Problem set 6: Quadratic differentials and the Bers embedding

Exercise 1 (Injectivity of the Bers embedding).

(a) We have seen in class that if $f : D \to \widehat{\mathbb{C}}$ is analytic with nowhere vanishing derivative and $g \in \mathrm{PGL}(2,\mathbb{C})$, then

$$\mathcal{S}(f \circ g) = \mathcal{S}(f)(g(z)) \cdot g'(z)^2 \quad \text{for } z \in D.$$

Prove the more general fact that if g is only analytic, then

$$\mathcal{S}(f \circ g) = \mathcal{S}(f)(g(z)) \cdot g'(z)^2 + \mathcal{S}(g)(z) \quad \text{for } z \in D.$$

Solution: This is still just a matter of applying the chain rule. We have

$$\begin{aligned}
(f \circ g)'(z) &= f'(g(z)) \cdot g'(z) \\
(f \circ g)''(z) &= f''(g(z)) \cdot g'(z)^2 + f'(g(z)) \cdot g''(z) \\
(f \circ g)'''(z) &= f'''(g(z)) \cdot g'(z)^3 + 3 \cdot f''(g(z)) \cdot g'(z) \cdot g''(z) + f'(g(z)) \cdot g'''(z).
\end{aligned}$$

So

$$\begin{aligned}
\left(\frac{(f \circ g)''(z)}{(f \circ g)'(z)}\right)^2 &= \left(\frac{f''(g(z)) \cdot g'(z)}{f'(g(z))} + \frac{g''(z)}{g'(z)}\right)^2 \\
&= \left(\frac{f''(g(z)) \cdot g'(z)}{f'(g(z))}\right)^2 + 2 \cdot \frac{f''(g(z)) \cdot g''(z)}{f'(g(z))} + \left(\frac{g''(z)}{g'(z)}\right)^2
\end{aligned}$$

$$\frac{(f \circ g)'''(z)}{(f \circ g)'(z)} = \frac{f'''(g(z)) \cdot g'(z)^2}{f'(g(z))} + 3 \cdot \frac{f''(g(z)) \cdot g''(z)}{f'(g(z))} + \frac{g'''(z)}{g'(z)}.$$

As such

$$\begin{aligned}
&\frac{(f \circ g)'''(z)}{(f \circ g)'(z)} - \frac{3}{2}\left(\frac{(f \circ g)''(z)}{(f \circ g)'(z)}\right)^2 \\
&\qquad = \left(\frac{f'''(g(z))}{f'(g(z))} - \frac{3}{2}\left(\frac{f''(g(z))}{f'(g(z))}\right)^2\right) \cdot g'(z)^2 + \frac{g'''(z)}{g'(z)} - \frac{3}{2}\left(\frac{g''(z)}{g'(z)}\right)^2,
\end{aligned}$$

which proves the claim.

(b) Prove that if $f : D \to \widehat{\mathbb{C}}$ is analytic with nowhere vanishing derivative, then f is a Möbius transformation if and only if

$$\mathcal{S}(f) = 0 \quad \text{on } D.$$

Hint: Write the Schwartzian derivative in terms of the function $u(z) = f''(z)/f'(z)$.

Solution: We use the hint:

$$\mathcal{S}(f)(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 = u'(z) - \frac{1}{2}u(z)^2.$$

So $\mathcal{S}(f)(z) = 0$ corresponds to the ODE

$$u'(z) - \frac{1}{2}u(z)^2 = 0$$

so

$$-u(z)^{-1} = \int \frac{du}{u^2} = \frac{1}{2}\int dz = \frac{1}{2}z + C.$$

In other words,

$$\frac{f''}{f'}(z) = (\log(f'(z)))' = u(z) = -\frac{2}{z-\alpha}$$

for some $\alpha \in \mathbb{C}$. Integrating once yields

$$\log(f'(z)) = -2\log(z-\alpha) + C \quad \Longrightarrow \quad f'(z) = \frac{A}{(z-\alpha)^2}$$

for some $A \in \mathbb{C}$. Integrating again we get that

$$f(z) = -\frac{A}{z-\alpha} + B$$

for some $B \in \mathbb{C}$, which is equivalent to f being a Möbius transformation.

