Exercise 1 (Beltrami differentials).

(a) Let S, X_1 and X_2 be Riemann surfaces and let

$$S \xrightarrow{f} X_1 \xrightarrow{g} X_2$$

be orientation preserving diffeomorphisms. Prove that:

$$\mu_g \circ f = \left(\frac{\partial f}{\partial z} \Big/ \overline{\left(\frac{\partial f}{\partial z}\right)} \right) \cdot \frac{\mu_{g\circ f} - \mu_f}{1 - \overline{\mu}_f \cdot \mu_{g \circ f}}.$$

<u>Solution:</u> We will write $f_z := \partial f/\partial z$ and $f_{\overline{z}} := \partial f/\partial \overline{z}$ in order to make the equations slightly shorter. The reader should however note that in general $\overline{f_z}$ does not equal $\overline{f}_z$ (but rather $\overline{f}_{\overline{z}}$). That is, attention should be paid to where the bar ends.

We compute, using the chain rule:

$$\mu_{g\circ f} = \frac{(g\circ f)_{\overline{z}}}{(g \circ f)_z} = \frac{(g_z \circ f)\cdot f_{\overline{z}} + (g_{\overline{z}} \circ f)\cdot \overline{f}_{\overline{z}}}{(g_z \circ f)\cdot f_z + (g_{\overline{z}} \circ f)\cdot \overline{f}_z}$$

So

$$\begin{aligned}\frac{f_z}{\overline{f_z}} \cdot \frac{\mu_{g\circ f} - \mu_f}{1-\overline{\mu}_f \cdot \mu_{g\circ f}} &= \frac{f_z}{\overline{f_z}} \cdot \frac{(g_z \circ f)\cdot f_{\overline{z}} + (g_{\overline{z}} \circ f)\cdot \overline{f}_{\overline{z}} - \frac{f_{\overline{z}}}{f_z}\cdot (g_z\circ f)\cdot f_z - \frac{f_{\overline{z}}}{f_z}\cdot (g_{\overline{z}}\circ f)\cdot \overline{f}_z}{(g_z \circ f)\cdot f_z + (g_{\overline{z}} \circ f)\cdot \overline{f}_z - \frac{\overline{f_{\overline{z}}}}{\overline{f_z}}\cdot (g_z\circ f)\cdot f_{\overline{z}} - \frac{\overline{f_{\overline{z}}}}{\overline{f_z}}\cdot (g_{\overline{z}}\circ f)\cdot \overline{f}_{\overline{z}}} \\ &= \frac{(g_{\overline{z}}\circ f)\cdot (|f_z|^2 - |f_{\overline{z}}|^2)}{(g_z\circ f)\cdot (|f_z|^2 - |f_{\overline{z}}|^2)} = \frac{(g_{\overline{z}}\circ f)}{(g_z \circ f)} = \mu_g \circ f.\end{aligned}$$

where we have used that $\overline{f}_{\overline{z}} = \overline{f_z}$ and $\overline{f}_z = \overline{f_{\overline{z}}}$.

(b) Prove the following lemma about compositions of quasiconformal maps: Suppose X, Y and Z are Riemann surfaces and $f : X \to Y$ and $g : Y \to Z$ are orientation preserving diffeomorphisms. Then the following holds:

(1) We have that

$$K_f \geq 1$$

with equality if and only if f is a biholomorphism.

(2) We have that

$$K_{g\circ f} \leq K_g \cdot K_f.$$

(3) Finally,

$$K_{f^{-1}} = K_f.$$

Hint for (2): Since $K_f(z)$ depends only on the Jacobian matrix $J_f(z)$ of f at z, this is a linear algebra question.

<u>Solution:</u> Recall that

$$K_f = \sup_{z \in X} K_f(z), \quad \text{where } K_f(z) = \frac{1 + |\mu_f(z)|}{1 - |\mu_f(z)|}.$$

We have seen that $\mu_f(z) = 0$ if and only if f is holomorphic at z. So $K_f(z) \geq 1$ with equality if and only if f is holomorphic at z. So this proves the inequality and also that in the equality case, f is holomorphic at all $z \in X$. Because f is invertible (and invertible holomorphic functions have holomorphic inverses), f is biholomorphic. This proves (1).

For (2), we use the hint. Write A for the Jacobian matrix of f and B for that of g, both with respect to some holomorphic coordinates on X, Y and Z. K_f can be computed as the ratio (major axis)/(minor axis) of the ellipse

$$\|A^{-1} \cdot z\| = 1$$

thinking of z as a real 2-dimensional vector $\begin{pmatrix} x \\ y \end{pmatrix}$. Using the standard inner product $\langle \cdot, \cdot, \rangle$ the equation for the ellipse is equivalent to

$$\langle \left(A^{-1}\right)^t \cdot A^{-1} z, z \rangle = 1$$

wher $\left(A^{-1}\right)^t$ denotes the transpose of A^{-1}. The matrix $\left(A^{-1}\right)^t \cdot A^{-1}$ is positive definite, so it has two orthogonal eigendirections (corresponding to the axes of the ellipse) and the ratio

$$K_f(z) = \frac{\text{major axis}}{\text{minor axis}} = \sqrt{\frac{\lambda_A^+}{\lambda_A^-}}$$

where λ_A^+ denotes the maximal eigenvalue of $\left(A^{-1}\right)^t \cdot A^{-1}$ and λ_A^- denotes the minimal eigenvalue of $\left(A^{-1}\right)^t \cdot A^{-1}$. Note that

$$\left(\lambda_A^+\right)^{1/2} = \|A^{-1}\|_\infty \quad \text{and} \quad \left(\lambda_A^-\right)^{-1/2} = \|A\|_\infty$$

(the latter holds because the top eigenvalue of $A \cdot A^t$ is the inverse of the bottom eigenvalue of $(A \cdot A^t)^{-1} = \left(A^{-1}\right)^t \cdot A^{-1}$. So

$$K_f(z) = \|A^{-1}\|_\infty \cdot \|A\|_\infty, \quad K_g(f(z)) = \|B^{-1}\|_\infty \cdot \|B\|_\infty$$

and

$$K_{g \circ f}(z) = \|B^{-1} A^{-1}\|_\infty \cdot \|AB\|_\infty.$$

This means that submultiplicativity of operator norms of matrices implies the inequality we're after.

