INVARIANCE AND LOCALIZATION

FOR CYCLIC HOMOLOGY

OF DG ALGEBRAS[1]

Bernhard Keller

May 13, 1996

Abstract. We show that two flat differential graded algebras whose derived categories are equivalent by a derived functor have isomorphic cyclic homology. In particular, 'ordinary' algebras over a field which are derived equivalent [48] share their cyclic homology, and iterated tilting [19] [3] preserves cyclic homology. This completes results of Rickard's [48] and Happel's [18]. It also extends well known results on preservation of cyclic homology under Morita equivalence [10], [39], [25], [26], [41], [42].

We then show that under suitable flatness hypotheses, an exact sequence of derived categories of DG algebras yields a long exact sequence in cyclic homology. This may be viewed as an analogue of Thomason-Trobaugh's [51] and Yao's [58] localization theorems in K-theory (cf. also [55]).

I am grateful to the referee for his careful reading of the manuscript.

Summary

This paper is concerned with cyclic homology of (unbounded, non-commutative) differential $\mathbf{Z}$-graded algebras. The case of positively graded DG algebras was first considered by Vigué-Burghelea [53] and T. Goodwillie [15]. We need the slightly more general setting to allow for the algebras appearing in Morita theory for derived categories. For simplicity, in this summary, we only state the results for the special case of 'ordinary' algebras. We point out however, that the range of possible applications is greatly enlarged if one admits general differential graded algebras.

Let k be a commutative ring. In this summary, all k-algebras are assumed to be projective over k. Let A and B be k-algebras. Consider the full subcategory rep (A, B) of the derived category of A-B-bimodules formed by the bimodule complexes X which when restricted to B become quasi-isomorphic to *perfect* complexes (i.e. finite complexes of finitely generated projective B-modules). Generalizing results of C. Kassel [25] [26] we show in (2.4) that each such complex X gives rise to a morphism in cyclic homology

$$\mathrm{HC}_*(X) : \mathrm{HC}_*(A) \to \mathrm{HC}_*(B).$$

This morphism is functorial in the sense that if we view A as an A-A-bimodule complex, then $\mathrm{HC}_*(A) = \mathbf{1}$ and if $Y \in \mathrm{rep}\,(B, C)$ then $\mathrm{HC}_*(X \otimes^{\mathbf{L}}_B Y) = \mathrm{HC}_*(Y) \circ \mathrm{HC}_*(X)$. This implies in particular that HC_* is an invariant for Morita equivalence of derived categories [48] [49], that is, if the derived functor $? \otimes^{\mathbf{L}}_A X : \mathcal{D}A \to \mathcal{D}B$ is an equivalence, then $\mathrm{HC}_*(X)$ is invertible.

Moreover, we show that $\mathrm{HC}_*(X)$ only depends on the class of X in the Grothendieck group of the triangulated category rep (A, B). These Grothendieck groups are naturally viewed as the morphism spaces of a category whose objects are all algebras. A *K-theoretic equivalence* is an isomorphism of this category. Thus, cyclic homology is invariant under K-theoretic equivalence. For example, a finite-dimensional algebra of finite global dimension over an algebraically closed field is K-theoretically equivalent to its largest semi-simple quotient (2.5). Thus, if k is an algebraically closed field, the cyclic homology of a finite-dimensional algebra A of finite global dimension only depends on the number of isomorphism classes of simple A-modules. This yields the 'no loops

[1] To appear in Journal of Pure and Applied Algebra

conjecture' in the algebraically closed case, which was first proved by H. Lenzing [36]. We refer to K. Igusa's article [22] for a proof under more general hypotheses.

The second part of the paper is concerned with the proof of the following 'localization theorem' (3.1): Let A, B and C be algebras over a field k (for simplicity). Suppose that $L \in \text{rep}\,(A,B)$ and $M \in \text{rep}\,(B,C)$ are such that the sequence

$$0 \to \mathcal{D}A \xrightarrow{?\otimes^{\mathbf{L}}_A L} \mathcal{D}B \xrightarrow{?\otimes^{\mathbf{L}}_B M} \mathcal{D}C \to 0$$

is exact, i.e. $\mathcal{D}A$ is identified with an épaisse subcategory of $\mathcal{D}B$ and $?\otimes^{\mathbf{L}}_B M$ induces an equivalence from $(\mathcal{D}B)/(\mathcal{D}A)$ onto $\mathcal{D}C$. Then the theorem states that there is a canonical long exact sequence

$$\mathrm{HC}_*\,(A) \xrightarrow{\mathrm{HC}_*\,(L)} \mathrm{HC}_*\,(B) \xrightarrow{\mathrm{HC}_*\,(M)} \mathrm{HC}_*\,(C) \to \mathrm{HC}_{*-1}\,(A).$$

This theorem may be viewed as an analogue in cyclic homology of the localization theorems of Thomason-Trobaugh [51] and Yao [58] (cf. also [55]). It is also a first step towards an excision theorem à la Wodzicki [57] in the context of derived categories (cf. 3.3 b).

Exact sequences of derived categories as considered above arise for example in the localization of rings with respect to multiplicative subsets admitting a calculus of fractions (4.1). They always yield a recollement setup [2] and conversely, by König's theorem [27], a recollement setup between derived categories of algebras yields an exact sequence of derived categories in the above sense.

We emphasize that our localization theorem does not supersede the result on central localization obtained by Geller–Reid–Weibel [14, Prop. A.3], Loday (unpublished, cf. however [37, 3.4]), Jon Bloch (unpublished), and Brylinski [5] (cf. also C. A. Weibel's recent book [56, 9.1.8.3], and [38, 1.1.17]). Neither does our theorem supersede the results on étale descent by Weibel–Geller [54]. We hope to establish the precise relationship with these results in a future paper.

Contents

1. Hochschild homology

1.1 Notations. We refer to section 7 for notations and basic results concerning DG algebras and their (relative) derived categories. Let k be a commutative ring and A a DG k-algebra (7.1). We write $\otimes$ for the tensor product of DG k-modules over k. The *bar resolution of* A is the chain complex $R\,(A)$ whose n-th component is the DG k-module $A \otimes A^{\otimes n} \otimes A$, $n \in \mathbf{N}$. The components $R\,(A)_n$ vanish for $n < 0$. The complex $R\,(A)$ is endowed with the differential given by

$$d(a_0 \otimes a_1 \otimes \ldots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \ldots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \ldots \otimes a_{n+1}.$$

The total complex of $R(A)$ will also be denoted by $R(A)$ (we always form the total complex using direct *sums*, not products). It is viewed as a DG A-A-bimodule. The multiplication map $A \otimes A \to A$ induces a morphism of DG bimodules

$$\varepsilon : R(A) \to A.$$

The *Hochschild complex* is the DG k-module

$$H(A) = R(A) \otimes_{A^e} A\,,$$

where $A^e = A^{\mathrm{op}} \otimes A$. Its homology is the *Hochschild homology of* A

$$\mathrm{HH}_n(A) = \mathrm{H}_n H(A)\,,\ n \in \mathbf{Z}.$$

These definitions agree with those in [38, 5.3.2].

In this context, Hochschild's interpretation [21] reads as follows: The mapping cone over $\varepsilon : R(A) \to A$ is contractile when considered as a right (or left) DG A-module. So, a fortiori, ε is a relative quasi-isomorphism (7.4) of DG $A^{\mathrm{op}} \otimes A$-modules. The filtration of $R(A)$ by the total complexes of the chains

$$\ldots \to 0 \to R(A)_p \to R(A)_{p-1} \to \ldots \to R(A)_0 \to 0 \to \ldots\,,\ p \in \mathbf{N}\,,$$

satisfies the hypotheses of lemma 7.5 and thus $R(A)$ is relatively closed. So by lemma 7.4, for any relative quasi-isomorphism $P \to A$ of DG bimodules with relatively closed P, we have a canonical homotopy equivalence $R(A) \xrightarrow{\sim} P$. Whence a canonical homotopy equivalence

$$H(A) \xrightarrow{\sim} P \otimes_{A^e} A.$$

and canonical isomorphisms $\mathrm{HH}_n\, A \xrightarrow{\sim} \mathrm{H}_n(P \otimes_{A^e} A)$. So $H(A)$ identifies with the image of A under the total relative left derived functor of the tensor product functor $? \otimes_{A^e} A$.

1.2 Comparison. Keep the assumptions of 1.1. For two DG A-modules L and M, we denote by $\mathcal{H}om_A(L, M)$ the cochain complex of k-modules whose n-th component consists of the morphisms of graded A-modules $f : L \to M$ which are homogeneous of degree n. The differential is given by $d(f) = d_M \circ f - (-1)^n f \circ d_L$. It is easy to check that $\mathcal{H}om_A(L, L)$ is a DG algebra (cf. example 7.1 b). If A and L are concentrated in degree 0, then so is $\mathcal{H}om_A(L, L)$ and its only non-vanishing component is $\mathrm{Hom}_A(L, L)$.

Let P and Q be two closed DG A-modules (7.4) and suppose that the smallest full triangulated subcategory of $\mathcal{H}A$ containing P and closed under forming direct summands contains Q as well.

Lemma.

a) The embedding

$$\mathcal{H}om_A(P, P) \to \mathcal{H}om_A(P \oplus Q, P \oplus Q)\,,\ f \mapsto \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix},$$

induces a homotopy equivalence

$$H(\mathcal{H}om_A(P, P)) \xrightarrow{\sim} H(\mathcal{H}om_A(P \oplus Q, P \oplus Q)).$$

b) The composition morphisms

$$\begin{array}{rcl} \mathcal{H}om_A(P, Q) \otimes_B \mathcal{H}om_A(Q, P) & \to & \mathcal{H}om_A(Q, Q) \text{ resp.} \\ \mathcal{H}om_A(P, Q) \otimes_B P & \to & Q \end{array}$$

where $B = \mathcal{H}om_A(P, P)$, are homotopy equivalences of DG k-modules, resp. DG A-modules.

Proof. a) Put $B = \mathcal{H}om_A(P,P)$ and $C = \mathcal{H}om_A(P \oplus Q, P \oplus Q)$. Let η be the composition

$$C \otimes_B R(B) \otimes_B C \xrightarrow{\varepsilon_*} C \otimes_B B \otimes_B C \xrightarrow{\mu} C.$$

The n-th component of $C \otimes_B R(B) \otimes_B C$ is isomorphic to $Ce \otimes B^{\otimes n} \otimes eC$, where $e \in C$ denotes the idempotent associated with P. This is clearly isomorphic to a direct summand of a module of the form $K \otimes C^e$ for some DG k-module K, where $C^e = C^{\mathrm{op}} \otimes C$. Using lemma 7.5 with the same filtration as above for $R(A)$ we see that $C \otimes_B R(B) \otimes_B C$ is relatively closed over C^e. We will prove that η is a relative quasi-isomorphism. Since the obvious morphism

$$(C \otimes_B R(B) \otimes_B C) \otimes_{C^e} C \to R(C) \otimes_{C^e} C$$

is compatible with the augmentations η and ε, it will then have to be a homotopy equivalence by lemma 7.4. The claim will follow because the composition

$$R(B) \otimes_{B^e} B \xrightarrow{\sim} (C \otimes_B R(B) \otimes_B C) \otimes_{C^e} C \to R(C) \otimes_{C^e} C$$

equals the canonical map $H(B) \to H(C)$.

Thus it remains to be proved that η is a relative quasi-isomorphism. For this, let U and V be arbitrary DG A-modules and consider the chain complex $R(U,V)$ with components

$$\mathcal{H}om_A(P,V) \otimes B^{\otimes n} \otimes \mathcal{H}om_A(U,P)\,,\ n \in \mathbf{N}\,,$$

and the differential

$$d(b_0 \otimes b_1 \otimes \ldots \otimes b_n \otimes b_{n+1}) = \sum_{i=0}^{n} (-1)^i b_0 \otimes \ldots \otimes b_{i-1} \otimes b_i b_{i+1} \otimes b_{i+2} \otimes \ldots \otimes b_{n+1}\,,$$

and denote by $T(U,V)$ the total complex of the mapping cone of the morphism

$$R(U,V) \to \mathcal{H}om_A(U,V)$$

induced by the composition

$$\mathcal{H}om_A(P,V) \otimes \mathcal{H}om_A(U,P) \to \mathcal{H}om_A(U,V).$$

It is clear that $R(P,P)$ identifies with $R(B)$ and $T(P,P)$ with the mapping cone over ε. Similarly, $T(P \oplus Q, P \oplus Q)$ identifies with the mapping cone over η. We have to show that it is k-contractile, i.e. vanishes as an object of $\mathcal{H}k$. Now we know that $T(P,P)$ vanishes in $\mathcal{H}k$. Let us view $T(P,?)$ as a triangle functor from $\mathcal{H}A$ to $\mathcal{H}k$. Its kernel is clearly a triangulated subcategory containing P and closed under forming direct summands. Hence by the assumption,the kernel contains Q as well and hence $P \oplus Q$. So the complex $T(P, P \oplus Q)$ is k-contractile. Now we consider $T(?, P \oplus Q)$ as a triangle functor $\mathcal{H}A \to (\mathcal{H}k)^{\mathrm{op}}$. As we have just seen, its kernel contains P. So by the assumption, it contains $P \oplus Q$ as well.

b) Let U and V be arbitrary DG A-modules and consider the composition morphism

$$\mathcal{H}om_A(P,V) \otimes_B \mathcal{H}om_A(U,P) \to \mathcal{H}om_A(U,V)$$

and the total complex of its mapping cone, which will be denoted by $T(U,V)$. For $U = V = P$, we clearly have an isomorphism of DG k-modules and thus $T(P,P)$ is contractile. So the kernel of $T(P,?)$ viewed as a functor $\mathcal{H}A \to \mathcal{H}k$ contains P. Since it is a full triangulated subcategory of $\mathcal{H}A$, the assumption then implies that it contains Q as well. So $T(P,Q)$ is contractile. By considering the kernel of $T(?,Q)$ we find in a similar way that $T(Q,Q)$ is contractile. The second homotopy equivalence is proved similarly.

2. Invariance of cyclic homology

2.1 Precyclic modules and mixed complexes. C. Kassel has defined the notion of a mixed complex [24] and associated a mixed complex with each cyclic module (cf. also [38, 2.5.13]). Following [42], we shall slightly modify this construction so as to make it functorial with respect to morphisms between cyclic modules which do not necessarily commute with the degeneracy operators. These arise from algebra homomorphisms which do not respect the unit. We use the notations and terminology of [38] (in particular, we use the term 'precyclic' for what has also been called 'semi-cyclic').

If C is a precyclic module (=cyclic module without degeneracy operators) we associate a mixed complex M to C as follows: The underlying DG module of M is the mapping cone over $(1-t)$ viewed as a morphism of complexes $(C, b') \to (C, b)$. So its underlying module is $C \oplus C$; it is endowed with the grading whose nth component is $C_n \oplus C_{n-1}$ and the differential is

$$\begin{bmatrix} b & 1-t \\ 0 & -b' \end{bmatrix}.$$

By definition, the operator $B : M \to M$ is

$$\begin{bmatrix} 0 & 0 \\ N & 0 \end{bmatrix}.$$

If C is endowed with degeneracy operators, one easily checks that the morphism $[1 \;\; (1-t)s]$ yields a morphism of mixed complexes between M and $(C, b, (1-t)sN)$, which is the usual mixed complex associated with C. This morphism is an homotopy equivalence of the underlying DG modules and hence induces isomorphisms in Hochschild and cyclic homology. Note that M is functorial with respect to morphisms of *precyclic* modules and that this does not hold for $(C, b, (1-t)sN)$.

2.2 The mixed derived category. Let us recall Kassel's interpretation [24] of mixed complexes: Let Λ be the DG algebra generated by an indeterminate ε of chain degree 1 with $\varepsilon^2 = 0$ and $d\varepsilon = 0$. The underlying complex of Λ is

$$\ldots 0 \to k\varepsilon \xrightarrow{0} k \to 0 \ldots.$$

Let C be a right DG module over Λ and put

$$Bc := (-1)^p c\varepsilon \, , \; bc := dc \, , \; c \in C^p .$$

Then (C, b, B) is an (unbounded) mixed complex, and in this way the category of (unbounded) mixed complexes identifies with the category of DG Λ-modules. The (Hochschild) homology $\mathrm{HH}_* C$ identifies with $\mathrm{H}^{-*}C$. By [24, Prop. 1.3], we have a canonical isomorphism

$$\mathrm{HC}_* C \xrightarrow{\sim} \mathrm{H}^{-*}(C \otimes^{\mathbf{L}}_{\Lambda} k) \, ,$$

where k denotes the trivial left Λ-module. So both, Hochschild and cyclic homology descend to cohomological functors on the derived category $\mathcal{D}\Lambda$. We use the notation $\mathcal{DM}ix = \mathcal{D}\Lambda$ and call this the *mixed derived category*. Note that despite the notation, this is *not* the derived category of the abelian category of mixed complexes (the objects of this category would be complexes of mixed complexes ...). We still denote by HH_* and HC_* the corresponding cohomological functors on the mixed derived category.

2.3 A bimodule category. Let A and B be DG k-algebras. Let $\mathrm{hrep}\,(A, B)$ be the full subcategory of the homotopy category $\mathcal{H}\,(A^{\mathrm{op}} \otimes B)$ formed by the DG bimodules X such that X_B is small (7.10) and closed (7.4) as a DG B-module. Clearly $\mathrm{hrep}\,(A, B)$ is a triangulated subcategory of $\mathcal{H}(A^{\mathrm{op}} \otimes B)$. Let Σ be the class of quasi-isomorphisms of $\mathrm{hrep}\,(A, B)$. It is worth noting that a morphism $s : X \to Y$ of Σ induces a homotopy equivalence $X_B \to Y_B$ of DG B-modules since both restrictions X_B and Y_B are closed as DG B-modules. Clearly, Σ is a multiplicative system in the sense of Verdier [52]. We define $\text{rep}\,(A,B)$ to be the localization of $\text{hrep}\,(A,B)$ at Σ (compare with [25] [26]). Observe that if C is a third DG k-algebra we have a well defined functor

$$\text{rep}\,(A,B) \times \text{rep}\,(B,C) \to \text{rep}\,(A,C)\,,\ (X,Y) \mapsto X \otimes_B Y.$$

Thanks to the following lemma, if A is closed as a DG k-module, then *we may regard any DG A-B-bimodule whose image in $\mathcal{D}B$ is small as an object of* $\text{rep}\,(A,B)$. The lemma also shows that if A and B are ordinary algebras which are projective over k, then Kassel's [25] [26] category $\text{Rep}\,(A,B)$ identifies with a full subcategory of $\text{rep}\,(A,B)$.

Lemma. *If A is closed as a DG k-module, then the canonical functor*

$$\mathcal{H}\,(A^{\text{op}} \otimes B) \to \mathcal{D}\,(A^{\text{op}} \otimes B)$$

induces an equivalence of $\text{rep}\,(A,B)$ *onto the full subcategory of* $\mathcal{D}\,(A^{\text{op}} \otimes B)$ *formed by the DG bimodules X such that X_B is a small in $\mathcal{D}B$. Moreover, if C is flat as a DG k-module, the following diagram is commutative up to canonical isomorphism*

$$\begin{array}{ccc} \text{rep}\,(A,B) \times \text{rep}\,(B,C) & \xrightarrow{\otimes_B} & \text{rep}\,(A,C) \\ \downarrow & & \downarrow \\ \mathcal{D}\,(A^{\text{op}} \otimes B) \times \mathcal{D}\,(B^{\text{op}} \otimes C) & \xrightarrow{\otimes_B^{\mathbf{L}}} & \mathcal{D}\,(A \otimes C). \end{array}$$

Proof. Since A is closed as a DG k-module, the bimodule $A \otimes B$ is closed as a DG B-module. It follows that X_B is closed for each closed DG bimodule X. It is then easy to check that the functor

$$\mathcal{D}\,(A^{\text{op}} \otimes B) \to \mathcal{H}\,(A^{\text{op}} \otimes B)\,,\ X \mapsto \boldsymbol{p}X$$

induces a quasi-inverse to the functor of the claim. To prove the second assertion, we consider the diagram

$$\begin{array}{ccc} \boldsymbol{p}X \otimes_B \boldsymbol{p}Y & \longrightarrow & X \otimes_B \boldsymbol{p}Y \\ \downarrow & & \downarrow \\ \boldsymbol{p}X \otimes_B Y & \longrightarrow & X \otimes_B Y. \end{array}$$

Here the top arrow is a quasi-isomorphism because C is flat as a DG k-module. The left vertical arrow is a quasi-isomorphism because A is flat as a DG k-module. The right vertical arrow is a quasi-isomorphism because X_B is closed as a DG B-module. Thus the bottom arrow is a quasi-isomorphism.

