A}(?, X).$$

The *category of dg modules* $\mathcal{C}(\mathcal{A})$ has as objects the dg $\mathcal{A}$-modules and as morphisms $L \to M$ the morphisms of dg functors (*cf.* 2.3). Note that $\mathcal{C}(\mathcal{A})$ is an abelian category and that a morphism $L \to M$ is an epimorphism (respectively a monomorphism) iff it induces surjections (respectively injections) in each component of $L(X) \to M(X)$ for each object $X$ of $\mathcal{A}$. A morphism $f : L \to M$ is a *quasi-isomorphism* if it induces an isomorphism in homology.

We have $\mathcal{C}(\mathcal{A}) = Z^0(\mathcal{C}_{dg}(\mathcal{A}))$, where, in the notations of 2.3, the dg category $\mathcal{C}_{dg}(\mathcal{A})$ is defined by

$$\mathcal{C}_{dg}(\mathcal{A}) = \mathcal{H}om(\mathcal{A}^{op}, \mathcal{C}_{dg}(k)).$$

We write $\mathcal{H}om(L, M)$ for the complex of morphisms from $L$ to $M$ in $\mathcal{C}_{dg}(\mathcal{A})$. For each $X \in \mathcal{A}$, we have a natural isomorphism

$$\mathcal{H}om(X^\wedge, M) \xrightarrow{\sim} M(X). \tag{1}$$

The *category up to homotopy of dg $\mathcal{A}$-modules* is

$$\mathcal{H}(\mathcal{A}) = H^0(\mathcal{C}_{dg}(\mathcal{A})).$$

The isomorphism (1) yields isomorphisms

$$\mathcal{H}(\mathcal{A})(X^\wedge, M[n]) \xrightarrow{\sim} H^n(\mathcal{H}om(X^\wedge, M)) \xrightarrow{\sim} H^n M(X)\ , \tag{2}$$

where $n \in \mathbb{Z}$ and $M[n]$ is the *shifted dg module* $Y \mapsto M(Y)[n]$.

If $\mathcal{A}$ is the dg category with one object associated with a $k$-algebra $B$, then a dg $\mathcal{A}$-module is the same as a complex of $B$-modules. More precisely, we have $\mathcal{C}(\mathcal{A}) = \mathcal{C}(B)$, $\mathcal{C}_{dg}(\mathcal{A}) = \mathcal{C}_{dg}(B)$ and $\mathcal{H}(\mathcal{A}) = \mathcal{H}(B)$. In this case, if $X$ is the unique object of $\mathcal{A}$, the dg module $X^\wedge$ is the complex formed by the free right $B$-module of rank one concentrated in degree 0.

3.2. **The derived category, resolutions.** The *derived category* $\mathcal{D}(\mathcal{A})$ is the localization of the category $\mathcal{C}(\mathcal{A})$ with respect to the class of quasi-isomorphisms. Thus, its objects are the dg modules and its morphisms are obtained from morphisms of dg modules by formally inverting [50] all quasi-isomorphisms. The projection functor $\mathcal{C}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ induces a functor $\mathcal{H}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ and the derived category could equivalently be defined as the localization of $\mathcal{H}(\mathcal{A})$ with respect to the class of all quasi-isomorphisms. Note that from this definition, it is not clear that the morphism classes of $\mathcal{D}(\mathcal{A})$ are sets or that $\mathcal{D}(\mathcal{A})$ is an additive category.

Call a dg module $P$ *cofibrant* if, for every surjective quasi-isomorphism $L \to M$, every morphism $P \to M$ factors through $L$. For example, for an object $X$ of $\mathcal{A}$, the dg module $X^\wedge$ is cofibrant. Call a dg module $I$ *fibrant* if, for every injective quasi-isomorphism $L \to M$, every morphism $L \to I$ extends to $M$. For example, if $E$ is an injective cogenerator of the category of $k$-modules and $X$ an object of $\mathcal{A}$, the dg module $\mathcal{H}om(\mathcal{A}(X, ?), E)$ is fibrant.

**Proposition 3.1.** a) *For each dg module $M$, there is a quasi-isomorphism $\mathbf{p}M \to M$ with cofibrant $\mathbf{p}M$ and a quasi-isomorphism $M \to \mathbf{i}M$ with fibrant $\mathbf{i}M$.*
b) *The projection functor $\mathcal{H}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ admits a fully faithful left adjoint given by $M \mapsto \mathbf{p}M$ and a fully faithful right adjoint given by $M \mapsto \mathbf{i}M$.*

One can construct $\mathbf{p}M$ and $\mathbf{i}M$ explicitly, as first done in [5] (*cf.* also [73]). We call $\mathbf{p}M \to M$ a *cofibrant resolution* and $M \to \mathbf{i}M$ a *fibrant resolution* of $M$. According to b), these resolutions are functorial in the category up to homotopy $\mathcal{H}(\mathcal{A})$ and we can compute morphisms in $\mathcal{D}(\mathcal{A})$ via

$$\mathcal{H}(\mathcal{A})(\mathbf{p}L, M) = \mathcal{D}(\mathcal{A})(L, M) = \mathcal{H}(\mathcal{A})(L, \mathbf{i}M).$$

In particular, for an object $X$ of $\mathcal{A}$ and a dg module $M$, the isomorphisms (2) yield

$$\mathcal{D}(\mathcal{A})(X^\wedge, M[n]) \xrightarrow{\sim} \mathcal{H}(\mathcal{A})(X^\wedge, M[n]) \xrightarrow{\sim} H^n M(X) \tag{3}$$

since $X^\wedge$ is cofibrant. The embedding $\mathcal{D}(\mathcal{A}) \to \mathcal{H}(\mathcal{A})$ provided by $\mathbf{p}$ also shows that the derived category is additive.

If $\mathcal{A}$ is associated with a $k$-algebra $B$ and $M$ is a right $B$-module considered as a complex concentrated in degree 0, then $\mathbf{p}M \to M$ is a projective resolution of $M$ and $M \to \mathbf{i}M$ an injective resolution. The proposition is best understood in the language of Quillen model categories [112]. We refer to [42] for a highly readable introduction and to [62] [61] for in-depth treatments. The proposition results from the following theorem, proved using the techniques of [62, 2.3].

**Theorem 3.2.** *The category $\mathcal{C}(\mathcal{A})$ admits two structures of Quillen model category whose weak equivalences are the quasi-isomorphisms:*

1) *The* projective *structure, whose fibrations are the epimorphisms. For this structure, each object is fibrant and an object is cofibrant iff it is a cofibrant dg module.*
2) *The* injective *structure, whose cofibrations are the monomorphisms. For this structure, each object is cofibrant and an object is fibrant iff it is a fibrant dg module.*

*For both structures, two morphisms are homotopic iff they become equal in the category up to homotopy $\mathcal{H}(\mathcal{A})$.*

3.3. **Exact categories, Frobenius categories.** Recall that an *exact category* in the sense of Quillen [111] is an additive category $\mathcal{E}$ endowed with a distinguished class of sequences

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0 \ ,$$

where $i$ is a kernel of $p$ and $p$ a cokernel of $i$. We will state the axioms these sequences have to satisfy using the terminology of [49]: The morphisms $p$ are called deflations, the morphisms $i$ inflations and the pairs $(i,p)$ conflations. The axioms are:

Ex0 The identity morphism of the zero object is a deflation.
Ex1 The composition of two deflations is a deflation.
Ex2 Deflations are stable under base change.
Ex2' Inflations are stable under cobase change.

As shown in [72], these axioms are equivalent to Quillen's and they imply that if $\mathcal{E}$ is small, then there is a fully faithful functor from $\mathcal{E}$ into an abelian category $\mathcal{E}'$ whose image is an additive subcategory closed under extensions and such that a sequence of $\mathcal{E}$ is a conflation iff its image is a short exact sequence of $\mathcal{E}'$. Conversely, one easily checks that an extension closed full additive subcategory $\mathcal{E}$ of an abelian category $\mathcal{E}'$ endowed with all conflations which become exact sequences in $\mathcal{E}'$ is always exact. The fundamental notions and constructions of homological algebra, and in particular the construction of the derived category, naturally extend from abelian to exact categories, *cf.* [101] and [74].

A *Frobenius category* is an exact category $\mathcal{E}$ which has enough injectives and enough projectives and where the class of projectives coincides with the class of injectives. In this case, the *stable category* $\underline{\mathcal{E}}$ obtained by dividing $\mathcal{E}$ by the ideal of morphisms factoring through a projective-injective carries a canonical structure of triangulated category, *cf.* [59] [57] [80] [54]. We write $\overline{f}$ for the image in $\underline{\mathcal{E}}$ of a morphism $f$ of $\mathcal{E}$. The suspension functor $S$ of $\underline{\mathcal{E}}$ is obtained by choosing a conflation

$$0 \longrightarrow A \longrightarrow IA \longrightarrow SA \longrightarrow 0$$

for each object $A$. Each triangle is isomorphic to a standard triangle $(\bar{\imath}, \overline{p}, \overline{e})$ obtained by embedding a conflation $(i,p)$ into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{p} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathbf{1}} & & \downarrow & & \downarrow{\scriptstyle e} & & \\ 0 & \longrightarrow & A & \longrightarrow & IA & \longrightarrow & SA & \longrightarrow & 0. \end{array}$$

3.4. **Triangulated structure.** Let $\mathcal{A}$ be a small dg category. Define a sequence

$$0 \longrightarrow L \xrightarrow{i} M \xrightarrow{p} N \longrightarrow 0$$

of $\mathcal{C}(\mathcal{A})$ to be a *conflation* if there is a morphism $r \in \mathcal{H}om(M,L)^0$ such that $ri = \mathbf{1}_L$ or, equivalently, a morphism $s \in \mathcal{H}om(N,M)$ such that $ps = \mathbf{1}_N$.

**Lemma 3.3.** a) *Endowed with these conflations, $\mathcal{C}(\mathcal{A})$ becomes a Frobenius category. The resulting stable category is canonically isomorphic to $\mathcal{H}(\mathcal{A})$. The suspension functor is induced by the shift $M \mapsto M[1]$.*

b) *Endowed with the suspension induced by that of $\mathcal{H}(\mathcal{A})$ and the triangles isomorphic to images of triangles of $\mathcal{H}(\mathcal{A})$ the derived category $\mathcal{D}(\mathcal{A})$ becomes a triangulated category. Each short exact sequence of complexes yields a canonical triangle.*

3.5. **Compact objects, Brown representability.** Let $\mathcal{T}$ be a triangulated category admitting arbitrary coproducts. Since the adjoint of a triangle functor is a triangle functor [80], the coproduct of triangles is then automatically a triangle. Moreover, $\mathcal{T}$ is *idempotent complete* [18], *i.e.* each idempotent endomorphism of an object of $\mathcal{T}$ is the composition of a section with a retraction. An object $C$ of $\mathcal{T}$ is *compact* if the functor $\mathcal{T}(C,?)$ commutes with arbitrary coproducts, *i.e.* for each family $(X_i)$ of objects of $\mathcal{T}$, the canonical morphism

$$\coprod \mathcal{T}(C, X_i) \to \mathcal{T}(C, \coprod X_i)$$

is invertible. The triangulated category $\mathcal{T}$ is *compactly generated* if there is a set $\mathcal{G}$ of compact objects $G$ such that an object $X$ of $\mathcal{T}$ vanishes iff we have $\mathcal{T}(G,X)=0$ for each $G\in\mathcal{G}$.

