Cluster algebras and cluster monomials

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Abstract. Cluster algebras were invented by Sergey Fomin and Andrei Zelevinsky at the beginning of the year 2000. Their motivations came from Lie theory and more precisely from the study of the so-called canonical bases in quantum groups and that of total positivity in algebraic groups. Since then, cluster algebras have been linked to many other subjects ranging from higher Teichmüller theory through discrete dynamical systems to combinatorics, algebraic geometry and representation theory. According to Fomin-Zelevinsky's philosophy, each cluster algebra should admit a 'canonical' basis, which should contain the cluster monomials. This led them to formulate, about ten years ago, the conjecture on the linear independence of the cluster monomials. In these notes, we give a concise introduction to cluster algebras and sketch the ingredients of a proof of the conjecture. The proof is valid for all cluster algebras associated with quivers and was obtained in recent joint work with G. Cerulli Irelli, D. Labardini-Fragoso and P.-G. Plamondon.

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1. Introduction

1.1. Context. Cluster algebras, invented [37] by Sergey Fomin and Andrei Zelevinsky around the year 2000, are commutative algebras whose generators and relations are constructed in a recursive manner. Among these algebras, there are the algebras of homogeneous coordinates on the Grassmannians, on the flag varieties and on many other varieties which play an important role in geometry and representation theory. Fomin and Zelevinsky's main aim was to set up a combinatorial framework for the study of the so-called canonical bases which these algebras possess [60] [78] and which are closely related to the notion of total positivity [79] [34] in the associated varieties. It has rapidly turned out that the combinatorics of cluster algebras also appear in many other subjects, for example in

- Poisson geometry [49] [50] [51] [52] [8] . . . ;
- discrete dynamical systems [25] [40] [57] [61] [65] [63] [74] . . . ;
- higher Teichmüller spaces [28] [29] [30] [31] [32] . . . ;
- combinatorics and in particular the study of polyhedra like the Stasheff associahedra [18] [19] [35] [36] [56] [73] [80] [82] [81] [96] . . . ;

- commutative and non commutative algebraic geometry and in particular the study of stability conditions in the sense of Bridgeland [9], Calabi-Yau algebras [53] [58] , Donaldson-Thomas invariants in geometry [59] [71] [72] [92] [99] … and in string theory [1] [2] [12] [13] [14] [43] [44] [45] …;
- in the representation theory of quivers and finite-dimensional algebras, cf. for example the survey articles [3] [5] [48] [65] [76] [94] [93] [95] …

as well as in mirror symmetry [54], KP solitons [70], hyperbolic 3-manifolds [84], … . We refer to the introductory articles [34] [39] [101] [102] [103] [104] and to the cluster algebras portal [33] for more information on cluster algebras and their links with other subjects in mathematics (and physics).

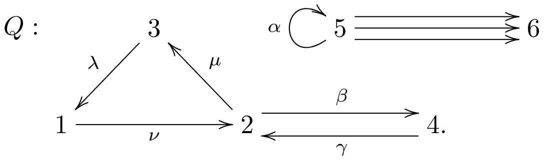
The link between cluster algebras and quiver representations follows the spirit of categorification: One tries to interpret cluster data as combinatorial (in some cases K-theoretic) invariants associated with categories of representations. Thanks to the rich structure of these categories, one can then hope to prove results on cluster algebras which seem beyond the scope of the purely combinatorial methods. At the end of this article, we present a recent result of this type, namely the linear independence of the cluster monomials conjectured in [37] and recently proved in [16] (for the skew-symmetric case). Our proof is based on a triangulated category constructed from a category of quiver representations, the so-called cluster category.

In section 2, we will review the definition of cluster algebras and Fomin-Zelevinsky's classification theorem for cluster-finite cluster algebras [38]. We will then state the theorem on the linear independence of cluster monomials. In section 3, we briefly describe the main tools used in its proof: the cluster category and the cluster character.

2. Cluster algebras

The cluster algebras we will be interested in are associated with antisymmetric matrices with integer coefficients. Instead of using matrices, we will use quivers (without loops and 2-cycles), since they are easy to visualize and well-suited to our later purposes.

