DERIVED INVARIANCE OF HIGHER STRUCTURES ON THE
HOCHSCHILD COMPLEX

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Abstract. We show that derived equivalences preserve the homotopy type
of the (cohomological) Hochschild complex as a $B_\infty$-algebra. More generally,
we prove that, as an object of the homotopy category of $B_\infty$-algebras, the
Hochschild complex is contravariant with respect to fully faithful derived tensor
functors. We also show that the Hochschild complexes of a Koszul algebra
and its dual are homotopy equivalent as $B_\infty$-algebras. In particular, their
Hochschild cohomologies are isomorphic as algebras, which is a recent result
by R.-O. Buchweitz [4], and as Lie algebras. Our methods also yield a derived
invariant definition of the Hochschild complex of an exact category.

1. Introduction

Let $k$ be a field. It is a theorem of D. Happel [14] and J. Rickard [28] that if two
(associative, unital) $k$-algebras $A$ and $B$ have equivalent derived categories
$\mathcal{D}A \cong \mathcal{D}B$,

then there is an isomorphism of graded algebras between their Hochschild coho-

mologies $\text{HH}^*(A, A)$ and $\text{HH}^*(B, B)$. More precisely, if $X$ is a complex of $A$-$B$-
bimodules such that the total derived functor

$? \otimes^L_A X : \mathcal{D}A \rightarrow \mathcal{D}B$

is an equivalence, then there is a canonical algebra isomorphism $f_X : \text{HH}^*(B, B) \cong \text{HH}^*(A, A)$ associated with $X$. It was shown in [17] that $f_X$ also respects the Lie

algebra structures given by the Gerstenhaber bracket. In this article, we refine and
generalize these results: the cup product and the Lie bracket on $\text{HH}^*(A, A)$ come

from certain operations on the Hochschild complex $C(A, A)$ itself. In fact, this

complex has many more operations, most of which do not descend to homology.
The situation is described precisely by saying that $C(A, A)$ is a $B_\infty$-algebra in the

sense of Getzler–Jones [9, 5.2], i.e. its bar construction $B(C(A, A))$ is endowed with

a differential and a multiplication which, together with the canonical comultipli-
cation, make $B(C(A, A))$ into a differential graded bialgebra. Let us consider the

homotopy category of $B_\infty$-algebras $\text{Ho}(B_\infty)$, i.e. the category obtained from the
category of $B_\infty$-algebras by formally inverting all morphisms which induce quasi-
isomorphisms in the underlying complexes. Our main result (3.2) is that in the
above situation, the morphism $f_X$ lifts to an isomorphism

$\varphi_X : C(B, B) \cong C(A, A)$

in the homotopy category of $B_\infty$-algebras. This refines the known invariance results.
Moreover, we succeed in constructing a (not necessarily invertible) morphism $\varphi_X$
under weaker hypotheses: for $\varphi_X$ to be defined, it suffices that the functor

$? \otimes^L_A X : \mathcal{D}A \rightarrow \mathcal{D}B$


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induces a fully faithful functor on the subcategory of perfect complexes, or, equivalently, that the canonical morphism

\[ A \to \text{RHom}_B(X, X) \]

is a quasi-isomorphism (3.3). If this is the case and, moreover, the canonical morphism

\[ B^{op} \to \text{RHom}_{A^{op}}(X, X) \]

is a quasi-isomorphism, then \( \varphi_X \) is invertible. This suffices to show that if \( A \) is a Koszul algebra and \( A' \) its Koszul dual (with the natural bigрадing), then we have an isomorphism

\[ C(A, A) \cong C(A', A') \]

in the homotopy category of Adams graded \( B_\infty \)-algebras (3.5). In particular, this yields a new proof of R.-O. Buchweitz’ theorem [4] that the Hochschild cohomology algebras of \( A \) and \( A' \) are isomorphic.

To prove these results, we work in the more general setting of dg categories (section 4). This proves to be extremely convenient thanks to the fact that each fully faithful dg functor \( F : A \to B \) induces a restriction map

\[ F^* : C(B, B) \to C(A, A) \]

which is compatible with the \( B_\infty \)-structures. Apart from this observation, our main technical tool is (the generalization to dg categories of) the homotopy bicartesian square (4.5)

\[
\begin{array}{ccc}
C(T, T) & \to & C(A, A) \\
\downarrow & & \downarrow \\
C(B, B) & \to & C(A, X, B)
\end{array}
\]

where \( A \) and \( B \) are dg algebras, \( X \) is a dg \( A-B \)-bimodule,

\[ T = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix} \]

the obvious dg algebra of upper triangular matrices and \( C(A, X, B) \) the canonical complex computing

\[ \text{RHom}_{A^{op} \otimes B}(X, X). \]

The long exact sequence associated with this square was discovered by D. Happel [14] (in the case where \( A = k \)) and has been further studied and generalized in [5], [25], [11], [6], [12], [10], [3]. Similar ideas appear in work on automorphism groups by Guil-Asensio–Saorín [13].

Using this square and the functoriality of \( A \mapsto C(A, A) \) with respect to fully faithful dg functors, we construct (4.6)

\[ \varphi_X : C(\mathcal{B}, \mathcal{B}) \to C(\mathcal{A}, \mathcal{A}), \]

for suitable \( A-B \)-bimodules \( X \), as a morphism of the homotopy category \( \text{Ho}(B_\infty) \). We prove its fundamental properties, notably its (partial) compatibility with tensor products of bimodules.

The generality of the construction in section 4 makes it possible to define (section 5) the Hochschild complex of a (small, \( k \)-linear) exact category \( \mathcal{E} \) in the sense of Quillen [26] in a way which is invariant under derived equivalences: We define

\[ C(\mathcal{E}, \mathcal{E}) = C(Q, Q), \quad Q = C^b(\mathcal{E})/Ac^b(\mathcal{E}). \]
where $C^b(\mathcal{E})/Ac^b(\mathcal{E})$ is the quotient dg category of the dg category of bounded complexes $C^b(\mathcal{E})$ by the dg category of acyclic bounded complexes $Ac^b(\mathcal{E})$. The existence and uniqueness of such a quotient (in a suitable category) was proved in [19] and [7]. The Hochschild cohomology and deformation theory of abelian categories is being studied by T. Lowen [23]. The relation of our approach to her work remains to be elucidated.

I thank Seokbong Seol for pointing out to that the Adams grading has to be taken into account in Theorem 3.5.