2.4 The cyclic functor. In analogy with Kassel's construction [25] [26], we define **ALG** to be the 'category' whose *objects* are the DG algebras A and whose *morphisms* $A \to B$ bijectively correspond to the isomorphism classes of DG A-B-bimodules ${}_AX_B$ of $\text{rep}\,(A,B)$ (we write 'category' since these classes usually do not form sets). The *identity* of A is the class of ${}_AA_A$. The *composition* of ${}_AX_B$ with ${}_BY_C$ is the tensor product ${}_AX \otimes_B Y_C$. Let **Alg** be the category with the same objects as **ALG** and whose morphisms are the k-linear maps of differential graded rings (not necessarily preserving the unit). A typical example of a morphism in **Alg** is the embedding

$$A \to \mathcal{H}om_A\,(A \oplus A, A \oplus A)\,,\ a \mapsto \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}.$$

We have the canonical functor $\mathbf{Alg} \to \mathbf{ALG}$ which associates with a morphism $\varphi : A \to B$ the bimodule $\varphi(1_A)B_B$ with the A-B-action given by $a.\varphi(1)b.b' := \varphi(a)bb'$. Note that ${}_\varphi B_B$ with the action $a.b.b' = \varphi(a)bb'$ is *not in general* a bimodule in our sense since $1 \otimes 1$ need not act by the identity. Note also that the functor $\mathbf{Alg} \to \mathbf{ALG}$ is *not* faithful. For example, it maps all inner automorphisms to the identity (if $\varphi : A \to A$ is conjugation by u then $a \mapsto ua$ defines an isomorphism of A-A-bimodules ${}_AA_A \xrightarrow{\sim} {}_\varphi A_A$).

Let A and B be two DG algebras. Denote by $K_0\,(A,B)$ the Grothendieck group of the triangulated category $\text{rep}\,(A,B)$. We define $\mathbf{ALG}_0$ to be the category whose objects are those of **ALG** and whose morphisms $A \to B$ bijectively correspond to elements of $K_0(A, B)$. The composition $K_0(A, B) \times K_0(B, C) \to K_0(A, C)$ is induced by the tensor product over B. A *K-theoretic equivalence* is an isomorphism of $\mathbf{ALG}_0$. We have a canonical functor $\mathbf{ALG} \to \mathbf{ALG}_0$ which is universal among functors F from $\mathbf{ALG}$ to an additive category which satisfy $F(Y) = F(X) + F(Z)$ for all triangles $X \to Y \to Z \to SX$ of $\operatorname{rep}(A, B)$.

Each DG algebra yields a cyclic module and each morphism of $\mathbf{Alg}$ yields a morphism of *precyclic* modules. By applying the construction of 2.2, we obtain a functor

$$C : \mathbf{Alg} \to \mathcal{DM}ix\,.$$

Our aim is to extend it to a functor defined on all of $\mathbf{ALG}$ and then to show that it descends to a functor defined on $\mathbf{ALG}_0$. We have to define the morphism $C(X)$ associated with a bimodule ${}_AX_B$ which is small and closed over B. For this, consider the morphisms of DG algebras

$$A \xrightarrow{\alpha_X} \mathcal{H}om_B(B \oplus X, B \oplus X) \xleftarrow{\beta_X} B$$

given by

$$\alpha_X(a)(b, x) = (0, ax) \text{ and } \beta_X(b')(b, x) = (b'b, 0).$$

By 7.10, the DG B-module X_B is contained in the smallest full triangulated subcategory of $\mathcal{H}B$ containing B_B and closed under forming direct summands. So by lemma 1.2, $C(\beta_X)$ is invertible in $\mathcal{DM}ix$. Thus we have a well-defined morphism

$$C(X) := C(\beta_X)^{-1} \circ C(\alpha_X).$$

in the mixed derived category.

Theorem.

a) *The morphism $C(X)$ only depends on the isomorphism class of X in* $\operatorname{rep}(A, B)$. *The assignment $X \mapsto C(X)$ defines a functor extending*

$$C : \mathbf{Alg} \to \mathcal{DM}ix$$

to the category $\mathbf{ALG}$. *This extension is unique.*

b) *If $X \to Y \to Z \to SX$ is a triangle of* $\operatorname{rep}(A, B)$, *then $C(Y) = C(X) + C(Z)$. Hence C induces a functor* $\mathbf{ALG}_0 \to \mathcal{DM}ix$. *In particular, cyclic homology is invariant under K-theoretic equivalence.*

Remark. The image of β_X in $\mathbf{ALG}$ is the class of the bimodule $\mathcal{H}om_B(B \oplus X, B)$. We claim that this is an invertible morphism of $\mathbf{ALG}$; indeed, its inverse is the class of $B \oplus X$ by lemma 1.2 b). So if Σ denotes the class of morphisms of $\mathbf{Alg}$ which are of the form β_X or which are homotopy equivalences of the underlying DG k-modules, then the canonical functor $\mathbf{Alg} \to \mathbf{ALG}$ makes all members of Σ invertible. The proof of the theorem will show that an arbitrary functor defined on $\mathbf{Alg}$ and making all members of Σ invertible extends to a unique functor on $\mathbf{ALG}$. We may therefore *view* $\mathbf{ALG}$ *as the localization of* $\mathbf{Alg}$ with respect to Σ.

Proof. a) Let X and X' be two isomorphic objects of $\operatorname{rep}(A, B)$. To prove that $C(X) = C(X')$, we may assume that we are given a quasi-isomorphism $s : X \to X'$ of DG bimodules inducing a split surjection of the underlying graded bimodules. Let

$$0 \to N \to X' \xrightarrow{s} X \to 0$$

be short exact. Note that the restriction N_B is contractile since s induces a homotopy equivalence $X'_B \to X_B$. Let $U \subset \mathcal{H}om_B(B \oplus X', B \oplus X')$ be the subalgebra formed by the f with $f(N) \subset N$. We have a k-split short exact sequence

$$0 \to U \to \mathcal{H}om_B(B \oplus X', B \oplus X') \to \mathcal{H}om_B(N, B \oplus X) \to 0.$$

The third term is contractile since N_B is contractile. Thus the inclusion $b_1 : U \to \mathcal{H}om_B(B \oplus X', B \oplus X')$ is an homotopy equivalence. Note that $\alpha_{X'}$ and $\beta_{X'}$ factor through b_1 and that moreover we have a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{\alpha_{X'}} & \mathcal{E}nd_B(B \oplus X') & \xleftarrow{\beta_{X'}} & B \\ \| & & \uparrow b_3 & & \| \\ A & \xrightarrow{a_2} & U & \xleftarrow{b_2} & B \\ \| & & \downarrow b_1 & & \| \\ A & \xrightarrow{\alpha_X} & \mathcal{E}nd_B(B \oplus X) & \xleftarrow{\beta_X} & B. \end{array}$$

Here $C(\beta_{X'})$, $C(\beta_X)$ and $C(b_1)$ are invertible. Therefore, the same holds for $C(b_2)$ and $C(b_3)$. Thus we have the identities

$$C(X') = C(\beta_{X'})^{-1} \circ C(\alpha_{X'}) = C(b_2)^{-1} \circ C(a_2) = C(\beta_X)^{-1} \circ C(\alpha_X) = C(X).$$

in the mixed derived category.

Let us show that $C(X \otimes_B Y) = C(Y) \circ C(X)$. Consider the following commutative diagram of DG algebras

$$\begin{array}{ccccc} & & & & C \\ & & & & \downarrow \beta_Y \\ & & B & \xrightarrow{\alpha_Y} & \mathcal{E}nd_C(C \oplus Y) \\ & & \downarrow \beta_X & & \downarrow \gamma \\ A & \xrightarrow{\alpha_X} & \mathcal{E}nd_B(B \oplus X) & \xrightarrow{\delta} & \mathcal{E}nd_C(C \oplus Y \oplus X \otimes_B Y) \end{array}$$

Here the morphism γ is the canonical inclusion and the morphism δ comes from the action of $\mathcal{E}nd_B(B \oplus X)$ on $(B \oplus X) \otimes_B Y \xrightarrow{\sim} Y \oplus (X \otimes_B Y)$. Since $C(\beta_X)$, $C(\beta_Y)$ and $C(\gamma\,\beta_Y)$ lie in Σ, we can conclude that we have

$$C(Y) \circ C(X) = C(\gamma\,\beta_Y)^{-1} C(\delta\,\alpha_X)$$

in $\mathcal{DM}ix$. On the other hand, the commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{\delta\alpha_X} & \mathcal{E}nd_C(C \oplus Y \oplus X \otimes_B Y) & \xleftarrow{\gamma\beta_Y} & C \\ \| & & \uparrow can & & \| \\ A & \xrightarrow{\alpha_{X\otimes Y}} & \mathcal{E}nd_C(C \oplus X \otimes_B Y) & \xleftarrow{\beta_{X\otimes Y}} & C, \end{array}$$

shows that $C(X \otimes_B Y) = C(\gamma\,\beta_Y)^{-1} C(\delta\,\alpha_X)$.

Let us show that ${}_AA_A$ is mapped to the identity in $\mathcal{DM}ix$. Indeed, we have

$$C({}_AA_A) = C({}_AA \otimes_A A_A) = C({}_AA_A) \circ C({}_AA_A)$$

by what we just proved. On the other hand, it is clear from lemma 1.2 that $C(\alpha_A)$ and hence $C({}_AA_A)$ becomes invertible in $\mathcal{DM}ix$. The claim follows. Note that the claim implies $C(\alpha_A) = C(\beta_A)$, which we will now use to show that $C({}_\varphi B_B) = C(\varphi)$. Put $X = {}_\varphi B_B$. Then by definition, we have $C(X) = C(\beta_X)^{-1} C(\alpha_X)$, where the morphisms

$$A \xrightarrow{\alpha_X} \mathcal{E}nd_B(B \oplus {}_\varphi B) \xleftarrow{\beta_X} B$$

are given by

$$\alpha_X : a \mapsto \begin{bmatrix} 0 & 0 \\ 0 & \varphi(a) \end{bmatrix} \text{ and } \beta_X : b \mapsto \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}.$$

We have $\alpha_X = \varphi \circ \alpha_A$, where φ is extended 'componentwise' to '2x2–matrices'. Since $C(\alpha_A) = C(\beta_A)$, we have $C(\varphi \circ \alpha_A) = C(\varphi \circ \beta_A)$ and $C(X) = C(\beta_X)^{-1} \circ C(\varphi \circ \beta_A)$. The claim is now clear from the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \beta_A \downarrow & & \downarrow \beta_X \\ \mathcal{E}nd_A(A \oplus A) & \xrightarrow{\varphi} & \mathcal{E}nd_B(B \oplus {}_\varphi B). \end{array}$$

To prove unicity, we have to prove that the diagram

$$\begin{array}{ccc} A & \stackrel{\mathrm{can}\,(\alpha_X)}{\longrightarrow} & \mathcal{E}nd_B\,(B \oplus X) \\ 1 \downarrow & & \uparrow \mathrm{can}\,(\beta_X) \\ A & \stackrel{X}{\longrightarrow} & B \end{array}$$

is commutative in **ALG**. Now we have

$$\mathrm{can}\,(\beta_X) = \mathcal{H}om_B\,(B \oplus X, B)\,,\ \mathrm{can}\,(\alpha_X) = \mathcal{H}om_B\,(B \oplus X, X).$$

By lemma 1.2 b), we have

$$\mathrm{can}\,(\beta_X)^{-1} = B \oplus X$$

and $\mathcal{H}om_B\,(B \oplus X, X)$ and $B \oplus X$ are inverse to each other in **ALG**. Thus

$$\mathrm{can}\,(\beta_X)^{-1} \circ \mathrm{can}\,(\alpha_X) = X.$$

b) We may assume that the triangle comes from a short exact sequence of DG bimodules

$$0 \to X \xrightarrow{i} Y \xrightarrow{p} Z \to 0$$

which splits as a sequence of graded bimodules. Let T be the algebra of "upper triangular 2×2 matrices" with coefficients in B

$$T := \begin{bmatrix} B & B \\ 0 & B \end{bmatrix}.$$

Endow the left A-module $X \oplus Y$ with the right T-action defined by

$$(x, y). \begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{bmatrix} := (x\, b_{11}, i(x\, b_{12}) + y\, b_{22}).$$

Then $X \oplus Y$ becomes an A-T-bimodule. It is easy to check that it is small and closed over T. Consider the sequence of T-B-bimodules

$$0 \to Te_{11} \to Te_{22} \to Te_{11}/Te_{22} \to 0.$$

If we tensor this sequence over T with $X \oplus Y$ we find the original sequence

$$0 \to X \xrightarrow{i} Y \xrightarrow{p} Z \to 0.$$

In particular, we have

$$C(Y) = C(Te_{22} \otimes_B (X \oplus Y)) = C(Te_{22}) \circ C(X \oplus Y)$$

and similarly for $C(X)$ and $C(Z)$. So it will be enough to prove the claim for $Te_{11}, Te_{22}, Te_{22}/Te_{11}$. Now by Kadison's argument [23], the two canonical projections $B \to T$ induce an isomorphism $C(T) \xrightarrow{\sim} C(B) \oplus C(B)$. It follows that the two canonical inclusions $B \to T$ induce an isomorphism $C(B) \oplus C(B) \to C(T)$. Using these isomorphisms one easily verifies that

$$C(Te_{22}) = C(Te_{11}) + C(Te_{22}/Te_{11}).$$

2.5 Finite-dimensional algebras of finite global dimension. Suppose that k is a field and that A is a finite-dimensional 'ordinary' k-algebra of finite global dimension. Suppose moreover that $\mathrm{Hom}_A(S, S) = k$ for each simple A-module S and that A/r is a product of copies of k, where r is the Jacobson radical of A. Let $E \subset A$ be a semisimple subalgebra such that $A = E \oplus r$. The last assertion of the following proposition was deduced by K. Igusa [22, Cor. 5.7] from results of T. Goodwillie [16] in the case of a field k of characteristic zero. An important special case was first proved by C. Cibils [7].

Proposition. *The inclusion $E \to A$ yields a K-theoretic equivalence $E \xrightarrow{\sim} A$. In particular, the canonical morphism $C(E) \to C(A)$ is an isomorphism in the mixed derived category and we have an isomorphism* $\mathrm{HC}_*(E) \xrightarrow{\sim} \mathrm{HC}_*(A)$.

Remark. For each i, let $P_i \to S_i$ be a projective cover and let $\mathrm{rad}\,(P_i, P_j)$ be the space of non-invertible maps $P_i \to P_j$. For each i, we have a canonical sequence

$$\bigoplus_{j=1}^{n} \mathrm{rad}\,(P_j, P_i) \otimes \mathrm{rad}\,(P_i, P_j) \xrightarrow{\mu} \mathrm{rad}\,(P_i, P_i) \to \mathrm{Ext}^1_A(S_i, S_i) \to 0\,,$$

where $\mu\,(f \otimes g) = fg$. According to a result of Cibils' [8, 2.1], we also have an exact sequence

$$\bigoplus_{i,j=1}^{n} \mathrm{rad}\,(P_j, P_i) \otimes \mathrm{rad}\,(P_i, P_j) \xrightarrow{\partial} \bigoplus_{i=1}^{n} \mathrm{rad}\,(P_i, P_i) \to \mathrm{HC}_0\,(A)/\mathrm{HC}_0\,(E) \to 0\,,$$

where $\partial(f \otimes g) = fg - gf$. We conclude that we have a surjection

$$\mathrm{HC}_0\,(A)/\mathrm{HC}_0\,(E) \to \bigoplus_{i=1}^{n} \mathrm{Ext}^1_A(S_i, S_i).$$

In particular, under the above hypotheses, the algebra A 'has no loops', i.e. we have $\mathrm{Ext}^1_A(S_i, S_i) = 0$ for all i. An even stronger statement was first proved by H. Lenzing in [36]. We refer the reader to [22] for a proof of the 'no loops conjecture' in more general situations.

Proof. Let $S_1, \ldots, S_n$ be a system of representatives of the isomorphism classes of simple A-modules and put $X = \bigoplus_{i=1}^n S_i$. We identify $\mathrm{Hom}_A(X, X)$ with E and we view X as an E-A-bimodule. Since A is finite-dimensional of finite global dimension, each simple A-module has a finite resolution by finitely generated projective A-modules. So X_A is small in $\mathcal{D}A$. For each i, let $P_i \to S_i$ be a projective cover. Put $P_i^* = \mathrm{Hom}_A(P_i, A)$ and let $Y = \bigoplus_{i=1}^n P_i^*$. We view Y as an A-E-bimodule.

We claim that $C(Y)$ is inverse to $C(X)$. Indeed, it is clear that

$$(\boldsymbol{p}S_i) \otimes_A P_j^* \xrightarrow{\sim} S_i \otimes_A P_j^* \xrightarrow{\sim} \left\{ \begin{array}{ll} 0 & i \neq j \\ k & i = j \end{array} \right. .$$

So $X \otimes_A Y \xrightarrow{\sim} E$ and $C(Y) \circ C(X) = 1$. It remains to be shown that the images of A and $Y \otimes_E X = \bigoplus_{i=1}^n P_i^* \otimes S_i$ in $K_0\,(A, A)$ coincide. For this we note first that a DG A-A-bimodule U is small iff U_A is small in $\mathcal{D}A$. Indeed, $A \otimes A$ is clearly small as a right A-module and hence each small DG A-A-bimodule is small as a right A-module. To prove the converse, note first that $A^{\mathrm{op}} \otimes A$ is of finite global dimension and hence that ${}_AA_A$ is small in $\mathcal{D}(A^{\mathrm{op}} \otimes A)$. Now the formula

$$\mathbf{R}\mathcal{H}om_{A-A}\,(U, V) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om_{A-A}\,(\,{}_AA_A, \mathbf{R}\mathcal{H}om_A\,(U_A, V_A))$$

shows that U is small if U_A is small. Thus $K_0\,(A, A)$ identifies with the Grothendieck group of the triangulated category of small objects in $\mathcal{D}(A^{\mathrm{op}} \otimes A)$ and hence with $K_0\,(A^{\mathrm{op}} \otimes A)$. Since A is finite-dimensional of finite global dimension and $\mathrm{Hom}_A(P_i, S_i) = k$ for all i, we have an isomorphism

$$K_0\,(A^{\mathrm{op}} \otimes A) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Z}}(K_0\,(A), K_0\,(A))$$

which sends $[U]$ to the map defined by $P \mapsto P \otimes_A U$, where P is a finitely generated projective A-module. Under this map, both A and $Y \otimes_E X$, correspond to the identity.

2.6 Equivalences in the flat case. Let A and B be DG k-algebras and X a DG A-B-bimodule which is closed over B. Recall that a DG k-module M is flat if the functor $? \otimes M$ preserves acyclicity. Since M may be unbounded to the right, this is not, in general, equivalent to requiring that each M_n, $n \in \mathbf{Z}$, be a flat k-module.

Theorem. *If A and B are flat as DG k-modules and*

$$? \otimes_A^{\mathbf{L}} X : \mathcal{D}A \to \mathcal{D}B$$

is an equivalence (cf. 7.6), then X is small over B and $C(X)$ is invertible.

Remarks. a) In the situation of the theorem, we have $\mathrm{HC}_* A \xrightarrow{\sim} \mathrm{HC}_* B$. In particular, two derived equivalent algebras [49] have isomorphic cyclic homology (this answers a question of J. Rickard's, who proved the corresponding statement for Hochschild homology in [49]).

b) If there is a quasi-isomorphism of DG algebras $\varphi : A \to B$, the functor $? \otimes_A^{\mathbf{L}} B : \mathcal{D}A \to \mathcal{D}B$ is an equivalence for arbitrary A and B (example 7.6). So the theorem does not hold without some flatness hypothesis.

c) It is immediate to verify that an invertible morphism X of **ALG** yields an equivalence $? \otimes_A^{\mathbf{L}} X : \mathcal{D}A \to \mathcal{D}B$. Conversely, if the functor $? \otimes_A^{\mathbf{L}} X$ is an equivalence and the algebras A and B are closed as DG k-modules (i.e. the functor $\mathcal{H}om_k(A, ?)$ preserves acyclicity), then X has an inverse as a morphism of **ALG**. To wit, it is given by the bimodule $X^T = \boldsymbol{p}\mathcal{H}om_B(X_B, B)$, where $\boldsymbol{p}$ has to be taken in $\mathcal{H}(A^{\mathrm{op}} \otimes B)$, cf. [31, 6.2]. So for the case where A and B are projective as DG k-modules, the theorem follows from theorem 2.4.

Proof. Note first that $X_B = A \otimes_A^{\mathbf{L}} X$ is small. Now consider the morphisms of DG algebras

$$A \xrightarrow{a_1} \mathcal{E}nd_B(X) \xrightarrow{a_2} \mathcal{E}nd_B(B \oplus X) \xleftarrow{\beta_X} B$$

where a_1 is given by the left action of A on X and a_2 is the canonical morphism, so that $\alpha_X = a_2\, a_1$. We will prove that $C(\alpha_X)$ lies in Σ. Let us first show that a_2 lies in Σ. Indeed, $X_B = A \otimes_A^{\mathbf{L}} X$ is a small generator of $\mathcal{D}B$ so that by lemma 7.10, the modules $P = X_B$ and $Q = B_B$ satisfy the hypotheses of lemma 1.2.