2.1. Quivers and quiver mutation. Let us recall that a *quiver* Q is an oriented graph. Thus, it is a quadruple given by a set Q_0 (the set of vertices), a set Q_1 (the set of arrows) and two maps $s : Q_1 \to Q_0$ and $t : Q_1 \to Q_0$ which take an arrow to its source respectively its target. Our quivers are 'abstract graphs' but in practice we draw them as in this example:



A *loop* in a quiver Q is an arrow α whose source coincides with its target; a 2-*cycle* is a pair of distinct arrows $\beta \neq \gamma$ such that the source of β equals the target of γ and vice versa. It is clear how to define 3-*cycles*, *connected components* .... A quiver is *finite* if both, its set of vertices and its set of arrows, are finite.

By convention, in the sequel, *by a quiver we always mean a finite quiver without loops nor* 2-*cycles whose set of vertices is the set of integers from* 1 *to* n *for some* $n \geq 1$. Up to an isomorphism fixing the vertices, such a quiver Q is given by the antisymmetric matrix $B = B_Q$ whose coefficient b_{ij} is the difference between the number of arrows from i to j and the number of arrows from j to i for all $1 \leq i, j \leq n$. Conversely, each antisymmetric matrix B with integer coefficients comes from a quiver.

Let Q be a quiver and k a vertex of Q. The *mutation* $\mu_k(Q)$ is the quiver obtained from Q as follows:

1) for each subquiver $i \xrightarrow{\beta} k \xrightarrow{\alpha} j$, we add a new arrow $[\alpha\beta] : i \to j$;

2) we reverse all arrows with source or target k;

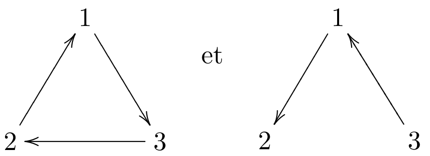
3) we remove the arrows in a maximal set of pairwise disjoint 2-cycles.

If B is the antisymmetric matrix associated with Q and B' the one associated with $\mu_k(Q)$, we have

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \ ; \\ b_{ij} + \operatorname{sgn}(b_{ik}) \max(0, b_{ik}b_{kj}) & \text{else.} \end{cases}$$

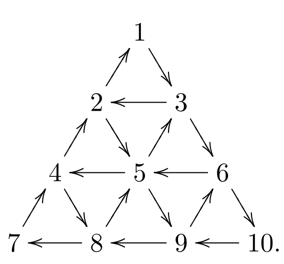
This is the *matrix mutation rule* for antisymmetric (more generally: antisymmetrizable) matrices introduced by Fomin-Zelevinsky in [37], cf. also [41].

One checks easily that μ_k is an involution. For example, the quivers



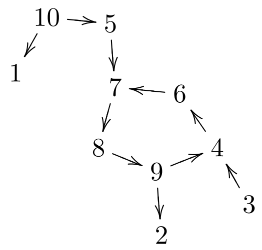
(2.1)

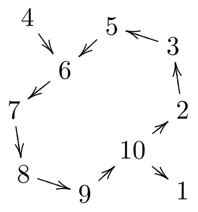
are linked by a mutation at the vertex 1. Notice that these quivers are drastically different: The first one is a cycle, the second one the Hasse diagram of a linearly ordered set. Now let us consider the quiver

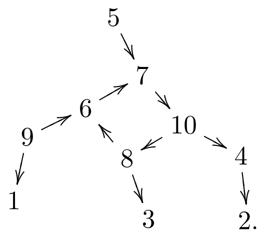


(2.2)

One can show [67] that it is impossible to transform it into a quiver without oriented cycles by a finite sequence of mutations. However, its mutation class (the set of all quivers obtained from it by iterated mutations) contains many quivers with just one oriented cycle, for example







In fact, in this example, the mutation class is finite and it can be completely computed using, for example, [64]: It consists of 5739 quivers up to isomorphism. The quiver (2.4) belongs to a family which appears in the study of the cluster algebra structure on the coordinate algebra of the subgroup of upper unitriangular matrices in $SL_n(\mathbb{C})$, *cf.* [47]. The study of coordinate algebras on varieties associated with reductive algebraic groups (in particular, double Bruhat cells) has provided a major impetus for the development of cluster algebras, *cf.* [7].