2. $B_\infty$-algebras

Let $k$ be a commutative ring and $C$ a $\mathbb{Z}$-graded $k$-module

\[ C = \bigoplus_{p \in \mathbb{Z}} CP. \]

The suspension $C[1]$ of $C$ is the $\mathbb{Z}$-graded $k$-module with $C[1]^p = C^{p+1}$ for all $p \in \mathbb{Z}$. Let $BC$ be the graded tensor $k$-coalgebra on the suspension $C[1]$ of $C$. We denote its comultiplication by $\Delta$, its counit by $\eta$ and its canonical augmentation by $\varepsilon$. A $B_\infty$-algebra [9] [1] structure on $C$ is the datum of a differential $d$ and a multiplication $\mu$ on $BC$ such that $(BC, d, \mu, \Delta, \varepsilon, \eta)$ is a differential graded bialgebra (it is then automatically a dg Hopf algebra).

The datum of the differential on $BC$ translates into the datum of an $A_\infty$-algebra structure on $C$ (in particular, $C$ itself carries a natural differential) while the multiplication $\mu$ corresponds to the datum of a family of morphisms

\[ C^{\otimes l} \otimes C^{\otimes m} \to C, \ l, m \geq 0, \]

satisfying a series of compatibility conditions among each other and with the $A_\infty$-structure.

Let $A$ be an associative unital dg $k$-algebra and $C = C(A, A)$ its Hochschild complex. It was shown by Getzler and Jones [9] that $C$ has a canonical $B_\infty$-algebra structure: The underlying $A_\infty$-structure on $C$ is the canonical dg algebra structure on $C$ (whose homology is the Hochschild cohomology algebra) whereas the multiplication $\mu$ corresponds to the brace operations [1], [16], [9, 5.2]. In particular, the $B_\infty$-structure on $C(A, A)$ determines the Gerstenhaber bracket and the cup product on the Hochschild cohomology $HH^*(A, A)$.

A natural explanation for the presence of the $B_\infty$-structure on the Hochschild complex $C(A, A)$ is provided by $A_\infty$-category theory [21] [22] [24] [20]: Let $A$ be the $A_\infty$-category with one object whose $A_\infty$-endomorphism algebra is $A$. Let $\text{Fun}(A, A)$ be the $A_\infty$-category of $A_\infty$-functors from $A$ to $A$ and let $1$ be the identical $A_\infty$-functor. Its $A_\infty$-endomorphism algebra in $\text{Fun}(A, A)$ is naturally isomorphic to the dg algebra $C(A, A)$

\[ \text{Hom}_{\text{Fun}(A, A)}(1, 1) = C(A, A) \]

and the associative multiplication on $BC(A, A)$ comes from the composition $A_\infty$-bifunctor

\[ \text{Fun}(A, A) \times \text{Fun}(A, A) \to \text{Fun}(A, A). \]

$B_\infty$-algebras can be considered as the algebras over a certain asymmetric operad (whose components are free graded $k$-modules of finite total rank). Therefore [15], the category of $B_\infty$-algebras admits the structure of a Quillen model category [27] [8] whose fibrations are the surjections and whose weak equivalences are the quasi-isomorphisms (i.e. the morphisms of $B_\infty$-algebras $C \to C'$ which induce isomorphisms in the homology of the underlying complexes). We define the homotopy category of $B_\infty$-algebras $\text{Ho}(B_\infty)$ to be homotopy category associated with this model category, i.e. the localization of the category of $B_\infty$-algebras with respect to the class of quasi-isomorphisms.
3. The invariance theorem

3.1. Notation and terminology. Let $k$ be a commutative ring. For a dg $k$-algebra $A$, we denote its derived category [18] by $\mathcal{D}A$. By $\text{per} A \subset \mathcal{D}A$ we denote the closure of the free $A$-module $A$ under shifts, extensions and passage to direct factors. The subcategory $\text{per} A$ thus [18, 5.3] consists precisely of the perfect (=compact=small) objects, i.e. the objects $X \in \mathcal{D}A$ such that the functor

$$\text{Hom}_{\mathcal{D}A}(X, ?) : \mathcal{D}A \to \text{Mod} k$$

commutes with infinite direct sums. A dg $A$-module $P$ is cofibrant if, for each surjective quasi-isomorphism of dg $A$-modules $M \to M'$, each morphism $L \to M'$ lifts to a morphism $L \to M$. For example, if $k$ is a field, each complex of vector spaces is a cofibrant dg $k$-module. If $k$ is a commutative ring, each right bounded complex of projective $k$-modules is a cofibrant dg $k$-module. More generally, up to homotopy equivalence, the cofibrant dg $A$-modules are precisely those having the property (P) of [18, 3.1].

3.2. The main theorem. Let $k$ be a commutative ring. Let $A$ and $B$ be dg algebras and let $X$ be a dg $A$-$B$-bimodule. Suppose that the dg $k$-modules underlying $A$, $B$ and $X$ are cofibrant. Recall [18, 4.2] that the functor

$$? \otimes^L A X : \text{per} A \to \mathcal{D}B$$

is fully faithful iff the canonical map

$$H^n A \to \text{Hom}_{\mathcal{D}B}(X, X[n])$$

is an isomorphism for all $n \in \mathbb{Z}$ iff the canonical morphism

$$A \to \text{RHom}_B(X, X)$$

is a quasi-isomorphism.

Theorem. If the functors

$$? \otimes^L A X : \text{per} A \to \mathcal{D}B \quad \text{and} \quad X \otimes^L B ? : \text{per}(B^{op}) \to \mathcal{D}(A^{op})$$

are fully faithful, there is a canonical isomorphism

$$\varphi_X : C(B, B) \to C(A, A)$$

in the homotopy category of $B_\infty$-algebras.

Corollary. If the functor

$$? \otimes^L A X : \mathcal{D}A \to \mathcal{D}B$$

is an equivalence, there is a canonical isomorphism

$$\varphi_X : C(B, B) \to C(A, A)$$

in the homotopy category of $B_\infty$-algebras.

The theorem and its corollary follow from the more general theorem 4.6 below.

3.3. Functoriality. In 4.6, we will construct a morphism $\varphi_X$ with suitable functoriality properties for a larger class of dg bimodules $X$: With the above notations, assume only that the functor

$$? \otimes^L B X : \text{per} A \to \mathcal{D}B$$

is fully faithful.