**Theorem 3.4** (Characterization of compact objects [143] [102])**.** *An object of $\mathcal{T}$ is compact iff it is a direct factor of an iterated extension of copies of objects of $\mathcal{G}$ shifted in both directions.*

**Theorem 3.5** (Brown representability [25] [1] [103])**.** *If $\mathcal{T}$ is compactly generated, a cohomological functor $F:\mathcal{T}^{op}\to \operatorname{Mod}\mathbb{Z}$ is representable iff it takes coproducts of $\mathcal{T}$ to products of* $\operatorname{Mod}\mathbb{Z}$*.*

A set of objects $\mathcal{G}$ *symmetrically generates* $\mathcal{T}$ [90] if we have

1) an object $X$ of $\mathcal{T}$ vanishes iff $\mathcal{T}(G,X)=0$ for each $G\in\mathcal{G}$ and
2) there is a set of objects $\mathcal{G}'$ such that a morphism $f:X\to Y$ of $\mathcal{T}$ induces surjections $\mathcal{T}(G,X)\to\mathcal{T}(G,Y)$ for all $G\in\mathcal{G}$ iff it induces injections $\mathcal{T}(Y,G')\to\mathcal{T}(X,G')$ for all $G'\in\mathcal{G}'$.

If $\mathcal{G}$ compactly generates $\mathcal{T}$, then we can take for $\mathcal{G}'$ the set of objects $G'$ defined by

$$\mathcal{T}(?,G')=\operatorname{Hom}_k(\mathcal{T}(G,?),E)\ ,\ G\in\mathcal{G}\ ,$$

where $E$ is an injective cogenerator of the category of $k$-modules. Thus, in this case, $\mathcal{G}$ also symmetrically generates $\mathcal{T}$.

**Theorem 3.6** (Brown representability for the dual [105] [90])**.** *If $\mathcal{T}$ is symmetrically generated, a homological functor $F:\mathcal{T}\to\operatorname{Mod}\mathbb{Z}$ is corepresentable iff it commutes with products.*

Let $\mathcal{A}$ be a small dg category. The derived category $\mathcal{D}(\mathcal{A})$ admits arbitrary coproducts and these are induced by coproducts of modules. Thanks to the isomorphisms

$$\mathcal{D}(\mathcal{A})(X^\wedge[n],M)\xrightarrow{\sim}H^{-n}M(X) \tag{4}$$

obtained from (3), each dg module $X^\wedge[n]$, where $X$ is an object of $\mathcal{A}$ and $n$ an integer, is compact. The isomorphism (4) also shows that a dg module $M$ vanishes in $\mathcal{D}(\mathcal{A})$ iff each morphism $X^\wedge[n]\to M$ vanishes. Thus the set $\mathcal{G}$ formed by the $X^\wedge[n]$, $X\in\mathcal{A}$, $n\in\mathbb{Z}$, is a set of compact generators for $\mathcal{D}(\mathcal{A})$. The *triangulated category* $\operatorname{tria}(\mathcal{A})$ *associated with* $\mathcal{A}$ is the closure in $\mathcal{D}(\mathcal{A})$ of the set of representable functors $X^\wedge$, $X\in\mathcal{A}$, under shifts in both directions and extensions. The *category of perfect objects* $\operatorname{per}(\mathcal{A})$ the closure of $\operatorname{tria}(\mathcal{A})$ under passage to direct factors in $\mathcal{D}(\mathcal{A})$. The above theorems yield the

**Corollary 3.7.** *An object of $\mathcal{D}(\mathcal{A})$ is compact iff it lies in* $\operatorname{per}(\mathcal{A})$*. A cohomological functor $\mathcal{D}(\mathcal{A})^{op}\to\operatorname{Mod}k$ is representable iff it takes coproducts of $\mathcal{D}(\mathcal{A})$ to products of* $\operatorname{Mod}k$*. A homological functor $\mathcal{D}(\mathcal{A})\to\operatorname{Mod}k$ is corepresentable iff it commutes with products.*

3.6. **Algebraic triangulated categories.** Let $\mathcal{T}$ be a $k$-linear triangulated category. We say that $\mathcal{T}$ is *algebraic* if it is triangle equivalent to $\underline{\mathcal{E}}$ for some $k$-linear Frobenius category $\mathcal{E}$. It is easy to see that each $k$-linear triangulated subcategory of an algebraic triangulated category is algebraic. We will see below that each Verdier localization of an algebraic triangulated category is algebraic (if we neglect a set-theoretic problem). Moreover, categories of complexes up to homotopy are algebraic, by 3.4. Therefore, 'all' triangulated categories occurring in algebra and geometry are algebraic (non algebraic ones occur in topology, *cf.* section 3.9). We wish to show that they can 'all' be described by dg categories.

Let $\mathcal{T}$ be a triangulated category and $\mathcal{G}$ a full subcategory. We make $\mathcal{G}$ into a graded category $\mathcal{G}_{gr}$ by defining

$$\mathcal{G}_{gr}(G, G') = \bigoplus_{n \in \mathbb{Z}} \mathcal{T}(G, S^n G').$$

We obtain a natural functor $\overline{F}$ from $\mathcal{T}$ to the category of graded $\mathcal{G}_{gr}$-modules by sending an object $Y$ of $\mathcal{T}$ to the $\mathcal{G}_{gr}$-module

$$X \mapsto \bigoplus_{n \in \mathbb{Z}} \mathcal{T}(X, S^n Y)$$

**Theorem 3.8** ([73]). *Suppose that $\mathcal{T}$ is algebraic. Then there is a dg category $\mathcal{A}$ such that $H^*(\mathcal{A})$ is isomorphic to $\mathcal{G}_{gr}$ and a triangle functor*

$$F : \mathcal{T} \to \mathcal{D}(\mathcal{A})$$

*such that the composition $H^* \circ F$ is isomorphic to $\overline{F}$. Moreover,*

a) *$F$ induces an equivalence from $\mathcal{T}$ to* tria$(\mathcal{A})$ *iff $\mathcal{T}$ coincides with its smallest full triangulated subcategory containing $\mathcal{G}$;*
b) *$F$ induces an equivalence from $\mathcal{T}$ to* per$(\mathcal{A})$ *iff $\mathcal{T}$ is idempotent complete (*cf. *section 3.5) and equals the closure of $\mathcal{G}$ under shifts in both directions, extensions and passage to direct factors;*
c) *$F$ is an equivalence $\mathcal{T} \xrightarrow{\sim} \mathcal{D}(\mathcal{A})$ iff $\mathcal{T}$ admits arbitrary coproducts and the objects of $\mathcal{G}$ form a set of compact generators for $\mathcal{T}$.*

Examples arise from commutative and non commutative geometry: A. Bondal and M. Van den Bergh show in [20] that if $X$ is a quasi-compact quasi-separated scheme, then the (unbounded) derived category $\mathcal{T} = \mathcal{D}_{qc}(X)$ of complexes of $\mathcal{O}_X$-modules with quasi-coherent homology admits a single compact generator $G$ and that moreover, $\mathrm{Hom}(G, G[n])$ vanishes except for finitely many $n$. Thus $\mathcal{T}$ is equivalent to the derived category of a dg category with one object whose endomorphism ring has bounded homology.

R. Rouquier shows in [122] (*cf.* also [91]) that if $X$ is a quasi-projective scheme over a perfect field $k$, then the derived category of coherent sheaves over $X$ admits a generator as a triangulated category (as in part b) and, surprisingly, that it is even of 'finite dimension' as a triangulated category: each object occurs as a direct factor of an object which admits a 'resolution' of bounded length by finite sums of shifts of the generator. Thus, the bounded derived category of coherent sheaves is equivalent to per$(\mathcal{A})$ for a dg category with one object whose endomorphism ring satisfies a strong regularity condition.

In [17], J. Block describes the bounded derived category of complexes of sheaves with coherent homology on a complex manifold $X$ as the category $H^0(\mathcal{A})$ associated with a dg category constructed from the Dolbeault dg algebra $(A^{0,\bullet}(X), \overline{\partial})$. This can be seen as an instance of a), where, for $\mathcal{G}$, we can take for example the category of coherent sheaves (*i.e.* complexes concentrated in degree 0). Note however that the term 'perfect derived category' is used with a different meaning in [17].

In the independently obtained [37], W. Dwyer and J. Greenlees give elegant descriptions via dg endomorphism rings of categories of complete, respectively torsion, modules. Their results are applied in a unifying study of duality phenomena in algebra and topology in [38].

One of the original motivations for the theorem was D. Happel's description [57] [58] of the bounded derived category of a finite-dimensional associative algebra of finite global dimension as the stable category of a certain Frobenius category. This in turn was inspired by Bernstein-Gelfand-Gelfand's [15] and Beilinson's [16] descriptions of the derived category of coherent sheaves on projective space.

A vast generalization of the theorem to non-additive contexts [128] is due to S. Schwede and B. Shipley [130], *cf.* also section 3.9 below.

3.7. **Well-generated algebraic triangulated categories.** A triangulated category $\mathcal{T}$ is *well-generated* [106] [89] if it admits arbitrary coproducts and a *good set of generators* $\mathcal{G}$, *i.e.* $\mathcal{G}$ is stable under shifts in both directions and satisfies

1) an object $X$ of $\mathcal{T}$ vanishes iff $\mathcal{T}(G,X)=0$ for each $G\in\mathcal{G}$,
2) there is a cardinal $\alpha$ such that each $G\in\mathcal{G}$ is *$\alpha$-small*, *i.e.* for each family of objects $X_i$, $i\in I$, of $\mathcal{T}$, each morphism
$$G\to\bigoplus_{i\in I}X_i$$
factors through a subsum $\bigoplus_{i\in J}X_i$ for some subset $J$ of $I$ of cardinality strictly smaller than $\alpha$,
3) for each family of morphisms $f_i : X_i\to Y_i$, $i\in I$, of $\mathcal{T}$ which induces surjections
$$\mathcal{T}(G,X_i)\to\mathcal{T}(G,Y_i)$$
for all $G\in\mathcal{G}$ and all $i\in I$, the sum of the $f_i$ induces surjections
$$\mathcal{T}(G,\bigoplus X_i)\to\mathcal{T}(G,\bigoplus Y_i)$$
for all $G\in\mathcal{G}$.

Clearly each compactly generated triangulated category is well-generated. A. Neeman proves in [106] that the Brown representability theorem holds for well-generated triangulated categories. This is one of the main reasons for studying them. Another important result of [106] is that if $\mathcal{T}$ is well-generated and $\mathcal{S}\to\mathcal{T}$ is a localization (*i.e.* a fully faithful triangle functor admitting a left adjoint whose kernel is generated by a *set* of objects) then $\mathcal{S}$ is well-generated. Thus each localization of a compactly generated triangulated category is well-generated and in particular, so is each localization of the derived category of a small dg category.

Here is another class of examples: Let $\mathcal{B}$ be a Grothendieck abelian category, *e.g.* the category of modules on a ringed space. Then, by the Popescu-Gabriel theorem [109] [96], $\mathcal{B}$ is the localization of the category of $\operatorname{Mod} A$ of $A$-modules over some ring $A$. One can deduce from this that the unbounded derived category of of the abelian category $\mathcal{B}$ (*cf.* [48] [142] [67]) is a localization of $\mathcal{D}(A)$ and thus is well-generated.

