Extremal problems and probabilistic methods in hyperbolic geometry

Mini course in the

Virtual Seminar on
Geometry and Topology

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LECTURE 1

A very brief reminder on hyperbolic geometry

In this lecture we introduce the main actors of this mini course: hyperbolic manifolds. We will state many facts without proof. For a more complete treatment, we refer to [**BP92, Rat06**].

1.1. Hyperbolic manifolds

There are multiple equivalent definitions of what a hyperbolic manifold is. The shortest of these is perhaps the following:

Definition 1.1.1. A *hyperbolic n-manifold* is an n-dimensional Riemannian manifold, whose metric is complete and has constant sectional curvature equal to -1.

The first example of such a manifold is hyperbolic n-space $\mathbb{H}^n$. The Killing–Hopf theorem states that there is a unique (up to isometry) simply connected hyperbolic n-manifold, so we may define $\mathbb{H}^n$ to be that manifold. A more concrete way of thinking of $\mathbb{H}^n$ is by specifying what is called a *model*: a concrete simply connected complete Riemannian manifold with a metric of constant sectional curvature -1. There are various models that are useful for various purposes. We mention:

- the *ball model*:
$$\mathbb{D}^n = \left\{ x \in \mathbb{R}^n \,:\, \sum_{i=1}^n x_i^2 < 1 \right\}, \quad ds^2 = 4\,\frac{dx_1^2 + \ldots + dx_n^2}{\left(1 - \sum_{i=1}^n x_i^2\right)^2},$$
- the *upper half space model*:
$$\mathbb{U}^n = \left\{ x \in \mathbb{R}^n \,:\, x_n > 0 \right\}, \quad ds^2 = \frac{dx_1^2 + \ldots + dx_n^2}{x_n^2},$$
- the *hyperboloid model*:
$$\mathbb{L}^n = \left\{ x \in \mathbb{R}^{n+1} \,:\, \langle x, x\rangle_{n,1} = -1,\ x_0 > 0 \right\},$$
where $\langle x, y\rangle_{n,1} = -x_0y_0 + x_1y_1 + \ldots + x_ny_n$, with the metric given by the restriction of $\langle \cdot, \cdot\rangle_{(n,1)}$ to
$$T_x\,\mathbb{L}^n = \left\{ y \in \mathbb{R}^{n+1} \,:\, \langle x, y\rangle_{n,1} = 0 \right\}.$$

We leave it to the reader to check that all these metrics indeed have constant sectional curvature -1.

Note that, using $\mathbb{H}^n$, we can alternatively define hyperbolic n-manifolds as

- complete Riemannian n-manifolds that are locally isometric to $\mathbb{H}^n$,
- or manifolds whose charts map to $\mathbb{H}^n$ and such that all chart transitions are restrictions of isometries of $\mathbb{H}^n$,

- or manifolds of the form $\Gamma\backslash\mathbb{H}^n$, where Γ is a discrete torsion-free group of isometries of $\mathbb{H}^n$.

1.2. Isometries

The isometry group $\mathrm{Isom}(\mathbb{H}^n)$ and orientation preserving isometry group $\mathrm{Isom}^+(\mathbb{H}^n)$ satisfy

$$\mathrm{Isom}(\mathbb{H}^n) \simeq \mathrm{PO}(1,n) := \left\{ A \in \mathrm{Mat}_{n+1}(\mathbb{R}) \ : \ \begin{array}{c} \langle x,y\rangle_{n,1} = \langle Ax,Ay\rangle_{n,1} \ \forall x,y \in \mathbb{R}^{n+1} \\ A\cdot \mathbb{L}^n = \mathbb{L}^n \end{array} \right\},$$

where $\mathrm{Mat}_n(\mathbb{R})$ denotes the set of $n\times n$ matrices with coefficients in $\mathbb{R}$, and

$$\mathrm{Isom}^+(\mathbb{H}^n) \simeq \mathrm{PSO}(1,n) := \{ A \in \mathrm{PO}(1,n) \ : \ \det(A) = 1 \}.$$

The fact that the groups $\mathrm{PO}(1,n)$ and $\mathrm{PSO}(1,n)$ act by isometries on $\mathbb{H}^n$ can be seen directly from the hyperboloid model $\mathbb{L}^n$. The proof of the fact that there are no other isometries can be found in [**Rat06**, Chapter 3] or [**BP92**, Chapter A].

