KOSZUL DUALITY AND CODERIVED CATEGORIES (AFTER K. LEFÈVRE)

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Abstract. This is a brief report on a part of Chapter 2 of K. Lefèvre's thesis [5]. We sketch a framework for Koszul duality [1] where the Koszul dual algebra is replaced by a coalgebra. This allows us to free ourselves from many assumptions (e.g. finiteness assumptions) and leads to clean statements about equivalences between the derived category and a suitably defined coderived category. These results are related to work by G. Fløystad [2]. Our approach is based on classical developments in topology [6] [4] and inspired by [3].

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1. Koszul duality (after [1])

Let k be a field and

$$A = k \oplus A_1 \oplus A_2 \oplus \cdots$$

a graded k-algebra with finite-dimensional components A_i, $i \in \mathbf{N}$. Let $\mathsf{Grmod}\, A$ denote the category of graded (right) A-modules (with the morphisms of degree 0). For a module $M \in \mathsf{Grmod}\, A$ and $n \in \mathbf{Z}$, the shifted module $M\langle n\rangle$ is defined by

$$M\langle n\rangle_p = M_{n+p}\ ,\ p \in \mathbf{Z}.$$

Assume that A is a *Koszul algebra*, *i.e.* that there is a projective resolution

$$\ldots \to P^{-i} \to \ldots \to P^0 \to 0 \qquad (1.0.1)$$

of the trivial module k in $\mathsf{Grmod}\, A$ such that P^{-i} is generated in degree i. We imagine such a resolution as a bigraded object, where the *differential degree* is drawn horizontally and the *Adams degree* (=internal degree) vertically. Put

$$E(A) = \bigoplus_{i=0}^{\infty} \mathsf{Ext}^i_{\mathsf{Grmod}\, A}(k, k\langle i\rangle)$$

and call this algebra the *Koszul dual algebra*.

Theorem. [1]

a) *The graded algebra $E(A)$ is Koszul and there is a canonical isomorphism $A \xrightarrow{\sim} E(E(A))$ (*cf.* section 2.8, Proposition 2.9.1 and Corollary 2.3.3 of [1]).*

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b) *There is a canonical equivalence of triangulated categories*

$$\mathcal{D}^{\downarrow}_{gr}(A) \xrightarrow{\sim} \mathcal{D}^{\uparrow}_{gr}(E(A)).$$

*It sends the simple k to the projective $E(A)$ and the graded k-dual of the free module A to the simple k (*cf.* Theorem 2.12.1 and Theorem 1.2.6 of* [1]*).*

Here $\mathcal{D}^{\downarrow}_{gr}(A)$ denotes the full subcategory of the unbounded derived category $\mathcal{D}(\mathsf{Grmod}\, A)$ of the abelian category $\mathsf{Grmod}\, A$ whose objects are the complexes K such that for some $N \gg 0$, we have

$$K^p_q \neq 0 \Rightarrow (p \geq -N \text{ or } p + q \leq N).$$

Analogously, $\mathcal{D}^{\uparrow}_{gr}(E(A))$ denotes the full subcategory of $\mathcal{D}(\mathsf{Grmod}\, E(A))$ whose objects are the complexes K such that for some $N \gg 0$, we have

$$K^p_q \neq 0 \Rightarrow (p \leq N \text{ or } p + q \geq -N).$$

As an example, let V be a finite-dimensional vector space and $A = SV$ the symmetric algebra on V. Then for the resolution 1.0.1, we can take the Koszul resolution

$$\ldots \to \Lambda^p V \otimes SV\langle -p \rangle \to \ldots \to \Lambda^0 V \otimes SV \to 0 \tag{1.0.2}$$

with the differential given by

$$d(v_1 \ldots v_p \otimes u) = \sum_{i=1}^{p} (-1)^{p-i} v_1 \ldots \widehat{v_i} \ldots v_p \otimes v_i u.$$

