Exercise 1 (Beltrami differentials and quasiconformal maps).

(a) Let S, X_1 and X_2 be Riemann surfaces and let

$$S \xrightarrow{f} X_1 \xrightarrow{g} X_2$$

be orientation preserving diffeomorphisms. Prove that:

$$\mu_g \circ f = \left(\frac{\partial f}{\partial z} \Big/ \overline{\left(\frac{\partial f}{\partial z} \right)} \right) \cdot \frac{\mu_{g \circ f} - \mu_f}{1 - \overline{\mu}_f \cdot \mu_{g \circ f}}.$$

(b) Prove the following lemma about compositions of quasiconformal maps: Suppose X, Y and Z are Riemann surfaces and $f : X \to Y$ and $g : Y \to Z$ are orientation preserving diffeomorphisms. Then the following holds:

(1) We have that

$$K_f \geq 1$$

with equality if and only if f is a biholomorphism.

(2) We have that

$$K_{g \circ f} \leq K_g \cdot K_f.$$

(3) Finally,

$$K_{f^{-1}} = K_f.$$

Hint for (2): Since $K_f(z)$ depends only on the Jacobian matrix $J_f(z)$ of f at z, this is a linear algebra question.

Exercise 2 (Grötschz's theorem) Suppose that $R_1 = [0, a] \times [0, 1]$ and $R_2 = [0, K \cdot a] \times [0, 1]$ are two rectangles in the plane, where $a > 0$ and $K \geq 1$. The goal of this exercise is to prove:

Theorem (Grötschz's theorem) Suppose R_1 and R_2 are as above and $f : R_1 \to R_2$ is a homeomorphism that is smooth and orientation preserving away from a finite number of points. Then

$$K_f \geq K$$

with equality if and only if f is the affine map

$$(x, y) \in R_1 \mapsto (K \cdot x, y) \in R_2.$$

(a) Writing $K_f(x,y)$ for the quasiconformal dilatation of f at $(x,y) \in R_1$, prove that

$$\left|\frac{\partial f}{\partial x}(x,y)\right|^2 \leq K_f(x,y) \cdot \det(J_f(x,y)), \tag{1}$$

where $J_f(x,y)$ denotes the Jacobian matrix of f at $(x,y) \in R_1$.

(b) Prove that:

$$\int_{R_1} \left|\frac{\partial f}{\partial x}(x,y)\right| dxdy \geq K \cdot \text{area}(R_1) \tag{2}$$

(c) Use the inequalities above to show that

$$(K \cdot \text{area}(R_1))^2 \leq K \cdot \text{area}(R_1) \cdot K_f \cdot \text{area}(R_1).$$

Thus yielding that $K_f \geq K$.

(d) We have seen during the course that the affine map realizes equality. Prove that this is the only such map.