Theorem. In the homotopy category of $B_\infty$-algebras, there is a canonical morphism

$$\varphi_X : C(B, B) \to C(A, A)$$

associated with $X$ such that
a) \( \varphi_X \) only depends on the isomorphism class of \( X \) in \( \mathcal{D}(A^{op} \otimes B) \),

b) if the functor

\[
X \otimes^L_B ? : \text{per}(B^{op}) \to \mathcal{D}(A^{op})
\]

is fully faithful, then \( \varphi_X \) is invertible,

c) if \( B = A \) and \( X = A \) then \( \varphi_X \) is the identity,

d) if \( X = Y \otimes^L_C Z \) for a dg algebra \( C \) which is cofibrant over \( k \) and for dg bimodules \( _AY_C \) and \( _CZ_B \) cofibrant over \( k \) and such that the functors

\[
? \otimes^L_A Y : \text{per} A \to DC \quad \text{and} \quad ? \otimes^L_C Z : \text{per} C \to DB
\]

are fully faithful, then \( \varphi_X = \varphi_Y \circ \varphi_Z \).

### 3.4. A remark on compositions.

If \( A, B, C \) are dg algebras which are cofibrant over \( k \) and \( _AY_C \) and \( _CZ_B \) are dg bimodules cofibrant over \( k \) such that the functors

\[
? \otimes^L_A Y : \text{per} A \to DC \quad \text{and} \quad ? \otimes^L_C Z : \text{per} C \to DB
\]

are fully faithful, it does not follow that the functor

\[
? \otimes^L_A X : \text{per} A \to DB, \quad \text{where} \quad X = Y \otimes^L_C Z,
\]

is fully faithful. For example, let \( k \) be a field and consider \( A = k[x] \), where \( x \) is of degree 0, \( B = A \) and \( C = k[\xi] \), where \( \xi \) is of degree 1 and \( \xi^2 = 0 \), \( d\xi = 0 \). Let \( Z \) be the bimodule whose underlying complex of \( k[x] \)-modules is

\[
(k \xi' \otimes k) \otimes k[x]
\]

where \( \xi' \) is of degree \(-1\) and

\[
d(\xi' \otimes 1) = 1 \otimes x, \quad \xi . (\xi' \otimes 1) = 1 \otimes 1.
\]

Then the functor \( ? \otimes^L_C Z \) takes the free module \( k[\xi] \) to \( k \) (concentrated in degree 0) and the trivial module \( k \) to the injective hull

\[
E = k[x, x^{-1}]/xk[x]
\]

of the trivial \( k[x] \)-module \( k \). Clearly the restriction of \( ? \otimes^L_C Z \) to \( \text{per} C \) is fully faithful. Let \( Y \) be the bimodule

\[
E \otimes k[\xi]
\]

with the differential determined by

\[
d(x^{-p} \otimes 1) = x^{-p+1} \otimes \xi.
\]

Then the functor \( ? \otimes^L_A Y \) takes the free module \( A = k[x] \) to the trivial module \( k \). The restriction of \( ? \otimes^L_A Y \) to \( \text{per} A \) is fully faithful. The functor

\[
? \otimes^L_A X : \text{per} A \to DB, \quad \text{where} \quad X = Y \otimes^L_C Z
\]

is isomorphic to the composition of \( ? \otimes^L_C Y \) with \( ? \otimes^L_B Z \). It takes the free module \( k[x] \) to the injective module \( E \). It is not fully faithful, since the endomorphism ring of \( E \) is the power series ring \( k[[x]] \).

Note finally that both, \( Y \) and \( Z \), satisfy the assumptions of theorem 3.3 b) so that \( A = B \) and \( C \) do have homotopy equivalent Hochschild \( B_\infty \)-algebras.
3.5. Application to Koszul algebras. In this section, we suppose that $k$ is a field. We consider bigraded $k$-modules

$$M = \bigoplus_{p,q \in \mathbb{Z}} M^p_q.$$  

The first index $p$ denotes the ‘differential grading’: all differentials are of degree $(1,0)$. The second index $q$ denotes the ‘Adams grading’. For a bimodule $M$, we have the differential shift $M[1]$ and the Adams shift $M(q)$, which are defined respectively by

$$(M[1])^p_q = M^{p+1}_q \quad \text{and} \quad (M(q))^p_q = M^p_{q+1}.$$  

If $A$ is a differential graded algebra endowed with an additional Adams grading, then the complex $C(A,A)$ inherits an Adams grading compatible with its $B_\infty$-structure.

Let $A = A_0 \oplus A_1 \ldots$

be a positively Adams-graded $k$-algebra (concentrated in differential degree 0) such that $A_0$ is a separable $k$-algebra (i.e. $A_0$ is a projective $A_0 \otimes A_0^{op}$-module) and all $A_p$ are finitely generated as right and as left $A_0$-modules. Suppose that in the category of graded $A$-modules (with degree 0 morphisms), the $A$-module $A_0$ admits a linear projective resolution, i.e. a projective resolution

$$\ldots \to P^{-p} \to P^{-p+1} \to \ldots \to P^{-1} \to P_0$$

such that $P^{-p}$ is generated in degree $p$ for all $p \in \mathbb{N}$. This means that $A$ is a Koszul algebra, cf. for example [2]. Let $A'$ be the Adams graded dg algebra with zero differential whose $(p,q)$-component is

$$\text{Ext}^p(A_0,A_0(q)).$$

Notice that by assumption, the algebra $A'$ is concentrated in bidegrees $(p,-p)$, $p \in \mathbb{N}$. R.-O. Buchweitz has shown [4] that there is an isomorphism

$$HH^*(A,A) \cong HH^*(A',A')$$

compatible with the cup product. We obtain the following stronger version of his result:

**Theorem.** There is a canonical isomorphism

$$\varphi : C(A,A) \cong C(A',A')$$

in the homotopy category of Adams graded $B_\infty$-algebras. In particular, $\varphi$ induces an isomorphism

$$(3.5.1) \quad HH^*(A,A) \cong HH^*(A',A')$$

compatible with the cup product and the Gerstenhaber bracket.

