RESOLUTIONS OF DG MODULES

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Following a request by Amnon Yekutieli, we explain the construction of the resolutions resolutions $\mathbf{p}M \to M$ and $M \to \mathbf{i}M$ of Theorems 3.1 and 3.2 of 'Deriving DG categories' in the special case of a module M over a dg algebra.

Let k be a commutative ring and A a dg k-algebra. We write $\mathcal{C}A$ for the category of dg (right) A-modules. Let M be a dg A-module. In section 1 below, we explain the construction of $\mathbf{p}M \to M$ and in section 2, the construction of $M \to \mathbf{i}M$.

1. The projective case

Our aims are as follows:

1) Construct an acyclic complex of dg A-modules

$$\ldots \to F_p \to \ldots \to F_1 \to F_0 \to M \to 0 \quad , p \geq 0 \quad ,$$

where each F_p is the direct sum of a free dg A-module and a contractible (semifree) dg A-module.

2) Show that we have an induced quasi-isomorphism

$$\mathrm{Tot}(F_*) \to M \quad ,$$

where Tot is the sum total dg A-module.

1) We have canonical isomorphisms

$$(\mathcal{C}A)(A[p], M) \xrightarrow{\sim} Z^p M \quad , p \in \mathbb{Z} \, .$$

Thus, we can construct a free dg A-module Q (a sum of modules $A[p]$) and a morphism $Q \to M$ inducing surjections

$$Z^p Q \to Z^p M$$

for all $p \in \mathbb{Z}$. We have canonical isomorphisms

$$(\mathcal{C}A)(I(A[p]), M) \xrightarrow{\sim} M^{p-1} \quad , p \in \mathbb{Z} \quad ,$$

where $I(A[p])$ denotes the mapping cone over the identity of $A[p]$. Thus we can construct a sum R of objects $I(A[p])$ and a morphsim $R \to M$ inducing surjections

$$R^p \to M^p \quad , p \in \mathbb{Z} \, .$$

We put $F_0 = Q \oplus R$ and let $F_0 \to M$ be be obvious morphism. We now let K be the kernel of $F_0 \to M$ and repeat the argument to get a morphism $F_1 \to K$, and so on.

Note that the exact sequence

$$0 \to K \to F_0 \to M \to 0$$

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induces exact sequences

$$
\begin{aligned}
&0 \to K^p \to F_0^p \to M^p \to 0\\
&0 \to Z^pK \to Z^pF_0 \to Z^pM \to 0\\
&0 \to B^pK \to B^pF_0 \to B^pM \to 0\\
&0 \to H^pK \to H^pF_0 \to H^pM \to 0.
\end{aligned}
$$

Thus, the complex

$$\ldots \to F_p \to \ldots \to F_1 \to F_0 \to M \to 0$$

is acyclic and induces acyclic complexes in Z^*, B^* and H^*.

2) We have to show that the complex

$$A = \mathrm{Tot}(\ldots \to F_1 \to F_0 \to M \to 0)$$

is acyclic. Let $\tau_{\leq p}$ denote the intelligent pth truncation functor. We have

$$\tau_{\leq p}M = (\ldots \to M^{p-1} \to Z^pM \to 0).$$

Notice that this is no longer an A-module but just a complex. The complex A is the union of its subcomplexes

$$A_p = \mathrm{Tot}(\ldots \to \tau_{\leq p}F_1 \to \tau_{\leq p}F_0 \to \tau_{\leq p}M \to 0).$$

To prove that A is acyclic, it is enough to show that each A_p is acyclic. Indeed, A_p is the total complex of a third quadrant double complex whose rows are acyclic because the complexes

$$\ldots \to Z^pF_1 \to Z^pF_0 \to Z^pM \to 0$$

and

$$\ldots \to F_1^q \to F_0^q \to M^q \to 0 \quad , q < p \quad ,$$

are acyclic.