To prove that $C(a_1)$ is invertible we first note that $A \to \mathcal{E}nd_B(X)$ is a quasi-isomorphism. Indeed, thanks to the canonical isomorphisms (cf. 7.4)

$$\mathrm{H}^n A \xleftarrow{\sim} \mathrm{Hom}_{\mathcal{D}A}(A, S^n A) \text{ and } \mathrm{H}^n \mathcal{H}om_B(X, S^n X) \xleftarrow{\sim} \mathrm{Hom}_{\mathcal{D}B}(X, S^n X)\,,$$

this follows from the full faithfulness of $? \otimes_A^{\mathbf{L}} X$. The morphism a_1 also induces quasi-isomorphisms between the tensor powers of A and of $\mathcal{E}nd_B(X)$. Indeed, A is flat as a DG k-module by assumption, and the same holds for $\mathcal{E}nd_B(X)$, by the following lemma.

2.7 Lemma. *Let B be a DG k-algebra which is flat as a DG k-module.*

a) If U and V are small closed DG B-modules, the functor $? \otimes_k \mathcal{H}om_B(U, V)$ preserves acyclicity.

b) Suppose moreover that k is coherent and of finite global dimension. If U and V are arbitrary closed DG B-modules, the functor $? \otimes_k \mathcal{H}om_B(U, V)$ preserves acyclicity.

Proof. a) The assertion is clear for $U = V = B$ since then $\mathcal{H}om_B(U, V) = B$ and B is flat over k. Now fix $U = B$. Since V is contained in the smallest triangulated subcategory of $\mathcal{H}B$ containing B and closed under direct summands (lemma 7.10), we can conclude that $? \otimes_k \mathcal{H}om_B(B, V)$ preserves acyclicity. If we now fix V and let U vary, we obtain the assertion.

b) If U is small, the class of V for which $\otimes \mathcal{H}om_B(U, V)$ preserves acyclicity is clearly closed under forming direct sums. By a) it therefore contains all closed V. Under our hypotheses on k, a product of flat DG k-modules is a flat DG k-module (cf. Appendix 8.3). This implies that for fixed closed V the class of U for which $\otimes \mathcal{H}om_B(U, V)$ preserves acyclicity is closed under direct sums. So this class contains all closed modules.

2.8 Triangular matrices. Let B be a DG k-algebra which is flat as a DG k-module.

Lemma. *If*

$$P \to B_B \to Q \to SP$$

is a triangle of $\mathcal{H}B$ *such that* P *and* Q *are small and closed and we have* $\mathrm{Hom}_{\mathcal{D}B}(P, S^n Q) = 0$ *for all* $n \in \mathbf{Z}$*, then the morphism*

$$[C(P)\ \ C(Q)]\ :\ C(\mathcal{E}nd_B\,(P)) \oplus C(\mathcal{E}nd_B\,(Q)) \longrightarrow C(\mathcal{E}nd_B\,(P \oplus Q)) \xleftarrow{\sim} C(B)$$

is an isomorphism of $\mathcal{DM}ix$*.*

Proof. Let $A = \mathcal{E}nd_B\,(P \oplus Q)$ and let $X = P \oplus Q$ viewed as a DG A-B-bimodule. Then the hypotheses of lemma 2.6 are clearly satisfied so that we have an isomorphism

$$C(X) : C(A) \xrightarrow{\sim} C(B).$$

Now let $A_0 \subset A$ be the DG subalgebra consisting of the morphisms f with $f(P) \subset P$. If we identify A with the 'algebra of matrices'

$$\begin{bmatrix} \mathcal{E}nd_B\,(P) & \mathcal{H}om_B\,(Q, P) \\ \mathcal{H}om_B\,(P, Q) & \mathcal{E}nd_B\,(Q) \end{bmatrix}$$

then A_0 corresponds to the subalgebra of 'upper triangular matrices'

$$\begin{bmatrix} \mathcal{E}nd_B\,(P) & \mathcal{H}om_B\,(Q, P) \\ 0 & \mathcal{E}nd_B\,(Q) \end{bmatrix}.$$

The inclusion $A_0 \to A$ is a quasi-isomorphism because $\mathcal{H}om_B\,(P, Q)$ is acyclic (since its n-th homology identifies with $\mathrm{Hom}_{\mathcal{D}B}(P, S^n Q)$). By lemma 2.7, the functors $? \otimes A_0$ and $? \otimes A$ preserve acyclicity and thus $C(A_0) \xrightarrow{\sim} C(A)$. The method of Kadison's [23] (cf. [38, 1.2.15]) shows that the inclusion

$$\begin{bmatrix} \mathcal{E}nd_B\,(P) & 0 \\ 0 & \mathcal{E}nd_B\,(Q) \end{bmatrix} \subset \begin{bmatrix} \mathcal{E}nd_B\,(P) & \mathcal{H}om_B\,(Q, P) \\ 0 & \mathcal{E}nd_B\,(Q) \end{bmatrix}$$

induces an isomorphism $C(\mathcal{E}nd_B\,(P)) \oplus C(\mathcal{E}nd_B\,(Q)) \xrightarrow{\sim} C(A_0)$. It is clear that the composition

$$C(\mathcal{E}nd_B\,(P)) \oplus C(\mathcal{E}nd_B\,(Q)) \xrightarrow{\sim} C(A_0) \xrightarrow{\sim} C(A) \xrightarrow{C(X)} C(B)$$

has the components $C(P)$ and $C(Q)$.

2.9 A split exact sequence. Let A, B and C be DG algebras which are flat as DG k-modules and

$$A \xrightarrow{L} B \xrightarrow{M} C$$

a sequence of **ALG** (recall from 2.4 that morphisms of **ALG** are isomorphism classes of certain DG bimodules; by abuse of notation, we will use the same symbol to refer to a bimodule and to the corresponding morphism of **ALG**). Write T_L for $? \otimes_A^{\mathbf{L}} L$ and T_M for $? \otimes_B^{\mathbf{L}} M$. Suppose that the associated sequence

$$0 \to \mathcal{D}A \xrightarrow{T_L} \mathcal{D}B \xrightarrow{T_M} \mathcal{D}C \to 0\,,$$

is *exact*, i.e. T_L is fully faithful, $T_M \circ T_L = 0$ and T_L induces an equivalence

$$\mathcal{D}B / T_L(\mathcal{D}A) \xrightarrow{\sim} \mathcal{D}C.$$

Proposition.

a) The object $\mathcal{H}om_B\,(L, B)$ *is small in* $\mathcal{D}A$ *if and only if* $\mathcal{H}om_C\,(M, C)$ *is small in* $\mathcal{D}B$*.*

b) If $\mathcal{H}om_B\,(L, B)$ *is small in* $\mathcal{D}A$*, the sequence*

$$C(A) \xrightarrow{C(L)} C(B) \xrightarrow{C(M)} C(C)$$

is split exact in $\mathcal{DM}ix$*.*

Proof. The functor T_L admits the right adjoint $H_L = \mathcal{H}om_B(L, ?)$ and the functor T_M the right adjoint $H_M = \mathcal{H}om_C(M, ?)$. It then follows from [52, Ch. 1, §2, n^o 6] that the adjunction morphisms fit into a triangle

$$T_L H_L B \to B \to H_M T_M B \to S T_L H_L.$$

and that

$$\mathrm{Hom}_{\mathcal{D}B}(T_L H_L B, S^n H_M T_M B) = 0$$

for all $n \in \mathbf{Z}$.

Let us prove a). Suppose that $H_L B$ is small. Then it belongs to the smallest triangulated subcategory of $\mathcal{D}A$ containing A and closed under forming direct summands (7.10). Since $L_B = T_L A$ is small, it follows that $T_L H_L B$ is small. By the triangle, it follows that $H_M T_M B$ is small. We claim that then $H_M C$ has to be small. Indeed, $T_M B$ is a small generator of $\mathcal{D}C$: It is a generator because $T_M : \mathcal{D}B \to \mathcal{D}C$ is the localization functor and it is small because T_M has the adjoint H_M which commutes with infinite sums since M_C is small. So C is contained in the smallest triangulated subcategory of $\mathcal{D}C$ containing $T_M B$ and closed under forming direct summands (7.10). Therefore, if $H_M T_M B$ is small, the same holds for $H_M C$.

Conversely, suppose that $H_M C$ is small. We have just seen that $T_M B$ is small. So it is contained in the smallest triangulated subcategory of $\mathcal{D}C$ containing C and closed under forming direct summands. So $H_M T_M C$ is small as well and so is $T_L H_L B$. Hence $H_L B$ is small, since T_L is fully faithful and commutes with infinite direct sums.

Let us prove b). Put $P = \boldsymbol{p} T_L H_L B$ and $Q = \boldsymbol{p} H_M T_M B$. We will show that the sequence $P \to B \to Q \to SP$ satisfies the assumptions of lemma 2.8 and that we have the following commutative diagram in **ALG**

$$\begin{array}{ccccc}
A & \xrightarrow{L} & B & \xrightarrow{M} & C \\
\downarrow & & \| & & \| \\
\mathcal{E}nd_B(L) & \xrightarrow{L} & B & \xrightarrow{M} & C \\
\mathcal{H}om_B(P, L)\downarrow & & \| & & \uparrow Q \otimes_B M \\
\mathcal{E}nd_B(P) & \xrightarrow{P} & B & \xleftarrow{Q} & \mathcal{E}nd_B(Q)
\end{array}$$

where the vertical morphisms are invertible. The assertion is then clear.

We have seen in the proof of a) that P and Q are small. So the assumptions of lemma 2.8 are satisfied.

We claim that P and L_B may be obtained from one another by shifts, extensions and forming direct summands. By lemma 1.2 b) this will imply that $\mathcal{H}om_B(P, L)$ and $\mathcal{H}om_B(L, P)$ are inverse to each other in **ALG**, and that the lower left square of the above diagram is commutative. It suffices to show that P and L_B are both small generators of $T_L(\mathcal{D}A) \xleftarrow{\sim} \mathcal{D}A$ (7.10). This is clear for L_B. We already know that P is small. To prove that it is a generator, take $M \in \mathcal{D}A$. Then $T_L M$ may be obtained from B by applying shifts, extensions and infinite sums. Hence the same holds for $T_L M \xrightarrow{\sim} T_L H_L T_L M$ with respect to $P \xrightarrow{\sim} T_L H_L B$.

Let us prove that $A \to \mathcal{E}nd_B(L)$ induces an isomorphism in Hochschild homology. Indeed, since T_L is fully faithful, the morphism $A \to \mathcal{E}nd_B(L)$ is a quasi-isomorphism so that our claim follows from lemma 2.7.

Finally, we have to prove that $Q \otimes_B M$ becomes invertible in $\mathcal{DM}ix$. By lemma 2.6, it is enough to show that the bimodule $Q \otimes_B M$ yields an equivalence $\mathcal{DE}nd_B(Q) \to \mathcal{D}C$. Indeed, this functor maps B to $T_M B$, which is clearly a small generator for $\mathcal{D}C \xleftarrow{\sim} \mathcal{D}B/T_L(\mathcal{D}A)$. Since we have $\mathrm{Hom}_{\mathcal{D}B}(T_L X, Q) = 0$ for all $X \in \mathcal{D}A$, we have

$$\mathrm{Hom}_{\mathcal{D}B}(Q, S^n Q) \xrightarrow{\sim} \mathrm{Hom}_{(\mathcal{D}B)/T_L(\mathcal{D}A)}(Q, S^n Q)$$

and therefore

$$\mathrm{H}^n \mathcal{E}nd_B(Q) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}B}(Q, S^n Q) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}C}(Q \otimes_B M, S^n Q \otimes_B M).$$

This implies that $Q \otimes_B M$ yields a fully faithful functor.

3. Localization for DG algebras

3.1 Statement of the theorem. We use the notations of section 2.4. Assume moreover that k is left coherent of finite global dimension and that A, B, and C are DG algebras which are flat as DG k-modules. Recall that we write T_L for the functor $? \otimes_A^{\mathbf{L}} L$ if L is a left A-module. The following theorem will be proved in sections 5 and 6.

Theorem. *If* $A \xrightarrow{L} B \xrightarrow{M} C$ *is a sequence of* **ALG** *such that the derived sequence*

$$0 \to \mathcal{D}A \xrightarrow{T_L} \mathcal{D}B \xrightarrow{T_M} \mathcal{D}C \to 0$$

is exact, then there is a canonical triangle

$$C(A) \xrightarrow{C(L)} C(B) \xrightarrow{C(M)} C(C) \to SC(A)$$

in the mixed derived category.

3.2 Remarks a) In particular, Hochschild and cyclic homology of the three algebras are related by a long exact sequence, thanks to the interpretation of Hochschild and cyclic homology as cohomological functors on the mixed derived category (cf. 2.2).

b) We point out that an exact sequence

$$0 \to \mathcal{D}A \xrightarrow{T_L} \mathcal{D}B \xrightarrow{T_M} \mathcal{D}C \to 0$$

always gives rise to a recollement setup in the sense of [2, 1.4]. The correspondence with the paradigmatic categories is as follows:

$$\mathcal{D}B \leftrightarrow \mathcal{D}^+(X)\,,\ \mathcal{D}A \leftrightarrow \mathcal{D}^+(U) \quad \text{(sic!)}\,,\ \mathcal{D}C \leftrightarrow \mathcal{D}^+(F) \quad ,$$

where X is a ringed topological space, F a closed subset of X, U the complementary open subset, $\mathcal{D}^+(X)$ the right bounded derived category of sheaves of modules over the ringed space X, etc. The correspondence between functors is as follows:

$$\begin{array}{lll} j_! \leftrightarrow ? \otimes_A^{\mathbf{L}} L\,, & j^* \leftrightarrow \mathbf{R}\mathcal{H}om_B(L, ?) \xrightarrow{\sim} ? \otimes_B^{\mathbf{L}} L^T\,, & j_* \leftrightarrow \mathbf{R}\mathcal{H}om_A(L^T, ?)\,, \\ i^* \leftrightarrow ? \otimes_B^{\mathbf{L}} M\,, & i_* \leftrightarrow \mathbf{R}\mathcal{H}om_C(M, ?) \xrightarrow{\sim} ? \otimes_C^{\mathbf{L}} M^T\,, & i_! \leftrightarrow \mathbf{R}\mathcal{H}om_B(M^T, ?)\,, \end{array}$$

where $L^T = \mathcal{H}om_B(L, B)$ and $M^T = \mathcal{H}om_C(M, C)$ (recall that L_B and M_C are small and closed).

Conversely, if A, B and C are 'ordinary' algebras which admit a recollement setup between their derived categories, then there is an exact sequence as above. This follows readily from König's theorem [27] using [32]. More precisely, if in König's notations we have a recollement setup

$$\mathcal{D}^-(\mathrm{Mod}\, B) \overset{\leftarrow}{\underset{\leftarrow}{\rightarrow}} \mathcal{D}^-(\mathrm{Mod}\, A) \overset{\leftarrow}{\underset{\leftarrow}{\rightarrow}} \mathcal{D}^-(\mathrm{Mod}\, C),$$

given by triples of adjoint functors $(i^*, i_*, i^!)$ and $(j_!, j^*, j_*)$, where

$$i_* : \mathcal{D}^-(\mathrm{Mod}\, B) \to \mathcal{D}^-(\mathrm{Mod}\, A) \text{ and } j^* : \mathcal{D}^-(\mathrm{Mod}\, A) \to \mathcal{D}^-(\mathrm{Mod}\, C)\,,$$

then we have an exact sequence

$$0 \leftarrow \mathcal{D}B \xleftarrow{T_M} \mathcal{D}A \xleftarrow{T_L} \mathcal{D}C \leftarrow 0$$

(note the reversal of the arrows), where M_B is isomorphic to $i^* A_A$ and L_A is isomorphic to $j_! B_B$ (we do not claim that $j_!$ is isomorphic to T_L or that i^* is isomorphic to T_M). So we have a triangle

$$SC(C) \leftarrow C(B) \leftarrow C(A) \leftarrow C(C)$$

in the mixed derived category. According to 2.9, this triangle splits if $\mathbf{R}\mathcal{H}om_B(M, B)$ is small over A. This latter condition holds for example if, in König's terminology, the recollement setup possesses a symmetric recollement [27, Thm. 3]. We refer to [27] and the references therein for a number of examples of recollement situations in the context of finite-dimensional algebras.

c) Suppose that k is a *field*. Let A and B be DG algebras and L a DG A-B-bimodule such that L_B is small and closed, so that L gives rise to a morphism of **ALG**. Suppose that $T_L : \mathcal{D}A \to \mathcal{D}B$ is fully faithful. Then, *one can find a DG algebra C and a B-C-bimodule M such that M_C is small and closed and the sequence*

$$0 \to \mathcal{D}A \xrightarrow{T_L} \mathcal{D}B \xrightarrow{T_M} \mathcal{D}C \to 0$$

is exact. Indeed, we may assume that L is closed over $A^{\mathrm{op}} \otimes B$ so that the functors $T_L = ? \otimes_A L$ and $H_L = \mathcal{H}om_B(L, ?)$ preserve acyclicity and $T_L K$ is closed for each DG A-module K (k is a field !). For a DG B-module N denote by $\Phi : T_L H_L N \to N$ the adjunction morphism. Let M' be the mapping cone over $\Phi : T_L H_L B \to B$. Note that M' is closed over B and that it inherits a *left* B-module structure. We put $C = \mathcal{H}om_B(M', M')$ and we choose for M a DG bimodule which is closed over $B^{\mathrm{op}} \otimes C$ and quasi-isomorphic to $\mathcal{H}om_C(M', C)$.

3.3 Examples. a) In the situation of 2.8, we can apply the theorem to $A = \mathcal{E}nd_B(\boldsymbol{p}P)$, $C = \mathcal{E}nd_B(\boldsymbol{p}Q)$ and the bimodules $M = \mathcal{H}om_B(P, B)$ and $L = \mathcal{H}om_B(Q, M)$.

b) The theorem also holds for small DG categories instead of DG algebras, as our proof will show. This allows us to establish the following link with M. Wodzicki's theorem [57]: Let B be an ordinary flat k-algebra and $I \subset B$ an ideal with idempotent local units (i.e. such that for each finite family of elements a_i of I there exists an idempotent $u = u^2 \in I$ such that $ua_i = a_i u = a_i$ for all i). We assume that I is flat over k as well. We can then consider the category $\mathcal{I}$ whose objects are the idempotents of I and whose morphisms $u \to u'$ are in bijection with uIu'. The ideal I yields an $\mathcal{I}$-B-bimodule and it is easy to check that the sequence

$$0 \to \mathcal{D}\mathcal{I} \xrightarrow{T_I} \mathcal{D}B \xrightarrow{T_{B/I}} \mathcal{D}(B/I) \to 0$$

is exact. Hence the theorem yields a triangle

$$C(\mathcal{I}) \to C(B) \to C(B/I) \to SC(\mathcal{I})$$

in the mixed derived category. We refer to 5.4 for the definition of $C(\mathcal{I})$. We have a canonical isomorphism $C(\mathcal{I}) \to C(I)$ and the sequences in cyclic and Hochschild homology induced by the triangle identify with those of [57]. Now suppose $I \subset B$ is an arbitrary H-unital ideal and $C = B/I$. Let $\mathcal{U} \subset \mathcal{D}B$ be the kernel of the functor $? \otimes_B^{\mathbf{L}} C : \mathcal{D}B \to \mathcal{D}C$. Then one can show that the sequence

$$0 \to \mathcal{U} \to \mathcal{D}B \to \mathcal{D}C \to 0$$

is an exact sequence of triangulated categories. In general, however, $\mathcal{U}$ need not be of the form $\mathcal{D}\mathcal{A}$ for a DG category $\mathcal{A}$. A counterexample is given in [33].

c) We refer to the next section for the example of localization of an ordinary algebra with respect to a multiplicative set.

4. Localization at a left denominator set

4.1 Rings of left fractions. As in section 3.1, suppose that k is coherent and of finite global dimension. Let B be an (ordinary) k-algebra which is flat over k. Suppose that $S \subset B$ is a left denominator set, i.e. it satisfies (cf. [12], [11])

- $1 \in S$, $SS \subset S$,
- For $s \in S$ and $b \in B$, there are $t \in S$ and $c \in B$ such that $cs = tb$.
- If $b \in B$ and $s \in S$ satisfy $bs = 0$ there is $t \in S$ such that $tb = 0$.

Then there is an algebra $B[S^{-1}]$ and an algebra homomorphism $B \to B[S^{-1}]$ universal among all algebra morphisms making the elements of S invertible. Moreover, the elements of $B[S^{-1}]$ may be taken to be left fractions $s^{-1}b$, i.e. equivalence classes $(s|b)$ of pairs (s,b) modulo the relation which identifies (s,b) with (s',b') if there are $c,c' \in B$ such that $cs = c's'$ belongs to S and $cb = c'b'$.