2.2. Seeds and mutations. Fix an integer $n \geq 1$. A *seed* is a pair (R, u), where

a) R is a finite quiver without loops or 2-cycles with vertex set $\{1, \ldots, n\}$;

b) u is a free generating set $\{u_1, \ldots, u_n\}$ of the field $\mathbb{Q}(x_1, \ldots, x_n)$ of fractions of the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$ in n indeterminates.

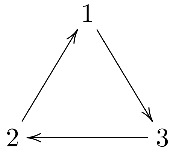
Notice that in the quiver R of a seed, all arrows between any two given vertices point in the same direction (since R does not have 2-cycles). Let (R, u) be a seed and k a vertex of R. The *mutation* $\mu_k(R, u)$ of (R, u) at k is the seed (R', u'), where

a) $R' = \mu_k(R)$; b) u' is obtained from u by replacing the element u_k with

$$u'_k = \frac{1}{u_k}\left(\prod_{\text{arrows } i \to k} u_i + \prod_{\text{arrows } k \to j} u_j\right). \tag{2.3}$$

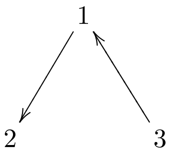
In the *exchange relation* (2.3), if there are no arrows from i with target k, the product is taken over the empty set and equals 1. It is not hard to see that $\mu_k(R, u)$ is indeed a seed and that μ_k is an involution: we have $\mu_k(\mu_k(R, u)) = (R, u)$.

2.3. Examples of seed mutations. Let R be the cyclic quiver

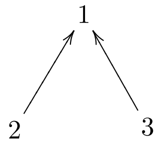


(2.4)

and $u = \{x_1, x_2, x_3\}$. If we mutate at $k = 1$, we obtain the quiver



and the set of fractions given by $u'_1 = (x_2+x_3)/x_1$, $u'_2 = u_2 = x_2$ and $u'_3 = u_3 = x_3$. Now, if we mutate again at 1, we obtain the original seed. This is a general fact: Mutation at k is an involution. If, on the other hand, we mutate (R', u') at 2, we obtain the quiver



and the set u'' given by $u''_1 = u'_1 = (x_2+x_3)/x_1$, $u'_2 = \frac{x_1+x_2+x_3}{x_1 x_2}$ and $u''_3 = u'_3 = x_3$.

2.4. Definition of cluster algebras. Let Q be a finite quiver without loops or 2-cycles with vertex set $\{1, \ldots, n\}$. Consider the seed (Q, x) consisting of Q and the set x formed by the variables $x_1, \ldots, x_n$. Following [37] we define

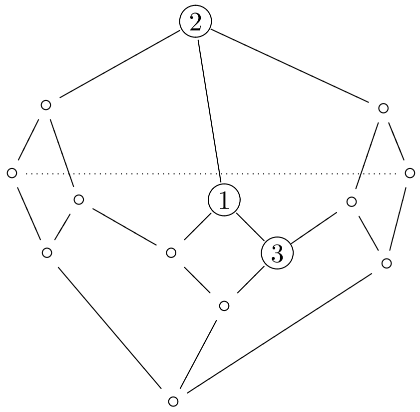
- the *clusters with respect to* Q to be the sets u appearing in seeds (R, u) obtained from (Q, x) by iterated mutation,
- the *cluster variables* for Q to be the elements of all clusters,
- the *cluster algebra* $\mathcal{A}_Q$ to be the $\mathbb{Q}$-subalgebra of the field $\mathbb{Q}(x_1, \ldots, x_n)$ generated by all the cluster variables.

Thus the cluster algebra consists of all $\mathbb{Q}$-linear combinations of monomials in the cluster variables. It is useful to define another combinatorial object associated with this recursive construction: The *exchange graph* associated with Q is the graph whose vertices are the seeds modulo simultaneous renumbering of the vertices and the associated cluster variables and whose edges correspond to mutations.

