Introduction to moduli spaces of Riemann surfaces

Lecture notes

Bram Petri

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Preface

These are the lecture notes for a course called *Introduction to moduli spaces of Riemann surfaces*, taught in January and February 2026 in the master's program *M2 de Mathématiques fondamentales* at Sorbonne University.

There are many references on various aspects of moduli spaces and Teichmüller spaces, like [**IT92**, **Bus10**, **GL00**, **Zor06**, **Hub06**, **FM12**, **Baa21**, **Wri15**]. All of these treat a lot more material than what we will have time for in the course, whence the present notes. Most of the material presented here is adapted from these references.

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LECTURE 1

Reminder on surfaces

Riemann surfaces are objects that appear everywhere in mathematics. Of course, they play an important role in complex analysis and in geometry but also for example in dynamics, number theory and combinatorics.

Their moduli spaces - the spaces that parameterize Riemann surface structures on a fixed surface - are also studied from many different points of view. The goal of this course is to understand the geometry and topology of these moduli spaces.

Before we get to any of this, we need to talk about surfaces themselves. So, today we will recall some of the basics on surfaces.

1.1. Preliminaries on surface topology

1.1.1. Examples and classification. A *surface* is a smooth two-dimensional manifold. We call a surface *closed* if it is compact and has no boundary. A surface is said to be of *finite type* if it can be obtained from a closed surface by removing a finite number of points and (smooth) open disks with disjoint closures. In what follows, we will always assume our surfaces to be orientable.

Example 1.1.1. To properly define a manifold, one needs to not only describe the set but also give smooth charts. In what follows we will content ourselves with the sets.

(a) The 2-*sphere* is the surface

$$\mathbb{S}^2 = \left\{ (x,y,z) \in \mathbb{R}^3 \,:\, x^2+y^2+z^2 = 1 \right\}.$$

(b) Let $\mathbb{S}^1$ denote the circle. The 2-*torus* is the surface

$$\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$$

(c) Given two (oriented) surfaces S_1, S_2, their *connected sum* $S_1 \# S_2$ is defined as follows. Take two closed sets $D_1 \subset S_1$ and $D_2 \subset S_2$ that are both diffeomorphic to closed disks, via diffeomorphisms

$$\varphi_i : \left\{ (x,y) \in \mathbb{R}^2 \,:\, x^2+y^2 \leq 1 \right\} \to D_i, \quad i=1,2,$$

so that φ_1 is orientation preserving and φ_2 is orientation reversing.

Then

$$S_1 \# S_2 = \left(S_1 \setminus \mathring{D}_1 \sqcup S_2 \setminus \mathring{D}_2 \right) / \sim$$

where $\mathring{D}_i$ denotes the interior of D_i for $i = 1, 2$ and the equivalence relation $\sim$ is defined by

$$\varphi_1(x, y) \sim \varphi_2(x, y) \quad \text{for all } (x, y) \in \mathbb{R}^2 \text{ with } x^2 + y^2 = 1.$$

The figure below gives an example.

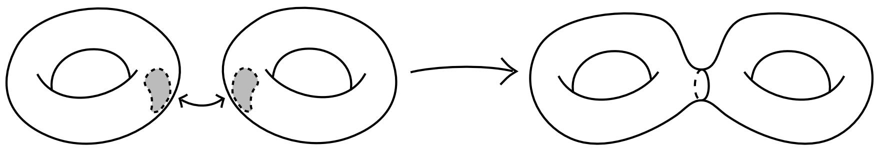

FIGURE 1. A connected sum of two tori.

Like our notation suggests, the manifold $S_1 \# S_2$ is independent (up to diffeomorphism) of the choices we make (the disks and diffeomorphisms φ_i). This is a non-trivial statement, the proof of which we will skip. Likewise, we will also not prove that the connected sum of surfaces is an associative operation and that $\mathbb{S}^2 \# S$ is diffeomorphic to S for all surfaces S.

A classical result from the 19^{th} century tells us that the three simple examples above are enough to understand all finite type surfaces up to diffeomorphism.

THEOREM 1.1.2 (Classification of closed surfaces). *Every closed orientable surface is diffeomorphic to the connected sum of a 2-sphere with a finite number of tori.*

Indeed, because the diffeormorphism type of a finite type surface does not depend on where we remove the points and open disks (another claim we will not prove), the theorem above tells us that an orientable finite type surface is (up to diffeomorphism) determined by a triple of positive integers (g, b, n), where

- g is the number of tori in the connected sum and is called the *genus* of the surface.
- b is the number of disks removed and is called the number of *boundary components* of the surface.
- n is the number of points removed and is called the number of *punctures* of the surface.