In other words, if Σ denotes the category whose objects are the B-module homomorphisms

$$\lambda(s) : B \to B\ ,\ b \mapsto sb\ ,\ s \in S$$

and whose morphisms are the commutative 'triangles'

$$\begin{array}{ccc} B & \stackrel{\lambda(s)}{\to} & B \\ \| & & \downarrow \lambda(b) \\ B & \stackrel{\lambda(t)}{\to} & B \end{array}$$

where $s,t \in S$, $b \in B$, then $B[S^{-1}]$ identifies with the colimit of the functor $F : \Sigma \to \operatorname{Mod} B, \lambda(s) \mapsto \operatorname{range}(\lambda(s))$. The axioms above imply that Σ is filtered. So $B[S^{-1}] = \varinjlim F$ is flat as a right B-module (and as a k-module).

We have a pair of adjoint functors

$$? \otimes_B B[S^{-1}] \qquad \begin{array}{c} \operatorname{Mod} B \\ \downarrow\ \uparrow \\ \operatorname{Mod} B[S^{-1}]. \end{array} \qquad \mathrm{res}$$

Note that the tensor product functor $? \otimes_B B[S^{-1}]$ is not exact in general; it is exact if S also satisfies the axioms for a right denominator set. Hence in general, $\operatorname{Mod} B[S^{-1}]$ will not identify with a localization of the abelian category $\operatorname{Mod} B$. However, as we will see below, the derived category $\mathcal{D}B[S^{-1}]$ always identifies with a localization of $\mathcal{D}B$.

Now for each $s \in S$, let $L(s)$ be the complex

$$\ldots \to 0 \to B \stackrel{\lambda(s)}{\to} B \to 0 \to \ldots$$

concentrated in (cochain) degrees 0 and 1, where $\lambda(s)$ denotes left multiplication by s. For $s,t \in S$, put $A_{s,t} = \mathcal{H}om_B\,(L(t), L(s))$ and let

$$A = \bigoplus_{s,t \in S} A_{s,t}.$$

The composition of graded maps makes A into a differential graded algebra (without unity), which identifies with a subalgebra of

$$\mathcal{H}om_B\,(\bigoplus_{t \in S} L(t), \bigoplus_{s \in S} L(s)).$$

In particular, the module $X = \bigoplus_{t \in S} L(t)$ has a natural structure of DG A-B-bimodule.

Proposition.

a) The sequence

$$0 \to \mathcal{D}A \stackrel{T_X}{\to} \mathcal{D}B \stackrel{T_{B[S^{-1}]}}{\to} \mathcal{D}B[S^{-1}] \to 0$$

is exact. There is a canonical triangle in the mixed derived category

$$C(A) \stackrel{C(X)}{\to} C(B) \stackrel{C(B[S^{-1}])}{\to} C(B[S^{-1}]) \to SC(A).$$

b) If $S' \subset S$ is a subset such that each $s \in S$ is product of elements of S' and if $A' = \bigoplus_{s,t \in S'} A_{s,t}$ and $X' = \bigoplus_{t \in S'} L(t)$, then statement a) also holds for A' and X' instead of A and X.

Remark. Of course, the DG algebra A is unique only up to derived equivalence. In more particular situations, there will be DG algebras with more concrete descriptions which are derived equivalent to A. We give two examples of this.

Examples. a) Let k be a field of characteristic zero, $B = k[x]$ and $S = \{1, x, x^2, ...\}$. Then of course $B[S^{-1}] = k[x, x^{-1}]$. There are (very) many DG algebras A such that one has an exact sequence

$$0 \to \mathcal{D}A \to \mathcal{D}k[x] \to \mathcal{D}k[x, x^{-1}] \to 0.$$

The most natural choice for A is probably the algebra whose underlying complex is

$$\ldots \to 0 \to k.1 \to k.\xi \to 0 \to \ldots$$

where 1 is in (cohomological) degree 0, ξ is in degree 1, the differential and the multiplication of A vanish. There is a DG A-B-bimodule X whose restriction to B is homotopy equivalent to a projective resolution

$$P = (0 \to k[x] \xrightarrow{x} k[x] \to 0)$$

of the trivial $k[x]$-module k concentrated in degree 0. Moreover, the action of $\xi \in A$ corresponds to a generator of $\mathrm{Ext}^1_{k[x]}(k, k)$. The derived functor associated with X yields a fully faithful embedding of $\mathcal{D}A$ into $\mathcal{D}B$ whose image is the triangulated subcategory with infinite sums generated by the trivial $k[x]$-module k. In fact, this subcategory is the kernel of $\mathcal{D}k[x] \to \mathcal{D}k[x, x^{-1}]$.

If one forgets the grading, A is just the algebra of dual numbers. By modification of the degrees one gets that $\mathrm{HC}_*(A) = k[u] \oplus k[w]$, as a graded k-module, where u is of (homological) degree 2 and the second factor $k[w]$ is concentrated in (homological) degree -1. In the associated sequence

$$\mathrm{HC}_*(A) \to \mathrm{HC}_*(k[x]) \to \mathrm{HC}_*(k[x, x^{-1}]) \to \mathrm{HC}_{*-1}(A)$$

the first arrow vanishes and the sequence splits.

b) Suppose that B is an ordinary commutative algebra. Let Y be the closed subset of $X = \mathrm{Spec}\,(B)$ defined by an ideal generated by a finite family $f = (f_1, \ldots, f_n)$ of elements of B and let U be the complement of Y. Let S be the multiplicative system generated by $f_1, \ldots, f_n$. Then $\mathcal{D}B$ and $\mathcal{D}(B_S)$ identify with the (unbounded) derived categories $\mathcal{D}X$ resp. $\mathcal{D}U$ of quasi-coherent sheaves on X resp. U (cf. [4]). For any $m > 0$, the kernel $\mathcal{U}$ of the quotient functor is generated [4] by the Koszul complex

$$L\,(f^m) = \bigotimes_{i=1}^{n} L\,(f_i^m).$$

We put

$$X_m = L\,(f^m) \qquad \text{and} \qquad A_m = \mathcal{H}om_B\,(L_m, L_m) = L_m \otimes_B L_m^*.$$

Then we have an exact sequence of the required type. Note that for each m we have an isomorphism $C(A_m) \to C(A_{m+1})$ induced by the bimodule $\mathcal{H}om_B\,(L_{m+1}, L_m)$.

Proof. a) Let $\mathcal{U} \subset \mathcal{D}B$ be the kernel of the functor

$$L = \mathbf{L}(? \otimes_B B[S^{-1}]) : \mathcal{D}B \to \mathcal{D}B[S^{-1}].$$

We will first show that L induces an equivalence from $\mathcal{D}B/\mathcal{U}$ onto $\mathcal{D}B[S^{-1}]$. For this let $R : \mathcal{D}B[S^{-1}] \to \mathcal{D}B$ be the restriction functor. We may view R as the right derived functor of the (exact) restriction functor at the module level. This latter functor is right adjoint to $? \otimes_B B[S^{-1}]$. Thus, R is right adjoint to L (cf. for example [35]). To prove that we have the equivalence $\mathcal{D}B/\mathcal{U} \to \mathcal{D}B[S^{-1}]$ it is therefore enough to show that R is fully faithful (8.1 a). In other words, we have to show that the adjunction morphism $LRM \to M$ is invertible for each $M \in \mathcal{D}B[S^{-1}]$. Since both, L and R, commute with infinite sums, it is enough to check this for the generator $B[S^{-1}]$ of $\mathcal{D}B[S^{-1}]$. In this case, the adjunction morphism is the canonical morphism

$$B[S^{-1}] \otimes_B^{\mathbf{L}} B[S^{-1}] \to B[S^{-1}].$$

It is invertible because $B[S^{-1}]$ is flat as a right B-module. Thus the sequence

$$0 \to \mathcal{U} \to \mathcal{D}B \overset{T_{B[S^{-1}]}}{\to} \mathcal{D}B[S^{-1}] \to 0$$

is exact. We will now show that T_X induces an equivalence $\mathcal{D}A \xrightarrow{\sim} \mathcal{U}$. By [31, 4.1], it is enough to show that the $L(s)$, $s \in S$, are closed and small and generate $\mathcal{U}$. Now, up to a shift, $L(s)$ is the mapping cone over $\lambda(s) : B \to B$ viewed as a morphism between B-modules concentrated in degree 0. So $L(s)$ is closed and small. Clearly, $L(s)$ belongs to $\mathcal{U}$ for each $s \in S$. Let $\mathcal{U}' \subset \mathcal{U}$ be the smallest triangulated subcategory of $\mathcal{U}$ containing the $L(s)$ and closed under forming infinite sums. We claim that $\mathcal{U}'$ contains the complex

$$L_\infty = (\ldots \to 0 \to B \to B[S^{-1}] \to 0 \to \ldots).$$

Indeed, for each $s \in S$, we have a canonical morphism $L(s) \to L_\infty$ given by the diagram

$$\begin{array}{ccc} B & \overset{\lambda(s)}{\to} & B \\ \| & & \downarrow \lambda(s^{-1}) \\ B & \to & B[S^{-1}] \end{array}$$

and these morphisms yield an isomorphism between L_∞ and the direct limit of the $L(s)$, $s \in S$. It follows that L_∞ belongs to $\mathcal{U}'$ by 8.2. On the other hand, L_∞ generates $\mathcal{U}$ by lemma 8.1 b) since B generates $\mathcal{D}B$.

b) By the octahedral axiom, the triangulated subcategory generated by the $L(s)$, $s \in S'$, contains all the $L(s), s \in S$. Now the claim follows from the proof of a).

4.2 Analytic isomorphisms. Keep the assumptions on k from section 3.1. Let B_1 and B_2 be two flat k-algebras and $S_1 \subset B_1$, $S_2 \subset B_2$ two sets of left denominators. Let A_1 and A_2 be the associated DG algebras constructed as in section 4.1 so that we have an exact sequence

$$0 \to \mathcal{D}A_1 \to \mathcal{D}B_1 \to \mathcal{D}B_1[S_1^{-1}] \to 0\,,$$

and similarly for B_2. Suppose that $f : B_1 \to B_2$ is an algebra homomorphism such that

a) $f(S_1) = S_2$ and

b) for each $s \in S_1$, the map f yields a quasi-isomorphism

$$\begin{array}{ccc} 0 \to B_1 & \overset{\lambda(s)}{\to} & B_1 \to 0 \\ \downarrow & & \downarrow \\ 0 \to B_2 & \overset{\lambda(s)}{\to} & B_2 \to 0 \end{array}$$

These conditions hold if f is an analytic isomorphism along S in the sense of Weibel-Yao [55]. Indeed, in this case, condition c) of the following lemma is satisfied by proposition 5.1 of [loc. cit.].

Lemma. *Condition b) is equivalent to the following condition*
c) f induces a quasi-isomorphism

$$\begin{array}{ccc} 0 \to B_1 & \to & B_1[S^{-1}] \to 0 \\ \downarrow & & \downarrow \\ 0 \to B_2 & \to & B_2[S^{-1}] \to 0. \end{array}$$

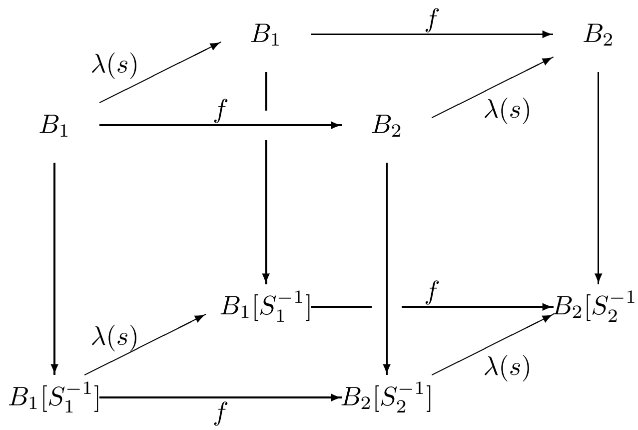
Proof. The complex

$$0 \to B_i \to B_i[S_i^{-1}] \to 0$$

is the filtered direct limit of the complexes

$$0 \to B_i \overset{\lambda(s)}{\to} B_i \to 0$$

in the category of complexes of B_i-modules (4.1). Thus condition b) implies c). To prove the converse, consider the cube



We want to show that the (total complex associated with the) top face is acyclic. Since the bottom face is contractile, it is enough to show that the whole cube is acyclic. This is clear since the front and the back face are acyclic by assumption.

4.3 Excision. Keep the notations and assumptions of section 4.2.

Lemma.

a) *The morphism f induces a quasi-isomorphism $A_1 \to A_2$. In particular, we have an isomorphism $C(A_1) \to C(A_2)$ in $\mathcal{DM}ix$.*

b) *There is a canonical 'Mayer-Vietoris triangle' in $\mathcal{DM}ix$*

$$C(B_1) \to C(B_2) \oplus C(B_1[S_1^{-1}]) \to C(B_2[S_2^{-1}]) \to SC(B_1).$$

Proof. The complex $(A_1)_{s,t}$ is isomorphic to the total complex associated with the square

$$\begin{array}{ccc} A_1 & \stackrel{\lambda(t)}{\to} & A_1 \\ \rho(s)\downarrow & & \downarrow\rho(s) \\ A_1 & \stackrel{\lambda(t)}{\to} & A_1 \end{array}$$

where $\rho(s)$ denotes right multiplication by s. By regarding the rows, we see that condition b) implies that f induces a quasi-isomorphism $(A_1)_{s,t} \to (A_2)_{s,t}$.

To prove b), we note that we have a morphism of triangles in $\mathcal{DM}ix$

$$\begin{array}{ccccccc} C(A_1) & \to & C(B_1) & \to & C(B_1[S_1^{-1}]) & \to & SC(A_1) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ C(A_2) & \to & C(B_2) & \to & C(B_2[S_2^{-1}]) & \to & SC(A_2) \end{array}$$

by proposition 4.1 and theorem 3.1. By a), the morphism $C(A_1) \to C(A_2)$ is invertible. The sequence appears as the Mayer-Vietoris sequence associated (cf. [2, 1.1.13]) with the octahedron over the composition

$$C(A_1) \to C(B_1) \to C(B_2).$$

5. Model categories

5.1 Motivation. To prove theorem 3.1, we introduce 'model categories', which are a slight generalization of categories of DG modules. We then prove the corresponding theorem for model categories.

5.2 Definitions. Let $\mathcal{T}$ be a triangulated category *with infinite sums*, i.e. for each family $(X_i)_{i\in I}$ of $\mathcal{T}$, the coproduct $\bigoplus_{i\in I} X_i$ exists in $\mathcal{T}$. It is then easy to check [30, 6.7] that the coproduct underlies a canonical triangle functor $\bigoplus : \prod_{i\in I} \mathcal{T}_i \to \mathcal{T}$, where $\mathcal{T}_i = \mathcal{T}$ for all i.

A *localizing subcategory* of $\mathcal{T}$ is a full triangulated subcategory $\mathcal{U}$ of $\mathcal{T}$ which is closed under forming infinite direct sums with respect to $\mathcal{T}$. A *set of generators for* $\mathcal{T}$ is a set of objects $\mathcal{X} \subset \mathcal{T}$ such that $\mathcal{T}$ coincides with its smallest localizing subcategory containing $\mathcal{X}$. An object $X \in \mathcal{T}$ is *small* if the functor $\mathrm{Hom}_{\mathcal{T}}(X, ?)$ commutes with infinite sums.

Let $\mathcal{E}$ be an exact category in the sense of Quillen [46]. We use the following terminology due to Gabriel-Roiter [13, §9]: admissible short exact sequence = conflation; admissible monomorphism = inflation; admissible epimorphism = deflation. We refer to [29, App. A] for a proof that Quillen's 'obscure axiom' is redundant and that each exact category fully and fully exactly embeds into an abelian category.

The morphism spaces of $\mathcal{E}$ will be denoted by $\mathcal{E}(X,Y)$ or $\mathrm{Hom}_{\mathcal{E}}(X,Y)$. Suppose that $\mathcal{E}$ is endowed with the following additional structure

S1 For all $X, Y \in \mathcal{E}$ we are given a DG k-module $\mathcal{H}om_{\mathcal{E}}(X,Y)$ such that the pair $(\mathrm{obj}\,\mathcal{E},\ \mathcal{H}om_{\mathcal{E}}(\,,\,))$ is a DG category.

S2 There is given a functorial morphism

$$\mathrm{Hom}_{\mathcal{E}}(X,Y) \to \mathcal{H}om_{\mathcal{E}}(X,Y)\,,\ X, Y \in \mathcal{E}\,,$$

which makes the identity into a DG functor from the exact category $\mathcal{E}$ viewed as a DG category concentrated in degree 0 to the DG category $(\mathrm{obj}\,\mathcal{E}\,,\ \mathcal{H}om_{\mathcal{E}}(\,,\,))$.

We assume that $\mathcal{E}$ is a *model category*, i.e. the following hold

P1 $\mathcal{E}$ is a Frobenius category [17] with infinite direct sums. The associated stable category $\underline{\mathcal{E}}$ admits a set of small generators.

P2 If $X \to Y \to Z$ is a conflation of $\mathcal{E}$, the sequences

$$\begin{array}{ccc} 0 \to \mathcal{H}om_{\mathcal{E}}(?,X) \to & \mathcal{H}om_{\mathcal{E}}(?,Y) & \to \mathcal{H}om_{\mathcal{E}}(?,Z) \to 0 \\ 0 \to \mathcal{H}om_{\mathcal{E}}(Z,?) \to & \mathcal{H}om_{\mathcal{E}}(Y,?) & \to \mathcal{H}om_{\mathcal{E}}(X,?) \to 0 \end{array}$$

split in the category of graded $\mathcal{E}$-modules (left resp. right modules with respect to the DG structure). Moreover, if I is projective-injective in $\mathcal{E}$, the DG $\mathcal{E}$-modules $\mathcal{H}om_{\mathcal{E}}(?,I)$ and $\mathcal{H}om_{\mathcal{E}}(I,?)$ are contractile.

P3 The canonical morphism (which is well defined by P2)

$$\underline{\mathcal{E}}(X,Y) \to \mathrm{H}^0\mathcal{H}om_{\mathcal{E}}(X,Y)$$

is invertible if X is small in $\underline{\mathcal{E}}$ and $Y \in \mathcal{E}$ arbitrary.

P4 For all $X, Y \in \underline{\mathcal{E}}$, the functor $? \otimes \mathcal{H}om_{\mathcal{E}}(X,Y)$ preserves acyclicity of DG k-modules.

If $\mathcal{E}'$ is another model category, a *model functor* $F : \mathcal{E} \to \mathcal{E}'$ is a pair consisting of an exact functor preserving projectivity and a DG functor such that the square

$$\begin{array}{rcl} \mathcal{E}(X,Y) & \longrightarrow & \mathcal{H}om_{\mathcal{E}}(X,Y) \\ F\downarrow & & \downarrow F \\ \mathcal{E}'(FX,FY) & \longrightarrow & \mathcal{H}om_{\mathcal{E}'}(FX,FY) \end{array}$$

commutes.

Denote by $\mathcal{E}^b$ the full subcategory of $\mathcal{E}$ whose objects are the ones whose images in $\underline{\mathcal{E}}$ are small, and by $\mathcal{S}$ a *stable skeleton* for $\mathcal{E}^b$, i.e. a small full subcategory of $\mathcal{E}^b$ whose image in $\underline{\mathcal{E}}$ is dense in the subcategory of small objects of $\underline{\mathcal{E}}$. By P1, a stable skeleton exists.

Lemma. *The functor*

$$\underline{\mathcal{E}} \to \mathcal{H}_p\mathcal{S}\ ,\ X \mapsto \mathcal{H}om_{\mathcal{E}}\,(?,X)|\mathcal{S}$$

is a triangle equivalence.

Proof. Note first that by P2, we have a well defined triangle functor $F : \underline{\mathcal{E}} \to \mathcal{HS}$ mapping X to $\mathcal{H}om_{\mathcal{E}}\,(?,X)|\mathcal{S}$. We claim that F commutes with direct sums. Let $(Y_i)_{i\in I}$ be a family of objects of $\mathcal{E}$. We have to check that

$$\mathrm{H}^n \bigoplus_{i\in I} \mathcal{H}om_{\mathcal{E}}\,(X,Y_i) \longrightarrow \mathrm{H}^n\mathcal{H}om_{\mathcal{E}}\,(X, \bigoplus_{i\in I} Y_i)$$

is bijective for each $n \in \mathbf{Z}$ and each $X \in \mathcal{S}$. By P1 and P2, we have canonical isomorphisms

$$\mathrm{H}^n\mathcal{H}om_{\mathcal{E}}\,(U,V) \xrightarrow{\sim} \mathrm{H}^{n-1}\mathcal{H}om_{\mathcal{E}}\,(U,SV)\ ,\ U,V \in \mathcal{E}\ ,$$

so that it is enough to consider the case $n = 0$. Since X is small in $\underline{\mathcal{E}}$, the claim then follows from the commutative diagram

$$\begin{array}{ccc} \mathrm{H}^0\mathcal{H}om_{\mathcal{E}}\,(X,\bigoplus Y_i) & \xleftarrow{\sim} & \underline{\mathcal{E}}\,(X,\bigoplus Y_i) \\ \uparrow & & \uparrow\sim \\ \bigoplus \mathrm{H}^0\mathcal{H}om_{\mathcal{E}}\,(X,Y_i) & \xleftarrow{\sim} & \bigoplus \underline{\mathcal{E}}\,(X,Y_i). \end{array}$$

By definition, F maps the $X \in \mathcal{S}$ to free $\mathcal{S}$-modules and it follows from $P2$ and $P3$ that F induces bijections

$$\underline{\mathcal{E}}\,(X,S^nY) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{HS}}(FX,S^nFY)\ ,\ X,Y \in \mathcal{S}.$$

By 'infinite devissage' (cf. [31, 4.2 b]), it follows that F is fully faithful. Since the $X \in \mathcal{S}$ generate $\underline{\mathcal{E}}$ and since they are mapped to free modules, the image of $\underline{\mathcal{E}}$ under F is contained in $\mathcal{H}_p\mathcal{S}$.