**Proof.** Let $X$ be the Koszul complex. It is a dg $A'$-$A$-bimodule, which, as a right dg $A$-module, is a projective resolution of $A_0$. By the assumption that $A_0$ admits a linear resolution, the complex

$$\text{RHom}_A(X,X(-q)[q])$$

has its homology concentrated in degree 0 for all $q \in \mathbb{Z}$. By the definition of $A'$, it follows that the canonical morphism

$$A'_q \to \text{Ext}^q(A_0,A_0(-q)) \to \text{RHom}_A(X,X(-q)[q])$$

is a quasi-isomorphism for all $q \in \mathbb{Z}$. Now we claim that the canonical morphism

$$A'^{op}_{q'} \to \text{RHom}_{(A')^{op}}(X,X(-q)[q])$$

is also a quasi-isomorphism for all $q \in \mathbb{Z}$. This results from the fact that $A'^{op}$ is Koszul with Koszul dual $(A')^{op}$ and Koszul complex $X$ viewed as a dg $A'^{op}$-$A'^{op}$-bimodule, cf. [2]. The assertion now follows from the ‘Adams graded’ variant of
Theorem 3.2. Alternatively, it follows from part b) of theorem 4.6 below applied to the dg category $\mathcal{A}$ with object set $\mathbb{Z}$ and morphism spaces

$$\mathcal{A}(i, j) = A_{j-i}, \ i, j \in \mathbb{Z},$$

(concentrated in degree 0), the dg category $\mathcal{B}$ defined analogously using $A'$ and the dg $\mathcal{A}$-$\mathcal{B}$-bimodule $\mathcal{X}$ defined by

$$\mathcal{X}(i, j) = X_{j-i}, \ j \in \mathcal{A}, i \in \mathcal{B}.$$

\[\sqrt{4}\]

4. Generalization and proof of the invariance theorem

4.1. From algebras to categories. Let $k$ be a commutative ring and $\mathcal{A}$ a dg $k$-category. By $k[\mathcal{A}]$, we denote the dg algebra

$$\bigoplus_{A, B \in \mathcal{A}} \mathcal{A}(A, B)$$

with matrix multiplication and the natural differential. This algebra does not have a unit but for each finite family of elements $a_i, i \in I$, of $k[\mathcal{A}]$, there is an element $e \in k[\mathcal{A}]$ of the form

$$e = 1_{A_1} \oplus \cdots \oplus 1_{A_n}$$

for certain $A_1, \ldots, A_n$ of $\mathcal{A}$, such that $ea_i = a_i = a_i e$ for all $i \in I$. By $\text{Modlu} k[\mathcal{A}]$, we denote the category of the right dg modules $M$ over $k[\mathcal{A}]$ which are locally unital, i.e. for each finite family of elements $m_i, i \in I$, of $M$, there is an element $e$ as above such that $me = m$. By $\text{Mod} \mathcal{A}$, we denote the category of right dg modules over $\mathcal{A}$. Then we have a canonical equivalence

$$\text{Mod} \mathcal{A} \to \text{Modlu} k[\mathcal{A}]$$

which takes a module $M$ to

$$\bigoplus_{A \in \mathcal{A}} M(A).$$

Its quasi-inverse takes a locally unital dg module $L$ to the module

$$\mathcal{A}^{op} \to \text{Mod} k, A \mapsto L.1_A.$$

This equivalence preserves quasi-isomorphisms. We define the derived category $\mathcal{D}k[\mathcal{A}]$ to be the localization of $\text{Modlu} k[\mathcal{A}]$ with respect to the quasi-isomorphisms. Thus we obtain an equivalence of derived categories

$$\mathcal{D} \mathcal{A} \to \mathcal{D}k[\mathcal{A}].$$

4.2. The Hochschild complex for categories and algebras. Keep the notations of the preceding section and assume moreover that $\mathcal{A}$ is cofibrant over $k$, i.e. such that $\mathcal{A}(A, B)$ is cofibrant over $k$ for all objects $A, B$ of $\mathcal{A}$. The Hochschild complex of $\mathcal{A}$ is the product total complex $C(\mathcal{A}, \mathcal{A})$ of the double complex with components which vanish for $p < 0$ and are equal to

$$\prod_{A_0, \ldots, A_p} \text{Hom}_k(\mathcal{A}(A_{p-1}, A_p) \otimes \cdots \otimes \mathcal{A}(A_0, A_1), \mathcal{A}(A_0, A_p))$$

for $p \geq 0$, where the product ranges over all sequences of $p$ objects of $\mathcal{A}$. The differential of a $p$-cochain $c$ is defined by the canonical formula. The Hochschild complex of $k[\mathcal{A}]$ is given by the product total complex $C(k[\mathcal{A}], k[\mathcal{A}])$ of the double complex with the components which vanish for $p < 0$ and are equal to

$$\text{Hom}_k(k[\mathcal{A}]^{\otimes p}, k[\mathcal{A}])$$

for $p \geq 0$. We have an injective morphism of complexes

$$(4.2.1) \quad C(\mathcal{A}, \mathcal{A}) \to C(k[\mathcal{A}], k[\mathcal{A}]).$$
Let \( I_\mathcal{A} \) denote the \( \mathcal{A} \)-\( \mathcal{A} \)-bimodule
\[(A, B) \mapsto \mathcal{A}(A, B), \ A, B \in \mathcal{A}.
\]

**Lemma.** The morphism \( 4.2.1 \) is a quasi-isomorphism of dg \( k \)-modules. Both sides are canonically isomorphic to
\[
\text{RHom}_{\mathcal{A}^{op} \otimes \mathcal{A}}(I_\mathcal{A}, I_\mathcal{A}).
\]

In particular, the \( n \)th homology of both sides is canonically isomorphic to
\[
HH^n(\mathcal{A}, \mathcal{A}) := \text{Hom}_{\mathcal{D}(\mathcal{A}^{op} \otimes \mathcal{A})}(I_\mathcal{A}, I_\mathcal{A}[n]), \ n \in \mathbb{Z}.
\]

**Proof.** The bar resolution of \( I_\mathcal{A} \) is the sum total complex \( BR(\mathcal{A}) \) of the double complex with components which vanish for \( p < 0 \) and equal
\[
\bigoplus_{A_0, \ldots, A_p} \mathcal{A}(A_p, -) \otimes \mathcal{A}(A_{p-1}, A_p) \otimes \cdots \otimes \mathcal{A}(A_0, A_1) \otimes \mathcal{A}(?, A_0)
\]
for \( p \geq 0 \). The differential is given by the canonical formula. The canonical morphism \( BR(\mathcal{A}) \to I_\mathcal{A} \) is a quasi-isomorphism and \( BR(\mathcal{A}) \) is cofibrant over \( \mathcal{A}^{op} \otimes \mathcal{A} \). Thus, we can compute
\[
\text{RHom}_{\mathcal{A}^{op} \otimes \mathcal{A}}(I_\mathcal{A}, I_\mathcal{A})
\]
as the image of \( BR(\mathcal{A}) \) under \( \text{Hom}_{\mathcal{A}^{op} \otimes \mathcal{A}}(?, I_\mathcal{A}) \). Now we have an isomorphism
\[
\text{Hom}(BR(\mathcal{A}), I_\mathcal{A}) \cong C(\mathcal{A}, \mathcal{A}).
\]
The category \( \text{Mod}(\mathcal{A}^{op} \otimes \mathcal{A}) \) is equivalent to \( \text{Modlu}(k[\mathcal{A}]^{op} \otimes k[\mathcal{A}]) \) and the equivalence \( F \) takes \( I_\mathcal{A} \) to the bimodule \( k[\mathcal{A}] \). It takes \( BR(\mathcal{A}) \) to a complex \( F(BR(\mathcal{A})) \) which naturally identifies with a quotient of the bar resolution \( BR(k[\mathcal{A}]) \) of \( k[\mathcal{A}] \).