In low dimensions there are two accidental isomorphisms

$$\mathrm{Isom}^+(\mathbb{H}^2) \simeq \mathrm{PSL}(2,\mathbb{R}) := \{ A \in \mathrm{Mat}_2(\mathbb{R}) \ : \ \det(A) = 1 \} \big/ \{\pm \mathrm{Id}_2\},$$

where Id_k denotes the $k\times k$ identity matrix, and

$$\mathrm{Isom}^+(\mathbb{H}^3) \simeq \mathrm{PSL}(2,\mathbb{C}) := \{ A \in \mathrm{Mat}_2(\mathbb{C}) \ : \ \det(A) = 1 \} \big/ \{\pm \mathrm{Id}_2\}.$$

The action of $\mathrm{PSL}(2,\mathbb{R})$ by isometries is that on $\mathbb{U}^2$ by linear fractional transformations – i.e. we see the upper hald plane as a subset of $\mathbb{C}$: $\mathbb{U}^2 = \{ z\in\mathbb{C} \ : \ \mathrm{Im}(z) > 0 \}$ and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az+b}{cz+d}$$

for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}(2,\mathbb{R})$, $z\in\mathbb{U}^2$.

The action of $\mathrm{PSL}(2,\mathbb{C})$ on $\mathbb{H}^3$ is harder to describe. First of all, $\mathrm{PSL}(2,\mathbb{C})$ acts on the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$ by linear fractional transformations – i.e.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az+b}{cz+d}$$

for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}(2,\mathbb{C})$, $z\in\widehat{\mathbb{C}}$. It turns out that every such map φ can be written as the composition of two inversions in circles $C_1, C_2 \subset \widehat{\mathbb{C}}$. If we see $\widehat{\mathbb{C}}$ as the boundary of $\mathbb{U}^3$, then these circles define two hemispheres H_1 and H_2 in $\mathbb{U}^3$. φ now acts on $\mathbb{U}^3$ by the composition of the inversions (reflections) in H_1 and H_2. For the details, we refer to [**Bea95**, Section 3.3].

1.3. Hyperbolic surfaces

Before we ask questions about hyperbolic manifolds, we need some examples of them. We start with surfaces.

One way to construct hyperbolic surfaces is using pairs of pants. In what follows, we sketch how this works. For details, see for instance [**Bus10**, Section 1.7].

For ease of drawing, we will work in the disk model $\mathbb{D}^2$. In this model, geodesics are exactly straight diagonals through the center of $\mathbb{D}^2$ and half-circles orthogonal to $\partial\,\mathbb{D}^2$. A *right- angled hexagon* $H \subset \mathbb{D}^2$ is a compact, simply connected set whose boundary is geodesic, except at exactly six points, at which the geodesic segments coming from the right and left meet at right angles. Figure 1 shows an example.

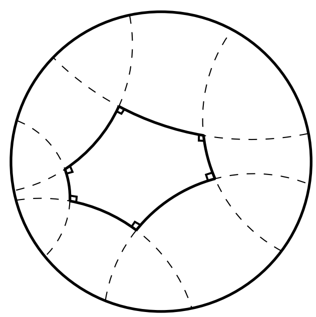

FIGURE 1. A right angled hexagon in $\mathbb{D}^2$

It turns out that, given three numbers $a, b, c > 0$, there exists a unique (up to isometry) right angled hexagon with three non-consecutive sides of lengts a, b and c.

Given two such hexagons, with the same side lengths, we can use three isometries to glue them along three non-consecutive sides of the same lengths, from which we obtain a hyperbolic pair of pants: a 2-sphere with three boundary components equipped with a hyperbolic metric. Figure 2 illustrates this gluing procedure.