2.5. The example A_3. Let us consider the quiver

$$Q:\ 1 \longrightarrow 2 \longrightarrow 3$$

obtained by endowing the Dynkin diagram A_3 with a linear orientation. By applying the recursive construction to the initial seed (Q, x) one finds exactly fourteen seeds (modulo simultaneous renumbering of vertices and cluster variables). These are the vertices of the exchange graph, which is isomorphic to the third Stasheff associahedron [98] [19]:



The vertex labeled 1 corresponds to (Q, x), the vertex 2 to $\mu_2(Q, x)$, which is given by

$$\begin{tikzcd} 1 \arrow[rr, bend left] & 2 \arrow[l] & 3 \arrow[l] \end{tikzcd} \; , \; \{x_1, \frac{x_1 + x_3}{x_2}, x_3\} \, ,$$

and the vertex 3 to $\mu_1(Q, x)$, which is given by

$$\begin{tikzcd} 1 & 2 \arrow[l] \arrow[r] & 3 \end{tikzcd} \; , \; \{\frac{1 + x_3}{x_1}, x_2, x_3\}.$$

We find a total of 9 cluster variables, namely

$$x_1 \, , \; x_2 \, , \; x_3, \frac{1 + x_2}{x_1} \, , \; \frac{x_1 + x_3 + x_2x_3}{x_1x_2} \, , \; \frac{x_1 + x_1x_2 + x_3 + x_2x_3}{x_1x_2x_3} \, ,$$
$$\frac{x_1 + x_3}{x_2} \, , \; \frac{x_1 + x_1x_2 + x_3}{x_2x_3} \, , \; \frac{1 + x_2}{x_3} \, .$$

Again we observe that all denominators are monomials. Notice also that $9 = 3+6$ and that 3 is the rank of the root system associated with A_3 and 6 its number of positive roots. Moreover, if we look at the denominators of the non trivial cluster variables (those other than x_1, x_2, x_3), we see a natural bijection with the positive roots

$$\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2, \alpha_2 + \alpha_3, \alpha_3$$

of the root system of A_3, where α_1, α_2, α_3 denote the three simple roots.

2.6. Cluster algebras with finitely many cluster variables. The phenomena observed in the above example are explained by the following key theorem: Let Q be a connected quiver. If its underlying graph is a simply laced Dynkin diagram Δ, we say that Q is a *Dynkin quiver of type* Δ.

Theorem 2.7 ([38]). a) *Each cluster variable of $\mathcal{A}_Q$ is a Laurent polynomial with integer coefficients [37].*

b) *The cluster algebra $\mathcal{A}_Q$ has only a finite number of cluster variables if and only if Q is mutation equivalent to a Dynkin quiver R. In this case, the underlying graph Δ of R is unique up to isomorphism and is called the* cluster type of Q.

c) *If Q is a Dynkin quiver of type Δ, then the non initial cluster variables of $\mathcal{A}_Q$ are in bijection with the positive roots of the root system Φ of Δ; more precisely, if $\alpha_1, \ldots, \alpha_n$ are the simple roots, then for each positive root $\alpha = d_1\alpha_1 + \cdots + d_n\alpha_n$, there is a unique non initial cluster variable X_α whose denominator is $x_1^{d_1} \ldots x_n^{d_n}$.*

2.8. Two conjectures on cluster algebras. A *cluster monomial* is a product of non negative powers of cluster variables belonging to the same cluster. The construction of a 'canonical basis' of the cluster algebra $\mathcal{A}_Q$ is an important and largely open problem, cf. for example [37] [97] [26] [15]. It is expected that such a basis should contain all cluster monomials. Whence the following conjecture.

Conjecture 2.9 ([37]). *The cluster monomials are linearly independent over the field $\mathbb{Q}$.*

If Q is a Dynkin quiver, one knows [11] that the cluster monomials form a basis of $\mathcal{A}_Q$. If Q is *acyclic*, i.e. does not have any oriented cycles, the conjecture follows from a theorem by Geiss-Leclerc-Schröer [46], who show the existence of a 'generic basis' containing the cluster monomials. The conjecture has also been shown for classes of cluster algebras with coefficients (cf. section 4.1 of [62]), for example in the papers [42] [46] [22] [21]. It is known for all cluster algebras associated with surfaces [36]. It was proved for all cluster algebras whose exchange matrix is of full rank in [24] and [89]. In fact, the condition on the rank may be dropped:

Theorem 2.10 ([16]). *The conjecture holds for all cluster algebras associated with quivers.*

We will outline the proof of the theorem below. In [16] it is proved more generally for all skew-symmetric cluster algebras with coefficients in a semi-field. Its generalization to skew-symmetrizable cluster algebras is still open. Before embarking on the proof, let us mention another important and largely open conjecture.