**Theorem 3.9** ([110]). *Let $\mathcal{T}$ be an algebraic triangulated category. Then $\mathcal{T}$ is well-generated iff it is a localization of $\mathcal{D}(\mathcal{A})$ for some small dg category $\mathcal{A}$. Moreover, if $\mathcal{T}$ is well-generated and $\mathcal{U}\subset\mathcal{T}$ a full small subcategory such that, for each $X\in\mathcal{T}$, we have*
$$X=0\Leftrightarrow\mathcal{T}(U,S^nX)=0 \text{ for all } n\in\mathbb{Z} \text{ and } U\in\mathcal{U}\,,$$
*then there is an associated localization $\mathcal{T}\to\mathcal{D}(\mathcal{A})$ for some small dg category $\mathcal{A}$ with $H^*(\mathcal{A})=\mathcal{U}_{gr}$.*

3.8. **Morita equivalence.** Let $\mathcal{A}$ and $\mathcal{B}$ be small dg categories. Let $X$ be an $\mathcal{A}$-$\mathcal{B}$-*bimodule*, *i.e.* a dg $\mathcal{A}^{op}\otimes\mathcal{B}$-module $X$. Thus $X$ is given by complexes $X(B,A)$, for all $A$ in $\mathcal{A}$ and $B$ in $\mathcal{B}$, and morphisms of complexes
$$\mathcal{B}(A,A')\otimes X(B,A)\otimes\mathcal{A}(B',B)\to X(B',A').$$
For each dg $\mathcal{B}$-module $M$, we obtain a dg $\mathcal{A}$-module
$$GM=\mathcal{H}om(X,M): A\mapsto\mathcal{H}om(X(?,A),M).$$

The functor $G : \mathcal{C}(\mathcal{B}) \to \mathcal{C}(\mathcal{A})$ admits a left adjoint $F : L \mapsto L \otimes_{\mathcal{A}} X$. These functors do not respect quasi-isomorphisms in general, but they form a Quillen adjunction (*cf.* section 3.9) and their derived functors

$$\mathbf{L}F : L \mapsto F(\mathbf{p}L) \text{ and } \mathbf{R}G : M \mapsto G(\mathbf{i}M)$$

form an adjoint pair of functors between $\mathcal{D}(\mathcal{A})$ and $\mathcal{D}(\mathcal{B})$.

**Lemma 3.10** ([73])**.** *The functor* $\mathbf{L}F : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ *is an equivalence if and only if*

a) *the dg* $\mathcal{B}$*-module* $X(?, A)$ *is perfect for all* $A$ *in* $\mathcal{A}$*,*
b) *the morphism*

$$\mathcal{A}(A, A') \to \mathcal{H}om(X(?, A), X(?, A'))$$

*is a quasi-isomorphism for all* $A$*,* $A'$ *in* $\mathcal{A}$ *and*
c) *the dg* $\mathcal{B}$*-modules* $X(?, A)$*,* $A \in \mathcal{A}$*, form a set of (compact) generators for* $\mathcal{D}(\mathcal{B})$*.*

For example, if $E : \mathcal{A} \to \mathcal{B}$ is a dg functor, then $X(B, A) = \mathcal{B}(B, E(A))$ defines a dg bimodule so that the above functor $G$ is the restriction along $E$. Then the lemma shows that $\mathbf{R}G$ is an equivalence iff $E$ is a quasi-equivalence. We loosely refer to the functor $\mathbf{L}F$ associated with a dg $\mathcal{A}$-$\mathcal{B}$-bimodule as a *tensor functor*.

**Theorem 3.11** ([73])**.** *The following are equivalent*
1) *There is an equivalence* $\mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ *given by a composition of tensor functors and their inverses.*
2) *There is a dg subcategory* $\mathcal{G}$ *of* $\mathcal{C}(\mathcal{B})$ *formed by cofibrant dg modules such that the objects of* $\mathcal{G}$ *form a set of compact generators for* $\mathcal{D}(\mathcal{B})$ *and there is a chain of quasi-equivalences*

$$\mathcal{A} \leftarrow \mathcal{A}' \to \cdots \leftarrow \mathcal{G}' \to \mathcal{G}$$

*linking* $\mathcal{A}$ *to* $\mathcal{G}$*.*

We say that $\mathcal{A}$ and $\mathcal{B}$ are *dg Morita equivalent* if the conditions of the theorem are satisfied. In this case, there is of course a triangle equivalence $\mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$. In general, the existence of such a triangle equivalence is not sufficient for $\mathcal{A}$ and $\mathcal{B}$ to be dg Morita equivalent, *cf.* section 3.9. The following theorem is therefore remarkable:

**Theorem 3.12** (Rickard [115])**.** *Suppose that* $\mathcal{A}$ *and* $\mathcal{B}$ *have their homology concentrated in degree* 0*. Then the following are equivalent:*

1) $\mathcal{A}$ *and* $\mathcal{B}$ *are dg Morita equivalent.*
2) *There is a triangle equivalence* $\mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$*.*
3) *There is a full subcategory* $\mathcal{T}$ *of* $\mathcal{D}(\mathcal{B})$ *such that*
   a) *the objects of* $\mathcal{T}$ *form a set of compact generators of* $\mathcal{D}(\mathcal{B})$*,*
   b) *we have* $\mathcal{D}(\mathcal{B})(T, T'[n]) = 0$ *for all* $n \neq 0$ *and all* $T$*,* $T'$ *of* $\mathcal{T}$*,*
   c) *there is an equivalence* $H^0(\mathcal{A}) \xrightarrow{\sim} \mathcal{T}$*.*

We refer to [73] or [36] for this form of the theorem. A subcategory $\mathcal{T}$ satisfying a) and b) in 3) is called a *tilting subcategory*, a concept which generalizes that of a tilting module. We refer to [113] [3] for the theory of tilting, from which this theorem arose and which provides huge classes of examples from the representation theory of finite-dimensional algebras and finite groups as well as from algebraic geometry, *cf.* also the appendix to [107] and [83] [116] [123].

3.9. **Topological Morita equivalence.** In recent years, Morita theory has been vastly generalized from algebraic triangulated categories to stable model categories in work due to S. Schwede and B. Shipley. This is based on the category of symmetric spectra as constructed in [63] (*cf.* [43] for a different construction of a symmetric monoidal model category for the category of spectra). We refer to [127] for an excellent exposition of these far-reaching results and their surprising applications in homotopy theory. In work by D. Dugger and B. Shippley, this 'topological Morita theory' has been applied to dg categories. We briefly describe their results and refer to [132] for a highly readable, more detailed survey.

The main idea is to replace the monoidal base category, the derived category of abelian groups $\mathcal{D}(\mathbb{Z})$, by a more fundamental category: the 'derived category of the category of sets', *i.e.* the homotopy category of spectra. To preserve higher homotopical information, one must not, of course, work at the level of derived categories but has to introduce model categories. So instead of considering $\mathcal{D}(\mathbb{Z})$, one considers its model category $\mathcal{C}(\mathbb{Z})$ of complexes of abelian groups and replaces it by a convenient model of the category of spectra: the category of symmetric spectra, which one might imagine as 'complexes of abelian groups up to homotopy'. We refer to [63] or [127] for the precise definition. As shown in [63], symmetric spectra form a *symmetric* monoidal category which carries a compatible Quillen model structure and whose homotopy category is equivalent to the homotopy category of spectra of Bousfield and Friedlander [24]. The tensor product is the *smash product* $\wedge$ and the unit object is the *sphere spectrum* $\mathbb{S}$. The unit object is cofibrant and the smash product induces a monoidal structure on the homotopy category of symmetric spectra. The *Eilenberg-MacLane functor* $H$ is a lax monoidal functor from the category of complexes $\mathcal{C}(k)$ to the category of symmetric spectra such that the homology groups of a complex $C$ become isomorphic to the homotopy groups of $HC$. Since $H$ is lax monoidal, if $A$ is a dg $\mathbb{Z}$-algebra, then $HA$ is naturally an algebra in the category of symmetric spectra and if $M$ is an $A$-module, then $HM$ becomes an $HA$-module. More generally, if $\mathcal{A}$ is a dg category over $\mathbb{Z}$, then $H\mathcal{A}$ becomes a *spectral category*, *i.e.* a category enriched in symmetric spectra, *cf.* [129] [131]. Each $\mathcal{A}$-module $M$ then gives rise to a *spectral module* $HM$ over $H\mathcal{A}$. The spectral modules over a spectral category form a Quillen model category [131]. Its homotopy category is a triangulated category which is not algebraic in general: for instance, in the homotopy category of 2-local spectra, the identity morphism of the cone over twice the identity of the sphere spectrum is of order four, but in each algebraic triangulated category, the identity of the cone on twice the identity of an object is of order two at most. A general method to prove that a triangulated category obtained from a suitable stable Quillen model category is not algebraic is to show that its [114] Hom-functor enriched in spectra does not factor through the canonical functor from the derived category of abelian groups to the homotopy category of spectra.

Recall that if $\mathcal{L}$ and $\mathcal{M}$ are Quillen model categories, a *Quillen adjunction* is given by a pair of adjoint functors $L : \mathcal{L} \to \mathcal{M}$ and $R : \mathcal{M} \to \mathcal{L}$ such that $L$ preserves cofibrations and $R$ fibrations. Such a pair induces an adjoint pair between the homotopy categories of $\mathcal{L}$ and $\mathcal{M}$. If the induced functors are equivalences, then $(L, R)$ is a *Quillen equivalence.* The model categories $\mathcal{L}$ and $\mathcal{M}$ are *Quillen equivalent* if they are linked by a chain of Quillen equivalences.

It was shown by A. Robinson [120], *cf.* also [130], that for an ordinary ring $R$, the unbounded derived category of $R$-modules is equivalent to the homotopy category of spectral modules over $HR$. This result is generalized and refined as follows:

**Theorem 3.13** (Shipley [131])**.** *If $\mathcal{A}$ is a dg category over $\mathbb{Z}$, the model categories of dg $\mathcal{A}$-modules and of spectral modules over $H\mathcal{A}$ are Quillen equivalent.*

This allows us to define two small dg categories $\mathcal{A}$ and $\mathcal{B}$ to be *topologically Morita equivalent* if their categories of spectral modules are Quillen equivalent.

**Proposition 3.14** ([132]). *Let $\mathcal{A}$ and $\mathcal{B}$ be two dg rings. Then statement a) implies b) and b) implies c):*

a) *$\mathcal{A}$ and $\mathcal{B}$ are dg Morita equivalent.*
b) *$\mathcal{A}$ and $\mathcal{B}$ are topologically Morita equivalent.*
c) *$\mathcal{D}(\mathcal{A})$ is triangle equivalent to $\mathcal{D}(\mathcal{B})$.*

It is remarkable that in general, these implications are strict. Examples which show this were obtained in recent joint work by D. Dugger and B. Shipley, *cf.* [132]. To show that c) does not imply b), they invoke Schlichting's example [126]: Let $p$ be an odd prime. The module categories over $A' = \mathbb{Z}/p^2$ and $B' = (\mathbb{Z}/p)[\varepsilon]/\varepsilon^2$ are Frobenius categories. Their stable categories are triangle equivalent (both are equivalent to the category of $\mathbb{Z}/p$-vector spaces with the identical suspension and the split triangles) but the $K$-theories associated with the stable module categories are not isomorphic. Since $K$-theory is preserved under topological Morita equivalence (*cf.* section 5.2 below), the dg algebras $A$ and $B$ associated (*cf.* section 3.6) with the canonical generators (corresponding to the one-dimensional vector space over $\mathbb{Z}/p$) of the stable categories of $A'$ and $B'$ cannot be topologically Morita equivalent.