DEFINITION 1.1.3. The triple (g, b, n) defined above will be called the *signature* of the surface. We will denote the corresponding surface by $\Sigma_{g,b,n}$ and will write $\Sigma_g = \Sigma_{g,0,0}$.

1.1.2. Euler characteristic. The Euler characteristic is a useful topological invariant of a surface. There are multiple ways to define it. We will use triangulations. A *triangulation* $\mathcal{T} = (V, E, F)$ of a closed surface S will be the data of a finite set of points $V = \{v_1, \ldots . v_k\} \subset S$ (called *vertices*), a finite set of arcs $E = \{e_1, \ldots, e_l\}$ with endpoints in the vertices (called edges) so that the complement $S \smallsetminus (\cup v_i \cup e_j)$ consists of a collection of disks $F = \{f_1, \ldots, f_m\}$ (called *faces*) that all connect to exactly 3 edges.

Note that a triangulation $\mathcal{T}$ here is a slightly more general notion than that of a simplicial complex (it's an example of what Hatcher calls a Δ-complex [Hat02, Page 102]). Figure 2 below gives an example of a triangulation of a torus that is not a simplicial complex.

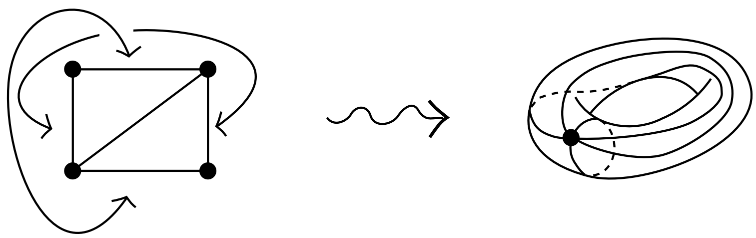

FIGURE 2. A torus with a triangulation

DEFINITION 1.1.4. S be a closed surface with a triangulation $\mathcal{T} = (V, E, F)$. The *Euler characteristic* of S is given by

$$\chi(S) = |V| - |E| + |F|.$$

Because $\chi(S)$ can be defined entirely in terms of singular homology (see [Hat02, Theorem 2.4] for details), it is a homotopy invariant. In particular this implies it should only depend on the genus of our surface S. Indeed, we have

LEMMA 1.1.5. *Let S be a closed connected and oriented surface of genus g. We have*

$$\chi(S) = 2 - 2g.$$

PROOF. Exercise: prove this using your favorite triangulation. □

For surfaces that are not closed, we can define

$$\chi(\Sigma_{g,b,n}) = 2 - 2g - b - n.$$

This can be computed with a triangulation as well. For surfaces with only boundary components, the usual definition still works. For surfaces with punctures there no longer is a finite triangulation, so the definition above no longer makes sense. There are multiple ways out. The most natural is to use the homological definition, which gives the formula above. Another option is to allow some vertices to be missing, that is, to allow edges to run between vertices and punctures. Both give the formula above.

1.2. Riemann surfaces

For the basics on Riemann surfaces, we refer to the lecture notes from the course by Elisha Falbel [Fal23] or any of the many books on them, like [Bea84, FK92]. For a text on complex functions of a single variable, we refer to [SS03].

1.2.1. Definition and first examples. A Riemann surface is a one-dimensional complex manifold. That is,

DEFINITION 1.2.1. A *Riemann surface* X is a connected Hausdorff topological space X, equipped with an open cover $\{U_\alpha\}_{\alpha\in A}$ of open sets and maps $\varphi_\alpha : U_\alpha \to \mathbb{C}$ so that

(1) $\varphi_\alpha(U_\alpha)$ is open and φ_α is a homeomorphism onto its image.

(2) For all $\alpha, \beta \in A$ so that $U_\alpha \cap U_\beta \neq \emptyset$ the map
$$\varphi_\alpha \circ (\varphi_\beta)^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$
is holomorphic.

The pairs $(U_\alpha, \varphi_\alpha)$ are usually called *charts* and the collection $((U_\alpha, \varphi_\alpha))_{\alpha\in A}$ is usually called an *atlas*.

Note that we do not a priori assume a Riemann surface X to be a second countable space. It is however a theorem by Radó that every Riemann surface is second countable (for a proof, see [**Hub06**, Section 1.3]). Moreover every Riemann surface is automatically orientable (see for instance [**GH94**, Page 18]).