(c) Let $S = \Gamma\backslash\mathbb{H}^2$ be a closed hyperbolic Riemann surface. Recall that

$$\Phi_{\text{Bers}} : \mathcal{T}(S) \to \mathcal{Q}(S^*)$$

is given by

$$\Phi_{\text{Bers}}([\mu]) = \mathcal{S}\left(f^{\widehat{\mu}}|_{\mathbb{H}^*}\right)$$

where $\widehat{\mu}$ is the Γ-invariant Beltrami differential on $\mathbb{C}$ that is obtained by extending μ by 0 on the lower half plane and $f^{\widehat{\mu}} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is the unique solution to the Beltrami equation for $\widehat{\mu}$ with $f^{\widehat{\mu}}(0) = 0$, $f^{\widehat{\mu}}(1) = 1$ and $f^{\widehat{\mu}}(\infty) = \infty$. Prove that, if $\Phi_{\text{Bers}}([\mu]) = \Phi_{\text{Bers}}([\nu])$, then

$$f^{\widehat{\mu}}|_{\overline{\mathbb{H}^*}} = f^{\widehat{\nu}}|_{\overline{\mathbb{H}^*}}$$

Solution: Suppose that $[\mu], [\nu] \in \mathcal{T}(S)$ and that

$$\Phi_{\text{Bers}}([\mu]) = \Phi_{\text{Bers}}([\nu]).$$

Now define the analytic map

$$F = \left(f^{\widehat{\nu}}|_{\mathbb{H}^*}\right) \circ \left(f^{\widehat{\mu}}|_{\mathbb{H}^*}\right)^{-1} : f^{\widehat{\nu}}(\mathbb{H}^*) \longrightarrow f^{\widehat{\mu}}(\mathbb{H}^*).$$

We have, for $z \in \mathbb{H}^*$,

$$\mathcal{S}\Big(f^{\widehat{\nu}}|_{\mathbb{H}^*}\Big)(z) = \mathcal{S}\Big(F \circ f^{\widehat{\mu}}|_{\mathbb{H}^*}\Big) \overset{\text{(a)}}{=} \mathcal{S}(F)\Big(f^{\widehat{\mu}}|_{\mathbb{H}^*}(z)\Big) \cdot \Big(f^{\widehat{\mu}}|_{\mathbb{H}^*}\Big)'(z)^2 + \mathcal{S}\Big(f^{\widehat{\mu}}|_{\mathbb{H}^*}\Big)(z).$$

Now using our assumption that $\mathcal{S}\Big(f^{\widehat{\nu}}|_{\mathbb{H}^*}\Big)(z) = \mathcal{S}\Big(f^{\widehat{\mu}}|_{\mathbb{H}^*}\Big)(z)$, we get that

$$\mathcal{S}(F) = 0$$

on $f^{\widehat{\mu}}(\mathbb{H}^*)$, which implies that F is a Möbius transformation. Since its extension to $\overline{\mathbb{H}^*}$ fixes 0, 1 and ∞, it needs to be the identity (on $\overline{\mathbb{H}^*}$), which proves the claim.

(d) Conclude that the Bers embedding is injective.

Solution: We have seen that, under the assumption that $\Phi_{\text{Bers}}([\mu]) = \Phi_{\text{Bers}}([\nu])$, the maps $f^{\widehat{\mu}}$ and $f^{\widehat{\nu}}$ coincide on $\mathbb{R}\cup\{\infty\}$. If we uniformize $S = \Gamma\backslash\mathbb{H}^2$. Then the points in Teichmüller space corresponding to f^μ and f^ν correspond to Fuchsian groups $f^\mu \cdot \Gamma \cdot (f^\mu)^{-1}$ and $f^\nu \cdot \Gamma \cdot (f^\nu)^{-1}$ respectively. The extensions to $\mathbb{R} \cup \{\infty\}$ of f^μ and f^ν coincide with $f^{\widehat{\mu}}$ and $f^{\widehat{\nu}}$ respectively. This means in particular, that on $\mathbb{R} \cup \{\infty\}$ the two representations of Γ coincide. That in turn means that they are the same (Möbius transformations that agree on $\mathbb{R} \cup \{\infty\}$ are equal).

Exercise 2 (Nehari's theorem). The goal of this exercise is to prove that for any injective analytic function $f : \mathbb{H}^* \to \mathbb{C}$ with non-vanishing derivative,

$$\|\mathcal{S}(f)\| = \sup_{z\in\mathbb{H}^*} \operatorname{Im}(z)^2 \cdot |\mathcal{S}(f)(z)| \leq \frac{3}{2}.$$

This in particular proves that the image of the Bers embedding is contained in a ball of radius $\frac{3}{2}$ in $\mathcal{Q}(S^*) \simeq \mathbb{C}^{3g-3}$.

(a) Write

$$\Delta^* = \{z \in \widehat{\mathbb{C}};\ |z| > 1\}$$

and suppose $F : \Delta^* \to \widehat{\mathbb{C}}$ is an injective analytic function. Given $r > 1$, let

$$C_r = \{z \in \mathbb{C};\ |z| = r\} \subset \Delta^*.$$

writing A_r for the (Euclidean) area of the bounded domain enclosed by $F(C_r)$, prove that

$$A_r = \frac{1}{2i} \int_{C_r} \overline{F(z)}\, dF(z)$$

Hint: Stokes's theorem.