To show that b) does not imply a), Dugger and Shipley consider two dg algebras $A$ and $B$ with homology isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2[2]$. The isomorphism classes of such algebras in the homotopy category of dg $\mathbb{Z}$-algebras are parametrized by the Hochschild cohomology group $HH^4_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/2)$. Their isomorphism classes in the homotopy category of $\mathbb{S}$-algebras are parametrized by the topological Hochschild cohomology group $THH^4_{\mathbb{S}}(\mathbb{Z}/2, \mathbb{Z}/2)$ as shown in [92]. The computation of the Hochschild cohomology group $HH^4_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/2)$ is elementary and, thanks to Franjou-Lannes-Schwartz' work [47], the topological Hochschild cohomology algebra

$$THH^*_{\mathbb{S}}(\mathbb{Z}/2, \mathbb{Z}/2)$$

is known. Dugger-Shipley then conclude by exhibiting a non-trivial element in the kernel of the canonical map

$$\Phi : HH^4_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/2) \to THH^4_{\mathbb{S}}(\mathbb{Z}/2, \mathbb{Z}/2).$$

The explicit description of the two algebras is given in [131] [36].

## 4. The homotopy category of small dg categories

4.1. **Introduction.** Invariants like $K$-theory, Hochschild homology, cyclic homology ... naturally extend from $k$-algebras to dg categories (*cf.* section 5). In analogy with the case of ordinary $k$-algebras, these extended invariants are preserved under dg Morita equivalence. However, unlike the module category over a $k$-algebra, the derived category of a dg category, even with its triangulated structure, does not contain enough information to compute the invariant (*cf.* the examples in section 3.9). Our aim in this section is to present a category obtained from that of small dg categories by 'inverting the dg Morita equivalences'. It could be called the 'homotopy category of enhanced (idempotent complete) triangulated categories' [21] or the 'Morita homotopy category of small dg categories' Hmo, as in [136]. Invariants like $K$-theory and cyclic homology factor through the Morita homotopy category.

The Morita homotopy category very much resembles the category of small, idempotent complete, triangulated categories. In particular, it admits 'dg quotients' [34], which correspond to Verdier localizations. Like these, they are characterized by a universal property.

The great advantages of the Morita homotopy category over that of small triangulated categories are that moreover, it admits *all (homotopy) limits and colimits* (like any homotopy category of a Quillen model category) and is *monoidal and closed*.

The Morita homotopy category Hmo is a full subcategory of the localization Hqe of the category of small dg categories with respect to the quasi-equivalences. The first step is therefore to analyze the larger category Hqe. Its morphism spaces are revealed by Toën's theorem 4.3 below.

4.2. **Inverting quasi-equivalences.** Let $k$ be a commutative ring and $\mathrm{dgcat}_k$ the category of small dg $k$-categories as in section 2.2. An analogue of the following theorem for simplicial categories is proved in [14].

**Theorem 4.1** ([137]). *The category* $\mathrm{dgcat}_k$ *admits a structure of cofibrantly generated model category whose weak equivalences are the quasi-equivalences and whose fibrations are the dg functors* $F : \mathcal{A} \to \mathcal{B}$ *which induce componentwise surjections* $\mathcal{A}(X,Y) \to \mathcal{B}(FX,FY)$ *for all* $X, Y$ *in* $\mathcal{A}$ *and such that, for each isomorphism* $v : F(X) \to Z$ *of* $H^0(\mathcal{B})$*, there is an isomorphism* $u$ *of* $H^0(\mathcal{A})$ *with* $F(u) = v$*.*

This shows in particular that the *localization* Hqe of $\mathrm{dgcat}_k$ with respect to the quasi-equivalences has small Hom-sets and that we can compute morphisms from $\mathcal{A}$ to $\mathcal{B}$ in the localization as morphisms modulo homotopy from a cofibrant replacement $\mathcal{A}_{cof}$ of $\mathcal{A}$ to $\mathcal{B}$ (note that all small dg categories are fibrant). In general, the cofibrant replacement $\mathcal{A}_{cof}$ is not easy to compute with but if $\mathcal{A}(X,Y)$ is cofibrant in $\mathcal{C}(k)$ and the unit morphisms $k \to \mathcal{A}(X,X)$ admit retractions in $\mathcal{C}(k)$ for all objects $X$, $Y$ of $\mathcal{A}$, for example if $k$ is a field, then for $\mathcal{A}_{cof}$, we can take the category with the same objects as $\mathcal{A}$ and whose morphism spaces are given by the 'reduced cobar-bar construction', *cf. e.g.* [34] [70]. The homotopy relation is then the one of [77, 3.3].

However, the morphism sets in the localization are much better described as follows: Consider two dg categories $\mathcal{A}$ and $\mathcal{B}$. If necessary, we replace $\mathcal{A}$ by a quasi-equivalent dg category so as to achieve that $\mathcal{A}$ is *$k$-flat*, *i.e.* the functor $\mathcal{A}(X,Y)\otimes ?$ preserves quasi-isomorphisms for all $X$, $Y$ of $\mathcal{A}$ (for example, we could take a cofibrant replacement of $\mathcal{A}$). Let $\mathrm{rep}(\mathcal{A},\mathcal{B})$ be the full subcategory of the derived category $\mathcal{D}(\mathcal{A}^{op}\otimes\mathcal{B})$ of $\mathcal{A}$-$\mathcal{B}$-bimodules formed by the bimodules $X$ such that the tensor functor

$$? \overset{L}{\otimes}_{\mathcal{A}} X : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$$

takes the representable $\mathcal{A}$-modules to objects which are isomorphic to representable $\mathcal{B}$-modules. In other words, we require that $X(?, A)$ is quasi-isomorphic to a representable $\mathcal{B}$-module for each object $A$ of $\mathcal{A}$. We call such a bimodule a *quasi-functor* since it yields a genuine functor

$$H^0(\mathcal{A}) \to H^0(\mathcal{B}).$$

We think of $\mathrm{rep}(\mathcal{A},\mathcal{B})$ as the 'category of representations up to homotopy of $\mathcal{A}$ in $\mathcal{B}$'.

**Theorem 4.2** (Toën [146]). *The morphisms from* $\mathcal{A}$ *to* $\mathcal{B}$ *in the localization of* $\mathrm{dgcat}_k$ *with respect to the quasi-equivalences are in natural bijection with the isomorphism classes of* $\mathrm{rep}(\mathcal{A},\mathcal{B})$*.*

The theorem has been in limbo for some time, *cf.* [75, 2.3] [77] [34]. It is due to B. Toën, as a corollary of a much more precise statement: Recall from [62, Ch. 5] that each model category $\mathcal{M}$ admits a mapping space bifunctor

$$\mathrm{Map} : \mathrm{Ho}(\mathcal{M})^{op} \times \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathrm{Sset})$$

such that we have, for example, the natural isomorphisms

$$\pi_0(\mathrm{Map}(X,Y)) = \mathrm{Ho}(\mathcal{M})(X,Y).$$

The spaces Map may also be viewed as the morphism spaces in the Dwyer-Kan localization [41] [39] [40] of $\mathcal{M}$ with respect to the class of weak equivalences, *cf.* [40] [61]. Now let $\mathcal{R}(\mathcal{A},\mathcal{B})$ be the category with the same objects as $\mathrm{rep}(\mathcal{A},\mathcal{B})$ and whose morphisms are the quasi-isomorphisms of dg bimodules. Thus, the category $\mathcal{R}(\mathcal{A},\mathcal{B})$ is a non full subcategory of the category of dg bimodules $\mathcal{C}(\mathcal{A}^{op}\otimes\mathcal{B})$.

**Theorem 4.3** (Toën [146])**.** *There is a canonical weak equivalence of simplicial sets between* $\mathrm{Map}(\mathcal{A},\mathcal{B})$ *and the nerve of the category* $\mathcal{R}(\mathcal{A},\mathcal{B})$*.*

The theorem allows one to compute the homotopy groups of the *classifying space* $|\,\mathrm{dgcat}\,|$ *of dg categories*, which is defined as the nerve of the category of quasi-equivalences between dg categories. Of course, the connected components of this space are in bijection with the isomorphism classes of Hqe. Now let $\mathcal{A}$ be a small dg category. Then the fundamental group of $|\,\mathrm{dgcat}\,|$ at $\mathcal{A}$ is the group of automorphisms of $\mathcal{A}$ in Hqe (*cf.* [111]). For example, if $\mathcal{A}$ is the category of bounded complexes of projective $B$-modules over an ordinary $k$-algebra $B$, then this group is the derived Picard group of $B$ as studied in [124] [78] [156]. For the higher homotopy groups, we have the

**Corollary 4.4** ([146])**.**
a) *The group* $\pi_2(|\,\mathrm{dgcat}\,|,\mathcal{A})$ *is the group of invertible elements of the dg center of* $\mathcal{A}$ *(=its zeroth Hochschild cohomology group).*
b) *For* $i\geq 2$*, the group* $\pi_i(|\,\mathrm{dgcat}\,|,\mathcal{A})$ *is the* $(2-i)$*-th Hochschild cohomology of* $\mathcal{A}$*.*

4.3. **Closed monoidal structure.** As we have observed in section 2.2, the category $\mathrm{dgcat}_k$ admits a tensor product $\otimes$ and an internal Hom-functor $\mathcal{H}om$. If $\mathcal{A}$ is cofibrant, then the functor $\mathcal{A}\otimes ?$ preserves weak equivalences so that the localization Hqe inherits a tensor product $\overset{L}{\otimes}$. However, the tensor product of two cofibrant dg categories is not cofibrant in general (in analogy with the fact that the tensor product of two non commutative free algebras is not non commutative free in general). By the adjunction formula

$$\mathcal{H}om(\mathcal{A},\mathcal{H}om(\mathcal{B},\mathcal{C})) = \mathcal{H}om(\mathcal{A}\otimes\mathcal{B},\mathcal{C})\ ,$$

it follows that even if $\mathcal{A}$ is cofibrant, the functor $\mathcal{H}om(\mathcal{A},?)$ cannot preserve weak equivalences in general and thus will not induce an internal Hom-functor in Hqe. Nevertheless, we have the

**Theorem 4.5** ([34] [146])**.** *The monoidal category* $(\mathrm{Hqe},\overset{L}{\otimes})$ *admits an internal* Hom-*functor* $\mathcal{RH}om$*. For two dg categories* $\mathcal{A}$ *and* $\mathcal{B}$ *such that* $\mathcal{A}$ *is k-flat, the dg category* $\mathcal{RH}om(\mathcal{A},\mathcal{B})$ *is isomorphic in* Hqe *to the dg category* $\mathrm{rep}_{dg}(\mathcal{A},\mathcal{B})$*,* i.e. *the full subcategory of the dg category of* $\mathcal{A}$*-*$\mathcal{B}$*-bimodules whose objects are those of* $\mathrm{rep}(\mathcal{A},\mathcal{B})$ *and which are cofibrant as bimodules.*

Thus we have equivalences (we suppose $\mathcal{A}$ $k$-flat)

$$H^0(\mathcal{RH}om(\mathcal{A},\mathcal{B})) = H^0(\mathrm{rep}_{dg}(\mathcal{A},\mathcal{B})) \xrightarrow{\sim} \mathrm{rep}(\mathcal{A},\mathcal{B}).$$

In terms of the internal Hom-functor $\mathcal{H}om$ of $\mathrm{dgcat}_k$, we have

$$H^0(\mathcal{RH}om(\mathcal{A},\mathcal{B})) = H^0(\mathcal{H}om(\mathcal{A},\mathcal{B}))[\Sigma^{-1}]\ ,$$

where $\Sigma$ is the set of morphisms $\phi : F\to G$ such that $\phi(A)$ is invertible in $H^0(\mathcal{B})$ for all objects $A$ of $\mathcal{A}$, *cf.* [75].