EXAMPLE 1.2.2. (a) The simplest example is of course $X = \mathbb{C}$ equipped with one chart: the identity map.

(b) We set $X = \mathbb{C} \cup \{\infty\} = \widehat{\mathbb{C}}$ and give it the topology of the one point compactification of $\mathbb{C}$, which is homeomorphic to the sphere $\mathbb{S}^2$. The charts are
$$U_0 = \mathbb{C}, \quad \varphi_0(z) = z$$
and
$$U_\infty = X \setminus \{0\}, \quad \varphi_\infty(z) = 1/z.$$
So $U_0 \cap U_\infty = \mathbb{C} \setminus \{0\}$ and
$$\varphi_0 \circ (\varphi_\infty)^{-1}(z) = 1/z \quad \text{for all } z \in \mathbb{C} \setminus \{0\}$$
which is indeed holomorphic on $\mathbb{C} \setminus \{0\}$. $\widehat{\mathbb{C}}$ is usually called the *Riemann sphere*.

(c) $\widehat{\mathbb{C}}$ can also be identified with the projective line
$$\mathbb{P}^1(\mathbb{C}) = \left(\mathbb{C}^2 \setminus \{(0,0\}\right) \Big/ \mathbb{C}^*,$$
where $\mathbb{C}^* \curvearrowright \mathbb{C}^2 \setminus \{(0,0)\}$ by $\lambda\cdot(z,w) = (\lambda\cdot z, \lambda\cdot w)$, for $\lambda \in \mathbb{C}^*$, $(z,w) \in \mathbb{C}^2 \setminus \{(0,0)\}$. Indeed, we may equip $\mathbb{P}^1(\mathbb{C})$ with two charts
$$U_0 = \{\, [z:w] \,:\, w \neq 0 \,\}, \quad \varphi_0([z:w]) = z/w$$
and
$$U_1 = \{\, [z:w] \,:\, z \neq 0 \,\}, \quad \varphi_1([z:w]) = w/z.$$
The map
$$[z:w] \mapsto \begin{cases} z/w & \text{if } w \neq 0 \\ \infty & \text{if } w = 0 \end{cases}$$
then defines a biholomorphism $\mathbb{P}^1(\mathbb{C}) \to \widehat{\mathbb{C}}$.

(d) Recall that a *domain* $D \subset \widehat{\mathbb{C}}$ is any connected and open set in $\widehat{\mathbb{C}}$. Any such domain inherits the structure of a Riemann surface from $\widehat{\mathbb{C}}$.

1.2.2. Automorphisms. To get a larger set of examples, we will consider quotients. First of all, we need the notion of a holomorphic map:

DEFINITION 1.2.3. Let X and Y be Riemann surfaces, equipped with atlasses $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ and $\{(V_\beta, \psi_\beta)\}_{\beta \in B}$ respectively. A function $f : X \to Y$ is called *holomorphic* if

$$\psi_\beta \circ f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap f^{-1}(V_\beta)) \to \psi_\beta(f(U_\alpha) \cap V_\beta)$$

is holomorphic for all $\alpha \in A, \beta \in B$ so that $f(U_\alpha) \cap V_\beta \neq \emptyset$. A bijective holomorphism is called a *biholomorphism* or *conformal*. $\mathrm{Aut}(X)$ will denote the *automorphism group* of X, the set of biholomorphisms $X \to X$.

The automorphism group of the Riemann sphere is

$$\mathrm{Aut}(\mathbb{P}^1(\mathbb{C})) = \mathrm{PGL}(2, \mathbb{C}) = \mathrm{GL}(2, \mathbb{C}) \Big/ \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \neq 0 \right\}.$$

It acts on $\mathbb{P}^1(\mathbb{C})$ through the projectivization of the linear action of $\mathrm{GL}(2, \mathbb{C})$ on $\mathbb{C}^2 \backslash \{(0,0)\}$. We can also descibe the action on $\widehat{\mathbb{C}}$. We have:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \begin{cases} \frac{az+b}{cz+d} & \text{if } z \neq -d/c \\ \infty & \text{if } z = -d/c \end{cases} \tag{1.2.1}$$

and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \infty = \begin{cases} \frac{a}{c} & \text{if } c \neq 0 \\ \infty & \text{if } c = 0. \end{cases}$$

These maps are called Möbius transformations.