2. The injective case

Let E be an injective cogenerator of the category of k-modules. We call cofree dg module a direct product of dg modules $\mathrm{Hom}_k(A[p], E)$. Let M be a dg module. Our aims are as follows:

1) Construct an acyclic complex of dg modules

$$0 \to M \to C^0 \to C^1 \to \ldots \quad ,$$

where each C^p is the direct sum of a cofree dg module and a contractible dg module.

2) Show that the induced morphism of dg modules

$$M \to \widehat{\mathrm{Tot}}(0 \to C^0 \to C^1 \to \ldots)$$

is a quasi-isomorphism, where $\widehat{\mathrm{Tot}}$ is the product total dg module.

1) We have canonical isomorphisms

$$(\mathcal{C}A)(M, \mathrm{Hom}_k(A[p], E)) \xrightarrow{\sim} (\mathcal{C}k)(M[p], E) \xrightarrow{\sim} \mathrm{Hom}_k(M^p/B^pM, E).$$

Thus, we can find a product Q of dg modules $\mathrm{Hom}_k(A[p], E)$ and a morphism of dg modules

$$M \to Q$$

inducing injections

$$M^p/B^pM \to Q^p/B^pQ \quad , p \in \mathbb{Z}\,.$$

We have canonical isomorphisms

$$(\mathcal{C}A)(M, I(\mathrm{Hom}_k(A[p], E))) \xrightarrow{\sim} \mathrm{Hom}_k(M^p, E) \quad ,$$

where $I(\mathrm{Hom}_k(A[p], E))$ is the mapping cone over the identity of $\mathrm{Hom}_k(A[p], E)$. Thus, we can find a product R of dg modules $I(\mathrm{Hom}_k(A[p], E))$ and a morphism of dg modules

$$M \to R$$

inducing injections

$$M^p \to R^p, p \in \mathbb{Z}\,.$$

We define $C^0 = Q \oplus R$ and let $M \to C^0$ be the natural morphism. We define the dg module S by the exact sequence

$$0 \to M \to C^0 \to S \to 0.$$

Notice that this sequence induces exact sequences in the components as well as exact sequences

$$0 \to M^p/B^pM \to C^{0p}/B^pC^0 \to S^p/B^pS \to 0.$$

We now apply the same argument to S instead of M to come up with a morphism $S \to C^1$. And so on. The result is a complex

$$0 \to M \to C^0 \to C^1 \to \dots$$

which induces acyclic complexes in the components as well as in the k-modules

$$0 \to M^p/B^pM \to C^{0p}/B^pC^0 \to C^{1p}/B^pC^1 \to \dots\,.$$

2) We have to show that the complex

$$A = \widehat{\mathrm{Tot}}(0 \to M \to C^0 \to C^1 \to \dots)$$

is acyclic. We denote by $\tau_{\geq p}$ the intelligent truncation functor. We have

$$\tau_{\geq p}M = (\dots \to 0 \to M^p/B^pM \to M^{p+1} \to M^{p+2} \to \dots).$$

Notice that this is no longer a dg A-module but just a complex. We consider the quotient complexes

$$A_p = \widehat{\mathrm{Tot}}(0 \to \tau_{\geq p}M \to \tau_{\geq p}C^0 \to \tau_{\geq p}C^1 \to \dots).$$

We claim that each A_p is acyclic. Indeed the non zero rows of A_p are

$$0 \to M^n \to C^{0n} \to C^{1n} \to \dots$$

for $n > p$ and

$$0 \to M^p/B^pM \to C^{0p}/B^pC^0 \to C^{1p}/B^pC^1 \to \dots\,.$$

These are acyclic and A_p itself is thus the total complex (no completion needed) of a first quadrant double complex with acyclic rows. Thus A_p is acyclic. The complex A is the inverse limit of the acyclic complexes A_p. By the Mittag-Leffler lemma, to show that it is acyclic, it suffices to show that the maps $A_{p+1} \to A_p$ induce surjections in the cycles. Since A_{p+1} and A_p are acyclic, the cycles coincide with the boundaries. But a surjection in the boundaries is implied by a surjection in the components and by construction, the maps $A_{p+1} \to A_p$ induce surjections in the components.