Yet another description can be given in terms of $A_\infty$-functors: Let $\mathcal{A}$ be a dg category such that the morphism spaces $\mathcal{A}(A,A')$ are cofibrant in $\mathcal{C}(k)$ and the unit maps

$k \to \mathcal{A}(A, A)$ admit retractions in $\mathcal{C}(k)$ for all objects $A$, $A'$ of $\mathcal{A}$. Then the dg category $\mathcal{RH}om(\mathcal{A}, \mathcal{B})$ is quasi-equivalent to the $A_\infty$-category of (strictly unital) $A_\infty$-functors from $\mathcal{A}$ to $\mathcal{B}$, *cf.* [86] [93] [98] [70]. Since $\mathcal{B}$ is a dg category, this $A_\infty$-category is in fact a dg category.

An important point of classical Morita theory is that for two rings $B$, $C$, there is an equivalence between the category of $B$-$C$-bimodules and the category of coproduct preserving functors from the category of $B$-modules to that of $C$-modules (note that here and in what follows, we need to consider 'large' categories and should introduce universes to make our statements rigorous . . . ). Similarly, if $\mathcal{A}$ is a small $k$-flat dg category, we consider the large dg category $\mathcal{D}_{dg}(\mathcal{A})$: it is the full dg subcategory of $\mathcal{C}_{dg}(\mathcal{A})$ whose objects are all the cofibrant dg modules. Thus we have an equivalence of categories

$$\mathcal{D}(\mathcal{A}) = H^0(\mathcal{D}_{dg}(\mathcal{A})).$$

This shows that if $\mathcal{B}$ is another dg category, then each quasi-functor $X$ in

$$\mathrm{rep}(\mathcal{D}_{dg}(\mathcal{A}), \mathcal{D}_{dg}(\mathcal{B}))$$

gives rise to a functor $\mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$. We say that the quasifunctor $X$ *preserves coproducts* if this functor preserves coproducts.

**Theorem 4.6** ([146])**.** *There is a canonical isomorphism in* Hqe

$$\mathcal{D}_{dg}(\mathcal{A}^{op} \otimes \mathcal{B}) \xrightarrow{\sim} \mathcal{RH}om_c(\mathcal{D}_{dg}(\mathcal{A}), \mathcal{D}_{dg}(\mathcal{B}))\,,$$

*where $\mathcal{RH}om_c$ denotes the full subcategory of $\mathcal{RH}om$ formed by the coproduct preserving quasifunctors.*

If we apply this theorem to $\mathcal{B} = \mathcal{A}$ and compare the endomorphism algebras of the identity functors on both sides, we see that the Hochschild cohomology (*cf.* section 5.4 below) of the small dg category $\mathcal{A}$ coincides with the Hochschild cohomology of the large dg category $\mathcal{D}_{dg}(\mathcal{A})$, which is quite surprising. An analogous result for Grothendieck abelian categories (in particular, module categories) is due to T. Lowen and M. Van den Bergh [97].

4.4. **Dg localizations, dg quotients, dg-derived categories.** Let $\mathcal{A}$ be a small dg category. Let $S$ be a set of morphisms of $H^0(\mathcal{A})$. Let us say that a morphism $R : \mathcal{A} \to \mathcal{B}$ of Hqe *makes $S$ invertible* if the induced functor

$$H^0(\mathcal{A}) \to H^0(\mathcal{B})$$

takes each $s \in S$ to an isomorphism.

**Theorem 4.7** ([146])**.** *There is a morphism $Q : \mathcal{A} \to \mathcal{A}[S^{-1}]$ of* Hqe *such that $Q$ makes $S$ invertible and each morphism $R$ of* Hqe *which makes $S$ invertible uniquely factors through $Q$.*

We call $\mathcal{A}[S^{-1}]$ the *dg localization of $\mathcal{A}$ at $S$*. Note that it is unique up to unique isomorphism in Hqe. It is constructed in [146] as a homotopy pushout

$$\begin{array}{ccc} \coprod_{s\in S} I & \longrightarrow & \mathcal{A} \\ \downarrow & & \downarrow \\ \coprod_{s\in S} k & \longrightarrow & \mathcal{A}[S^{-1}], \end{array}$$

where $I$ denotes the dg $k$-category freely generated by one arrow $f : 0 \to 1$ of degree 0 with $df = 0$ and left vertical arrow is induced by the morphisms $I \to k$ which sends $f$ to

1. The universal property of $Q : \mathcal{A} \to \mathcal{A}[S^{-1}]$ admits refined forms, namely, $Q$ induces an equivalence of categories

$$\mathrm{rep}(\mathcal{A}[S^{-1}], \mathcal{B}) \xrightarrow{\sim} \mathrm{rep}_S(\mathcal{A}, \mathcal{B}) ,$$

an isomorphism of Hqe

$$\mathrm{rep}_{dg}(\mathcal{A}[S^{-1}], \mathcal{B}) \xrightarrow{\sim} \mathrm{rep}_{dg,S}(\mathcal{A}, \mathcal{B}) ,$$

and a weak equivalence of simplicial sets

$$\mathrm{Map}(\mathcal{A}[S^{-1}], \mathcal{B}) \xrightarrow{\sim} \mathrm{Map}_S(\mathcal{A}, \mathcal{B}).$$

Here $\mathrm{rep}_S$ and $\mathrm{rep}_{dg,S}$ denote the full subcategories of quasi-functors whose associated functors $H^0(\mathcal{A}) \to H^0(\mathcal{B})$ make $S$ invertible and $\mathrm{Map}_S$ the union of the connected components containing these quasi-functors.

An important variant is the following: Let $\mathcal{N}$ be a set of objects of $\mathcal{A}$. Let us say that a morphism $Q : \mathcal{A} \to \mathcal{B}$ of Hqe *annihilates* $\mathcal{N}$ if the induced functor

$$H^0(\mathcal{A}) \to H^0(\mathcal{B})$$

takes all objects of $\mathcal{N}$ to zero objects (*i.e.* objects whose identity morphism vanishes in $H^0(\mathcal{B})$).

**Theorem 4.8** ([77] [34])**.** *There is a morphism $Q : \mathcal{A} \to \mathcal{A}/\mathcal{N}$ of* Hqe *which annihilates $\mathcal{N}$ and is universal among the morphisms annihilating $\mathcal{N}$.*

We call $\mathcal{A}/\mathcal{N}$ the *dg quotient of $\mathcal{A}$ by $\mathcal{N}$*. If $\mathcal{A}$ is $k$-flat (*cf.* section 4.2), then $\mathcal{A}/\mathcal{N}$ admits a beautiful simple construction [34]: One adjoins to $\mathcal{A}$ a contracting homotopy for each object of $\mathcal{N}$. The general case can be reduced to this one or treated using orthogonal subcategories [77]. The dg quotient has refined universal properties analogous to those of the dg localization. In particular, the morphism $\mathcal{A} \to \mathcal{A}/\mathcal{N}$ induces an equivalence [34]

$$\mathrm{rep}(\mathcal{A}/\mathcal{N}, \mathcal{B}) \to \mathrm{rep}_{\mathcal{N}}(\mathcal{A}, \mathcal{B}) ,$$

where $\mathrm{rep}_{\mathcal{N}}$ denotes the full subcategory of quasi-functors whose associated functors $H^0(\mathcal{A}) \to H^0(\mathcal{B})$ annihilate $\mathcal{N}$.

Dg quotients yield functorial dg versions of Verdier localizations [151]. For example, if $\mathcal{E}$ is a small abelian (or, more generally, exact) category, we can take for $\mathcal{A}$ the dg category of bounded complexes $\mathcal{C}^b_{dg}(\mathcal{E})$ over $\mathcal{E}$ and for $\mathcal{N}$ the dg category of bounded acyclic complexes $\mathcal{A}c^b_{dg}(\mathcal{E})$. Then we obtain the *dg-derived category*

$$\mathcal{D}^b_{dg}(\mathcal{E}) = \mathcal{C}^b_{dg}(\mathcal{E})/\mathcal{A}c^b_{dg}(\mathcal{E})$$

so that we have

$$\mathcal{D}^b(\mathcal{E}) = H^0(\mathcal{D}^b_{dg}(\mathcal{E})).$$

More generally, every localization pair [77] (=Frobenius pair [125]) gives rise to a dg category. After taking the necessary set-theoretic precautions, we also obtain a dg-derived category

$$\mathcal{D}_{dg}(\mathcal{E}) = \mathcal{C}_{dg}(\mathcal{E})/\mathcal{A}c_{dg}(\mathcal{E})$$

which refines the *unbounded* derived category of a $k$-linear Grothendieck abelian category $\mathcal{E}$. For a quasi-compact quasi-separated scheme $X$, let us write $\mathcal{D}_{dg}(X)$ for $\mathcal{D}_{dg}(\mathcal{E})$, where $\mathcal{E}$ is the Grothendieck abelian category of quasi-coherent sheaves on $X$. The following theorem shows that dg functors between dg derived categories are much more closely related to geometry than triangle functors between derived categories, *cf.* [19] [108].

**Theorem 4.9** ([146]). *Let $X$ and $Y$ be quasi-compact separated schemes over $k$ such that $X$ is flat over* Spec $k$. *Then we have a canonical isomorphism in* Hqe

$$\mathcal{D}_{dg}(X \times_k Y) \xrightarrow{\sim} \mathcal{RH}om_c(\mathcal{D}_{dg}(X), \mathcal{D}_{dg}(Y)) ,$$

*where $\mathcal{RH}om_c$ denotes the full subcategory of $\mathcal{RH}om$ formed by the coproduct preserving quasi-functors. Moreover, if $X$ and $Y$ are smooth and projective over* Spec $k$, *we have a canonical isomorphism in* Hqe

$$\mathrm{par}_{dg}(X \times_k Y) \xrightarrow{\sim} \mathcal{RH}om(\mathrm{par}_{dg}(X), \mathrm{par}_{dg}(Y))$$

*where* $\mathrm{par}_{dg}$ *denotes the full dg subcategory of $\mathcal{D}_{dg}$ whose objects are the perfect complexes.*

4.5. **Pretriangulated dg categories.** Let $\mathcal{A}$ be a small dg category. We say that $\mathcal{A}$ is *pretriangulated* or *exact* if the image of the Yoneda functor

$$Z^0(\mathcal{A}) \to \mathcal{C}(\mathcal{A}) \, , \; X \mapsto X^\wedge$$

is stable under shifts in both directions and extensions (in the sense of the exact structure of section 3.4). Equivalently, for all objects $X$, $Y$ of $\mathcal{A}$ and all integers $n$, the object $X^\wedge[n]$ is isomorphic to $X[n]^\wedge$ and the cone over a morphism $f^\wedge : X^\wedge \to Y^\wedge$ is isomorphic to $C(f)^\wedge$ for unique objects $X[n]$ and $C(f)$ of $Z^0(\mathcal{A})$. If $\mathcal{A}$ is exact, then $Z^0(\mathcal{A})$ becomes a Frobenius subcategory of $\mathcal{C}(\mathcal{A})$ and $H^0(\mathcal{A})$ a triangulated subcategory of $\mathcal{H}(\mathcal{A})$. If $\mathcal{B}$ is an exact dg category and $\mathcal{A}$ an arbitrary dg category, then $\mathcal{H}om(\mathcal{A}, \mathcal{B})$ is exact (whereas $\mathcal{A} \otimes \mathcal{B}$ is not, in general).