Finally, we observe that

$$\mathrm{PGL}(2, \mathbb{C}) \simeq \mathrm{PSL}(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{C}, \quad ad - bc = 1 \right\} \Big/ \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

1.2.3. Quotients. Many subgroups of $\mathrm{Aut}(\mathbb{P}^1(\mathbb{C}))$ give rise to Riemann surfaces:

THEOREM 1.2.4. *Let $D \subset \widehat{\mathbb{C}}$ be a domain and let $G < \mathrm{PSL}(2, \mathbb{C})$ such that*

(1) $g(D) = D$ for all $g \in G$

(2) If $g \in G \setminus \{e\}$ then the fixed points of g lie ourside of D.

(3) For each compact subset $K \subset D$, the set

$$\{ g \in G : g(K) \cap K \neq \emptyset \}$$

is finite.

Then the quotient space

$$D/G$$

has the structure of a Riemann surface.

A group that satisfies the second condition is said to act *freely* on D and a group that satisfies the thirs condition is said to act *properly discontinuously* on D. The proof of this theorem will be part of the exercises.

1.2.4. Tori. The theorem from the previous section gives us a lot of new examples. The first is that of tori. Consider the elements

$$g_1 := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, g_\tau := \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} \in \mathrm{PSL}(2, \mathbb{C}),$$

for some $\tau \in \mathbb{C}$ with $\mathrm{Im}(\tau) > 0$, acting on the domain $\mathbb{C} \subset \widehat{\mathbb{C}}$ by

$$g_1(z) = z + 1 \quad \text{and} \quad g_\tau(z) = z + \tau$$

for all $z \in \mathbb{C}$.

We define the group

$$\Lambda_\tau = \langle g_1, g_\tau \rangle < \mathrm{PSL}(2, \mathbb{C}).$$

A direct computation shows that

$$\begin{bmatrix} 1 & p + q\tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & r + s\tau \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & p + q + (r + s)\tau \\ 0 & 1 \end{bmatrix},$$

for all $p, q, r, s \in \mathbb{Z}$, from which it follows that

$$\Lambda_\tau = \left\{ \begin{bmatrix} 1 & n + m\tau \\ 0 & 1 \end{bmatrix} : m, n \in \mathbb{Z} \right\} \simeq \mathbb{Z}^2.$$

Let us consider the conditions from Theorem 1.2.4. (1) is trivially satisfied: Λ_τ preserves $\mathbb{C}$. Any non-trivial element in Λ_τ is of the form

$$\begin{bmatrix} 1 & n + m\tau \\ 0 & 1 \end{bmatrix}$$

and hence only has the point $\infty \in \widehat{\mathbb{C}}$ as a fixed point, which gives us condition (2). To check condition (3), suppose $K \subset \mathbb{C}$ is compact. Write $d_K = \sup \{ |z - w| \, : \, z, w \in K \} < \infty$. Given $g \in \Lambda_\tau$, write

$$T_g = \inf \{ |gz - z| \, : \, z \in \mathbb{C} \}$$

for the *translation length* of g. Note that $T_g = |gz - z|$ for all $z \in \mathbb{C}$ (this is quite special to quotients of $\mathbb{C}$). We have

$$\{ g \in \Lambda_\tau \, : \, g(K) \cap K \neq \emptyset \} \subset \{ g \in \Lambda_\tau \, : \, T_g \leq 2d_K \}$$

and the latter is finite. So $\mathbb{C}/\Lambda_z$ is indeed a Riemann surface.

We claim that this is a torus. One way to see this is to note that the quotient map $\pi : \mathbb{C} \to \mathbb{C}/\Lambda_\tau$ restricted to the convex hull

$$\begin{aligned} \mathcal{F} &= \mathrm{conv}(\{0, 1, \tau, 1 + \tau) \\ &:= \{ \lambda_1 + \lambda_2\tau + \lambda_3(1 + \tau) \, : \, \lambda_1, \lambda_1, \lambda_3 \in [0, 1], \lambda_1 + \lambda_2 + \lambda_3 \leq 1 \} \end{aligned}$$

is surjective. Figure 3 shows a picture of $\mathcal{F}$. On $\mathring{\mathcal{F}}$, π is also injective. So to understand what the quotient looks like, we only need to understand what happens to the sides of $\mathcal{F}$.

Since the quotient map identifies the left hand side of $\mathcal{F}$ with the right hand side and the top with the bottom, the quotient is a torus.