Conjecture 2.11 ([38]). *The cluster variables are Laurent polynomials with non negative integer coefficients in the variables of each cluster.*

For quivers with two vertices, an explicit and manifestly positive formula for the cluster variables is given in [77]. The technique of monoidal categorification developed by Leclerc [75] and Hernandez-Leclerc [55] has recently allowed to prove the conjecture first for the quivers of type A_n and D_4, cf. [55], and then for each bipartite quiver [85], i.e. a quiver where each vertex is a source or a sink. The positivity of all cluster variables with respect to the initial seed of an acyclic quiver is shown by Fan Qin [91] and by Nakajima [85, Appendix]. This is also proved by Efimov [27], who moreover shows the positivity of all cluster variables belonging to an acyclic seed with respect to the initial variables of an arbitrary quiver. Efimov combines the techniques of [72] with those of [83]. A proof of the full conjecture for acyclic quivers using Nakajima quiver varieties is announced by Kimura–Qin [69]. The conjecture has been shown in a combinatorial way by Musiker-Schiffler-Williams [82] for all the quivers associated with triangulations of surfaces (with boundary and marked points) and by Di Francesco-Kedem [25] for the quivers and the cluster variables associated with the T-system of type A, with respect to the initial cluster.

We refer to [39] and [41] for numerous other conjectures on cluster algebras and to [24], cf. also [83] and [90] [89], for the solution of a large number of them using additive categorification.

3. On the proof of the independence conjecture

3.1. The proper Laurent property. Let x be a set of indeterminates $x_1, \ldots, x_n$. Recall that a *monomial* in x is a product of non negative powers of the x_i. A *Laurent monomial* in x is a product of integer powers of the x_i. It is *proper* if at least one of the factors x_i appears with a strictly negative exponent. A *proper Laurent polynomial* in x is a $\mathbb{Q}$-linear combination of proper Laurent monomials. Notice that the space of Laurent polynomials in x decomposes as the direct sum

$$P(x) \oplus L(x)$$

of the space of polynomials $P(x)$ and the space $L(x)$ of proper Laurent polynomials.

Now let Q be a quiver and $\mathcal{A}_Q$ the associated cluster algebra. Following [17], we define $\mathcal{A}_Q$ to have the *proper Laurent property* if, for all clusters u and v of $\mathcal{A}_Q$, each monomial in v containing at least one variable not in u with an exponent > 0 is a proper Laurent polynomial in u. Thus, these monomials lie in $L(u)$ and there cannot be any linear relation between them and the monomials in u. This yields the following lemma.

Lemma 3.2 ([17]). *If $\mathcal{A}_Q$ has the proper Laurent property, then its cluster monomials are linearly independent.*

In the following section, we describe explicit expressions for the cluster monomials which allow one to check that the proper Laurent property holds for the cluster algebra $\mathcal{A}_Q$ associated with an arbitrary quiver.

3.3. Cluster categories. Let Q be a quiver (in the sense of our convention in section 2.1). Let W be a generic potential on Q in the sense of [23]. The cluster category $\mathcal{C}_{Q,W}$ of (Q, W) was defined by Claire Amiot [4]. She showed that for acyclic quivers Q, which have $W = 0$ as their only potential, her definition extended the classical one by Buan–Marsh–Reineke–Reiten–Todorov [6]. The cluster category is a triangulated category in the sense of Verdier [100]. This means that $\mathcal{C}_{Q,W}$ is additive and endowed with

a) a *suspension functor* $\Sigma : \mathcal{C}_{Q,W} \xrightarrow{\sim} \mathcal{C}_{Q,W}$

b) a class of *triangles*, *i.e.* sequences of the form

$$U \longrightarrow V \longrightarrow W \longrightarrow \Sigma U\ .$$

These data have to satisfy suitable axioms, cf. [100]. An object X of a triangulated category is *rigid* if we have $\mathsf{Hom}(X, \Sigma X) = 0$. The category $\mathcal{C}_{Q,W}$ is endowed with a canonical rigid object Γ which decomposes as a direct sum of n indecomposable summands $\Gamma_1, \ldots, \Gamma_n$ with local endomorphism algebras. The full subcategory $\mathsf{add}\,\Gamma$ is formed by all finite direct sums of these summands. Its (split) Grothendieck group $K_0(\mathsf{add}\,\Gamma)$ has the classes of the Γ_i as a basis and thus canonically identifies with $\mathbb{Z}^n$. The endomorphism algebra of Γ is isomorphic to the Jacobian algebra $J(Q, W)$, *cf.* [23]. As in [90, Definition 3.9], define $\mathcal{P}$ to be the full subcategory of all objects X of $\mathcal{C}_{Q,W}$ such that