If $\mathcal{A}$ is an arbitrary small dg category, there is a universal dg functor

$$\mathcal{A} \to \mathrm{pretr}(\mathcal{A})$$

to a pretriangulated dg category $\mathrm{pretr}(\mathcal{A})$, *i.e.* a functor inducing an equivalence

$$\mathcal{H}om(\mathcal{A}, \mathcal{B}) \xrightarrow{\sim} \mathcal{H}om(\mathrm{pretr}(\mathcal{A}), \mathcal{B})$$

for each exact dg category $\mathcal{B}$. The dg category $\mathrm{pretr}(\mathcal{A})$ is the *pretriangulated hull of $\mathcal{A}$* constructed explicitly in [22], *cf.* also [34] [136].

For any dg category $\mathcal{A}$, the category $H^0(\mathrm{pretr}(\mathcal{A}))$ is equivalent to the triangulated subcategory of $\mathcal{HA}$ generated by the representable dg modules. The functor pretr preserves quasi-equivalences and induces a left adjoint to the inclusion of the full subcategory of exact dg categories into the homotopy category Hqe. If $\mathcal{B}$ is pretriangulated, then so is $\mathcal{RH}om(\mathcal{A}, \mathcal{B})$ for each small dg category $\mathcal{A}$ and we have

$$\mathcal{RH}om(\mathrm{pretr}(\mathcal{A}), \mathcal{B}) \xrightarrow{\sim} \mathcal{RH}om(\mathcal{A}, \mathcal{B}).$$

4.6. **Morita closed dg categories, exact sequences.** A dg functor $F : \mathcal{A} \to \mathcal{B}$ between small dg categories is a *Morita morphism* if it induces an equivalence $\mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{A})$. Each quasi-equivalence is a Morita morphism (*cf.* section 3.8) and so is the canonical morphism $\mathcal{A} \to \mathrm{pretr}(\mathcal{A})$ from $\mathcal{A}$ to its pretriangulated hull.

**Theorem 4.10** ([136]). *The category* $\mathrm{dgcat}_k$ *admits a structure of cofibrantly generated model category whose weak equivalences are the Morita morphisms and whose cofibrations are the same as those of the canonical model structure on* $\mathrm{dgcat}_k$ (cf. *theorem 4.1).*

A dg category $\mathcal{A}$ is *Morita closed* (or *triangulated* in the terminology of [147]) iff it is fibrant with respect to this model structure. This is the case iff the canonical functor $H^0(\mathcal{A}) \to \mathrm{per}(\mathcal{A})$ is an equivalence iff $\mathcal{A}$ is pretriangulated and $H^0(\mathcal{A})$ is idempotent complete (*cf.* section 3.5). We write $\mathcal{A} \to \mathrm{per}_{dg}(\mathcal{A})$ for a fibrant replacement of $\mathcal{A}$ and then have

$$\mathrm{per}(\mathcal{A}) = H^0(\mathrm{per}_{dg}(\mathcal{A})).$$

We write Hmo *for the localization* of $\mathrm{dgcat}_k$ with respect to the Morita morphisms. Then the functor $\mathcal{A} \mapsto \mathrm{per}_{dg}(\mathcal{A})$ yields a right adjoint of the quotient functor $\mathrm{Hqe} \to \mathrm{Hmo}$ and induces an equivalence from Hmo onto the subcategory of Morita closed dg categories in Hqe, *cf.* [136]. The category Hmo is pointed: The dg category with one object and one morphism is both initial and terminal. Moreover, Hmo admits all finite coproducts (they are induced by the disjoint unions) and these are isomorphic to products.

Let

$$\mathcal{A} \xrightarrow{I} \mathcal{B} \xrightarrow{P} \mathcal{C} \tag{5}$$

be a sequence of Hqe such that $PI = 0$ in Hmo.

**Theorem 4.11.** *The following are equivalent*

i) *In* Hmo*, $I$ is a kernel of $P$ and $P$ a cokernel of $I$.*
ii) *The morphism $I$ induces an equivalence of* $\mathrm{per}(\mathcal{A})$ *onto a thick subcategory of* $\mathrm{per}(\mathcal{B})$ *and $P$ induces an equivalence of the idempotent closure* [8] *of the Verdier quotient with* $\mathrm{per}(\mathcal{C})$*.*
iii) *The functor $I$ induces an equivalence of $\mathcal{D}(\mathcal{A})$ with a thick subcategory of $\mathcal{D}(\mathcal{B})$ and $P$ identifies the Verdier quotient with $\mathcal{D}(\mathcal{C})$.*

The theorem is proved in [77]. The equivalence of ii) and iii) is a consequence of Thomason-Trobaugh's localization theorem [143] [102] [106]. We say that (5) is an *exact sequence* of Hmo if the conditions of the theorem hold. For example, if $X$ is a quasi-compact quasi-separated scheme, $U \subset X$ a quasi-compact open subscheme and $Z = X \setminus U$, then the sequence

$$\mathrm{par}_{dg}(X \text{ on } Z) \to \mathrm{par}_{dg}(X) \to \mathrm{par}_{dg}(U)$$

is an exact sequence of Hmo by the results of [143, Sect. 5], where $\mathrm{par}_{dg}(X)$ denotes the dg quotient of the category of perfect complexes (viewed as a full dg subcategory of the category of complexes of $\mathcal{O}_X$-modules) by its subcategory of acyclic perfect complexes and $\mathrm{par}_{dg}(X \text{ on } Z)$ the full subcategory of perfect complexes supported on $Z$.

4.7. **Dg categories of finite type.** Let $\mathcal{M}$ be a cofibrantly generated model category and $I$ a small category. Recall that the category of functors $\mathcal{M}^I$ is again a cofibrantly generated model category (with the componentwise weak equivalences). Thus, the diagonal functor $\mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{M}^I)$ admits a left adjoint, the *homotopy colimit functor*, and a right adjoint, the *homotopy limit functor*. An object $X$ of $\mathcal{M}$ is *homotopically finitely presented* if, for each filtered direct system $Y_i$, $i \in I$, of $\mathcal{M}$, the canonical morphism

$$\operatorname{hocolim} \mathrm{Map}(X, Y_i) \to \mathrm{Map}(X, \operatorname{hocolim} Y_i)$$

is a weak equivalence of simplicial sets. The category $\mathcal{M}$ is *homotopically locally finitely presented* if, in $\mathrm{Ho}(\mathcal{M})$, each object is the homotopy colimit of a filtered direct system (in $\mathcal{M}$) of homotopically finitely presented objects.

For example [60], the category of dg algebras is homotopically locally finitely presented and a dg algebra is homotopically finitely presented iff, in the homotopy category, it is a retract of a non commutative free graded algebra $k\langle x_1, \dots, x_n\rangle$ endowed with a differential such that $dx_i$ belongs to $k\langle x_1, \dots, x_{i-1}\rangle$ for each $1 \leq i \leq n$. A dg category is *of finite type* if it is dg Morita equivalent to a homotopically finitely presented dg algebra.

**Theorem 4.12** ([147])**.** *The category of small dg categories endowed with the canonical model structure whose weak equivalences are the Morita morphisms is homotopically locally finitely presented and a dg category is homotopically finitely presented iff it is of finite type.*

A dg category $\mathcal{A}$ is *smooth* if the bimodule $(X,Y)\mapsto \mathcal{A}(X,Y)$ is perfect in $\mathcal{D}(\mathcal{A}^{op}\overset{L}{\otimes}\mathcal{A})$. This property is invariant under dg Morita equivalence. The explicit description of the homotopically finitely presented dg algebras shows that a dg category of finite type is smooth. Conversely [147], a dg category $\mathcal{A}$ is of finite type if it is smooth and *proper*, *i.e.* dg Morita equivalent to a dg algebra whose underlying complex of $k$-modules is perfect.

4.8. **Moduli of objects in dg categories.** Let $T$ be a small dg category. In [147], B. Toën and M. Vaquié introduce and study the $D^-$-stack (in the sense of [148]) of objects in $T$. By definition, this $D^-$-stack is the functor

$$\mathcal{M}_T : \mathrm{Scalg} \to \mathrm{Sset}$$

which sends a simplicial commutative $k$-algebra $A$ to the simplicial set

$$\mathrm{Map}(T^{op}, \mathrm{per}_{dg}(NA))\;,$$

where $NA$ is the commutative dg $k$-algebra obtained from $A$ by the Dold-Kan equivalence. They show that if $T$ is a dg category of finite type, then this $D^-$-stack is locally geometric and locally of finite presentation. Moreover, if $E : T \to \mathrm{per}_{dg}(k)$ is a $k$-point of $\mathcal{M}_T$, then the tangent complex of $\mathcal{M}_T$ at $E$ is given by

$$\mathcal{T}_{\mathcal{M}_T,E} \xrightarrow{\sim} \mathcal{RH}om(E,E)[1].$$

In particular, if $E$ is quasi-isomorphic to a representable $x^\wedge$, then we have

$$\mathcal{T}_{\mathcal{M}_T,E} \xrightarrow{\sim} T(x,x)[1].$$

It follows that the restriction of $\mathcal{M}_T$ to the category of commutative $k$-algebras is a locally geometric $\infty$-stack in the sense of C. Simpson [135]. Here are three consequences derived from these results in [147]:

1) If $T$ is a dg category over a field $k$ and is smooth, proper and Morita closed, then the sheaf associated with the presheaf

$$R \mapsto \mathrm{Aut}_{\mathrm{Hqe}_R}(T\otimes_k R)\;,$$

on the category of commutative $k$-algebras is a group scheme locally of finite type over $k$ (*cf.* [156] for the case where $T$ is an algebra).

2) If $X$ is a smooth proper scheme over a commutative ring $k$, then the $\infty$-stack of perfect complexes on $X$ is locally geometric.

3) If $A$ is a (non commutative) $k$-algebra over a field $k$, then the $\infty$-stack of bounded complexes of finite-dimensional $A$-modules is locally geometric if either $A$ is the path algebra of a finite quiver or a finite-dimensional algebra of finite global dimension.

4.9. **Dg orbit categories.** Let $\mathcal{A}$ be a dg category and $F : \mathcal{A}\to\mathcal{A}$ an automorphism of $\mathcal{A}$ in Hqe. Let us assume for simplicity that $F$ is given by a dg functor $\mathcal{A}\to\mathcal{A}$. The *dg orbit category* $\mathcal{A}/F^{\mathbb{Z}}$ has the same objects as $\mathcal{A}$ and the morphisms defined by

$$(\mathcal{A}/F^{\mathbb{Z}})(X,Y) = \bigoplus_{d\in\mathbb{Z}} \mathrm{colim}_n\, \mathcal{A}(F^nX, F^{n+d}Y).$$

The projection functor $P : \mathcal{A}\to\mathcal{A}/F^{\mathbb{Z}}$ is endowed with a canonical morphism $\phi : PF \to P$ which becomes invertible in $H^0(\mathcal{A}/F^{\mathbb{Z}})$ and the pair $(P,\phi)$ is the solution of a universal problem, *cf.* [79]. The category $H^0(\mathcal{A})/F^{\mathbb{Z}}$ is defined analogously. It is isomorphic to $H^0(\mathcal{A}/F^{\mathbb{Z}})$ and can be thought of as the 'category of orbits' of the functor $F$ acting in $H^0(\mathcal{A})$.