Property (3) follows from the fact that $K_f(z) = \|A^{-1}\|_\infty \cdot \|A\|_\infty$.

Exercise 2 (Grötschz's theorem) Suppose that $R_1 = [0, a] \times [0, 1]$ and $R_2 = [0, K \cdot a] \times [0, 1]$ are two rectangles in the plane, where $a > 0$ and $K \geq 1$. The goal of this exercise is to prove:

Theorem (Grötschz's theorem) Suppose R_1 and R_2 are as above and $f : R_1 \to R_2$ is a homeomorphism that is smooth and orientation preserving away from a finite number of points. Then

$$K_f \geq K$$

with equality if and only if f is the affine map

$$(x, y) \in R_1 \mapsto (K \cdot x, y) \in R_2.$$

(a) Writing $K_f(x, y)$ for the quasiconformal dilatation of f at $(x, y) \in R_1$, prove that

$$\left| \frac{\partial f}{\partial x}(x, y) \right|^2 \leq K_f(x, y) \cdot \det(J_f(x, y)), \tag{1}$$

where $J_f(x, y)$ denotes the Jacobian matrix of f at $(x, y) \in R_1$.

Solution: Writing

$$M = \sup \left\{ \left| df_{(x,y)}(v) \right| ;\ v \in T^1_{(x,y)} R_1 \right\}$$

and

$$m = \inf \left\{ \left| df_{(x,y)}(v) \right| ;\ v \in T^1_{(x,y)} R_1 \right\},$$

We have $K_f(x, y) = M/m$, $\det(J_f(x, y)) = M \cdot m$ and $|\frac{\partial f}{\partial x}(x, y)|^2 \leq M^2$, which proves the claim.

(b) Prove that:

$$\int_{R_1} \left| \frac{\partial f}{\partial x}(x, y) \right| dxdy \geq K \cdot \text{area}(R_1) \tag{2}$$

Solution: This is the observation that for almost all $y \in [0, 1]$, $\int_0^a |\frac{\partial f}{\partial x}(x, y)| dx \geq K \cdot a$, by the substitution rule. Integrating with respect to y gives the desired inequality.

(c) Use the inequalities above to show that

$$(K \cdot \text{area}(R_1))^2 \leq K \cdot \text{area}(R_1) \cdot K_f \cdot \text{area}(R_1).$$

Thus yielding that $K_f \geq K$.

Solution: We have

$$
\begin{aligned}
(K \cdot \operatorname{area}(R_1))^2 &\overset{(2)}{\leq} \left(\int_{R_1} \left| \frac{\partial f}{\partial x}(x,y) \right| dxdy \right)^2 \\
&\overset{(1)}{\leq} \left(\int_{R_1} \sqrt{K_f(x,y)} \cdot \sqrt{\det(J_f(x,y))} dxdy \right)^2 \\
&\overset{\text{Cauchy–Schwarz}}{\leq} \int_{R_1} K_f(x,y) dxdy \cdot \int_{R_1} \det(J_f(x,y)) dxdy \\
&\leq \operatorname{area}(R_2) \cdot K_f \cdot \operatorname{area}(R_1) \\
&= K \cdot \operatorname{area}(R_1) \cdot K_f \cdot \operatorname{area}(R_1).
\end{aligned}
$$

(d) We have seen during the course that the affine map realizes equality. Prove that this is the only such map.

Solution: We observe that

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \overline{z}}$$

and recall from the course that

$$K_f(z) = \frac{\left|\frac{\partial f}{\partial z}\right| + \left|\frac{\partial f}{\partial \overline{z}}\right|}{\left|\frac{\partial f}{\partial z}\right| - \left|\frac{\partial f}{\partial \overline{z}}\right|} \quad \text{and} \quad \det(J_f(z)) = \left|\frac{\partial f}{\partial z}\right|^2 - \left|\frac{\partial f}{\partial \overline{z}}\right|^2.$$

So

$$\sqrt{K_f(z) \cdot \det(J_f(z))} = \left|\frac{\partial f}{\partial z}\right| + \left|\frac{\partial f}{\partial \overline{z}}\right|.$$

This means that equality in the first inequality above means that

$$\left|\frac{\partial f}{\partial z} + \frac{\partial f}{\partial \overline{z}}\right| = \left|\frac{\partial f}{\partial z}\right| + \left|\frac{\partial f}{\partial \overline{z}}\right|$$

almost everywhere and thus that the argument of $\partial f / \partial z$ and $\partial f / \partial \overline{z}$ is the same almost everywhere.

Equality in the last inequality means that $K_f(x,y)$ is constant almost everywhere and equal to K. Combining the above, we see that μ_f coincides with the Beltrami coefficient of the affine map f_0. The result of Exercise 1(a), applied to $f \circ f_0^{-1}$ now gives that this map is conformal. An argument similar to that of the proof of Proposition 7.1.9 from the course now allows us to conclude.