Then $E(A)$ identifies with the exterior algebra $\Lambda(DV)$ on the k-dual space DV of V. According to the theorem, we have an equivalence

$$F : \mathcal{D}^{\downarrow}_{gr}(SV) \xrightarrow{\sim} \mathcal{D}^{\uparrow}_{gr}(\Lambda DV).$$

Can the equivalence F be extended to an equivalence $\widetilde{F}$ between the whole derived categories ?

$$\begin{array}{ccc}
\mathcal{D}^{\downarrow}_{gr}(SV) & \xrightarrow{F} & \mathcal{D}^{\uparrow}_{gr}(\Lambda DV) \\
\downarrow & & \downarrow \\
\mathcal{D}(\mathsf{Grmod}\, SV) & \overset{\widetilde{F}}{\dashrightarrow} & \mathcal{D}(\mathsf{Grmod}\, \Lambda DV)
\end{array}$$

Suppose that such an equivalence $\widetilde{F}$ exists. It is not hard to see that it has to take the free module SV to the doubly shifted trivial module $k\langle n \rangle[-n]$, where the braces [ ] indicate a shift of the differential degree. This is impossible since SV is compact in $\mathcal{D}(\mathsf{Grmod}\, SV)$, *i.e.* the functor

$$\mathsf{Hom}_{\mathcal{D}(\mathsf{Grmod}\, A)}(SV, ?) : \mathcal{D}(\mathsf{Grmod}\, SV) \to \mathsf{Mod}\, k$$

commutes with infinite direct sums, but the object k is non compact in $\mathcal{D}(\mathsf{Grmod}\, \Lambda DV)$.

In what follows, our aim is to present a setting where we free ourselves from the following restrictions

- A is (Adams-)graded with finite-dimensional components,
- A is a Koszul algebra,
- there is an equivalence only between certain subcategories of the derived categories.

2. Data

2.1. **An algebra.** Let A be a differential graded (=dg) algebra, *i.e.* A is an associative unital $\mathbf{Z}$-graded algebra

$$A = \bigoplus_{p \in \mathbf{Z}} A^p$$

endowed with a differential d of degree $+1$ such that the Leibniz rule holds: We have

$$d(ab) = d(a)b + (-1)^p a d(b)$$

for all $a \in A^p$ and all $b \in A$. Note that A is differentially graded but not Adams graded. We assume that A is endowed with an augmentation, *i.e.* a morphism of dg algebras $\varepsilon : A \to k$. Therefore A decomposes as

$$A = k \oplus \overline{A}\,,$$

where $\overline{A}$ is the kernel of ε. Note that we do not require the components of A to be finite-dimensional.

2.2. **A coalgebra.** Let C be a dg coalgebra, *i.e.* C is a $\mathbf{Z}$-graded coalgebra endowed with a differential d of degree $+1$ such that

$$\Delta \circ d = (d \otimes \mathbf{1} + \mathbf{1} \otimes d) \circ \Delta.$$

We assume that C is endowed with a co-augmentation, *i.e.* a morphism of dg coalgebras $\varepsilon : k \to C$. Then C decomposes as

$$C = k \oplus \overline{C}\,,$$

where $\overline{C}$ is the cokernel of ε. Moreover, we assume that C is *cocomplete*, which means that

$$\overline{C} = \bigcup_{n \geq 2} \mathsf{ker}(\overline{C} \to \overline{C}^{\otimes n})\,,$$

i.e. each element of $\overline{C}$ is annihilated by a high enough iterate of the map induced by the comultiplication of C. Note that this implies that the k-dual algebra DC is a complete local algebra.

2.3. **A twisting cochain.** Let $\tau : C \to A$ be a *twisting cochain*, *i.e.* τ is a k-linear homogeneous map of degree $+1$ such that $\varepsilon_A \circ \tau \circ \varepsilon_C = 0$ and

$$d \circ \tau + \tau \circ d + \mu \circ (\tau \otimes \tau) \circ \Delta = 0\,,$$

where μ is the multiplication of A. In other words, the map τ satisfies $\varepsilon(\tau) = 0$ and $d(\tau) + \tau * \tau = 0$ in the dg convolution algebra $\mathsf{Hom}_k(C, A)$ of homogeneous maps from C to A.