Remarks. a) Suppose that k is coherent of finite global dimension and let $\mathcal{A}$ be a small DG category such that $\mathcal{A}\,(A,B)$ is a flat DG k module for all $A,B \in \mathcal{A}$. Let $\mathcal{C}_p\mathcal{A}$ denote the preimage of $\mathcal{H}_p\mathcal{A}$ in $\mathcal{CA}$. Then $\mathcal{E} = \mathcal{C}_p\mathcal{A}$ is a model category in the obvious way (cf. sections 1 and 2 of [31] and lemma 8.3). In particular, each DG algebra A which is flat as a DG k-module gives rise to the model category $\mathcal{C}_pA$.

b) It was proved in [31, 4.3] that if $\mathcal{E}_0$ is an exact category with property P1, there is always a triangle equivalence $\underline{\mathcal{E}_0} \xrightarrow{\sim} \underline{\mathcal{E}}$ where $\mathcal{E}$ is a model category.

5.3 Filtered objects. Let $\mathcal{E}$ be a model category. We define $\mathrm{Fil}\,(\mathcal{E})$, the *category of filtered objects*, to be the category whose *objects* are the inflations

$$X_0 \xrightarrow{i} X_1$$

of $\mathcal{E}$, and whose *morphisms* $f : X \to X'$ are commutative diagrams of $\mathcal{F}$

$$\begin{array}{ccc} X_0 & \xrightarrow{i} & X_1 \\ f_0 \downarrow & & \downarrow f_1 \\ X_0' & \xrightarrow{i'} & X_1'. \end{array}$$

By definition, a sequence $X \xrightarrow{i} Y \xrightarrow{p} Z$ is a *conflation* of $\mathrm{Fil}\,(\mathcal{E})$ if each of its components (i_0,p_0) and (i_1,p_1) is a conflation of $\mathcal{E}$. For $X,X' \in \mathrm{Fil}\,(\mathcal{E})$, we define $\mathcal{H}om_{\mathrm{Fil}\,(\mathcal{E})}\,(X,X')$ to be the DG submodule of

$$\mathcal{H}om_{\mathcal{E}}\,(X_0,X_0') \oplus \mathcal{H}om_{\mathcal{E}}\,(X_1,X_1')$$

consisting of the (u_0,u_1) with $i'\,u_0 = u_1\,i$.

Lemma. $\mathrm{Fil}\,(\mathcal{E})$ *is a model category.*

Remark. By definition, the *category of cofiltered objects* $\mathrm{Cof}(\mathcal{E})$ is the category of deflations $X_1 \xrightarrow{p} X_2$ of $\mathcal{E}$ endowed with the analogous exact structure and the DG structure such that $\mathcal{H}om_{\mathrm{Cof}(\mathcal{E})}(X, X')$ is formed by the (u_1, u_2) in

$$\mathcal{H}om_{\mathrm{Cof}(\mathcal{E})}(X_1, X_1') \oplus \mathcal{H}om_{\mathrm{Cof}(\mathcal{E})}(X_2, X_2')$$

such that $u_2\, p = p'\, u_1$. We have a model functor

$$\mathrm{Fil}(\mathcal{E}) \to \mathrm{Cof}(\mathcal{E}),\ (X_0 \xrightarrow{i} X_1) \mapsto (X_1 \to \mathrm{Cok}\, i)$$

which is an equivalence both of exact categories and of DG categories. Note that $\mathrm{Cof}(\mathcal{E}) \neq \mathrm{Fil}(\mathcal{E}^{\mathrm{op}})$ and that $\mathcal{E}^{\mathrm{op}}$ is not even a model category (the notion of small object is not self-dual).

Proof. It is easy to see that $\mathrm{Fil}(\mathcal{E})$ is a Frobenius category (cf. [29, section 5]) and that an object of $\mathrm{Fil}(\mathcal{E})$ is projective-injective iff its components are projective-injective. It is clear that $\mathrm{Fil}(\mathcal{E})$ has infinite direct sums. We claim that it is generated by the objects $(X \xrightarrow{1} X)$ and $(0 \to X)$, where X ranges over the small objects of $\underline{\mathcal{E}}$. These objects are small since we have

$$\begin{array}{rcl} \underline{\mathrm{Fil}(\mathcal{E})}((X \xrightarrow{1} X), Y) & \xrightarrow{\sim} & \underline{\mathcal{E}}(X, Y_0) \\ \underline{\mathrm{Fil}(\mathcal{E})}((0 \to X), Y) & \xrightarrow{\sim} & \underline{\mathcal{E}}(X, Y_1). \end{array}$$

Clearly, the localizing subcategory they generate contains all objects $(Y \xrightarrow{1} Y)$ and $(0 \to Y)$, $Y \in \mathcal{E}$. The claim follows since for each $Y \in \mathrm{Fil}(\mathcal{E})$ we have the conflation

$$\begin{array}{ccccc} Y_0 & \xrightarrow{1} & Y_0 & \to & 0 \\ \| & & \downarrow i & & \downarrow \\ Y_0 & \xrightarrow{j} & Y_1 & \to & \mathrm{Cok}\, j. \end{array}$$

In the sequel, we will write

$$Y_\lambda \to Y \to Y^\rho$$

for this conflation.

To prove P2, let $X \to Y \to Z$ be a conflation of $\mathrm{Fil}(\mathcal{E})$. We form the diagram

$$\begin{array}{ccccc} X_\lambda & \to & Y_\lambda & \to & Z_\lambda \\ \downarrow & & \downarrow & & \downarrow \\ X & \to & Y & \to & Z \\ \downarrow & & \downarrow & & \downarrow \\ X^\rho & \to & Y^\rho & \to & Z^\rho \end{array}$$

whose rows and columns are conflations of $\mathrm{Fil}(\mathcal{E})$. We have to show that the middle row of the following diagram is split exact as a sequence of graded $\mathrm{Fil}(\mathcal{E})$-modules.

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 \to & & \mathcal{H}om_{\mathrm{Fil}(\mathcal{E})}(?, X_\lambda) & \to & \mathcal{H}om_{\mathrm{Fil}(\mathcal{E})}(?, Y_\lambda) & \to & \mathcal{H}om_{\mathrm{Fil}(\mathcal{E})}(?, Z_\lambda) & \to 0 & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 \to & & \mathcal{H}om_{\mathrm{Fil}(\mathcal{E})}(?, X) & \to & \mathcal{H}om_{\mathrm{Fil}(\mathcal{E})}(?, Y) & \to & \mathcal{H}om_{\mathrm{Fil}(\mathcal{E})}(?, Z) & \to 0 & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 \to & & \mathcal{H}om_{\mathrm{Fil}(\mathcal{E})}(?, X^\rho) & \to & \mathcal{H}om_{\mathrm{Fil}(\mathcal{E})}(?, Y^\rho) & \to & \mathcal{H}om_{\mathrm{Fil}(\mathcal{E})}(?, Z^\rho) & \to 0 & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

It will be enough to show that this holds for the top row, the bottom row, and all the columns. For $U = (U_0 \xrightarrow{u} U_1) \in \mathrm{Fil}(\mathcal{E})$, we have

$$\mathcal{H}om_{\mathrm{Fil}(\mathcal{E})}(U, X_\lambda) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{E}}(U_1, X_0) \text{ and } \mathcal{H}om_{\mathrm{Fil}(\mathcal{E})}(U, X_\rho) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{E}}(\mathrm{Cok}\, u, X_1)$$

so that it is clear that the top row and the bottom row split. Let us show that the columns split. To do this for the first column, we choose a splitting ρ of the morphism

$$\mathcal{H}om_{\mathcal{E}}(?, X_0) \to \mathcal{H}om_{\mathcal{E}}(?, X_1).$$

Then it is easy to check that

$$(u_0, u_1) \mapsto (u_0, \rho(u_1))$$

defines a splitting of

$$\mathcal{H}om_{\mathrm{Fil}(\mathcal{E})}(U, X_\lambda) \to \mathcal{H}om_{\mathrm{Fil}(\mathcal{E})}(U, X)$$

(which of course is functorial in $U \in \mathrm{Fil}(\mathcal{E})$). The assertion for the sequence

$$0 \to \mathcal{H}om_{\mathrm{Fil}(\mathcal{E})}(Z, ?) \to \mathcal{H}om_{\mathrm{Fil}(\mathcal{E})}(Y, ?) \to \mathcal{H}om_{\mathrm{Fil}(\mathcal{E})}(X, ?) \to 0$$

is proved similarly.

If I is projective-injective in $\mathrm{Fil}(\mathcal{E})$ then it is a direct sum of two objects of the form $(I_0 \xrightarrow{\sim} I_1)$ and $(0 \to I_1)$. Now we have for $X = (X_0 \xrightarrow{i} X_1)$

$$\begin{array}{lcl} \mathcal{H}om_{\mathrm{Fil}(\mathcal{E})}((I_0 \xrightarrow{\sim} I_1), X) & \xrightarrow{\sim} & \mathcal{H}om_{\mathcal{E}}(I_0, X_0) \\ \mathcal{H}om_{\mathrm{Fil}(\mathcal{E})}((0 \to I_1), X) & \xrightarrow{\sim} & \mathcal{H}om_{\mathcal{E}}(I_1, X_1) \\ \mathcal{H}om_{\mathrm{Fil}(\mathcal{E})}(X, I_0 \xrightarrow{\sim} I_1)) & \xrightarrow{\sim} & \mathcal{H}om_{\mathcal{E}}(X_1, I_1) \\ \mathcal{H}om_{\mathrm{Fil}(\mathcal{E})}(X, 0 \to I_1) & \xrightarrow{\sim} & \mathcal{H}om_{\mathcal{E}}(\mathrm{Cok}\, i, I_1) \end{array}$$

so that the second condition of P2 is now clear.

It is enough to prove $P3$ for X of the form $(X_0 \xrightarrow{\sim} X_1)$ resp. $(0 \to X_1)$. In the first case, we have a commutative square

$$\begin{array}{rcl} \underline{\mathrm{Fil}(\mathcal{E})}(X, Y) & \to & \mathrm{H}^0\mathcal{H}om_{\mathrm{Fil}(\mathcal{E})}(X, Y) \\ \sim\downarrow & & \downarrow\sim \\ \underline{\mathcal{E}}(X_0, Y_0) & \xrightarrow{\sim} & \mathrm{H}^0\mathcal{H}om_{\mathcal{E}}(X_0, Y_0) \end{array}$$

and in the second case, we have a commutative square

$$\begin{array}{rcl} \underline{\mathrm{Fil}(\mathcal{E})}(X, Y) & \to & \mathrm{H}^0\mathcal{H}om_{\mathrm{Fil}(\mathcal{E})}(X, Y) \\ \sim\downarrow & & \downarrow\sim \\ \underline{\mathcal{E}}(X_1, Y_1) & \xrightarrow{\sim} & \mathrm{H}^0\mathcal{H}om_{\mathcal{E}}(X_1, Y_1). \end{array}$$

Finally, to prove P4, we may assume that X is of the form $(X_0 \xrightarrow{\sim} X_1)$ or $(0 \to X_1)$ and similarly for Y. The assertion then easily follows from the corresponding statement for $\mathcal{E}$.

5.4 Hochschild and Cyclic homology of model categories. We leave it to the reader to generalize the definitions and results of sections 1 and 2 from DG algebras to small DG categories. In the sequel, we will assume this generalization has been carried out. For example, the *precyclic chain complex* $C(\mathcal{A})$ associated with a small DG category $\mathcal{A}$ has the components $C(\mathcal{A})_n$ given by

$$\bigoplus \mathcal{H}om_{\mathcal{A}}(A_{n-1}, A_n) \otimes \mathcal{H}om_{\mathcal{A}}(A_{n-2}, A_{n-1}) \otimes \ldots \otimes \mathcal{H}om_{\mathcal{A}}(A_1, A_2) \otimes \mathcal{H}om_{\mathcal{A}}(A_n, A_1)$$

where the sum ranges over all sequences $A_1, A_2, \ldots, A_n$ of objects of $\mathcal{A}$. Note that this sum is well defined since $\mathcal{A}$ is small. The cyclic operator and the degeneracy operators are given by the usual formulae. The homology of the total complex is the *Hochschild-Mitchell homology* of $\mathcal{A}$ (cf. [44]).

Let $\mathcal{E}$ be a model category. For each stable skeleton $\mathcal{S}$ of $\mathcal{E}^b$ we have a precyclic chain complex $C(\mathcal{S})$ (defined using the DG structure of $\mathcal{E}$) and if $\mathcal{S} \subset \mathcal{S}'$ are two skeleta, we have a canonical morphism $C(\mathcal{S}) \to C(\mathcal{S}')$. Lemma 1.2 shows that it is invertible in $\mathcal{DM}ix$. In particular, if $\mathcal{S}$ and $\mathcal{S}'$ are two stable skeleta, we always have a canonical isomorphism $C(\mathcal{S}) \to C(\mathcal{S}')$ of $\mathcal{DM}ix$ defined by the commutative diagram

$$\begin{array}{rcl} C(\mathcal{S}) & \xrightarrow{\sim} & C(\mathcal{S} \cup \mathcal{S}') \\ \downarrow & & \| \\ C(\mathcal{S}') & \xrightarrow{\sim} & C(\mathcal{S} \cup \mathcal{S}') \end{array}$$

So if we define $C(\mathcal{E}^b) := C(\mathcal{S})$ for some fixed stable skeleton $\mathcal{S}$ of $\mathcal{E}^b$, we obtain a precyclic object well defined up to canonical isomorphism in $\mathcal{DM}ix$.

It is important to note that this construction is different from the one used by R. McCarthy in [42] to define 'naive' cyclic homology HC^s_*. Indeed, our construction entirely relies on the the differential graded structure of $\mathcal{E}$, which is not present in the categories considered by R. McCarthy.

Let $F : \mathcal{E} \to \mathcal{E}'$ be a model functor preserving smallness of objects, i.e. taking $\mathcal{E}^b$ to $\mathcal{E}'^b$. We complete the image of a stable skeleton $\mathcal{S}$ of $\mathcal{E}^b$ under F to a stable skeleton $\mathcal{S}'$ of $\mathcal{E}'^b$. Then F yields a morphism of precyclic objects $C(\mathcal{S}) \to C(\mathcal{S}')$. Thus, by composition with the canonical isomorphism, F yields a well defined morphism $C(F) : C(\mathcal{E}^b) \to C(\mathcal{E}'^b)$ of $\mathcal{DM}ix$. One easily checks that this yields a functor from model categories to $\mathcal{DM}ix$.

Suppose that $F : \mathcal{E} \to \mathcal{E}'$ is a model functor inducing an equivalence $\underline{\mathcal{E}} \to \underline{\mathcal{E}'}$. Then by lemma 5.2, the morphism

$$\mathcal{H}om_{\mathcal{E}}(X, Y) \to \mathcal{H}om_{\mathcal{E}'}(FX, FY)$$

is a quasi-isomorphism for all $X, Y \in \mathcal{E}^b$. By property P4, this implies that if $\mathcal{S}$ is some stable skeleton of $\mathcal{E}^b$, the induced morphism of precyclic chain complexes

$$C(\mathcal{S}) \to C(F\mathcal{S})$$

is a quasi-isomorphism, hence that

$$C(F) : C(\mathcal{E}^b) \to C(\mathcal{E}'^b)$$

is invertible in $\mathcal{DM}ix$.

Let $F_1, F_2 : \mathcal{E} \to \mathcal{E}'$ be two model functors and $\varphi : F_1 \to F_2$ a morphism of the underlying functors between exact categories such that

$$\varphi X : F_1 X \to F_2 X$$

is a deflation which becomes invertible in $\underline{\mathcal{E}'}$ for each $X \in \mathcal{E}$.

Lemma. *We have* $C(F_1) = C(F_2)$ *in* $\mathcal{DM}ix$.

Proof. Let $\mathcal{E}'_+ \subset \mathrm{Cof}(\mathcal{E}')$ be the full subcategory on the deflations $X = (X_1 \to X_2)$ which become invertible in $\underline{\mathcal{E}'}$ (equivalently, which have a projective-injective kernel). It is easy to see that $\mathcal{E}'_+$ with the structure inherited from $\mathrm{Cof}(\mathcal{E}')$ becomes a model category. Let P_1, P_2 and F be the model functors

$$\begin{array}{rcl} P_1 : \mathcal{E}'_+ \to \mathcal{E}' & , & X \mapsto X_1 \\ P_2 : \mathcal{E}'_+ \to \mathcal{E}' & , & X \mapsto X_2 \\ F : \mathcal{E} \to \mathcal{E}'_+ & , & X \mapsto (F_1 X \stackrel{\varphi X}{\to} F_2 X). \end{array}$$

Clearly, $P_1 F = F_1$ and $P_2 F = F_2$. Since $C(?)$ is a functor, it suffices to prove that $C(P_1) = C(P_2)$ in $\mathcal{DM}ix$. Now let D be the model functor

$$D : \mathcal{E} \to \mathcal{E}' \, , \; X \mapsto (X \stackrel{1}{\to} X).$$

It is easy to see that D induces an equivalence $\underline{\mathcal{E}} \to \underline{\mathcal{E}'}$. Thus $C(D)$ is invertible in $\mathcal{DM}ix$. Since $P_1 D = P_2 D$ (both are the identity), we can conclude that $C(P_1) = C(P_2)$.

5.5 DG algebras. Suppose that k is coherent of finite global dimension and let A be a DG algebra which is flat as a DG k module. Let $\mathcal{E}$ the model category $\mathcal{C}_p A$ (cf. remark a) in 5.2). If we compute $C(\mathcal{E}^b)$ using a stable skeleton containing A_A, we obtain a morphism $C(A) \to C(\mathcal{E}^b)$ of precyclic modules whose image becomes invertible in $\mathcal{DM}ix$. To see this, we use lemma 1.2 and the fact that $C(?)$ viewed as a functor from DG categories to precyclic modules commutes with direct limits.

If ${}_AX_B : A \to B$ is a morphism of **ALG**, then $? \otimes_A X : \mathcal{C}_p A \to \mathcal{C}_p B$ is a model functor preserving smallness of objects and we have a commutative diagram of $\mathcal{DM}ix$

$$\begin{array}{ccc} C(A) & \xrightarrow{C(X)} & C(B) \\ \downarrow & & \downarrow \\ C((\mathcal{C}_p A)^b) & \xrightarrow{C(?\otimes_A X)} & C((\mathcal{C}_p B)^b) \end{array}$$

whose vertical morphisms are invertible. Thus the functor C defined in this section extends the one of section 2.4. We leave it to the reader to adapt these remarks to the case of DG categories.

5.6 The localization theorem for model categories. Let

$$\mathcal{E} \xrightarrow{F} \mathcal{F} \xrightarrow{G} \mathcal{G}$$

be a sequence of model categories and model functors commuting with direct sums and preserving smallness of objects.

Suppose that $\underline{F} : \underline{\mathcal{E}} \to \underline{\mathcal{F}}$ is fully faithful, that $\underline{G}\,\underline{F} = 0$ and that $\underline{G}$ induces an equivalence $\underline{\mathcal{F}}/\underline{F}(\underline{\mathcal{E}}) \xrightarrow{\sim} \underline{\mathcal{G}}$.

Theorem. *There is a canonical triangle*

$$C(\mathcal{E}^b) \xrightarrow{C(F)} C(\mathcal{F}^b) \xrightarrow{C(G)} C(\mathcal{G}^b) \to SC(\mathcal{E}^b)$$

in the mixed derived category.

By section 5.5, this generalizes theorem 3.1.

6. Proof of the localization theorem for model categories

6.1 Lifting the exact sequence. By assumption, the sequence of stable categories

$$0 \to \underline{\mathcal{E}} \xrightarrow{F} \underline{\mathcal{F}} \xrightarrow{G} \underline{\mathcal{G}} \to 0$$

is exact, but the sequence

$$0 \to \mathcal{E} \xrightarrow{F} \mathcal{F} \xrightarrow{G} \mathcal{G} \to 0$$

need not be exact in any sense. Usually, we will even have $G\,F \neq 0$. We shall therefore replace $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{G}$ by model categories with equivalent stable categories so as to have an 'exact sequence' even before the transition to the associated stable categories. This would not be possible if we were to work with categories of DG modules only.