We have a morphism of resolutions
\[
BR(k[\mathcal{A}]) \to F(BR(\mathcal{A})).
\]
The induced morphism
\[
\text{Hom}_{k[\mathcal{A}]^{op} \otimes k[\mathcal{A}]}(BR(k[\mathcal{A}]), k[\mathcal{A}]) \cong \text{Hom}_{k[\mathcal{A}]^{op} \otimes k[\mathcal{A}]}(F(BR(\mathcal{A})), k[\mathcal{A}])
\]
identifies with the inclusion \( 4.2.1 \). It is a quasi-isomorphism of dg \( k \)-modules because both sides compute
\[
\text{RHom}_{k[\mathcal{A}]^{op} \otimes k[\mathcal{A}]}(k[\mathcal{A}], k[\mathcal{A}]).
\]

\( \square \)

### 4.3. Functoriality of the categorical Hochschild complex

Keep the notations of the previous paragraph and let \( \mathcal{B} \) be another dg category which is cofibrant over \( k \). Let \( F : \mathcal{A} \to \mathcal{B} \) be a fully faithful dg functor. The restriction along \( F \) yields an obvious morphism
\[
F^* : C(\mathcal{B}, \mathcal{B}) \to C(\mathcal{A}, \mathcal{A})
\]
which is compatible with the differential and, indeed, with the structure of \( B_\infty \)-algebra, as one easily checks. Compositions of fully faithful functors clearly yield compositions of the restrictions. This functoriality property is a great advantage, which we gain in working with categories rather than with algebras.

The morphism \( F^* \) has a natural interpretation in the derived category: The restriction functor
\[
\mathcal{D}(\mathcal{B}^{op} \otimes \mathcal{B}) \to \mathcal{D}(\mathcal{A}^{op} \otimes \mathcal{A})
\]
admits a fully faithful left adjoint, which we still denote by \( F \). It takes \( \mathcal{A}(B, ?) \otimes \mathcal{A}(?, A) \) to \( \mathcal{B}(FB, ?) \otimes \mathcal{A}(?, FA) \). If we have a Hochschild \( n \)-cochain \( c \) of \( \mathcal{B} \), it corresponds to a morphism \( c : I_B \to I_B[n] \). Now we have a functorial isomorphism
\[
M \otimes_B I_B \cong M
\]
for each $M \in D(B^{op} \otimes B)$, since $I_B$ is the neutral object for the monoidal structure given by tensoring over $B$. Thus $c$ gives rise to a functorial morphism $Mc : M \to M[n]$. In particular, we obtain
\[(FI_A)c : FI_A \to FI_A[n].\]
By the full faithfulness of $F$, this yields a morphism $I_A \to I_A[n]$. This is the morphism that corresponds to the Hochschild cochain $F''c$.

4.4. Bimodule morphisms. Let $A$ and $B$ be dg algebras which are cofibrant over $k$. Let $X$ be an $A$-$B$-bimodule which is cofibrant over $k$. Let $C(A, X, B)$ be the product total complex of the complex whose components vanish for $p < 0$ and equal
\[\prod \text{Hom}_k(A(A_0, A_1) \otimes \cdots \otimes A(A_0, A_1) \otimes X(B_{0}, A_0) \otimes B(B_{m-1}, B_{m}) \otimes \cdots \otimes B(B_{0}, B_{1}), X)\]
for $p \geq 0$, where the inner product ranges over all objects $A_0, \ldots, A_t$ of $A$ and all objects $B_0, \ldots, B_m$ of $B$ and $t + m = p$. The differential is given by the same formula as the Hochschild differential. We define a morphism of complexes
\[\alpha : C(A, A) \to C(A, X, B)\]
by taking a cochain
\[c \in C^p(A, A)\]
to the map whose only non trivial components are the
\[A(A_{p-1}, A_{p}) \otimes \cdots \otimes A(A_0, A_1) \otimes X(B_0, A_0) \otimes X(B_0, A_0), u \otimes x \mapsto c(u)x.\]
Similarly, we define a morphism
\[\beta : C(B, B) \to C(A, X, B).\]
Let us interpret $C(A, X, B)$ and the morphisms $\alpha$ and $\beta$ in the derived categories: One easily checks that $C(A, X, B)$ is isomorphic to
\[\text{RHom}_{A^{op} \otimes B}(X, X).\]
Let us suppose (without restriction of generality) that $X$ is cofibrant over $A^{op} \otimes B$. Then $X_B$ is cofibrant over $B$ (because $A$ is cofibrant over $k$). The left action of $A$ on $X$ yields morphisms
\[A(A, B) \to \text{Hom}_B(X(?; A), X(?; B)), A, B \in A,\]
which yield a morphism of $A$-$A$-bimodules
\[\lambda : I_A \to \text{Hom}_B(X, X) = \text{RHom}_B(X, X).\]
This induces a morphism
\[\lambda_* : \text{RHom}_{A^{op} \otimes A}(I_A, I_A) \to \text{RHom}_{A^{op} \otimes A}(I_A, \text{RHom}_B(X, X)).\]
The right hand side is canonically isomorphic to $\text{RHom}_{A^{op} \otimes B}(X, X)$. It is not hard to check that the following diagram is commutative:
\[\begin{array}{ccc}
C(A, A) & \sim \rightarrow & \text{RHom}_{A^{op} \otimes A}(I_A, I_A) \\
\downarrow \alpha & & \downarrow \lambda_* \\
C(A, X, B) & \sim \rightarrow & \text{RHom}_{A^{op} \otimes B}(X, X) \sim \rightarrow \text{RHom}_{A^{op} \otimes A}(I_A, \text{RHom}_B(X, X)).
\end{array}\]
It follows that $\alpha$ is a quasi-isomorphism if $\lambda$ is an isomorphism in $D(A^{op} \otimes A)$. A similar diagram links $\beta$ to $\rho_*$, where
\[\rho : I_{B^{op}} \to \text{RHom}_{A^{op}}(X, X)\]
is given by the right action of $B$ on $X$.\]
4.5. Triangular matrices. We keep the notations of the previous paragraph. Let \( \mathcal{G} = \mathcal{G}(\mathcal{A}, X, \mathcal{B}) \) be the dg category whose set of objects is the disjoint union of the sets of objects of \( \mathcal{A} \) and \( \mathcal{B} \) and whose morphisms are defined by

\[
\mathcal{G}(A, A') = \mathcal{A}(A, A') \quad \mathcal{G}(B, B') = \mathcal{B}(B, B') \quad \mathcal{G}(B, A) = X(B, A) \quad \mathcal{G}(A, B) = 0
\]

for all objects \( A, A' \) of \( \mathcal{A} \) and \( B, B' \) of \( \mathcal{B} \). Note that we have fully faithful functors \( i_A : \mathcal{A} \to \mathcal{G} \) and \( i_B : \mathcal{B} \to \mathcal{G} \).