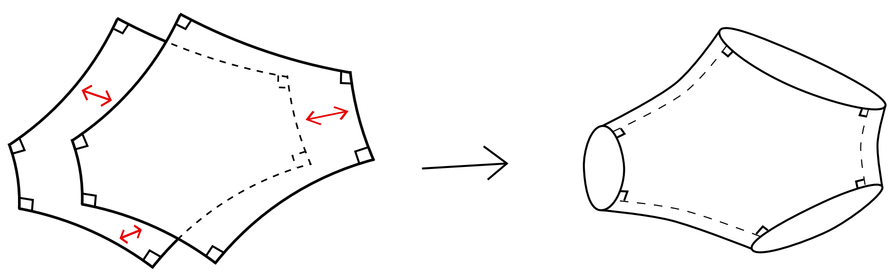

FIGURE 2. Gluing two right-angled hexagons into a pair of pants

It turns out that the hyperbolic metric on a pair of pants is determined (up to isometry) by the lengths of its three boundary components.

Finally, given two copies P_1 and P_2 of the same hyperbolic pair of pants, we may glue them together using three isometries between their boundary components. The result is a genus 2 surface. Figure 3 shows how this gluing works.

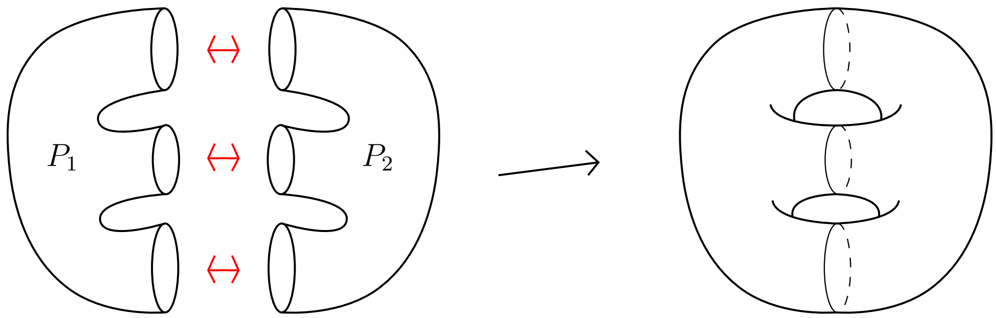

FIGURE 3. Gluing two pair of pants into a surface of genus 2

Note that we have multiple choices to make in this gluing process. First of all, we have to choose three boundary lengths for our pairs of pants. On top of that, we have to choose the three isometries between the pairs of boundary components. These lengths are naturally parametrized by $\mathbb{R}_+^3$. The three isometries can be parametrized by three copies of the circle. We will however choose to create some multiplicity and parameterize these by $\mathbb{R}$. So all in all, we get a space

$$\mathcal{T}_2 = \mathbb{R}_+^3 \times \mathbb{R}^3$$

of hyperbolic surfaces of genus 2 – called *Teichmüller space* of surfaces of genus 2. It turns out that this space contains a copy of *every* isometry type of hyperbolic surface of genus 2. In fact, it contains many copies of each isometry type (not just because of the multiplicity we introduced in the second half of the coordinates). The quotient in which all isometric pairs of surfaces is identified is called the *Moduli space* $\mathcal{M}_2$ of hyperbolic surfaces of genus 2.

Of course, a similar construction works for any genus. An Euler characteristic computation tells us that we need $2g-2$ pairs of pants for a closed surface of genus g. This gives us a Teichmüller space

$$\mathcal{T}_g = \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$$

and a moduli space $\mathcal{M}_g$. The parameters we used to construct $\mathcal{T}_g$ are called *Fenchel-Nielsen coordinates*, the first $3g-3$ are called the *length coordinates* and the last $3g-3$ coordinates are called the *twist coordinates*.

Remark 1.3.1. The reason we pick the twists in $\mathbb{R}$ and not in the circle is that we then have a homeomorphism

$$\mathcal{T}_g \to \mathcal{T}(S) = \left\{ (X,f) \; : \; \begin{array}{c} X \text{ a Riemann surface} \\ f : S \to X \text{ a homeomorphism} \end{array} \right\} \Big/ \sim$$

where S is a Riemann surface of genus g and $(X,f) \sim (Y,g)$ if and only if there exists an holomorphism $\varphi : X \to Y$ such that $g^{-1} \circ \varphi \circ f : S \to S$ is isotopic to the identity. The topology on $\mathcal{T}(S)$ is induced by quasiconformal maps: (X,f) and (Y,g) are close if and only if there exists a map $h : X \to Y$ such that $g^{-1} \circ h \circ f : S \to S$ is "close" to a conformal map. See for instance [**Hub06, IT92**] for proper definitions.