Let $\mathcal{T} \subset \underline{\mathcal{F}}$ be the essential image of $\underline{F}$ and $\mathcal{T}^{\perp} \subset \underline{\mathcal{F}}$ the full subcategory on the objects $Y \in \underline{\mathcal{F}}$ with $\mathrm{Hom}_{\underline{\mathcal{F}}}(X, Y) = 0$ for all $X \in \mathcal{T}$. Let $\mathcal{F}_1 \subset \mathrm{Fil}\,(\mathcal{F})$ be the category whose objects are the admissible monomorphisms

$$X_0 \xrightarrow{i} X_1$$

of $\mathcal{F}$ such that $X_0 \in \mathcal{T}$ and $\mathrm{Cok}\, i \in \mathcal{T}^{\perp}$. We endow $\mathcal{F}_1$ with the exact structure and the DG structure inherited from $\mathrm{Fil}\,(\mathcal{F})$. We will prove below that $\mathcal{F}_1$ is a model category.

Let $\mathcal{E}_1 \subset \mathcal{F}_1$ be the full subcategory on the objects $X = (X_0 \xrightarrow{i} X_1)$ with invertible i (invertible in $\mathcal{F}$!) and let $\mathcal{G}_1 \subset \mathcal{F}_1$ be the full subcategory on the X with $X_0 = 0$. Both inherit from $\mathcal{F}_1$ the structure of model categories. The inclusion $\mathcal{G}_1 \subset \mathcal{F}_1$ admits the right adjoint $X \mapsto X^{\rho}$ mapping X to $0 \to \mathrm{Cok}\, i$. The functors

$$\begin{array}{ll} \mathcal{F}_1 \to \mathcal{F}\,, & X \mapsto X_1 \\ \mathcal{E} \to \mathcal{E}_1\,, & X \mapsto (FX \xrightarrow{1} FX) \\ \mathcal{G}_1 \to \mathcal{G}\,, & (0 \to X_1) \mapsto GX_1 \end{array}$$

fit into a diagram of model categories and functors

$$\begin{array}{ccccc} \mathcal{E}_1 & \xrightarrow{\text{incl}} & \mathcal{F}_1 & \xrightarrow{\rho} & \mathcal{G}_1 \\ \uparrow & & \downarrow & & \downarrow \\ \mathcal{E} & \xrightarrow{F} & \mathcal{F} & \xrightarrow{G} & \mathcal{G}. \end{array}$$

Here vertical functors induce equivalences in the associated stable categories, the left hand square is commutative, and the right hand square is commutative up to the natural transformation

$$G(X_1) \to G(\operatorname{Cok}(X_0 \to X_1))$$

which is a deflation for each $X \in \mathcal{F}_1$ and becomes invertible in $\underline{\mathcal{G}}$. By lemma 5.4 this diagram yields a truly commutative diagram in $\mathcal{DM}ix$ and we may thus replace the given sequence by the sequence $\mathcal{E}_1 \to \mathcal{F}_1 \to \mathcal{G}_1$.

Lemma. *$\mathcal{F}_1$ is a model category, and the functor $\mathcal{F}_1 \to \mathcal{F}$, $X \mapsto X_1$ induces an equivalence $\underline{\mathcal{F}_1} \xrightarrow{\sim} \underline{\mathcal{F}}$.*

Proof. It is easy to see that $\mathcal{F}_1$ is a Frobenius category. Let us prove that the functor $F : \mathcal{F}_1 \to \mathcal{F}$, $X \mapsto X_1$ induces an equivalence $\underline{\mathcal{F}_1} \xrightarrow{\sim} \underline{\mathcal{F}}$. We claim that F is essentially surjective. By Brown's representability theorem (cf. e.g. [31, 5.2]) the inclusion $\mathcal{T} \to \underline{\mathcal{F}}$ admits a right adjoint. Therefore (cf. [28, 1.1]), for each $M \in \mathcal{T}$, there is a triangle

$$M_{\mathcal{T}} \to M \to M^{\mathcal{T}^{\perp}} \to SM_{\mathcal{T}}$$

of $\underline{\mathcal{F}}$ with $M_{\mathcal{T}} \in \mathcal{T}$ and $M^{\mathcal{T}^{\perp}} \in \mathcal{T}^{\perp}$. Let $f : M_{\mathcal{T}} \to M$ be a preimage in $\mathcal{F}$ of the morphism $M_{\mathcal{T}} \to M$ of $\underline{\mathcal{F}}$. Let $j : M \to I$ be an inflation with injective I. Then $i = [f \;\; j]^t : M_{\mathcal{T}} \to M \oplus I$ is an object of $\mathcal{F}_1$ whose image in $\underline{\mathcal{F}}$ is isomorphic to M. To prove that F is fully faithful, we note that for each $Y \in \mathcal{F}_1$, we have the conflation

$$\begin{array}{ccccc} Y_0 & \xrightarrow{1} & Y_0 & \to & 0 \\ \| & & \downarrow i & & \downarrow \\ Y_0 & \xrightarrow{i} & Y_1 & \to & \operatorname{Cok} i \end{array}$$

By devissage it is therefore enough to check that F induces bijections

$$\underline{\mathcal{F}_1}\,(X, X') \xrightarrow{\sim} \underline{\mathcal{F}}\,(X_1, X_1')$$

when X and X' are of the type $Y_0 \xrightarrow{\sim} Y_1$ or $0 \to Y_1$. For example, if $X = (X_0 \xrightarrow{\sim} X_1)$ and $X' = (0 \to X_1')$ then clearly $\mathcal{F}_1\,(X, X') = 0$. But since $X_1 \xleftarrow{\sim} X_0 \in \mathcal{T}$ and $X_1' \xrightarrow{\sim} \operatorname{Cok}(0 \to X_1') \in \mathcal{T}^{\perp}$, we also have $\underline{\mathcal{F}}\,(X_1, X_1') = 0$ as well. The other three cases are left to the reader.

It is now clear that $\mathcal{F}_1$ satisfies P1. Properties P2, P3, and P4 immediately carry over from $\operatorname{Fil}(\mathcal{F})$ to $\mathcal{F}_1$.

6.2 Plan of the proof for the lifted sequence. We keep the notations of 6.1. We fix stable skeleta $\mathcal{R} \subset \mathcal{E}_1^b$, $\mathcal{S} \subset \mathcal{F}_1^b$, $\mathcal{T} \subset \mathcal{G}_1^b$ with $\mathcal{R} \subset \mathcal{S}$ and $X^{\rho} \in \mathcal{T}$ for each $X \in \mathcal{S}$. If we use these to compute the corresponding precyclic chain complexes, then the inclusion $\mathcal{E}_1 \subset \mathcal{F}_1$ and the functor $\rho : \mathcal{F}_1 \to \mathcal{G}_1$ induce morphisms of precyclic chain complexes

$$C(\mathcal{E}_1^b) \overset{C(\text{incl})}{\to} C(\mathcal{F}_1^b) \overset{C(\rho)}{\to} C(\mathcal{G}_1^b) \tag{1}$$

whose composition is zero, since $X^{\rho} = 0$ for each $X \in \mathcal{E}_1$. By our flatness assumptions, $C(\text{incl})$ is injective. To prove the assertion of theorem 5.6, it will then be enough to show that the induced morphism $\operatorname{Cok} C(\text{incl}) \to C(\mathcal{G}_1^b)$ becomes invertible in $\mathcal{DM}ix$. We will prove this by exhibiting an exact sequence of $\mathcal{S}$-$\mathcal{S}$-bimodules whose image under the relative left derived functor of the tensor product $? \otimes_{\mathcal{S}^e} I$, where $I\,(X, Y) = \mathcal{S}\,(X, Y)$, identifies with the image of the sequence (1) under the totalizing functor. The details of the general argument are given in 7.7.

6.3 The bimodule sequence. For given $Y \in \mathcal{F}_1$, we have the conflation

$$Y_\lambda \to Y \to Y^\rho$$

given by

$$\begin{array}{ccccc} Y_0 & \xrightarrow{1} & Y_0 & \to & 0 \\ \| & & \downarrow i & & \downarrow \\ Y_0 & \xrightarrow{i} & Y_1 & \to & \operatorname{Cok} i. \end{array}$$

If we apply $\mathcal{H}om_{\mathcal{F}_1}(X, ?)$ to this conflation we obtain an exact sequence

$$0 \to \mathcal{H}om_{\mathcal{F}_1}(X, Y_\lambda) \to \mathcal{H}om_{\mathcal{F}_1}(X, Y) \to \mathcal{H}om_{\mathcal{F}_1}(X, Y^\rho) \to 0.$$

Now we let X and Y vary in $\mathcal{S}$, and view the above sequence as a sequence of $\mathcal{S}$-$\mathcal{S}$-bimodules. We will now choose suitable relatively acyclic resolutions of the terms of the sequence.

6.4 Resolution of the second and the third term. For the second term, we take the bar resolution with respect to $\mathcal{S}$.

For the third term, note that we have an isomorphism

$$\mathcal{H}om_{\mathcal{F}_1}(X, Y^\rho) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{G}_1}(X^\rho, Y^\rho).$$

We take the bar resolution with respect to $\mathcal{T}$, and view its terms as $\mathcal{S}$-$\mathcal{S}$-bimodules via the functor ρ. These terms are of the form

$$\bigoplus \mathcal{H}om_{\mathcal{G}_1}(G_n, Y^\rho) \otimes \mathcal{H}om_{\mathcal{G}_1}(G_{n-1}, G_n) \otimes \ldots \otimes \mathcal{H}om_{\mathcal{G}_1}(G_0, G_1) \otimes \mathcal{H}om_{\mathcal{G}_1}(X^\rho, G_0),$$

where the sum runs over all $G_0, \ldots, G_n$ of $\mathcal{T}$. They are relatively acyclic for I by lemma 7.8. Indeed, we have the isomorphism of $\mathcal{S}$-modules

$$\mathcal{H}om_{\mathcal{G}_1}(X^\rho, G_0) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{F}_1}(X, G_0)$$

and here the right hand side is closed as an $\mathcal{S}$-module by lemma 5.2.

6.5 Resolution of the first term We consider the subcomplex of the bar resolution over $\mathcal{S}$ whose terms are the

$$\bigoplus \mathcal{H}om_{\mathcal{E}_1}(E_n, Y_\lambda) \otimes \mathcal{H}om_{\mathcal{E}_1}(E_{n-1}, E_n) \otimes \ldots \otimes \mathcal{H}om_{\mathcal{E}_1}(E_0, E_1) \otimes \mathcal{H}om_{\mathcal{F}_1}(X, E_0),$$

where the $E_0, \ldots, E_n$ run through $\mathcal{R}$. Since $\mathcal{H}om_{\mathcal{E}_1}(X, E_0)$ is free over $\mathcal{S}$, they are relatively acyclic for I by lemma 7.8. If we had $Y_\lambda \in \mathcal{E}_1^b$, we could assume $Y \in \mathcal{R}$, and the sequence would admit a splitting over $\mathcal{R}$. However, in general, Y_λ will not be small in $\underline{\mathcal{E}_1}$ (nor in $\underline{\mathcal{F}_1}$). To prove that the sequence always yields a relative resolution we replace Y_λ by a variable $Z \in \mathcal{E}_1$ and consider the total complex of the augmented sequence

$$\ldots \to \bigoplus \mathcal{H}om_{\mathcal{E}_1}(E_0, Z) \otimes \mathcal{H}om_{\mathcal{F}_1}(X, E_0) \to \mathcal{H}om_{\mathcal{F}_1}(X, Z) \to 0 \to \ldots$$

as a triangle functor of $Z \in \underline{\mathcal{E}_1}$ with values in $\mathcal{H}k$. Since X and the E_n are small, this functor commutes with direct sums. It vanishes for $Z \in \mathcal{R}$. So it vanishes for each $Z \in \underline{\mathcal{E}_1}$ and in particular for Y_λ. This proves the assertion.

6.6 Image under the tensor product We now compute the tensor products over $\mathcal{S}^e$ with I of each of the relative resolutions we constructed.

The image of the bar resolution of the middle term is the Hochschild complex, as is well known.

We would like to show that the image of the resolution of the third and the first term also identify with the corresponding Hochschild complexes.

The terms of the first resolution are sums of objects of the form

$$\mathcal{H}om_{\mathcal{E}_1}(E_n, Y_\lambda) \otimes L \otimes \mathcal{H}om_{\mathcal{F}_1}(X, E_0)$$

where X and Y denote 'variable objects' of $\mathcal{S}$ and L is some DG k-module. The tensor product over $\mathcal{S}^e$ with I is isomorphic to

$$\mathcal{H}om_{\mathcal{F}_1}(X, E_0) \otimes_{\mathcal{S}} \mathcal{H}om_{\mathcal{E}_1}(E_n, Y_\lambda) \otimes L.$$

Since $\mathcal{H}om_{\mathcal{F}_1}(?, E_0)$ is free over $\mathcal{S}$, it is clear that the canonical morphism

$$\mathcal{H}om_{\mathcal{F}_1}(X, E_0) \otimes_{\mathcal{S}} \mathcal{H}om_{\mathcal{E}_1}(E_n, Y_\lambda) \to \mathcal{H}om_{\mathcal{E}_1}(E_n, E_0)$$

is an isomorphism. It is now trivial to check that we do obtain the Hochschild complex with respect to $\mathcal{R}$.

The terms of the third resolution are sums of objects of the form

$$\mathcal{H}om_{\mathcal{G}_1}(G_n, Y^\rho) \otimes L \otimes \mathcal{H}om_{\mathcal{G}_1}(X^\rho, G_0)$$

where X and Y denote 'variable objects' of $\mathcal{S}$ and L is some DG k-module. The tensor product over $\mathcal{S}^e$ with I is isomorphic to

$$\mathcal{H}om_{\mathcal{G}_1}(X^\rho, G_0) \otimes_{\mathcal{S}} \mathcal{H}om_{\mathcal{G}_1}(G_n, Y^\rho) \otimes L.$$

We would like to show that the canonical morphism

$$\mathcal{H}om_{\mathcal{G}_1}(X^\rho, G_0) \otimes_{\mathcal{S}} \mathcal{H}om_{\mathcal{G}_1}(G_n, Y^\rho) \to \mathcal{H}om_{\mathcal{G}_1}(G_n, G_0)$$

is an homotopy equivalence. Now we have an isomorphism

$$\mathcal{H}om_{\mathcal{G}_1}(X^\rho, G_0) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{F}_1}(X, G_0),$$

and $G_0 \in \mathcal{F}_1$ (though G_0 is not necessarily small in $\underline{\mathcal{F}_1}$). Moreover $\mathcal{H}om_{\mathcal{G}_1}(G_n, Y^\rho)$ viewed as a functor of Y is a model functor $\mathcal{F}_1 \to \mathcal{C}k$ which induces a functor $\underline{\mathcal{F}_1} \to \mathcal{H}k$ commuting with direct sums. Now the claim follows by the

Lemma. *If $M : \mathcal{F}_1 \to \mathcal{C}k$ is a model functor inducing a functor $\underline{\mathcal{F}_1} \to \mathcal{H}k$ commuting with direct sums, then the canonical morphism*

$$\mathcal{H}om_{\mathcal{F}_1}(?, X) \otimes_{\mathcal{S}} M \to MX$$

is an homotopy equivalence for each $\mathcal{F}_1$.

Proof. Indeed for $X \in \mathcal{S}$, the canonical morphism is an isomorphism of complexes. Since both sides yield triangle functors $\underline{\mathcal{F}_1} \to \mathcal{H}k$ commuting with direct sums, the claim follows at once.

6.7 Conclusion. Let L', M' and N' be the relative resolutions we have constructed. They are linked by canonical morphisms

$$L' \xrightarrow{i} M' \to N'$$

which, after tensoring over $\mathcal{S}^e$ with I, yield the morphisms of precyclic chain complexes

$$C(\mathcal{E}_1^b) \xrightarrow{C(\text{incl})} C(\mathcal{F}_1^b) \to C(\mathcal{G}_1^b)$$

induced by the functors $\mathcal{E}_1 \to \mathcal{F}_1 \to \mathcal{G}_1$. By our flatness hypotheses, $C(\text{incl})$ is injective. According to 7.7, in order to conclude that $C(\mathcal{G}_1^b)$ is quasi-isomorphic to the cokernel of $C(\text{incl})$, we have to check that i splits in the category of graded k-modules and that $\operatorname{Cok} i$ is still relatively acyclic for I.

Now indeed, L' has the components

$$\bigoplus \mathcal{H}om_{\mathcal{E}_1}(E_n, Y_\lambda) \otimes \mathcal{H}om_{\mathcal{E}_1}(E_{n-1}, E_n) \otimes \ldots \otimes \mathcal{H}om_{\mathcal{E}_1}(E_0, E_1) \otimes \mathcal{H}om_{\mathcal{F}_1}(X, E_0).$$

Since we have

$$\mathcal{H}om_{\mathcal{E}_1}(E_n, Y_\lambda) \cong \mathcal{H}om_{\mathcal{F}_1}(E_n, Y),$$

and since $\mathcal{R} \subset \mathcal{S}$, the morphism $L' \to M'$ sends this isomorphically onto a partial sum of the component

$$\bigoplus \mathcal{H}om_{\mathcal{F}_1}(F_n, Y) \otimes \mathcal{H}om_{\mathcal{F}_1}(F_{n-1}, F_n) \otimes \ldots \otimes \mathcal{H}om_{\mathcal{F}_1}(F_0, F_1) \otimes \mathcal{H}om_{\mathcal{F}_1}(X, F_0)$$

of M', where the $F_0, \ldots, F_n$ run through $\mathcal{S}$. In particular, i is k-split and its cokernel still has relatively acyclic components for I.

7. Differential graded algebras, derived categories

7.1 DG algebras. Let k be a commutative ring and A a *differential graded k-algebra* (=DG algebra), i.e. a $\mathbf{Z}$-graded associative k-algebra with one

$$A = \coprod_{p \in \mathbf{Z}} A^p$$

endowed with a k-linear differential $d : A \to A$ which is homogeneous of degree 1 (i.e. $dA^p \subset A^{p+1}$ for each p) and satisfies the graded Leibniz rule

$$d(ab) = (da)\, b + (-1)^p a\, db\ ,\ \forall\, a \in A^p\ ,\ \forall\, b \in A.$$

It turns out to be convenient *not* to impose any a priori finiteness conditions on A. In particular, we do not assume that A is a chain algebra as in [6] or [15].

Examples. a) If B is an 'ordinary' k-algebra, it gives rise to a DG algebra A defined by

$$A^p = \begin{cases} B & p = 0 \\ 0 & p \neq 0. \end{cases}$$

Conversely, any DG algebra A which is *concentrated in degree* 0 (i.e. A^p=0 for all $p \neq 0$) is obtained in this way from an 'ordinary' algebra.

b) If B is a k-algebra and

$$M = (\ldots \to M^i \xrightarrow{d} M^{i+1} \to \ldots)\ ,\ i \in \mathbf{Z}\ ,\ dd = 0$$

a complex of right B-modules, we consider the DG algebra $A = \mathcal{H}om_B\,(M, M)$ with the components

$$A^p = \prod_{-i+j=p} \mathrm{Hom}_B(M^i, M^j)$$

and the differential defined by

$$d(f^i) = (d \circ f^i - (-1)^p f^{i+1} \circ d)\ ,\ (f^i) \in A^p.$$

Note that even if $M^i = 0$ for all $i \gg 0$, there may occur non-vanishing components of A in arbitrarily small *and* arbitrarily large degrees.

7.2 DG modules. A *differential graded module over A* (=DG A-module) is a $\mathbf{Z}$-graded right A-module

$$M = \coprod_{p \in \mathbf{Z}} M^p$$

endowed with a k-linear differential $d : M \to M$ which is homogeneous of degree 1 and satisfies the graded Leibniz rule

$$d(ma) = (dm)\, a + (-1)^p m\, da\ ,\ \forall\, m \in M^p\ ,\ \forall\, a \in A.$$

We sometimes use the notation M_p for the component M^{-p}. *Differential graded left A-modules* are defined similarly. The Leibniz rule then reads

$$d(am) = (da)\, m + (-1)^p a\, (dm)\ ,\ \forall\, a \in A^p\ ,\ \forall\, m \in M.$$

A *morphism* of DG A-modules $f : M \to N$ is a morphism of the underlying graded A-modules which is homogeneous of degree 0 and commutes with the differential.

Examples. a) In the situation of example 7.1 a), the category of DG A-modules identifies with the category of differential complexes of right B-modules.

b) In the situation of example 7.1 b), each complex N of right B-modules gives rise to a DG A-module $\mathcal{H}om_B(M,N)$ endowed with the A-action $(g^j)(f^i) = (g^{i+p} \circ f^i)$, where $(g^j) \in \mathrm{Hom}_B(M,N)^q$ and $(f^i) \in A^p$. On the other hand, M becomes itself a DG left A-module for the action $(f^i)(m^j) = (f^i(m^i))$.