Let us now assume that $k$ is a field. Let $Q$ be a quiver (=oriented graph) whose underlying graph is a Dynkin graph of type $A$, $D$ or $E$. Let $\mathrm{mod}\, kQ$ be the abelian category of finite-dimensional representations of $Q$ over $k$ (*cf. e.g.* [49] [4]). Let $\mathcal{A} = \mathcal{D}^b_{dg}(\mathrm{mod}\, Q)$ and

$F : \mathcal{A} \to \mathcal{A}$ an automorphism in Hqe. We say that $F$ *acts properly* if no indecomposable object of $\mathcal{D}^b(\operatorname{mod} kQ)$ is isomorphic to its image under $F$. For example, if $\Sigma$ is the *Serre functor* of $\mathcal{A}$, defined by the bimodule

$$(X, Y) \mapsto \mathcal{H}om_k(\mathcal{A}(Y, X), k) ,$$

then $\Sigma$ acts properly and, more generally, if $S$ is the suspension functor, then $S^{-d}\Sigma$ acts properly for each $d \in \mathbb{N}$ unless $Q$ is reduced to a point.

**Theorem 4.13** ([79]). *If $F$ acts properly, the orbit category $\mathcal{D}^b_{dg}(\operatorname{mod} kQ)/F^{\mathbb{Z}}$ is Morita closed and thus $\mathcal{D}^b(\operatorname{mod} kQ)/F^{\mathbb{Z}}$ is canonically triangulated.*

In the particular case where $F = S^{-d}\Sigma$, the triangulated category $H^0(\mathcal{A}/F^{\mathbb{Z}})$ is Calabi-Yau [86] of CY-dimension $d$ (*cf.* [79]). For $d = 1$, the category $H^0(\mathcal{A}/F^{\mathbb{Z}})$ is equivalent to the category of finite-dimensional projective modules over the preprojective algebra (*cf.* [53] [31] [119]) associated with the Dynkin graph underlying $Q$. For $d = 2$, one obtains the *cluster category* associated with the Dynkin graph. This category was introduced in [27] for type $A$ and in [6] in the general case. It serves in the representation-theoretic approach (*cf. e.g.* [6] [26] [52]) to the study of cluster algebras [44] [45] [12] [46]. It seems likely [2] that if $k$ is algebraically closed, the theorem yields all Morita closed dg categories whose associated triangulated categories have finite-dimensional morphism spaces and only finitely many isoclasses of indecomposables. In particular, for each $d \in \mathbb{N}$, those among these categories which are Calabi-Yau of fixed CY-dimension $d$ would then be parametrized by the simply laced Dynkin diagrams.

## 5. Invariants

5.1. **Additive invariants.** Let $\mathrm{Hmo}_0$ be the category with the same objects as Hmo and where morphisms $\mathcal{A} \to \mathcal{B}$ are given by elements of the Grothendieck group of the triangulated category $\operatorname{rep}(\mathcal{A}, \mathcal{B})$. The composition is induced from that of Hmo. The category $\mathrm{Hmo}_0$ is additive and endowed with a canonical functor $\mathrm{Hmo} \to \mathrm{Hmo}_0$ (*cf.* [23] for a related construction). One can show [136] that a functor $F$ defined on Hmo with values in an additive category factors through $\mathrm{Hmo} \to \mathrm{Hmo}_0$ iff for each exact dg category $\mathcal{A}$ endowed with full exact dg subcategories $\mathcal{B}$ and $\mathcal{C}$ which give rise to a semi-orthogonal decomposition $H^0(\mathcal{A}) = (H^0(\mathcal{B}), H^0(\mathcal{C}))$ in the sense of [22], the inclusions induce an isomorphism $F(\mathcal{B}) \oplus F(\mathcal{C}) \xrightarrow{\sim} F(\mathcal{A})$. We then say that $F$ *is an additive invariant.* The most basic additive invariant is given by $F\mathcal{A} = K_0(\operatorname{per}\mathcal{A}))$. In $\mathrm{Hmo}_0$, it becomes a corepresentable functor: $K_0(\operatorname{per}(\mathcal{A})) = \mathrm{Hmo}_0(k, \mathcal{A})$. As we will see below, the $K$-theory spectrum and all variants of cyclic homology are additive invariants. This is of interest since non isomorphic objects of Hmo can become isomorphic in $\mathrm{Hmo}_0$. For example, if $k$ is an algebraically closed field, each finite-dimensional algebra of finite global dimension becomes isomorphic to a product of copies of $k$ in $\mathrm{Hmo}_0$ (*cf.* [75]) but it is isomorphic to such a product in Hmo only if it is semi-simple.

5.2. **$K$-theory.** Let $\mathcal{A}$ be a small dg $k$-category. Its $K$-theory $K(\mathcal{A})$ is defined by applying Waldhausen's construction [152] to a suitable category with cofibrations and weak equivalences: here, the category is that of perfect $\mathcal{A}$-modules, the cofibrations are the morphisms $i : L \to M$ of $\mathcal{A}$-modules which admit retractions as morphisms of graded $\mathcal{A}$-modules (*i.e.* the inflations of section 3.4) and the weak equivalences are the quasi-isomorphisms. This construction can be improved so as to yield a functor $K$ from $\mathrm{dgcat}_k$ to the homotopy category of spectra. As in [143], from Waldhausen's results [152] one then obtains the following

**Theorem 5.1.** a) [36] *The map $\mathcal{A} \mapsto K(\mathcal{A})$ yields a well-defined functor on* Hmo.

b) *Applied to the bounded dg-derived category* $\mathcal{D}^b_{dg}(\mathcal{E})$ *of an exact category* $\mathcal{E}$*, the* $K$*-theory defined above agrees with Quillen* $K$*-theory.*
c) *The functor* $\mathcal{A} \mapsto K(\mathcal{A})$ *is an additive invariant. Moreover, each short exact sequence* $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ *of* Hmo (cf. *section 4.6) yields a long exact sequence*

$$\ldots \to K_i(\mathcal{A}) \to K_i(\mathcal{B}) \to K_i(\mathcal{C}) \to \ldots \to K_0(\mathcal{B}) \to K_0(\mathcal{A}).$$

Part a) can be improved on: In fact, D. Dugger and B. Shipley show in [36] that $K$-theory is even preserved under *topological* Morita equivalence. Part c) can be improved on by defining *negative* $K$*-groups* and showing that the exact sequence extends indefinitely to the right. We refer to [125] for the most recent results, which include the case of dg categories. By combining part a) with Rickard's theorem 3.12, one obtains the invariance of the $K$-theory of rings under triangle equivalences between their derived categories. By combining a) and b), one obtains the invariance of the $K$-theory of abelian categories under equivalences between their derived categories which come from isomorphisms of Hmo (or, more generally, from topological Morita equivalences). In fact, according to A. Neeman's results [104] the $K$-theory of an abelian category is even determined by the underlying triangulated category of its derived category, *cf.* [107] for a survey of his work.

Of course, any invariant defined for small triangulated categories applied to the perfect derived category yields an invariant of small dg categories. For example, Balmer-Witt groups (*cf.* [7] for a survey), defined for dg categories $\mathcal{A}$ endowed with a suitable involution $\mathcal{A} \xrightarrow{\sim} \mathcal{A}^{op}$ in Hmo, yield such invariants.

5.3. **Hochschild and cyclic homology.** Let $\mathcal{A}$ be a small $k$-flat $k$-category. Following [100] the *Hochschild chain complex* of $\mathcal{A}$ is the complex concentrated in homological degrees $p \geq 0$ whose $p$th component is the sum of the

$$\mathcal{A}(X_p, X_0) \otimes \mathcal{A}(X_p, X_{p-1}) \otimes \mathcal{A}(X_{p-1}, X_{p-2}) \otimes \cdots \otimes \mathcal{A}(X_0, X_1)\ ,$$

where $X_0, \ldots, X_p$ range through the objects of $\mathcal{A}$, endowed with the differential

$$d(f_p \otimes \ldots \otimes f_0) = f_{p-1} \otimes \cdots \otimes f_0 f_p + \sum_{i=1}^{p} (-1)^i f_p \otimes \cdots \otimes f_i f_{i-1} \otimes \cdots \otimes f_0.$$

Via the cyclic permutations

$$t_p(f_{p-1} \otimes \cdots \otimes f_0) = (-1)^p f_0 \otimes f_{p-1} \otimes \cdots \otimes f_1$$

this complex becomes a precyclic chain complex and thus gives rise [74, Sect. 2] to a *mixed complex* $C(\mathcal{A})$ in the sense of [68], *i.e.* a dg module over the dg algebra $\Lambda = k[B]/(B^2)$, where $B$ is of degree $-1$ and $dB = 0$. As shown in [68], all variants of cyclic homology [95] only depend on $C(\mathcal{A})$ considered in $\mathcal{D}(\Lambda)$. For example, the cyclic homology of $\mathcal{A}$ is the homology of the complex $C(\mathcal{A}) \overset{L}{\otimes}_\Lambda k$.

If $\mathcal{A}$ is a $k$-flat differential graded category, its mixed complex is the sum-total complex of the bicomplex obtained as the natural re-interpretation of the above complex. If $\mathcal{A}$ is an arbitrary dg $k$-category, its Hochschild chain complex is defined as the one of a $k$-flat (*e.g.* a cofibrant) resolution of $\mathcal{A}$.

**Theorem 5.2** ([76] [77])**.** a) *The map* $\mathcal{A} \mapsto C(\mathcal{A})$ *yields an additive functor* $\mathrm{Hmo}_0 \to \mathcal{D}(\Lambda)$*. Moreover, each exact sequence of* Hmo (cf. *section 4.6) yields a canonical triangle of* $\mathcal{D}(\Lambda)$*.*
b) *If* $A$ *is a* $k$*-algebra, there is a natural isomorphism* $C(A) \xrightarrow{\sim} C(\mathrm{per}_{dg}(A))$*.*
c) *If* $X$ *is a quasi-compact separated scheme, there is a natural isomorphism* $C(X) \xrightarrow{\sim} C(\mathrm{par}_{dg}(X))$*, where* $C(X)$ *is the cyclic homology of* $X$ *in the sense of* [94] [154] *and* $\mathrm{par}_{dg}(X)$ *the dg category defined in section 4.6.*

The second statement in a) may be viewed as an excision theorem analogous to [155]. We refer to the recent proof [28] of Weibel's conjecture [153] on the vanishing of negative $K$-theory for an application of the theorem. The algebraic description of *topological* Hochschild (co-)homology [133] would suggest that it is also preserved under topological Morita equivalence but no reference seems to exist as yet.

The endomorphism algebra $\mathcal{RH}om_\Lambda(k,k)$ is quasi-isomorphic to $k[u]$, where $u$ is of degree 2 and $d(u) = 0$. It acts on $C(\mathcal{A}) \overset{L}{\otimes}_\Lambda k$ and this action is made visible in the isomorphism

$$C(\mathcal{A}) \overset{L}{\otimes}_\Lambda k = C(\mathcal{A}) \otimes k[u]$$

where $u$ is of degree 2 and the differential on the right hand complex is given by

$$d(x \otimes f) = d(x) \otimes f + (-1)^{|x|} xB \otimes uf.$$

The following 'Hodge–de Rham conjecture' is true for the dg category of perfect complexes on a smooth projective variety or over a finite-dimensional algebra of finite global dimension. It is wide open in the general case.