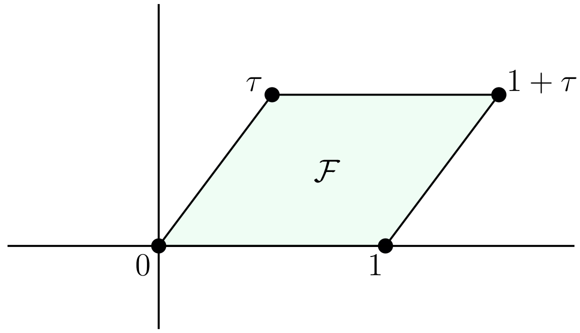

FIGURE 3. A fundamental domain for the action $\Lambda_\tau \curvearrowright \mathbb{C}$.

We can also prove that $\mathbb{C}/\Lambda_\tau$ is a torus by using the fact that for all $z \in \mathbb{C}$ there exist unique $x, y \in \mathbb{R}$ so that

$$z = x + y\tau.$$

The map $\mathbb{C}/\Lambda_\tau \to \mathbb{S}^1 \times \mathbb{S}^1$ given by

$$[x + y\tau] \mapsto (e^{2\pi i x}, e^{2\pi i y})$$

is a homeomorphism.

Note that we have not yet proven whether all these tori are distinct as Riemann surfaces. But it will turn out later that many of them are.

1.2.5. Hyperbolic surfaces. Set $\mathbb{H}^2 = \{ z \in \mathbb{C} : \mathrm{Im}(z) > 0 \}$, the upper half plane. It turns out that the automorphism group of $\mathbb{H}^2$ is $\mathrm{PSL}(2, \mathbb{R})$. We will see a lot more about this later during the course, but for now we will just note that there are many subgroups of $\mathrm{PSL}(2, \mathbb{R})$ that satisfy the conditions of Theorem 1.2.4.

It also turns out that $\mathrm{PSL}(2, \mathbb{R})$ is exactly the group of orientation preserving isometries of the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

This is a complete metric of constant curvature -1. So, this means that all these Riemann surfaces naturally come equipped with a complete metric of constant curvature -1. We will prove some of these statements and treat a first example in the first problem sheet.

1.3. The uniformization theorem and automorphism groups

The Riemann mapping theorem tells us that any pair of simply connected domains in $\mathbb{C}$ that are both not all of $\mathbb{C}$ are biholomorphic. In the early 20^{th} century this was generalized by Koebe and Poincaré to a classification of *all* simply connected Riemann surfaces:

THEOREM 1.3.1 (Uniformization theorem). *Let X be a simply connected Riemann surface. Then X is biholomorphic to exactly one of*

$$\widehat{\mathbb{C}}, \quad \mathbb{C} \quad \text{or} \quad \mathbb{H}^2.$$

PROOF. See for instance [FK92, Chapter IV]. □

This theorem implies that we can see obtain every Riemann surface as a quotient of one of three Riemann surfaces. Before we formally state this, we record the following fact:

PROPOSITION 1.3.2.
- $\mathrm{Aut}(\widehat{\mathbb{C}}) = \mathrm{PSL}(2,\mathbb{C})$ *acting by Möbius transformations,*
- $\mathrm{Aut}(\mathbb{C}) = \{\, \varphi : z \mapsto az+b \;:\; a \in \mathbb{C}\setminus\{0\}, b \in \mathbb{C}\,\} \simeq \mathbb{C} \rtimes \mathbb{C}^*$,
- $\mathrm{Aut}(\mathbb{H}^2) = \mathrm{PSL}(2,\mathbb{R})$ *acting by Möbius transformations.*

PROOF. See for instance [Bea84, Chapter 5] or [IT92, Section 2.3]. □

Note that in all three cases, we have

$$\mathrm{Aut}(X) = \left\{\, g \in \mathrm{Aut}(\widehat{\mathbb{C}}) \;:\; g(X) = X \,\right\},$$

that is, all the automorphisms of $\mathbb{C}$ and $\mathbb{H}^2$ extend to $\widehat{\mathbb{C}}$. However, not all automorphisms of $\mathbb{H}^2$ extend to $\mathbb{C}$.

