Problem set 6: Quadratic differentials and the Bers embedding

Exercise 1 (Injectivity of the Bers embedding).

(a) We have seen in class that if $f : D \to \widehat{\mathbb{C}}$ is analytic with nowhere vanishing derivative and $g \in \mathrm{PGL}(2, \mathbb{C})$, then

$$\mathcal{S}(f \circ g) = \mathcal{S}(f)(g(z)) \cdot g'(z)^2 \quad \text{for } z \in D.$$

Prove the more general fact that if g is only analytic, then

$$\mathcal{S}(f \circ g) = \mathcal{S}(f)(g(z)) \cdot g'(z)^2 + \mathcal{S}(g)(z) \quad \text{for } z \in D.$$

(b) Prove that if $f : D \to \widehat{\mathbb{C}}$ is analytic with nowhere vanishing derivative, then f is a Möbius transformation if and only if

$$\mathcal{S}(f) = 0 \quad \text{on } D.$$

Hint: Write the Schwartzian derivative in terms of the function $u(z) = f''(z)/f'(z)$.

(c) Let $S = \Gamma\backslash\mathbb{H}^2$ be a closed hyperbolic Riemann surface. Recall that

$$\Phi_{\mathrm{Bers}} : \mathcal{T}(S) \to \mathcal{Q}(S^*)$$

is given by

$$\Phi_{\mathrm{Bers}}([\mu]) = \mathcal{S}\left(f^{\widehat{\mu}}|_{\mathbb{H}^*}\right)$$

where $\widehat{\mu}$ is the Γ-invariant Beltrami differential on $\mathbb{C}$ that is obtained by extending μ by 0 on the lower half plane and $f^{\widehat{\mu}} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is the unique solution to the Beltrami equation for $\widehat{\mu}$ with $f^{\widehat{\mu}}(0) = 0$, $f^{\widehat{\mu}}(1) = 1$ and $f^{\widehat{\mu}}(\infty) = \infty$. Prove that, if $\Phi_{\mathrm{Bers}}([\mu]) = \Phi_{\mathrm{Bers}}([\nu])$, then

$$f^{\widehat{\mu}}|_{\overline{\mathbb{H}^*}} = f^{\widehat{\nu}}|_{\overline{\mathbb{H}^*}}$$

(d) Conclude that the Bers embedding is injective.

Exercise 2 (Nehari's theorem). The goal of this exercise is to prove that for any injective analytic function $f : \mathbb{H}^* \to \mathbb{C}$ with non-vanishing derivative,

$$\|\mathcal{S}(f)\| = \sup_{z \in \mathbb{H}^*} \mathrm{Im}(z)^2 \cdot |\mathcal{S}(f)(z)| \leq \frac{3}{2}.$$

This in particular proves that the image of the Bers embedding is contained in a ball of radius $\frac{3}{2}$ in $\mathcal{Q}(S^*) \simeq \mathbb{C}^{3g-3}$.

(a) Write

$$\Delta^* = \{z \in \widehat{\mathbb{C}};\ |z| > 1\}$$

and suppose $F : \Delta^* \to \widehat{\mathbb{C}}$ is an injective analytic function. Given $r > 1$, let

$$C_r = \{z \in \mathbb{C};\ |z| = r\} \subset \Delta^*.$$

writing A_r for the (Euclidean) area of the bounded domain enclosed by $F(C_r)$, prove that

$$A_r = \frac{1}{2i} \int_{C_r} \overline{F(z)}\, dF(z)$$

Hint: Stokes's theorem.

(b) Suppose now that F admits the expansion

$$F(z) = z + \sum_{k \geq 0} b_k \cdot z^{-k}, \quad z \in \Delta^*.$$

Show that

$$\sum_{k \geq 1} k \cdot |b_k|^2 \leq 1.$$

This is called **Bieberbach's area theorem**.

(c) Prove that

$$\lim_{z \to \infty} |z^4 \cdot \mathcal{S}(F)(z)| = 6 \cdot |b_1| \leq 6.$$

(d) Now let $f : \mathbb{H}^* \to \widehat{\mathbb{C}}$ be an analytic function with nowhere vanishing derivative and let $z_0 = x_0 + iy_0 \in \mathbb{H}^*$ such that $f(z_0) \neq \infty$. Moreover, let $T : \mathbb{H}^* \to \Delta^*$ denote the Möbius transformation given by

$$T(z) = \frac{z - \overline{z_0}}{z - z_0}.$$

Moreover, define $F : \Delta^* \to \widehat{\mathbb{C}}$ by

$$F(z) = \frac{2iy_0 f'(z_0)}{f(T^{-1}(z)) - f(z_0)}$$

Use Exercise 1(a) to write the Schwartzian derivative of f at z_0 using F and then conclude, using the result of (c) that

$$|\mathcal{S}(f)(z_0)| \leq \frac{3}{2 \cdot y_0^2}.$$

(e) Use a Möbius transformation to deal with the case when $f(z_0) = \infty$, thus completing the proof of Nehari's theorem.