2.4. **Example.** Let V be an (arbitrary) vector space and $A = SV$ the symmetric algebra on V considered as a dg algebra concentrated in differential degree 0. The exterior algebra ΛV, graded such that V is in degree -1, admits a unique structure of super Hopf algebra such that $\Delta(v) = v \otimes 1 + 1 \otimes v$ for all $v \in V$. Let C be the graded coalgebra ΛV endowed with the differential $d = 0$. Thus, the underlying complex of C is

$$\ldots \to \Lambda^p V \to \ldots \to \Lambda^1 V \to \Lambda^0 V \to 0 \to \ldots\,,$$

where the elements of V are of degree -1. We define the twisting cochain $\tau : C \to A$ to have only one non-vanishing component and this component identifies $V \subset \Lambda V$ with $V \subset SV$. Is is trivial to check the conditions of the preceding paragraphs.

3. Adjoint functors

We will construct a pair of adjoint functors between the category of dg modules over A and the category of certain dg comodules over C. Recall that a *dg A-module* is a $\mathbf{Z}$-graded A-module M endowed with a differential of degree $+1$ such that

$$d(ma) = d(m)a + (-1)^p md(a)$$

for all $m \in M^p$ and all $a \in A$. The notion of *dg C-comodule* is defined dually. A dg C-comodule N is *cocomplete* if it is the union of the kernels of the iterates

$$N \to N \otimes \overline{C}^{\otimes n}$$

of the map induced by the comultiplication.

For a dg A-module M, we define the *twisted tensor product* $M \otimes_\tau C$ to be the dg C-comodule whose underlying graded comodule is the graded tensor product $M \otimes_k C$ and whose differential is

$$d = d_M \otimes \mathbf{1} + \mathbf{1} \otimes d_C + (\mu \otimes \mathbf{1})(\mathbf{1} \otimes \tau \otimes \mathbf{1})(\mathbf{1} \otimes \Delta) ,$$

where μ is the multiplication of M. Similarly, for a dg C-comodule N, one defines the dg A-module $N \otimes_\tau A$.

In the above example 2.4, the twisted tensor product $C \otimes_\tau A$ is nothing but the Koszul complex 1.0.2.

Lemma. *We have adjoint functors*

$$\begin{array}{c} \{\textit{dg } A\textit{-modules}\} \\ L = ? \otimes_\tau A \uparrow \downarrow ? \otimes_\tau C = R \\ \{\textit{cocomplete dg } C\textit{-comodules}\} \end{array}$$

4. (Co-)derived categories

Let $\mathcal{D}A$ be the *derived category of A*, *i.e.* the localization of the category of dg A-modules at the class of all quasi-isomorphisms.

To define the coderived category of C, we will need to replace the quasi-isomorphisms by a different class of morphisms. To define these, we need the *cobar construction* ΩC: this is the graded tensor algebra on the shift $\overline{C}[-1]$ endowed with the unique differential such that for each homogeneous element $c \in \overline{C}[-1]$, we have

$$d(c) = -d_C(c) + \sum (-1)^{|c_{(1)}|} c_{(1)} \otimes c_{(2)} ,$$

where we have used Sweedler's notation and $|c_{(1)}|$ denotes the degree of $c_{(1)}$. The cobar construction is endowed with the *canonical twisting cochain* $\tau_0 : C \to \Omega C$ given by the evident map. We define a morphism $f : M \to N$ of dg C-comodules to be a *weak equivalence* if its image under the functor $? \otimes_{\tau_0} \Omega C$ defined in the preceding section is a quasi-isomorphism. In other words, the morphism

$$f \otimes_{\tau_0} \Omega C : M \otimes_{\tau_0} \Omega C \to N \otimes_{\tau_0} \Omega C$$

should be a quasi-isomorphism. We define the *coderived category $\mathcal{D}C$ of C* to be the localization of the category of cocomplete dg C-comodules at the class of all weak equivalences.