7.3 The homotopy category. Let $f : M \to N$ be a morphism of DG A-modules. We say that f *is null-homotopic* if we have $f = dr + rd$, where $r : M \to N$ is a morphism of the underlying graded A-modules which is homogeneous of degree -1. A DG module is *contractile* if its identity morphism is null-homotopic.

The *homotopy category* $\mathcal{H}A$ has the DG A-modules as *objects*. Its *morphisms* are classes $\overline{f}$ of morphisms f of DG A-modules modulo null-homotopic morphisms. Isomorphisms of $\mathcal{H}A$ are called *homotopy equivalences*.

Define the *suspension functor* $S : \mathcal{H}A \to \mathcal{H}A$ by

$$(SM)^p = M^{p+1} \ , \ d_{SM} = -d_M \ , \ \mu_{SM}(m,a) = \mu_M(m,a) \ ,$$

for $m \in M$ and $a \in A$, where μ_M and μ_{SM} are the multiplication maps of the respective modules. Define a *standard triangle of* $\mathcal{H}A$ to be a sequence

$$L \xrightarrow{\overline{f}} M \xrightarrow{\overline{g}} Cn(f) \xrightarrow{\overline{h}} SL \ ,$$

where $f : L \to M$ is a morphism of DG modules, $Cn(f) = M \oplus SL$ as a graded k-module,

$$d_{Cn(f)} = \begin{bmatrix} d_M & f \\ 0 & d_{SL} \end{bmatrix} \ , \ \mu_{Cn(f)}(\begin{bmatrix} m \\ l \end{bmatrix}, a) = \begin{bmatrix} ma \\ la \end{bmatrix} \ ,$$

for $m \in M$, $l \in L^p$, the morphism g is the canonical injection $M \to Cn(f)$, and $-h$ (note the sign) is the canonical projection. As usual, $Cn(f)$ is called the *mapping cone* over f.

Lemma. *Endowed with the suspension functor S and the triangles isomorphic to standard triangles, the category $\mathcal{H}A$ becomes a triangulated category in the sense of Verdier* [52]. *Moreover, each short exact sequence of DG A-modules*

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

which splits in the category of graded A-modules gives rise to a triangle of $\mathcal{H}A$

$$L \xrightarrow{\overline{f}} M \xrightarrow{\overline{g}} N \xrightarrow{\overline{rds}} SL \ ,$$

where r and s are morphisms of graded A-modules satisfying $rf = 1_L$, $gs = 1_N$, $rs = 0$.

In the situation of example 7.1 a), the category $\mathcal{H}A$ identifies with the homotopy category of complexes of right B-modules. To prove the lemma, one may proceed as in [20]. Alternatively [31], one can make the category of DG modules into a Frobenius category whose associated stable category identifies with $\mathcal{H}A$, which therefore automatically carries a triangulated structure [17].

7.4 Derived categories, Resolutions. A DG A-module N is *acyclic* (resp. *relatively acyclic*) if we have $\mathrm{H}^*N = 0$ (resp. if the underlying DG k-module of N is contractile). Here, as always, H^*N denotes the $\mathbf{Z}$-graded k-module with components

$$H^pN = \mathrm{Ker}\,(d : N^p \to N^{p+1})/dN^{p-1}.$$

A morphism of DG A-modules $s : M \to M'$ is a *(relative) quasi-isomorphism* if its mapping cone $Cn(s)$ is (relatively) acyclic.

By definition, the *(relative) derived category of A* is the localization (cf. [52])

$$\mathcal{D}A := (\mathcal{H}A)[\Sigma^{-1}] \ , \ (\mathcal{D}_{rel}A = (\mathcal{H}A)[\Sigma_{rel}^{-1}]) \ ,$$

where Σ (resp. Σ_{rel}) denotes the class of all homotopy classes of (relative) quasi-isomorphisms. In the situation of example 7.1 a), the category $\mathcal{D}A$ identifies with the (unbounded) derived category of the category of right B-modules. If k is a field, we have $\Sigma = \Sigma_{rel}$ and $\mathcal{D}A = \mathcal{D}_{rel}A$.

The (relative) derived category inherits by localization a triangulated structure from $\mathcal{H}A$. So by definition, each triangle of $\mathcal{H}A$ maps to a triangle of $\mathcal{D}A$ and $\mathcal{D}_{rel}A$. Moreover, if

$$0 \to L \xrightarrow{i} M \xrightarrow{p} N \to 0$$

is a short exact sequence of DG A-modules, then p induces a canonical quasi-isomorphism

$$Cn(i) \to N \; ,$$

where $Cn(i)$ is the mapping cone (resp. a canoncial relative quasi-isomorphism, if the sequence has k-split components). Thus the sequence yields a canonical triangle

$$L \to M \to N \to SN$$

of $\mathcal{D}A$ (resp. $\mathcal{D}_{rel}A$).

It is not hard to check that $\mathcal{D}A$ *and* $\mathcal{D}_{rel}A$ *have infinite direct sums* and that these are given by the ordinary sums of DG A-modules.

Let A_A denote the free DG A-module on one generator. Let M be any A-module. Then it is easy to check that the map

$$\mathrm{Hom}_{\mathcal{H}A}(A_A, N) \to \mathrm{H}^0 M \; , \; \overline{f} \mapsto \overline{f(1)}$$

is bijective. In particular, each quasi-isomorphism $s : M \to M'$ induces a bijection

$$\mathrm{Hom}_{\mathcal{H}A}(A_A, M) \xrightarrow{\overline{s}_*} \mathrm{Hom}_{\mathcal{H}A}(A_A, M'). \tag{2}$$

As an immediate consequence, we have a bijection

$$\mathrm{Hom}_{\mathcal{H}A}(A_A, M) \xrightarrow{\sim} \varinjlim \mathrm{Hom}_{\mathcal{H}A}(A_A, M') = \mathrm{Hom}_{\mathcal{D}A}(A_A, M). \tag{3}$$

Here $\varinjlim$ is taken over the filtering category of quasi-isomorphims $\overline{s} : M \to M'$ with domain M. We note in passing that this implies

$$\mathrm{Hom}_{\mathcal{D}A}(A_A, M) \xrightarrow{\sim} \mathrm{H}^0 M \tag{4}$$

A DG A-module sharing the two equivalent properties (2) and (3) with A_A is called *closed* ('having property (P)' in the terminology of [31]). We denote by $\mathcal{H}_pA$ the full subcategory of $\mathcal{H}A$ formed by the closed objects. Property (3) combined with the 5-lemma shows that the mapping cone over a morphism of closed objects is still closed. So $\mathcal{H}_pA$ is a triangulated subcategory of $\mathcal{H}A$.

Similarly, if K is an (ordinary) k-module and M a DG A-module, the canonical map

$$\mathrm{Hom}_{\mathcal{H}A}(K \otimes_k A_A, M) \to \mathrm{H}^0\mathrm{Hom}_k(K, M)$$

is bijective. As above, we conclude that we have a bijection

$$\mathrm{Hom}_{\mathcal{H}A}(K \otimes_k A_A, M) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}_{rel}A}(K \otimes_k A_A, M). \tag{5}$$

A DG A-module sharing this property with $K \otimes_k A_A$ is called *relatively closed*. We denote by $\mathcal{H}_{p,rel}A$ the full subcategory of $\mathcal{H}A$ formed by the relatively closed objects. It is a triangulated subcategory of $\mathcal{H}A$.

Proposition.

a) *For each $M \in \mathcal{H}A$, there is a (relative) quasi-isomorphism*

$$\boldsymbol{p}M \to M \;\; (\textit{resp. } \boldsymbol{p}_{rel}M \to M \;)$$

where $\boldsymbol{p}M$ is closed (resp. $\boldsymbol{p}_{rel}M$ is relatively closed). If we have two (relative) quasi-isomorphisms $\varphi : P \to M$ and $\varphi' : P' \to M$ with (relatively) closed P and P', there is a unique homotopy equivalence $\psi : P \xrightarrow{\sim} P'$ such that $\varphi' \; \psi = \varphi$.

b) *The assignment $M \mapsto \boldsymbol{p}M$ may be completed to a triangle functor which commutes with infinite sums and induces a triangle equivalence $\mathcal{D}A \xrightarrow{\sim} \mathcal{H}_p A$. Similarly, the assignment $M \mapsto \boldsymbol{p}_{rel}M$ yields a triangle equivalence $\mathcal{D}_{rel}A \xrightarrow{\sim} \mathcal{H}_{p,rel}A$.*

In the situation of example 7.1 a), if M is concentrated in degree 0, then $\boldsymbol{p}M$ may be chosen as a projective resolution of M^0. If M is a right bounded complex, $\boldsymbol{p}M$ is a 'projective resolution of the complex M' (cf. [20]). For arbitrary M over an 'ordinary' k-algebra, $\boldsymbol{p}M$ is a K-projective resolution in the sense of [50]. The proof for an arbitrary DG algebra may be found in [31]. The proof in the relative case is completely analogous.

7.5 Closed objects. Keep the assumptions of 7.4. In the absolute case, the following proposition results from [31, sect. 3]. The relative case is proved similarly.

Proposition. *A DG A-module is closed (resp. relatively closed) if and only if it is homotopy equivalent to a DG module P admitting a filtration*

$$0 = F_{-1} \subset F_0 \subset F_1 \subset \dots F_p \subset F_{p+1} \dots \subset P\ ,\ p \in \mathbf{N}$$

such that

(F1) P is the union of the F_p, $p \in \mathbf{N}$,

(F2) the inclusion morphism $F_{p-1} \subset F_p$ splits in the category of graded A-modules, $\forall\, p \in \mathbf{N}$,

(F3) each subquotient F_p/F_{p-1} is isomorphic as a DG module to a direct summand of a direct sum of DG modules $S^n A_A$, $n \in \mathbf{N}$ (resp. $S^n K \otimes_k A_A$, where K is a DG k-module and $n \in \mathbf{N}$).

Note that (F1) and (F2) imply that the following *sequence* $(*)$ is split exact in the category of graded A-modules and thus (lemma 7.4) produces a triangle in $\mathcal{HA}$

$$\coprod_{p\in\mathbf{N}} F_p \xrightarrow{\Phi} \coprod_{q\in\mathbf{N}} F_q \xrightarrow{\text{can}} P\ ;$$

here Φ has the components

$$F_p \xrightarrow{[1\ -\iota]^t} F_p \oplus F_{p+1} \xrightarrow{\text{can}} \coprod_{q\in\mathbf{N}} F_q\ ,\ \iota = \text{incl}\,.$$

By lemma 7.3 it follows that $\mathcal{H}_p A$ (resp. $\mathcal{H}_{p,rel}A$) coincides with its smallest full triangulated subcategory containing A_A (resp. $K \otimes_k A$ for each k-module K) and closed under infinite sums. By proposition 7.4 b), the same holds for $\mathcal{D}A$ (resp. $\mathcal{D}_{rel}A$). This gives rise to an 'induction principle' as illustrated by the following fact: If $\mathcal{T}$ is a triangulated category admitting infinite sums and $F_1, F_2 : \mathcal{D}A \to \mathcal{T}$ are two triangle functors commuting with infinite sums, then *a morphism $\mu : F_1 \to F_2$ of triangle functors is invertible if (and only if) $\mu A_A : F_1 A_A \to F_2 A_A$ is invertible.* Indeed, the full subcategory of $\mathcal{D}A$ formed by the DG modules U with invertible μU is a triangulated subcategory by the 5-lemma, contains A_A by assumption, and is closed under infinite sums since F_1 and F_2 commute with infinite sums.

7.6 Left derived tensor functors. Let A and B be DG algebras, and ${}_A X_B$ a DG A-B-*bimodule*, i.e.

$$X = \coprod_{p\in\mathbf{Z}} X^p$$

is a graded left A-module and a graded right B-module, the two actions commute and coincide on k, and X is endowed with a homogeneous k-linear differential d of degree 1 satisfying

$$d(axb) = (da)\,xb + (-1)^p a\,(dx)\,b + (-1)^{p+q} ax\,(db)$$

for all $a \in A^p$, $x \in X^q$, $b \in B$. We define the DG *algebra* $A^{\mathrm{op}} \otimes_k B$ by

$$(A^{\mathrm{op}} \otimes_k B)^n = \coprod_{p+q=n} A^p \otimes B^q ,$$
$$d(a \otimes b) = (da) \otimes b + (-1)^p a \otimes (db) ,$$
$$(a \otimes b)(a' \otimes b') = (-1)^{qp'+pp'} a'a \otimes bb' ,$$

for all $a \in A^p$, $b \in B^q$, $a' \in A^{p'}$, $b' \in B$. We may then view X as a (right) DG $A^{\mathrm{op}} \otimes_k B$-module via

$$x.(a \otimes b) = (-1)^{rp} axb , \ \forall\, x \in X^r , \ \forall\, a \in A^p , \ \forall\, b \in B.$$

Let M be a DG A-module. We define $M \otimes_k X$ to be the DG B-module with the action of B on X and with the DG structure

$$(M \otimes_k X)^n = \coprod_{p+q=n} M^p \otimes_k X^q , \ d(m \otimes x) = (dm) \otimes x + (-1)^p m \otimes (dx) ,$$

for all $m \in M^p$, $x \in X$. The k-submodule generated by all differences $ma \otimes x - m \otimes ax$ is stable under d and under multiplication by elements of B. So $M \otimes_A X$, the quotient modulo this submodule, is a well defined DG B-module, which is moreover functorial in M and X. We call M a *flat DG A-module* if $M \otimes_A N$ is acyclic for each acyclic left DG A-module N (cf. 8.3). The functor $? \otimes_A X$ yields a triangle functor $\mathcal{H}A \to \mathcal{H}B$ which will be denoted by the same symbol. We define the left derived functor

$$? \otimes_A^{\mathbf{L}} X : \mathcal{D}A \to \mathcal{D}B$$

by $N \otimes_A^{\mathbf{L}} X := (\boldsymbol{p}N) \otimes_A X$. If the canonical morphism $N \otimes_A^{\mathbf{L}} X \to N \otimes_A X$ is a quasi-isomorphism, then N is called *acyclic for* X. Note that $? \otimes_A^{\mathbf{L}} X$ commutes with infinite sums since $\boldsymbol{p}$ and $? \otimes_A X$ do. The following lemma is proved in [34].

Lemma. *The functor $F = ? \otimes_A^{\mathbf{L}} X$ is an equivalence if and only if the following conditions hold*

a) F induces bijections

$$\mathrm{Hom}_{\mathcal{D}A}(A, S^n A) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}B}(X_B, S^n X_B) , \ \forall\, n \in \mathbf{Z} ,$$

b) the functor $\mathrm{Hom}_{\mathcal{D}B}(X_B, ?)$ *commutes with infinite sums,*

c) the smallest full triangulated subcategory of $\mathcal{D}B$ containing X_B and closed under infinite sums coincides with $\mathcal{D}B$.

Example. Suppose that $\varphi : B \to A$ is a quasi-isomorphism, i.e. a morphism of DG algebras inducing an isomorphism $\mathrm{H}^*B \xrightarrow{\sim} \mathrm{H}^*A$. Then

$$? \otimes_A^{\mathbf{L}} A_B : \mathcal{D}A \to \mathcal{D}B \text{ and } ? \otimes_B^{\mathbf{L}} A_A : \mathcal{D}B \to \mathcal{D}A$$

are equivalences.

7.7 Relative derived tensor functors. Let A and B be DG algebras, and ${}_A X_B$ a DG A-B-bimodule (cf. 7.6). We define the *relative left derived functor*

$$? \otimes_A^{\mathbf{L}\mathrm{rel}} X : \mathcal{D}_{rel}A \to \mathcal{D}_{rel}B$$

by $N \otimes_A^{\mathbf{L}\mathrm{rel}} X := (\boldsymbol{p}_{rel}N) \otimes_A X$. If the canonical morphism $N \otimes_A^{\mathbf{L}\mathrm{rel}} X \to N \otimes_A X$ is an homotopy equivalence, then N is called *relatively acyclic* for X.

Suppose that we have a commutative diagram of $\mathcal{H}A$

$$\begin{array}{ccc} L' & \to & L \\ \downarrow & & \downarrow \\ M' & \to & M , \end{array}$$

where L' and M' are relatively acyclic and the horizontal morphisms are relative quasi-isomorphisms. Then we can compute the image of $L \to M$ under $? \otimes_A^{\mathbf{L}\mathrm{rel}} X$ as $L' \otimes_A X \to M' \otimes_A X$. Indeed, the composition

$$\boldsymbol{p}_{rel} L' \to L' \to L$$

is a relative quasi-isomorphism with relatively closed domain and the resulting relative quasi-isomorphism $L' \otimes_A^{\mathbf{L}\mathrm{rel}} X \xrightarrow{\sim} L \otimes_A^{\mathbf{L}\mathrm{rel}} X$ is compatible with morphisms by the following commutive diagram of $\mathcal{H}A$

$$\begin{array}{ccccc} \boldsymbol{p}_{rel} L' & \to & L' & \to & L \\ \downarrow & & \downarrow & & \downarrow \\ \boldsymbol{p}_{rel} M' & \to & M' & \to & M. \end{array}$$

Now let

$$0 \to L \to M \to N \to 0$$

be a short exact sequence of DG A-modules which admits a splitting in the category of graded k-modules. Suppose that it fits into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & L & \to & M & \to & N & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & L' & \xrightarrow{i} & M' & \to & N' & & \end{array}$$

whose vertical morphisms are relative quasi-isomorphisms and whose second row is a complex with relatively acyclic terms for X. Then (L, M, N) fits into a canonical triangle of $\mathcal{D}_{rel}A$ and hence $(L' \otimes_A X, M' \otimes_A X, N' \otimes_A X)$ fits into a triangle of $\mathcal{D}_{rel}B$. However, we will need to know that this triangle comes from a canonical short exact sequence of B-modules.

For this, suppose that i and $i \otimes_A X$ are both split monomorphic as morphisms of graded k-modules and that $\mathrm{Cok}\, i$ is relatively acyclic for X as well. Then clearly $\mathrm{Cok}\, i \to N$ and hence $\mathrm{Cok}\, i \to N'$ are relative quasi-isomorphisms. Therefore $(\mathrm{Cok}\, i) \otimes_A X \to N' \otimes_A X$ is a relative quasi-isomorphism and, since $? \otimes_A X$ commutes with cokernels, the canonical map $\mathrm{Cok}\,(L' \otimes_A X \to M' \otimes_A X) \to (\mathrm{Cok}\, i) \otimes_A X$ is an isomorphism. So we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & L' \otimes_A X & \xrightarrow{i \otimes_A X} & M' \otimes_A X & \to & N' \otimes_A X & & \\ & & \| & & \| & & \uparrow & & \\ 0 & \to & L' \otimes_A X & \xrightarrow{i \otimes_A X} & M' \otimes_A X & \to & \mathrm{Cok}\,(i \otimes_A X) & \to & 0 \end{array}$$

where the last vertical morphism is a relative quasi-isomorphism.

7.8 Relatively acyclic objects for Hochschild homology. Let A be a DG algebra and put $A^e = A^{\mathrm{op}} \otimes A$. Then A becomes a left A^e-module in a canonical way. Recall that $\otimes$ without index means tensor product over k.

Lemma. *If P is a closed DG A-module and M an arbitrary left DG A-module, then $M \otimes P$ is relatively acyclic (7.7) for the A^e-k-bimodule A.*

Proof. We have to construct a relative quasi-isomorphism $\boldsymbol{p}_{rel}(P \otimes M) \to P \otimes M$ with relatively closed $\boldsymbol{p}_{rel}(P \otimes M)$ over A^e and show that the induced morphism

$$\boldsymbol{p}_{rel}(M \otimes P) \otimes_{A^e} A \to (M \otimes P) \otimes_{A^e} A$$

is an homotopy equivalence over k. For this, we choose a relative quasi-isomorphism $\boldsymbol{p}_{rel} M \to M$ with relatively acyclic $\boldsymbol{p}_{rel} M$ over A. Then clearly $(\boldsymbol{p}_{rel} M) \otimes P \to M \otimes P$ is a relative quasi-isomorphism with relatively closed $(\boldsymbol{p}_{rel} M) \otimes P$ over A^e. So we put $\boldsymbol{p}_{rel}(P \otimes M) = P \otimes \boldsymbol{p}_{rel} M$. We then have to show that the induced morphism

$$(\boldsymbol{p}_{rel} M) \otimes P) \otimes_{A^e} A \to (M \otimes P) \otimes_{A^e} A$$

is an homotopy equivalence. Now if U is a left DG A-module and V a right DG A-module, then we have a canonical isomorphism of DG A-modules

$$(U \otimes V) \otimes_{A^e} A \xrightarrow{\sim} U \otimes_A V.$$

Using this isomorphism we are reduced to showing that

$$P \otimes_A \boldsymbol{p}_{rel} M \to P \otimes_A M$$

is an homotopy equivalence. This is clear for $P = A$ since $\boldsymbol{p}_{rel} M \to M$ is a relative quasi-isomorphism. It then follows for any P by the structure of closed objects (7.4).