- there exists a triangle $T_1^X \to T_0^X \to X \to \Sigma T_1^X$, with T_0^X and T_1^X in $\mathsf{add}\,\Gamma$;
- there exists a triangle $T_{\Sigma X}^0 \to T_{\Sigma X}^1 \to \Sigma X \to \Sigma T_{\Sigma X}^0$, with $T_{\Sigma X}^0$ and $T_{\Sigma X}^1$ in $\mathsf{add}\,\Gamma$; and
- the space $\mathsf{Hom}_{\mathcal{C}}(\Gamma, \Sigma X)$ is finite-dimensional.

Define the *index* [20] of an object X of $\mathcal{P}$ as

$$\mathsf{ind}\,\Gamma X = [T_0^X] - [T_1^X] \in K_0(\mathsf{add}\,\Gamma) \cong \mathbb{Z}^n.$$

3.4. Proof via the cluster character. Let us keep the notations of section 3.3. The *cluster character* [10] [86] [88] [90] is the map $CC : \mathrm{Obj}(\mathcal{P}) \longrightarrow \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ defined by

$$CC(X) = \mathbf{x}^{\mathsf{ind}\,\Gamma X} \sum_e \chi\Big(\mathsf{Gr}_e(\mathsf{Hom}_{\mathcal{C}}(\Gamma, \Sigma X))\Big)\, x^{Be}\ , \tag{3.1}$$

where

- B is the skew-symmetric matrix associated to Q;
- $\mathsf{Gr}_e(\mathsf{Hom}_{\mathcal{C}}(\Gamma, \Sigma X))$ is the *quiver Grassmannian* (see Section 2.3 of [10]). It is the projective variety whose closed points are the submodules with dimension vector e of the $\mathsf{End}_{\mathcal{C}}(\Gamma)$-module $\mathsf{Hom}_{\mathcal{C}}(\Gamma, \Sigma X)$;
- χ is the Euler–Poincaré characteristic of the underlying topological space.

The following proposition shows that the cluster category is 'not much larger' than the category $\mathsf{mod}\,J(Q, W)$ of finite-dimensional (right) modules over the Jacobian algebra. In the special case where this algebra is finite-dimensional, the proposition follows from a result of [66].

Proposition 3.5 ([90]). *The functor* $F = \mathsf{Hom}_{\mathcal{C}}(\Gamma, \Sigma(?))$ *induces an equivalence of additive categories*

$$\mathcal{P}/(\Gamma) \xrightarrow{\sim} \mathsf{mod}\,J(Q, W)\ ,$$

where (Γ) *denotes the ideal of morphisms factoring through a direct sum of copies of* Γ.

The *mutation* of certain rigid objects like Γ inside $\mathcal{C}_{Q,W}$ is studied in [68]. It is a categorical lift of the mutation operation on seeds. A *reachable object* is a direct factor of an object obtained by iterated mutation from Γ. Any reachable object is rigid and lies in the subcategory $\mathcal{P}$ [90, Section 3.3]. The following theorem generalizes results obtained in [10] [11] [86].

Theorem 3.6 ([90]). *The cluster character CC induces a surjection from the set of isomorphism classes of (rigid) reachable objects of $\mathcal{P}$ onto the set of all cluster monomials of $\mathcal{A}_Q$.*

The theorem shows that each cluster monomial is given by the explicit formula (3.1). Now one can use this formula and the homological properties of the category $\mathcal{P}$ to show the proper Laurent property for the cluster algebra $\mathcal{A}_Q$, *cf.* [16], and thus the linear independence of the cluster monomials. Once this is established, it is not hard to deduce the following corollary.

Corollary 3.7. *The surjection of Theorem 3.6 is a bijection.*

The beauty of this last result is only diminished by the condition of 'reachability'. However, in general, this cannot be removed as shown in example 4.3 of [87].

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