Proposition. *We have a pair of induced adjoint functors*

$$\begin{array}{c} \mathcal{D}A \\ L = ? \otimes_\tau A \uparrow \downarrow ? \otimes_\tau C = R \\ \mathcal{D}C \end{array}$$

Theorem. [5, Ch. 2] *The following are equivalent*

(i) *The functors L and R are equivalences.*

(ii) *The canonical morphism*

$$A \otimes_\tau C \otimes_\tau A \to A$$

is a quasi-isomorphism.

(iii) *The map τ induces a quasi-isomorphism $\Omega C \to A$.*

Moreover, in this case, A is determined by C up to quasi-isomorphism; C is determined by A up to weak equivalence[1]*; we have*

$$H_* C = \mathsf{Tor}^A_*(k,k) \text{ and } H^* A = \mathsf{Ext}^*_C(k,k).$$

We define (A, C, τ) to be a *Koszul-Moore triple* if the conditions (i)-(iii) hold. The theorem shows that for each given dg coalgebra C, there is at least one Koszul-Moore triple $(\Omega C, C, \tau_0)$. Dually, one can show that the bar construction [4] yields a Koszul-Moore triple for each given algebra A.

Let us consider the example 2.4, where $A = SV$, $C = \Lambda V$ and τ is the natural morphism. Then (A, C, τ) is indeed a Koszul-Moore triple by condition (ii): here, the canonical morphism is the bimodule Koszul resolution of $A = SV$. Thus we do have

$$\mathcal{D}(A) \xrightarrow{\sim} \mathcal{D}(C).$$

Suppose that $\dim V < \infty$. Then $\dim C < \infty$ and each C-comodule is cocomplete. Moreover, we have an isomorphism of categories

$$\{\text{dg } C\text{-comodules}\} \xrightarrow{\sim} \{\text{left } DC\text{-modules}\}$$

given by sending a dg C-comodule N to the dg left DC-module with the same underlying space and whose multiplication is given by the natural composition

$$DC \otimes N \to DC \otimes N \otimes C \to DC \otimes C \otimes N \to N.$$

So we obtain an equivalence

$$\mathcal{D}(SV) \xrightarrow{\sim} \{\text{left dg } \Lambda(DV)\text{-modules}\}[\mathcal{W}^{-1}]$$

where $\mathcal{W}$ denotes the class of morphisms which corresponds to the weak equivalences. This equivalence sends a dg SV-module M to the left $\Lambda(DV)$-module $N \otimes \Lambda V$ endowed with the Koszul differential.

To see that the weak equivalences form a strictly smaller class than the quasi-isomorphisms, let us further specialize to the case where V is of dimension 1, *i.e.* $V = kx$ and $SV = k[x]$. We obtain an equivalence

$$\mathcal{D}(k[x]) \xrightarrow{\sim} \{\text{dg left } k[\xi]\text{-modules}\}[\mathcal{W}^{-1}]$$

where ξ is of degree 1 and $d\xi = 0$. The equivalence sends the module

$$k_\lambda = k[x]/(x-\lambda)\,,\ \lambda \in k\,,$$

to the dg module

$$\ldots \to 0 \to k \xrightarrow{d} k \to 0 \to \ldots$$

concentrated in degrees -1 and 0, where d is the multiplication by λ and ξ acts by the graded endomorphism of degree 1 which is given by the identity of k. Note that for each $\lambda \neq 0$, the image of k_λ is quasi-isomorphic to 0, which corresponds to the fact that

$$\mathsf{Tor}^{k[x]}_*(k, k_\lambda) = 0.$$

However, the image of k_λ is never weakly equivalent to 0, since $k_\lambda \neq 0$.

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[1]A morphism of dg augmented coalgebras $f : C \to C'$ is a weak equivalence if Ωf is a quasi-isomorphism

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