7.9 Relative equivalences. The following result is given for completeness. We shall neither prove it nor use it. Let A and B be DG algebras, and ${}_A X_B$ a DG A-B-bimodule (cf. 7.6).

Lemma. *The functor $F = ? \otimes_A^{\mathbf{L}\mathrm{rel}} X$ is an equivalence if and only if the following conditions hold*

a) For each k-module K the canonical morphism

$$A \otimes_k K \to \mathcal{H}om_B (X_B, K \otimes_k X_B) ,$$

is a homotopy equivalence of DG k-modules.

b) the functor $\mathrm{Hom}_{\mathcal{D}_{rel} B}(X_B, ?)$ *commutes with infinite sums,*

c) the smallest full triangulated subcategory of $\mathcal{D}B$ containing $K \otimes_k X_B$ for each k-module K and closed under infinite sums coincides with $\mathcal{D}_{rel} B$.

In interpreting condition a) note that as an object of $\mathcal{H}k$ the complex $\mathcal{H}om_B (M, N)$ is well defined and functorial in $M, N \in \mathcal{D}_{rel} A$. The existence of an equivalence between the relative derived categories of A and B turns out to be too restrictive a hypothesis for our purposes.

7.10 Small objects. Let A be a DG algebra. A closed object P of $\mathcal{H}A$ is *small* if the functor

$$\mathrm{Hom}_{\mathcal{H}A}(P, ?) : \mathcal{H}A \to \mathrm{Mod}\, k$$

commutes with direct sums. It is a *generator* of $\mathcal{H}_p A$ if the smallest full triangulated subcategory of $\mathcal{H}_p A$ containing P and closed under infinite direct sums coincides with $\mathcal{H}_p A$. For example $P = A_A$ is a small generator, or more generally, in the situation of lemma 7.6, $P = \boldsymbol{p} X_A$ is a small generator. It is remarkable that two small generators are always obtained from each other by a finite sequence of 'finitistic' constructions (no infinite sums are needed), as it is made precise in the following proposition.

Proposition.

a) If P is a small generator of $\mathcal{H}_p A$, then the full subcategory of $\mathcal{H}_p A$ formed by the small objects coincides with the smallest full triangulated category of $\mathcal{H}_p A$ containing P and closed under forming direct summands.

b) If P is small, then the functor

$$\mathcal{H}om_A (P, ?) : \mathcal{H}_p A \to \mathcal{H}k$$

commutes with direct sums.

Statement a) is proved in [31, 5.3] and goes back to [45] resp. [47]. Statement b) clearly holds for $P = A_A$ and hence, by a), for any small P.

8. Appendix

8.1 Localization and adjoints. Suppose that $\mathcal{S}$ and $\mathcal{T}$ are triangulated categories and

$$\begin{array}{c} \mathcal{T} \\ L \downarrow\uparrow R \\ \mathcal{S} \end{array}$$

a pair of adjoint triangle functors.

Lemma.

a) If R is fully faithful, the sequence

$$0 \to \operatorname{Ker} L \to \mathcal{T} \xrightarrow{L} \mathcal{S} \to 0$$

is exact.

b) If $\mathcal{S}$ and $\mathcal{T}$ admit infinite direct sums, R commutes with infinite direct sums and $\mathcal{T}$ is generated (5.2) by the object G, then $\operatorname{Ker} L$ *is generated by the object G' occuring in the triangle*

$$G' \to G \to RLG \to SG'.$$

Proof. a) The canonical functor $\mathcal{T}/\operatorname{Ker} L \to \mathcal{S}$ is clearly essentially surjective. To prove that it is fully faithful, it is enough to show that the restriction of the localization functor to $\operatorname{Im} R$ is fully faithful. By prop. 5.3 on page 286 of [52], this follows from the fact that $\operatorname{Hom}_{\mathcal{T}}(N, RLX) = 0$ for each $N \in \operatorname{Ker} L$, $X \in \mathcal{T}$.

b) Put $\mathcal{N} = \operatorname{Ker} L$. For each $X \in \mathcal{T}$, define $X_{\mathcal{N}}$ by the triangle

$$X_{\mathcal{N}} \to X \to RLX \to SX_{\mathcal{N}}.$$

One checks that $X_{\mathcal{N}}$ belongs to $\mathcal{N}$ and that it represents the restriction of the functor $\operatorname{Hom}_{\mathcal{T}}(?, X)$ to $\mathcal{N}$. Hence the functor $X \mapsto X_{\mathcal{N}}$ defines a right adjoint to the inclusion functor. Such an adjoint becomes a triangle functor in a canonical way [30, 6.7]. Moreover, the fact that the identity functor and RL commute with infinite sums implies that the functor $X \mapsto X_{\mathcal{N}}$ commutes with infinite sums. Finally, this functor is clearly essentially surjective since it is right adjoint to a fully faithfull functor. It follows that if G generates $\mathcal{T}$ then $G_{\mathcal{N}}$ generates $\mathcal{N}$, the image of $\mathcal{T}$ under the functor $X \mapsto X_{\mathcal{N}}$.

8.2 Filtered direct limits. Let A be a DG algebra and $\mathcal{U} \subset \mathcal{D}A$ a full triangulated category stable with respect to forming infinite sums. Let $(M_i)_{i \in I}$ be a filtered direct system in the category of DG modules such that $M_i \in \mathcal{U}$ for each $i \in I$.

Lemma. *The direct limit $M = \varinjlim M_i$ belongs to $\mathcal{U}$.*

Proof. Since I is filtered, there is a resolution

$$\ldots \to C_1 \to C_0 \to M \to 0$$

of M by DG A-modules which are direct sums of copies of the M_i. Hence M is quasi-isomorphic to a DG module admitting a countable filtration whose subquotients belong do $\mathcal{U}$. By the 'Milnor triangle' [43] (cf. for example [31, 3.1]) we conclude that M belongs to $\mathcal{U}$.

8.3 Products of flat DG k-modules. Let k be a ring (associative with 1). A theorem of S. U. Chase asserts that if (and only if) k is left coherent, then every product of flat right k-modules is flat [1, 19.20]. We need the analogous assertion for flat DG k-modules. First recall from (7.6) that a DG k-module M is flat if $M \otimes N$ is acyclic for each acyclic left k-module N. If M is a bounded complex, it is flat as a DG module if its components are flat k-modules. However, for unbounded complexes M this will not suffice in general. The proof of the following lemma shows among other things that if k is of finite global dimension, then M is flat as a DG module iff in the homotopy category it is an extension of a purely acyclic module by a closed module.

Lemma. *Suppose that k is left coherent and of finite global dimension. Then every product of flat DG k-modules is flat as a DG k-module.*

Proof. If N is a complex of k-modules we denote by $\sigma_{\geq n} N$ resp. $\tau_{\leq n} N$ the subcomplexes

$$(\ldots 0 \to 0 \to N^n \to N^{n+1} \to \ldots) \text{ resp. } (\ldots \to N^{n-2} \to N^{n-1} \to \mathrm{Z}^n N \to 0 \ldots).$$

First step: Each acyclic DG k-module N is homotopy equivalent to a direct limit (in the category of DG modules) of left bounded acyclic DG modules. Indeed, let

$$0 \to N \to I^0 \to I^1 \to \ldots \to I^p \to I^{p+1} \to \ldots$$

be a Cartan-Eilenberg resolution [40, XII, 11] of N, i.e. a complex of DG modules such that the induced sequences

$$0 \to N^q \to I^{0,q} \to I^{1,q} \to \ldots \to I^{p,q} \to I^{p+1,q} \to \ldots$$
$$0 \to \mathrm{Z}^q\, N \to \mathrm{Z}^q\, I^0 \to \mathrm{Z}^q\, I^1 \to \ldots \to \mathrm{Z}^q\, I^p \to \mathrm{Z}^q\, I^{p+1} \to \ldots$$

are injective resolutions for each q. Since k has finite global dimension, we may suppose that $I^p = 0$ for all $p \gg 0$. Since each $\mathrm{Z}^q\, I^p$ is injective, the I^p are contractile. We conclude that N is homotopy equivalent to N', the total complex associated with

$$0 \to N \to I^0 \to I^1 \to \ldots \to I^p \to I^{p+1} \to \ldots.$$

We define N'_r as the total complex associated with

$$0 \to \sigma_{\geq r} N \to \sigma_{\geq r} I^0 \to \ldots \to \sigma_{\geq r} I^p \to \sigma_{\geq r} I^{p+1} \to \ldots$$

Then N' is clearly isomorphic to the direct limit of the N'_r.

2nd step: Each DG k-module with flat components is flat as a DG k-module. Let X be a DG module with flat components and N an acyclic left DG k-module. The complex X is a direct limit of the left bounded complexes $\sigma_{\geq n} X$, $n \to -\infty$. So in order to show that $X \otimes_k N$ is acyclic, we may suppose that X is itself left bounded. By the first step, N is homotopy equivalent to a direct limit of left bounded acyclic complexes. We may assume that N itself is left-bounded. After these reductions it is clear that $X \otimes_k N$ is acyclic since it is the total complex of a first quadrant complex with acyclic columns.

3rd step: Let N be a DG k-module. Then $N \otimes_k M$ is acyclic for each DG k-module M iff this holds for each DG k-module M concentrated in degree 0. Suppose that $N \otimes_k M$ is acyclic for each DG k-module M concentrated in degree 0. Then it also holds if M is bounded (by devissage) and if M is right bounded (by passage to the limit over the $\sigma_{\geq n} M$, $n \to -\infty$). Finally, it holds for arbitrary M by passage to the limit of the $\tau_{\leq n} M$, $n \to \infty$.

4th step: Let N by a DG k-module. Then $N \otimes_k M$ is acyclic for each DG k-module M iff each sequence

$$0 \to \mathrm{Z}^n\, N \to N^n \to \mathrm{Z}^{n+1}\, N \to 0\,,\ n \in \mathbf{Z}\,,$$

is pure exact in the sense of P. M. Cohn [9, p. 383] (i.e. its tensor product with an arbitrary left k-module is exact). By the third step, the condition is sufficient. Conversely, suppose that $N \otimes_k M$ is acyclic for each k-module M. Taking $M = k$ we see that N is acyclic. Therefore we have $\mathrm{B}^n\, N \xrightarrow{\sim} \mathrm{Z}^n\, N$ for each $n \in \mathbf{Z}$. If M is arbitrary, we see that we have $(\mathrm{Z}^n\, N) \otimes M \xrightarrow{\sim} \mathrm{Z}^n\,(N \otimes M)$ by considering the diagram

$$\begin{array}{ccc} (\mathrm{B}^n\, N) \otimes M & \xrightarrow{\sim} & (\mathrm{Z}^n\, N) \otimes M \\ \downarrow & & \downarrow \\ \mathrm{B}^n\,(N \otimes M) & \xrightarrow{\sim} & \mathrm{Z}^n\,(N \otimes M), \end{array}$$

whose left vertical arrow is clearly an isomorphism.

5th step: The assertion. Let X_i, $i \in I$, be a family of flat DG k-modules. For each i we choose a triangle

$$P_i \to X_i \to N_i \to SP_i$$

where N_i is acyclic, and P_i is closed. We may and will assume that P_i has projective components. If M is any left DG k-module, we have a commutative square

$$\begin{array}{ccc} P_i \otimes_k M & \to & X_i \otimes_k M \\ \uparrow & & \uparrow \\ P_i \otimes_k \boldsymbol{p}M & \to & X_i \otimes_k \boldsymbol{p}M \end{array}$$

whose bottom morphism and whose vertical morphisms are quasi-isomorphisms. Thus $N_i \otimes M$ is acyclic. By the third step, N_i is spliced up from pure exact sequences. Now products of pure exact sequences are pure exact (by [9, thm. 2.4]), so that $(\prod_{i\in I} N_i) \otimes_k M$ is acyclic as well for each DG module M, by the third step. In particular, if N is an acyclic left DG k-module, the third term of the sequence

$$(\prod_{i\in I} P_i) \otimes_k N \to (\prod_{i\in I} X_i) \otimes_k N \to (\prod_{i\in I} N_i) \otimes_k N \to S(\prod_{i\in I} P_i) \otimes_k N$$

is acyclic. By Chase's theorem and the second step, the same holds for the first term. The assertion follows.

References

[1] F. W. Anderson, K. R. Fuller, *Rings and categories of modules*, Graduate Texts in Mathematics **13**, Springer-Verlag, 1974.

[2] A. A. Beilinson, J. Bernstein, P. Deligne, *Faisceaux pervers*, Astérisque **100**, 1982.

[3] K. Bongartz, *Tilted algebras*, Representations of Algebras, Puebla 1980, Springer LNM **903** (1981), 26-38.

[4] M. Bökstedt, A. Neeman, *Homotopy limits in triangulated categories*, Comp. Math. **86** (1993), 209–234.

[5] J.-L. Brylinski, *Central localization in Hochschild homology*, J. Pure and Appl. Math. **57** (1989), 1–4.

[6] D. Burghelea, *Cyclic homology and the algebraic K-theory of spaces I*, proc. Summer Inst. on Alg. K-theory, Boulder Colorado 1983, Contemp. Math. **95** I (1986), 89–115.

[7] C. Cibils, *Hochschild homology of a quiver whose algebra has no oriented cycles*, Representation Theory of finite-dimensional algebras, Proceedings, Ottawa 1984, Springer, 1986, 55–59.

[8] C. Cibils, *Cyclic and Hochschild homology of 2-nilpotent algebras*, K-theory **4** (1990), 131–141.

[9] P. M. Cohn, *On the free product of associative rings*, Math. Z. **71**, 1959, 380–398.

[10] A. Connes, *Non-commutative differential geometry*, Publ. Math. IHES **62** (1985), 257–360.

[11] C. Faith, *Algebra: Rings, Modules and Categories I*, Grundlehren **190**, Springer-Verlag, 1973.

[12] P. Gabriel, M. Zisman, *Calculus of fractions and homotopy theory*, Ergebnisse der Mathematik, Springer-Verlag, 1967.

[13] P. Gabriel, A. Roiter, *Representations of finite-dimensional algebras*, Encyclopaedia of the Mathematical Sciences, Vol. 73, A. I. Kostrikin and I. V. Shafarevich (Eds.), Algebra VIII, Springer 1992.

[14] S. Geller, L. Reid, C. Weibel, *The cyclic homology and K-theory of curves*, J. reine angew. Math. **393** (1989), 39–90.

[15] T. Goodwillie, *Cyclic Homology, Derivations, and the free Loopspace*, Topology **24** (1985), 187–215.

[16] T. Goodwillie, *Relative algebraic K-theory and Cyclic Homology*, Ann. of Math. **124** (1986), 347–402.

[17] D. Happel, *On the derived Category of a finite-dimensional Algebra*, Comment. Math. Helv. **62**, 1987, 339–389.

[18] D. Happel, *Hochschild cohomology of finite-dimensional algebras*, Séminaire d'algèbre Paul Dubreil et Marie-Paule Malliavin, 39^e année (Paris 1987/1988), Springer LNM **1404** (1989), 108–126.

[19] D. Happel, C. M. Ringel, *Tilted algebras*, Representations of algebras, Puebla 1980, Trans. Am. Math. Soc. **274** (1982), 399-443.

[20] R. Hartshorne, *Residues and Duality*, Springer LNM **20**, 1966.

[21] G. Hochschild, *Relative homological algebra*, Trans. AMS **82** (1956), 246–269.

[22] K. Igusa, *Notes on the no loops conjecture*, J. Pure and Appl. Alg. **69** (1990), 161–176.

[23] L. Kadison, A relative cyclic cohomology theory useful for computation, C. R. Acad. Sci. Paris Sér. A-B **308** (1989), 569–573.

[24] Chr. Kassel, *Cyclic homology, comodules and mixed complexes*, J. Alg. **107** (1987), 195–216.

[25] Chr. Kassel, *K-théorie algébrique et cohomologie cyclique bivariantes*, C. R. Acad. Sci. Paris Sér. 1 **306** (1988), 799–802.

[26] Chr. Kassel, *Caractère de Chern bivariant*, K-Theory **3** (1989), 367–400.

[27] S. König, *Tilting complexes, perpendicular categories and recollements of derived module categories of rings*, J. Pure Appl. Algebra **73** (1991), 211–232.

[28] B. Keller, D. Vossieck, *Aisles in derived categories*, Bull. Soc. Math. Belg. **40** (1982), 2, ser. A, 239–253.

[29] B. Keller, *Chain complexes and stable categories*, Manus. math. **67** (1990), 379–417.

[30] B. Keller, *Derived categories and universal problems*, Comm. in Alg. **19** (1991), 699–747.

[31] B. Keller, *Deriving DG categories*, Ann. scient. Ec. Norm. Sup., 4^e série **27** (1994), 63–102.

[32] B. Keller, *A remark on tilting theory and DG algebras*, Manus. Math. **79** (1993), 247–252.

[33] B. Keller, *A remark on the generalized smashing conjecture*, Manus. Math. **84** (1994), 193–198.

[34] B. Keller, *Tilting theory and differential graded algebras*, Proceedings of the Nato ARW on Representations of algebras and related topics, edited by V. Dlab and L. Scott, Kluwer, 1994.

[35] B. Keller, *On the construction of triangle equivalences*, Lectures at the Workshop on derived equivalences (Pappenheim 1994), Preprint, 1994.

[36] H. Lenzing, *Nilpotente Elemente in Ringen von endlicher globaler Dimension*, Math. Z. **108** (1969), 313-324.

[37] J.-L. Loday, *Cyclic homology: a survey*, Banach Center Publications (Warsaw) **18** (1986), 285–307.

[38] J.-L. Loday, *Cyclic Homology*, Grundlehren 301, Springer-Verlag, 1992.

[39] J.-L. Loday, D. Quillen, *Cyclic homology and the Lie algebra homology of matrices*, Comment. Math. Helv. **59** (1984), 565–591.

[40] S. Maclane, *Homology*, Springer-Verlag, 1963.

[41] R. McCarthy, *L'équivalence de Morita et l'homologie cyclique*, C. R. Acad. Sci. Paris Sér. A-B **307** (1988), 211–215.

[42] R. McCarthy, *The cyclic homology of an exact category*, J. Pure and Appl. Alg. **93** (1994), 251–296.

[43] J. Milnor, *On Axiomatic Homology Theory*, Pacific J. Math. **12** (1962), 337-341.

[44] B. Mitchell, *Rings with several objects*, Avd. in Math. **8** (1972), 1–161.

[45] A. Neeman, *The Connection between the K-theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel*, Ann. Scient. ENS **25** (1992), 547–566.

[46] D. Quillen, *Higher Algebraic K-theory I*, Springer LNM **341**, 1973, 85–147.

[47] D. C. Ravenel, *Localization with respect to certain periodic homology theories*, Amer. J. of Math., **106**, 1984, 351–414.

[48] J. Rickard, *Morita theory for Derived Categories*, J. London Math. Soc. **39** (1989), 436–456.

[49] J. Rickard, *Derived equivalences as derived functors*, J. London Math. Soc. **43** (1991), 37–48.

[50] N. Spaltenstein, *Resolutions of unbounded complexes*, Compositio Mathematica **65** (1988), 121–154.

[51] R. W. Thomason, T. Trobaugh, *Higher Algebraic K-Theory of Schemes and of Derived Categories*, in *The Grothendieck Festschrift*, III, Birkhäuser, Progress in Mathematics **87** (1990), p. 247–436.

[52] J.-L. Verdier, *Catégories dérivées, état 0*, SGA 4 1/2, Springer LNM **569**, 1977, 262–311.

[53] M. Vigué-Poirrier, D. Burghelea, *A model for cyclic homology and algebraic K-theory of 1-connected topological spaces*, J. Diff. Geom. **22** (1985), 243–253.

[54] C. A. Weibel, S. C. Geller, *Etale descent for Hochschild and cyclic homology*, Comment. Math. Helv. **66** (1991), 368–388.

[55] C. A. Weibel, Dongyuan Yao, *Localization for the K-theory of Noncommutative rings*, Algebraic K-theory, commutative algebra, and algebraic geometry (Santa Margherita Ligure, 1989), Contemporary Mathematics **125** (1992), 219–230.

[56] C. A. Weibel, *Introduction to homological algebra* (Cambridge Studies in Adv. Math. 38), Cambridge University Press, 1994.

[57] M. Wodzicki, *Excision in cyclic homology and in rational algebraic K-theory*, Ann. of Math. **129** (1989), 591–639.

[58] Dongyuan Yao, *Higher algebraic K-theory of admissible abelian categories and localization theorems*, J. Pure Appl. Alg. **77** (1992), 263–339.

Bernhard Keller
U.F.R. de Mathématiques
U.R.A. 748 du CNRS
Université Paris 7
2, place Jussieu,
75251 Paris Cedex 05, France
keller@mathp7.jussieu.fr
http://www.mathp7.jussieu.fr/ura748/personalpages/keller