1.4. 3-manifolds

The situation in higher dimensions is wildly different than that of surfaces. One of the main reasons for this is the following theorem:

Theorem 1.4.1 (Mostow–Prasad rigidity theorem)**.** *Let $n \geq 3$ and let M and N be hyperbolic n-manifolds of finite volume. If $\pi_1(M) \simeq \pi_1(N)$ then M and N are isometric.*

This theorem in particular implies that there are no interesting deformation spaces of hyperbolic structures on a fixed smooth manifold of dimension more than 2. Together with the fact that the fundamental group of a hyperbolic manifold is finitely presented, it also implies there are only countably many hyperbolic manifolds of finite volume (up to isometry).

Like Teichmüller theory, hyperbolic 3-manifolds is a vast subject and there is no way to do justice to it in a short introduction. So instead of trying to, we will just (have to) content ourselves with some examples of hyperbolic 3-manifolds.

One good source of hyperbolic 3-manifolds is knot complements. A *knot* in the 3-sphere $\mathbb{S}^3$ is a smooth embedding $K : \mathbb{S}^1 \to \mathbb{S}^3$. Figure 4 shows an example: the figure eight knot.

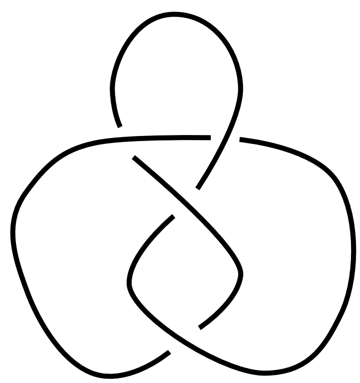

FIGURE 4. The figure eight knot.

Riley [**Ril75a, Ril75b**] discovered that the complement of the figure eight knot in $\mathbb{S}^3$ admits the structure of a hyperbolic 3-manifold of finite volume. In fact, Thurston later on determined exactly which knot complements (many of them) admit hyperbolic structures [**Thu82**].

This, again using work by Thurston, also leads to many examples of closed hyperbolic 3-manifolds, via the process of Dehn filling. Very briefly, every knot complement is homotopic to a 3-manifold M with one boundary component, homeomorphic to a 2-torus. If we now fix a solid torus T and a diffeomorphism $f : \partial T \to \partial M$, we obtain a closed 3-manifold M_f. Thurston proved that, if M itself admits a hyperbolic structure, then so does M_f for "most" choices of f [**Thu78**] (see also [**BP92**, Chapter E]).

1.5. Higher dimensions

In dimension higher than 3, hyperbolic manifolds are much harder to come by. There are still countably infinitely many closed (or of finite volume) hyperbolic n-manifolds for all $n \geq 4$, but they are much less well understood. One big difference with lower dimensional manifold is Wang's theorem, which states:

Theorem 1.5.1 ([**Wan72**]). *The number of hyperbolic n-manifolds of volume $\leq v$ (up to isometry) is finite.*

First of all there are *arithmetic* hyperbolic manifolds. These are manifolds whose fundamental group is an arithmetic subgroup of $\mathrm{Isom}^+(\mathbb{H}^n)$. We will not go into the (lengthy) definition of what an arithmetic group is, but very roughly, they come from taking the integral points in an algebraic group. Prototypical examples are $\mathrm{PSL}(2,\mathbb{Z}) < \mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{Z}[i]) < \mathrm{PSL}(2,\mathbb{C})$ and their finite index subgroups. These two examples give rise to 2- and 3- orbifolds and manifolds (via their subgroups), but arithmetic groups exist in all dimensions.

Not every hyperbolic n-manifold is arithmetic. In dimension 4 and above this is a celebrated theorem due to Gromov – Piatetski-Shapiro [**GPS88**]. Their proof is constructive. It goes by taking two arithmetic hyperbolic n-manifolds M_1 and M_2 that both contain an embedded copy of a fixed hyperbolic $(n-1)$-manifold, cutting the n-manifolds along these $(n-1)$-manifolds and gluing the resulting blocks together along their boundary. It turns out that if M_1 and M_2 are not *commensurable* (i.e. do not have a common finite degree cover), then the result will be non-arithmetic. Several similar cut-and-paste constructions are by now known [**BT11, Rai13, GL14**].

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