For example, if \( \mathcal{A} \) and \( \mathcal{B} \) each have one object and \( A \) and \( B \) denote the endomorphism algebras of the unique objects of \( \mathcal{A} \) and \( \mathcal{B} \), then \( \mathcal{G} \) can be visualized as

\[
\begin{array}{ccc}
1 & \xrightarrow{X} & 2 \\
\downarrow & \nearrow & \downarrow \\
A & \quad & B
\end{array}
\]

and \( k[G] \) is the algebra of upper triangular matrices

\[
\begin{pmatrix}
a & x \\
0 & b
\end{pmatrix}, \quad a \in A, \quad b \in B, \quad x \in X.
\]

Recall that, in a triangulated category, a homotopy bicartesian square is a commutative square

\[
\begin{array}{ccc}
U & \longrightarrow & V \\
\downarrow & & \downarrow \\
W & \longrightarrow & Z
\end{array}
\]

endowed with a morphism \( Z \to U[1] \) such that the sequence

\[
U \longrightarrow V \oplus W \longrightarrow Z \longrightarrow U[1]
\]

is a triangle.

**Lemma.** The complex \( C(\mathcal{G}, \mathcal{G}) \) is isomorphic to the mapping cylinder on the morphism

\[
\begin{pmatrix} \alpha \\ \beta \end{pmatrix} : C(\mathcal{A}, \mathcal{A}) \oplus C(\mathcal{B}, \mathcal{B}) \to C(\mathcal{A}, X, \mathcal{B}),
\]

where \( \alpha \) and \( \beta \) are the morphisms defined in 4.4. Therefore we have a homotopy bicartesian square

\[
\begin{array}{ccc}
C(\mathcal{G}, \mathcal{G}) & \xrightarrow{i_A^*} & C(\mathcal{A}, \mathcal{A}) \\
\downarrow & & \downarrow \alpha \\
C(\mathcal{B}, \mathcal{B}) & \xrightarrow{i_B^*} & C(\mathcal{A}, X, \mathcal{B})
\end{array}
\]

in the derived category of dg \( k \)-modules. In particular, we have a long exact sequence

\[
\ldots \to HH^n(\mathcal{G}, \mathcal{G}) \to HH^n(\mathcal{A}, \mathcal{A}) \oplus HH^n(\mathcal{B}, \mathcal{B}) \to \text{Hom}_{D(\mathcal{A}^{\text{op}} \otimes \mathcal{B})}(X, X[n]) \to \ldots
\]

The proof of the lemma is an easy computation left to the reader. We refer to the introduction for the history of the lemma. We stress that \( i_A^* \) and \( i_B^* \) are morphisms of \( B_\infty \)-algebras, as we have seen in 4.3, and that \( \alpha \) and \( \beta \) have simple interpretations in the derived categories, as seen in 4.4.
4.6. **The main theorem on the Hochschild complex of a dg category.** We keep the assumptions of paragraph 4.4. Assume that the functor
\[ ? \otimes^L_A X : \text{per} A \to DB \]
is fully faithful. Then the morphism \( \lambda \)
\[ I_A \to \mathbb{R}\text{Hom}_B(X, X) \]
of paragraph 4.4 is invertible and thus the morphism \( \alpha \) is a quasi-isomorphism. By the bicartesian square 4.5.1, the morphism \( i_B \) is a quasi-isomorphism. Thus it becomes invertible in the homotopy category of \( B_\infty \)-algebras. We define
\[ \varphi_X = i_A \circ (i_B)^{-1}. \]

**Theorem.**

a) \( \varphi_X \) only depends on the isomorphism class of \( X \) in \( D(A^\text{op} \otimes B) \).

b) If the functor
\[ X \otimes^L_B ? : \text{per}(B^\text{op}) \to D(A^\text{op}) \]
is fully faithful, then \( \varphi_X \) is invertible. This holds in particular if
\[ ? \otimes^L_A X : DA \to DB \]
is an equivalence.

c) If \( F : A \to B \) is a fully faithful dg functor and the bimodule \( X \) is defined by
\[ X(B, A) = B(B, FA), \ A \in A, \ B \in B, \]
then \( \varphi_X = F^* \) in \( \text{Ho}(B_\infty) \). In particular, if \( B = A \) and \( X = I_A \), then \( \varphi_X \) is the identity.

d) If \( C \) is a dg category cofibrant over \( k \) and \( Y \) a dg \( B \)-\( C \)-bimodule cofibrant over \( k \) such that the functors
\[ ? \otimes^L_B Y : \text{per} B \to DC \text{ and } ? \otimes^L_C Z : \text{per} A \to DC \]
are fully faithful, where \( Z = X \otimes_B Y \), then \( \varphi_Z = \varphi_X \circ \varphi_Y \).

Before proving the theorem, let us record the following useful consequence: Let \( F : A \to B \) be a dg functor which is not necessarily fully faithful but which induces quasi-isomorphisms
\[ \mathcal{A}(A, B) \to B(FA, FB), \ A, B \in A. \]
Let \( X_F \) be the bimodule
\[ (A, B) \mapsto B(B, FA), \ A, B \in A. \]
Then the functor
\[ ? \otimes^L_A X_F : \text{per} A \to DB \]
takes \( \mathcal{A}(?, A) \) to \( B(?, FA) \). Therefore it is fully faithful (and takes \( \text{per} A \) to \( \text{per} B \)). Hence we have a well defined morphism
\[ \varphi_F := \varphi_{X_F} : C(B, B) \to C(A, A) \]
in \( \text{Ho}(B_\infty) \). Moreover, if \( F \) induces an equivalence \( H^0 A \to H^0 B \), then the associated tensor functor is and equivalence \( DA \to DB \) and so \( \varphi_F \) is invertible.