Solution: Writing $F_z = \partial F/\partial z$ (and using that $\partial F/\partial \overline{z} = 0$) we get

$$dF(z) = F_z(z)dz \quad \text{and} \quad d\overline{F}(z) = \overline{F_z(z)}d\overline{z}.$$

So

$$d(\overline{F} \cdot dF) = d\overline{F} \wedge dF = |F_z(z)|^2 \cdot d\overline{z} \wedge dz.$$

Now we use that $d\overline{z} \wedge dz = 2i dx \wedge dy$ and that $|F_z(z)|^2 = \det(J_F)$ (as we have computed multiple times during the course). This means that, by Stokes's theorem

$$\int_{C_r} \overline{F(z)}\, dF(z) = 2i \int_{D_r} \det(J_F) \cdot dxdy = 2i \cdot A_r,$$

where D_r denotes the bounded region enclosed by $F(C_r)$.

(b) Suppose now that F admits the expansion

$$F(z) = z + \sum_{k \geq 0} b_k \cdot z^{-k}, \quad z \in \Delta^*.$$

Show that

$$\sum_{k \geq 1} k \cdot |b_k|^2 \leq 1.$$

This is called **Bieberbach's area theorem**.

Solution: Filling in the result from (a), we have

$$\begin{aligned} 0 &\leq A_r \\ &= \frac{1}{2i} \int_{C_1} (re^{-i\theta} + \sum_{k \geq 0} \overline{b_k} \cdot r^{-k} \cdot e^{ki\theta}) \cdot d\left(re^{i\theta} + \sum_{k \geq 0} b_k \cdot r^{-k} \cdot e^{-ki\theta} \right) \\ &= \pi \cdot \left(r^2 - \sum_{k \geq 1} k \cdot |b_k|^2 \cdot r^{-k} \right). \end{aligned}$$

Since the latter expression is non-negative for all $r < 1$, we can take the limit and obtain the claim.

(c) Prove that

$$\lim_{z \to \infty} |z^4 \cdot \mathcal{S}(F)(z)| = 6 \cdot |b_1| \leq 6.$$

Solution: One computes that

$$\mathcal{S}(F)(z) = -\frac{6}{z^4} b_1 + \sum_{k \geq 5} \frac{c_k}{z^k}$$

for some $c_k \in \mathbb{C}$ and all $z \in \Delta^* - \{\infty\}$. As such

$$\lim_{z \to \infty} |z^4 \cdot \mathcal{S}(F)(z)| = 6 \cdot |b_1|,$$

which is bounded by 6 by (b).

(d) Now let $f : \mathbb{H}^* \to \widehat{\mathbb{C}}$ be an analytic function with nowhere vanishing derivative and let $z_0 = x_0 + iy_0 \in \mathbb{H}^*$ such that $f(z_0) \neq \infty$. Moreover, let $T : \mathbb{H}^* \to \Delta^*$ denote the Möbius transformation given by

$$T(z) = \frac{z - \overline{z_0}}{z - z_0}.$$

Moreover, define $F : \Delta^* \to \widehat{\mathbb{C}}$ by

$$F(z) = \frac{2iy_0 f'(z_0)}{f(T^{-1}(z)) - f(z_0)}$$

Use Exercise 1(a) to write the Schwartzian derivative of f at z_0 using F and then conclude, using the result of (c) that

$$|\mathcal{S}(f)(z_0)| \leq \frac{3}{2 \cdot y_0^2}.$$

Solution: From Exercise 1(a) we obtain that

$$\mathcal{S}(f)(z) = \mathcal{S}(F)(T(z)) \cdot T'(z)^2.$$

One computes $T'(z)^2 = -\frac{4y_0^2}{(z-\bar{z}_0)^4} \cdot T(z)^4$. Moreover observe that $T(z_0) = \infty$. As such

$$\begin{aligned}
|\mathcal{S}(f)(z_0)| &= \lim_{z \to z_0} |\mathcal{S}(F)(T(z)) \cdot T'(z)^2| \\
&= \lim_{w \to \infty} |w^4 \cdot \mathcal{S}(F)(w)| \cdot \lim_{z \to z_0} \frac{4y_0^2}{|z - \overline{z_0}|^4}] \\
&\leq \frac{3}{2 \cdot y_0^2}.
\end{aligned}$$

(e) Use a Möbius transformation to deal with the case when $f(z_0) = \infty$, thus completing the proof of Nehari's theorem.

Solution: From Exercise 1(a) it follows that

$$\mathcal{S}(f)(z_0) = \mathcal{S}(g \circ (1/f))(z_0) = \mathcal{S}(1/f)(z_0),$$

where $g(w) = 1/w$. This means that we can apply the argument above to $1/f$.