**Conjecture 5.3** ([33] [85])**.** *If $\mathcal{A}$ is a smooth proper dg category over a field $k$ of characteristic 0, then the homology of $C(\mathcal{A}) \otimes k[u]/(u^n)$ is a flat $k[u]/(u^n)$-module for all $n \geq 1$.*

5.4. **Hochschild cohomology.** Let $\mathcal{A}$ be a small cofibrant dg category. Its cohomological Hochschild complex $C(\mathcal{A}, \mathcal{A})$ is defined as the product-total complex of the bicomplex whose 0th column is

$$\prod \mathcal{A}(X_0, X_0)\,,$$

where $X_0$ ranges over the objects of $\mathcal{A}$, and whose $p$th column, for $p \geq 1$, is

$$\prod \mathcal{H}om_k(\mathcal{A}(X_{p-1}, X_p) \otimes \mathcal{A}(X_{p-2}, X_{p-1}) \otimes \cdots \otimes \mathcal{A}(X_0, X_1), \mathcal{A}(X_0, X_p))$$

where $X_0, \ldots, X_p$ range over the objects of $\mathcal{A}$. The horizontal differential is given by the Hochschild differential. This complex carries rich additional structure: As shown in [55], it is a $B_\infty$-algebra, *i.e.* its bar construction carries, in addition to its canonical differential and comultiplication, a natural *multiplication* which makes it into a dg bialgebra. The $B_\infty$-structure contains in particular the cup product and the Gerstenhaber bracket, which both descend to the Hochschild cohomology

$$HH^*(\mathcal{A}, \mathcal{A}) = H^* C(\mathcal{A}, \mathcal{A}).$$

The Hochschild cohomology is naturally interpreted as the homology of the complex

$$\mathcal{H}om(\mathbf{1}_\mathcal{A}, \mathbf{1}_\mathcal{A})$$

computed in the dg category $\mathcal{RH}om(\mathcal{A}, \mathcal{A})$, where $\mathbf{1}_\mathcal{A}$ denotes the identity functor of $\mathcal{A}$ (*i.e.* the bimodule $(X, Y) \mapsto \mathcal{A}(X, Y)$). Then the cup product corresponds to the composition (whereas the Gerstenhaber bracket has no obvious interpretation). Each $c \in HH^n(\mathcal{A}, \mathcal{A})$ gives rise to morphisms $cM : M \to M[n]$ of $\mathcal{D}(\mathcal{A})$, functorial in $M \in \mathcal{D}(\mathcal{A})$. Another interpretation links the Hochschild cohomology of $\mathcal{A}$ to the derived Picard group and to the higher homotopy groups of the category of quasi-equivalences between dg categories, *cf.* section 4.2.

A natural way of obtaining the $B_\infty$-algebra structure on $C(\mathcal{A}, \mathcal{A})$ is to consider the $A_\infty$-category of $A_\infty$-functors from $\mathcal{A}$ to itself [86] [93] [98]. Here, the $B_\infty$-algebra $C(\mathcal{A}, \mathcal{A})$ appears as the endomorphism algebra of the identity functor (*cf.* [70]).

Note that $C(\mathcal{A}, \mathcal{A})$ is not functorial with respect to dg functors. However, if $F : \mathcal{A} \to \mathcal{B}$ is a fully faithful dg functor, it clearly induces a restriction map

$$F^* : C(\mathcal{B}, \mathcal{B}) \to C(\mathcal{A}, \mathcal{A})$$

and this map is compatible with the $B_\infty$-structure. This can be used to construct [71] a morphism

$$\phi_X : C(\mathcal{B},\mathcal{B}) \to C(\mathcal{A},\mathcal{A})$$

in the homotopy category of $B_\infty$-algebras associated with each dg $\mathcal{A}$-$\mathcal{B}$-bimodule $X$ such that the functor

$$? \overset{L}{\otimes}_{\mathcal{A}} X : \mathrm{per}(\mathcal{A}) \to \mathcal{DB}$$

is fully faithful. If moreover the functor $X \overset{L}{\otimes}_{\mathcal{B}} ? : \mathrm{per}(\mathcal{B}^{op}) \to \mathcal{D}(\mathcal{A}^{op})$ is fully faithful, then $\phi_X$ is an isomorphism. In particular, the Hochschild complex becomes a functor

$$\mathrm{Hmo}_{\mathrm{ff}}^{op} \to \mathrm{Ho}(B_\infty)\ ,$$

where $\mathrm{Ho}(B_\infty)$ is the homotopy category of $B_\infty$-algebras and $\mathrm{Hmo}_{\mathrm{ff}}$ the (non full) subcategory of Hmo whose morphisms are the quasi-functors $X \in \mathrm{rep}(\mathcal{A},\mathcal{B})$ such that

$$? \overset{L}{\otimes}_{\mathcal{A}} X : \mathrm{per}(\mathcal{A}) \to \mathrm{per}(\mathcal{B})$$

is fully faithful. We refer to [97] for the closely related study of the Hochschild complex of an abelian category.

Let us suppose that $k$ is a field of characteristic 0. Endowed with the Gerstenhaber bracket the Hochschild complex $C(\mathcal{A},\mathcal{A})$ becomes a differential graded Lie algebra and this Lie algebra 'controls the deformations of the $A_\infty$-category $\mathcal{A}$', *cf. e.g.* [88]. Here the $A_\infty$-structures $(m_n)$, $n \geq 0$, may have a non trivial term $m_0$. Some (but not all) Hochschild cocyles also correspond to deformations of $\mathcal{A}$ as an object of Hmo. To be precise, let $k[\varepsilon]$ be the algebra of dual numbers and consider the reduction functor

$$R : \mathrm{Hmo}_{k[\varepsilon]} \to \mathrm{Hmo}_k\ ,\ \mathcal{B} \mapsto \mathcal{B} \overset{L}{\otimes}_{k[\varepsilon]} k.$$

A *first order Morita deformation* of $\mathcal{A}$ is a pair $(\mathcal{A}', \phi)$ formed by a dg $k[\varepsilon]$-category $\mathcal{A}'$ and an isomorphism $\phi : R\mathcal{A}' \to \mathcal{A}$ of $\mathrm{Hmo}_k$. An equivalence between such deformations is given by an isomorphism $\psi : \mathcal{A}' \to \mathcal{A}''$ such that $\phi' R\psi = \phi$. Then one can show [51] that the equivalence classes of first order Morita deformations of $\mathcal{A}$ are in natural bijection with the classes $c \in HH^2(\mathcal{A},\mathcal{A})$ such that the induced morphism $cP : P \to P[2]$ is nilpotent in $H^*\mathcal{H}om(P,P)$ for each perfect $\mathcal{A}$-module $P$. If $\mathcal{A}$ is proper or, more generally, if $H^n\mathcal{A}(?,X)$ vanishes for $n \gg 0$ for all objects $X$ of $\mathcal{A}$, then this condition holds for all Hochschild 2-cocycles $c$. On the other hand, if $\mathcal{A}$ is given by the dg algebra $k[u,u^{-1}]$, where $u$ is of degree 2 and $du = 0$, then it does not hold for the cocycle $u \in HH^2(\mathcal{A},\mathcal{A})$.

5.5. **Fine structure of the Hochschild complexes.** The Hochschild cochain complex of a dg category carries a natural homotopy action of the little squares operad. This is the positive answer to a question by P. Deligne [29] which has been obtained, for example, in [99] [87] [13] .... Hochschild cohomology acts on Hochschild homology and this action comes from a homotopy action of the Hochschild cochain complex, viewed as a homotopy algebra over the little squares, on the Hochschild chain complex. This is the positive answer to a series of conjectures due to B. Tsygan [149] and Tamarkin-Tsygan [138]. It has recently been obtained by B. Tsygan and D. Tamarkin [150]. Together, the two Hochschild complexes endowed with these structures yield a *non commutative calculus* [141] analogous to the differential calculus on a smooth manifold. The link with classical calculus on smooth commutative manifolds is established through M. Kontsevich's formality theorem [84] [139] for Hochschild cochains and in [134] (*cf.* also [32]) for Hochschild chains.

Clearly, these finer structures on the Hochschild complexes are linked to the category of dg categories and its simplicial enrichment given by the Dwyer-Kan localization as developped in [146]. At the end of the introduction to [146], the reader will find a more detailed discussion of these links, *cf.* also [82]. A precise relationship is announced in [140].

5.6. **Derived Hall algebras.** Let $\mathcal{A}$ be a *finitary* abelian category, *i.e.* such that the underlying sets of $\mathcal{A}(X, Y)$ and $\mathrm{Ext}^1(X, Y)$ are finite for all objects $X$, $Y$ of $\mathcal{A}$. The *Ringel-Hall algebra* $\mathcal{H}(\mathcal{A})$ is the free abelian group on the isomorphism classes of $\mathcal{A}$ endowed with the multiplication whose structure constants are given by the Hall numbers $f^Z_{XY}$, which count the number of subobjects of $Z$ isomorphic to $X$ and such that $Z/X$ is isomorphic to $Y$, *cf.* [30] for a survey. Thanks to Ringel's famous theorem [117] [118], for each simply laced Dynkin diagram $\Delta$, the *positive part* of the Drinfeld-Jimbo quantum group $U_q(\Delta)$ (*cf. e.g.* [69]) is obtained as the (generic, twisted) Ringel-Hall algebra of the abelian category of finite-dimensional representations of a quiver $\vec{\Delta}$ with underlying graph $\Delta$. Since Ringel's discovery, it has been pointed out by several authors, *cf. e.g.* [66], that an extension of the construction of the Ringel-Hall algebra to the derived category of the representations of $\vec{\Delta}$ might yield the *whole* quantum group. However, if one tries to mimick the construction of $\mathcal{H}(\mathcal{A})$ for a triangulated category $\mathcal{T}$ by replacing short exact sequences by triangles one obtains a multiplication which fails to be associative, *cf.* [66] [64].

A solution to this problem has recently been proposed by B. Toën in [144]. He obtains an explicit formula for the structure constants $\phi^Z_{XY}$ of an associative multiplication on the rational vector space generated by the isomorphism classes of any triangulated category $\mathcal{T}$ which appears as the perfect derived category $\mathrm{per}(T)$ of a proper dg category $T$ over a finite field $k$. The resulting $\mathbb{Q}$-algebra is the *derived Hall algebra* $\mathcal{DH}(T)$ *of* $T$. The formula for the structure constants reads as follows:

$$\phi^Z_{XY} = \sum_f |\,\mathrm{Aut}(f/Z)|^{-1} \prod_{i>0} |\,\mathrm{Ext}^{-i}(X, Z)|^{(-1)^i} |\,\mathrm{Ext}^{-i}(X, X)|^{(-1)^{i+1}}| \ ,$$

where $f$ ranges over the set of orbits of the group $\mathrm{Aut}(X)$ in the set of morphisms $f : X \to Z$ whose cone is isomorphic to $Y$, and $\mathrm{Aut}(f/Z)$ denotes the stabilizer of $f$ under the action of $\mathrm{Aut}(X)$. The proof of associativity is inspired by methods from the study of higher moduli spaces [148] [146] [147] and by the homotopy theoretic approach to $K$-theory [111]. From the formula, it is immediate that $\mathcal{DH}(T)$ is preserved under triangle equivalences $\mathrm{per}(T) \xrightarrow{\sim} \mathrm{per}(T')$. Another consequence is that if $\mathcal{A}$ is the heart of a non degenerate $t$-structure [11] on $\mathrm{per}(T)$, then the Ringel-Hall algebra of $\mathcal{A}$ appears as a subalgebra of $\mathcal{DH}(T)$. The derived Hall algebra of the derived category of representations of $\vec{\Delta}$ over a finite field appears closely related to the constructions of [66]. Its precise relation to the quantum group $U_q(\Delta)$ remains to be investigated.

Notice that like the $K_0$-group, the derived Hall algebra only depends on the underlying triangulated category of $\mathrm{per}(T)$. On the other hand, it clearly contains much finer information than $K_0$. One would expect that geometric versions of the derived Hall algebra, as defined in [145, 3.3] will depend on finer data.

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