If \( G : B \to C \) is another functor inducing quasi-isomorphisms in the morphism complexes, then clearly \( GF \) also has this property; and
\[ X_F \otimes^L_B X_G \to X_{GF} \]
so that we have
\[ \varphi_{GF} = \varphi_F \circ \varphi_G. \]
Proof. Let us prove the first statement of b). We note that under the assumption, the morphism $\beta$ is a quasi-isomorphism. So by the homotopy bicartesian square 4.5.1, the morphism $i^*A$ is invertible and hence $\varphi_X$ is invertible. Let us prove the second statement of b). For a dg $B$-module $M$, denote by $\text{Hom}_B(M, B)$ the dg $B^{op}$-module

$$B \mapsto \text{Hom}_B(M, B(?), B), \ B \in B.$$ 

Then the transposition functor

$$\text{Tr}_B = \text{RHom}_B(?, B) : DB \to D(B^{op})^{op}$$

induces an equivalence

$$\text{per}_{B} \to \text{per}(B^{op})^{op}.$$ 

Notice that the functor $\text{RHom}_B(X, ?) : DB \to DA$ is an equivalence, since it is left adjoint to the equivalence $X \otimes_{A}^{L} ?$. For $Q \in \text{per}_B$, we have natural isomorphisms

$$X \otimes_B^{L} \text{RHom}_B(Q, B) \to \text{RHom}_B(Q, X) \to \text{RHom}_A(\text{RHom}_B(X, Q), \text{RHom}_B(X, X)) \to \text{RHom}_A(\text{RHom}_B(X, Q), A).$$

Here we use that $Q$ is perfect for the first isomorphism, that $\text{RHom}_B(X, ?)$ is fully faithful for the second, and that $\text{RHom}_B(X, ?)$ takes $X(?, A)$ to $A(?, A)$ for the third. We deduce that we have a natural isomorphism

$$(X \otimes_B^{L} ?) \circ \text{Tr}_B \to \text{Tr}_A \circ \text{RHom}_B(X, ?)$$

of functors $\text{per}_B \to D(A^{op})^{op}$. This shows that the functor

$$X \otimes_B^{L} ? : \text{per}(B^{op}) \to D(A^{op})$$

is fully faithful. Let us now prove d) under the additional assumption that $X$ is cofibrant over $B$ and that $Z = X \otimes_B Y$. The general case follows from a), which we will prove later. We consider the following diagram of dg categories:

Here, the symbol $A \leftarrow B$ denotes $G(A, X, B)$ and similarly for $B \leftarrow C$ and $A \leftarrow C$, which denotes $G(A, X \otimes_B Y, C)$. The symbol $A \leftarrow B \leftarrow C$ denotes the category $U$ whose set of objects is the disjoint union of the objects of $A$, $B$ and $C$ and whose only possibly non zero morphisms are given by

$$U(A, A') = A(A, A'), \ U(B, A) = X(B, A), \ U(C, A) = Z(C, A),$$

$$U(B, B') = B(B, B'), \ U(C, B) = Y(C, B), \ U(C, C') = C(C, C').$$

All arrows of the diagram denote the obvious fully faithful dg functors. The diagram therefore yields a commutative diagram in $\text{Ho}(B_{\infty})$. The composition $\varphi_X \circ \varphi_Y$ is obtained from the arrows on the lower right rim whereas $\varphi_Z$ comes from the long skew arrows. Note that $i_1$, $i_2$ and $i_3$ induce isomorphisms in $\text{Ho}(B_{\infty})$. We can
conclude that \( \varphi_Z = \varphi_X \circ \varphi_Y \) once we show that \( i_4 \) induces an isomorphism. For this, we reinterpret \( U \) as \( G(A, U, B \leftarrow C) \), where \( U \) is the bimodule with \( U(B, A) = X(B, A) \) and \( U(C, A) = Z(C, A) \).

The functor

\[ ? \otimes_A^L U : \text{per}_A \to D(B \leftarrow C) \]

equals the composition

\[ \text{per}_A \otimes_A^L X \to D(B) \to D(B \leftarrow C) \]

Since both functors are fully faithful, so is \( ? \otimes_A^L U \). It follows that \( i_4 \) induces an invertible morphism in \( \text{Ho}(B_\infty) \) and we are done.

We deduce the last statement of c): Indeed, for \( X = I_A \), we have

\[ \varphi_X = \varphi_{I_A} = \varphi_X \circ \varphi_X. \]

Since \( \varphi_X \) is invertible, this implies that it is the identity. Let us now prove the first statement of c): With notations as above, we consider the following diagram of dg categories and fully faithful functors

\[ 
\begin{array}{ccc}
(A \leftarrow B) & \longrightarrow & B \\
\uparrow & & \downarrow F \\
A & \longrightarrow & A \\
\end{array}
\]

The lower right angle yields \( \varphi_{I_A} \), which equals the identity, as we have just proved. The upper right angle yields \( \varphi_X \) for the bimodule \( X \) associated with \( F \). The commutativity of the image of the diagram in \( \text{Ho}(B_\infty) \) implies that \( \varphi_X = F^* \).

Before we prove a), let us first extend \( F \mapsto \varphi_F \) to dg functors which are not necessarily fully faithful. Let \( F : A \to B \) be a dg functor. Let \( X_F \) denote the associated bimodule defined by

\[ X_F(B, A) = B(B, FA). \]

Suppose that

\[ ? \otimes_A^L X_F : \text{per}_A \to DB \]

is fully faithful. For example, this is the case if \( F \) induces a fully faithful functor \( H^*A \to H^*B \). We have a well defined morphism

\[ \varphi_F := \varphi_{X_F} : C(B, B) \to C(A, A) \]

in \( \text{Ho}(B_\infty) \). We have just proved that if \( F \) is itself fully faithful, then \( \varphi_F = F^* \).

Now suppose that \( G : B \to C \) is another dg functor such that

\[ ? \otimes_B^L X_G : \text{per}_B \to DC \]

is fully faithful. The bimodule \( X_F \) is cofibrant over \( B \) and we have

\[ X_F \otimes_B X_G = X_{G \circ F}. \]

Moreover, the associated functor \( \text{per}_A \to DC \) is fully faithful since the functor

\[ ? \otimes_A^L X_F : \text{per}_A \to DB \]

takes \( \text{per}_A \) to \( \text{per}_B \). Therefore, it follows from the special case of c) which we have proved that

\[ \varphi_{G \circ F} = \varphi_F \circ \varphi_G. \]
We conclude that $\varphi_F$ is in particular defined and functorial in the dg functors $F : A \to B$ which induce fully faithful functors $H^*A \to H^*B$.

Let us now prove a). Let $X$ be as in a) and let $f : X \to X'$ be a quasi-isomorphism. Then $f$ yields an obvious dg functor

$$F : G(A, X, B) \to B(A, X', B),$$

which is not fully faithful, in general, but which induces a fully faithful functor in homology. Now we consider the following diagram of dg categories and dg functors:

$$
\begin{array}{ccc}
A & \xleftarrow{X} & B \\
\downarrow F & & \downarrow F \\
(A & \xleftarrow{X'} & B)
\end{array}
$$

By what we have just seen, this diagram has a well defined and commutative image in $\text{Ho}(B_\infty)$. In the image, the two left hand arrows become invertible, hence so does $\varphi_F$. The composition of the upper arrows yields $\varphi_X$ and the composition of the lower arrows yields $\varphi'_X$.

5. The Hochschild complex of an exact category

In this section, for simplicity, we will suppose that $k$ is a field.

5.1. Exact dg categories and their quotients. We refer to [19, 2.1] for the notion of an exact dg category. The simplest example of such a category is the dg category of complexes over an additive category. If $A$ is an exact dg category, it is in particular a Frobenius category and its associated stable category $A$ is triangulated. The stable category is also called the associated triangulated category. If $A$ is the category of complexes over an additive category, the associated triangulated category is the homotopy category.

If $E$ is an exact category, it gives rise to two exact dg categories, namely the category $C^b(E)$ of bounded complexes over $E$ and its full subcategory $Ac^b(E)$ whose objects are the acyclic complexes. We will need the dg quotient category $C^b(B)/Ac^b(E)$.

Let us recall how dg quotient categories are characterized in general: Let $U$ be a universe containing an infinite set. A category is $U$-small if the set of its morphisms belongs to $U$.

We assume that the ground field $k$ belongs to $U$. The ‘strict’ category $M^b_{str}$ has as objects the $U$-small exact dg categories. Its morphisms are the dg functors. The category $M^b$ is obtained from $M^b_{str}$ by localization at the class of all dg functors $F : A \to B$ such that $F$ induces an equivalence in the associated triangulated categories.

**Theorem.** [19] Let $B$ be an exact dg category, $A \subset B$ a full exact dg subcategory and $I : A \to B$ the inclusion. Then there is a $U$-small exact dg category $B/A$ and a morphism

$$Q : B \to B/A$$

of $M^b$ such that $QI = 0$ and that for each morphism $F : B \to C$ of $M^b$ such that $FI = 0$, there is a unique morphism $\overline{F} : B/A \to C$ such that $F = \overline{F} \circ Q$.

The theorem follows by combining Theorem 4.6, Lemma 4.2 and Proposition 4.1 of [19]. A stronger statement was proved by Drinfeld in [7]. He gives a 2-universal property of the quotient (instead of a 1-universal property as in the theorem). The theorem shows that the dg quotient $B/A$ is indeed a quotient in the category $M^b$. In particular, it is unique and functorial in this category. Let us now show that, as an object of $\text{Ho}(B_\infty)$, the Hochschild complex of an object of $M^b$ is well defined.
For this, we note that, by the remark following theorem 4.6, each morphism $F : A \to B$ of $\mathcal{M}^b_{str}$ which induces a fully faithful functor in the associated triangulated categories yields a well defined morphism

$$\varphi_F : C(B, B) \to C(A, A)$$

in $\mathcal{H}o(B_{\infty})$ and $\varphi_F$ is invertible if $F$ induces an equivalence in the triangulated categories.

Now if $F : A \to B$ is a morphism of the localization $\mathcal{M}^b$ inducing a fully faithful functor in the triangulated categories, then, by construction, $F$ is the composition of functors inducing fully faithful functors in the triangulated categories and formal inverses of functors inducing equivalences in the triangulated categories. By the preceding remark, we again obtain a well-defined morphism

$$\varphi_F : C(B, B) \to C(A, A)$$

in $\mathcal{H}o(B_{\infty})$ and $\varphi_F$ is invertible if $F$ induces an equivalence in the triangulated categories.

We deduce that there is a well defined functor

$$(\mathcal{M}^b)^{op} \to \mathcal{H}o(B_{\infty}), \ A \mapsto C(A, A)$$

defined on the subcategory of $\mathcal{M}^b$ with the same objects as $\mathcal{M}^b$ and whose morphisms are those morphisms which induce fully faithful functors in the triangulated categories.

### 5.2. The Hochschild complex of a small exact category

We keep the notations of the preceding paragraph. For a $U$-small $k$-linear exact category $\mathcal{E}$, we can now define

$$C(\mathcal{E}, \mathcal{E}) = C(Q, Q), \text{ where } Q = \mathcal{C}^b(\mathcal{E})/\mathcal{A}^b(\mathcal{E}).$$

We obtain a well-defined object of $\mathcal{H}o(B_{\infty})$. If $F : \mathcal{E} \to \mathcal{E}'$ is an exact functor between exact categories which induces a fully faithful functor in the derived categories, then $F$ yields a well-defined morphism

$$F^* : C(\mathcal{E}', \mathcal{E}') \to C(\mathcal{E}, \mathcal{E})$$

in $\mathcal{H}o(B_{\infty})$.

**Theorem.** If $F$ induces an equivalence up to factors

$$\mathcal{D}^b(\mathcal{E}) \to \mathcal{D}^b(\mathcal{E}'),$$

i.e. a fully faithful functor such that each object of $\mathcal{D}^b(\mathcal{E}')$ is a direct factor of an object in the image, then $F^*$ is an isomorphism

$$C(\mathcal{E}', \mathcal{E}') \cong C(\mathcal{E}, \mathcal{E})$$

in $\mathcal{H}o(B_{\infty})$.

**Proof.** Let $Q = \mathcal{C}^b(\mathcal{E})/\mathcal{A}^b(\mathcal{E})$ and define $Q'$ similarly. By the assumption, $F$ induces a morphism $Q \to Q'$ in $\mathcal{M}^b$ which induces an equivalence up to factors in the associated triangulated categories. It follows that the induced functor

$$F^+ : \mathcal{D}Q \to \mathcal{D}Q'$$

induces an equivalence up to factors between the subcategories of compact objects $\text{per}(Q) \to \text{per}(Q')$. But then $F^+$ has to be an equivalence. So the assertion follows from part b) of theorem 4.6.
References