COROLLARY 1.3.3. *Let X be a Riemann surface. Then there exists a group $G < \mathrm{Aut}(D)$, where D is exactly one of $\mathbb{C}$, $\widehat{\mathbb{C}}$ or $\mathbb{H}^2$ so that*

- *G acts freely and properly discontinuously on D and*
- *$X = D/G$ as a Riemann surface.*

PROOF. Let $\widetilde{X}$ denote the universal cover of X and $\pi_1(X)$ its fundamental group. The fact that X is a Riemann surface, implies that $\widetilde{X}$ can be given the structure of a Riemann surface too, so that $\pi_1(X)$ acts freely and properly discontinuously on $\widetilde{X}$ by biholomorphisms (see for instance [IT92, Lemma 2.6]) and such that

$$\widetilde{X}/\pi_1(X) = X.$$

Since $\widetilde{X}$ is simply connected, it must be biholomorphic to exactly one of $\mathbb{C}$, $\widehat{\mathbb{C}}$ or $\mathbb{H}^2$. □

1.4. Quotients of the three simply connected Riemann surfaces

Now that we know that we can obtain all Riemann surfaces as quotients of one of three simply connected Riemann surfaces, we should start looking for interesting quotients.

1.4.1. Quotients of the Rieman sphere. It turns out that for the Riemann sphere there are none:

PROPOSITION 1.4.1. *Let X be a Riemann surface. The universal cover of X is biholomorphic to $\widehat{\mathbb{C}}$ if and only if X is biholomorphic to $\widehat{\mathbb{C}}$.*

PROOF. The "if" part is clear. For the "only if" part, note that every element in $\mathrm{PSL}(2,\mathbb{C})$ has at least one fixed point on $\widehat{\mathbb{C}}$ (this either follows by direct computation or from the fact that orientation-preserving self maps of the sphere have at least one fixed point, by the Brouwer fixed point theorem [Mil65, Problem 6]). Since, by assumption

$$X = \widehat{\mathbb{C}}/G,$$

where G acts properly discontinuously and freely, we must have $G = \{e\}$. □

1.4.2. Quotients of the plane. In Section 1.2.4, we have already seen that in the case of the complex plane, the list of quotients is a lot more interesting: there are tori. This however turns out to be almost everything:

PROPOSITION 1.4.2. *Let X be a Riemann surface. The universal cover of X is biholomorphic to $\mathbb{C}$ if and only if X is biholomorphic to either $\mathbb{C}$, $\mathbb{C} \setminus \{0\}$ or*

$$\mathbb{C}/\left\langle \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix} \right\rangle$$

for some $\lambda, \mu \in \mathbb{C} \setminus \{0\}$ that are linearly independent over $\mathbb{R}$.

PROOF. First suppose $X = \mathbb{C}/G$. We claim that, since G acts properly discontinuously, G is one of the following three forms:

(1) $G = \{e\}$
(2) $G = \langle \varphi_b \rangle$, where $\varphi_b(z) = z + b$ for some $b \in \mathbb{C} \setminus \{0\}$
(3) $G = \langle \varphi_{b_1}, \varphi_{b_2} \rangle$ where $b_1, b_2 \in \mathbb{C}$ are independent over $\mathbb{R}$.

To see this, we first prove that G cannot contain any automorphism $z \mapsto az + b$ for $a \neq 1$. Indeed, if $a \neq 1$ then $b/(1-a)$ is a fixed point for this map, which would contradict freeness of the action. Moreover, since $z \mapsto z + b_1$ and $z \mapsto z + b_2$ commute for all $b_1, b_2 \in \mathbb{C}$, G is a free abelian group and

$$G \cdot z = \{\, z + b \,:\, \varphi_b \in G \,\}\,.$$

In particular, if G contains $\{z \mapsto z + b_1, z \mapsto z + b_2, z \mapsto z + b_3\}$ for $b_1, b_2, b_3 \in \mathbb{C}$ that are independent over $\mathbb{Q}$, then $\mathrm{span}_{\mathbb{Z}}(b_1, b_2, b_3)$ is dense in $\mathbb{C}$. This means that we can find a sequence $((k_i, l_i, m_i))_i$ such that

$$\varphi_{b_1}^{k_i} \circ \varphi_{b_2}^{l_i} \circ \varphi_{b_3}^{m_i}(z) \to z \quad \text{as } i \to \infty,$$

thus contradicting proper discontinuity. On a side note, we could have also used the classification of surfaces (of potentially infinite type) in the last step: there is no surface that has $\mathbb{Z}^k$ for $k \geq 3$ as a fundamental group.

We have already seen that the third case gives rise to tori. In the second case, the surface is biholomorphic to $\mathbb{C} \setminus \{0\}$. Indeed, the map

$$[z] \in \mathbb{C}/\langle \varphi_b \rangle \quad \mapsto \quad e^{2\pi i z/b} \in \mathbb{C} \setminus \{0\}$$

is a biholomorphism.

Now let us prove the converse. For $X = \mathbb{C}$ the statement is clear. Likewise, for $X = \mathbb{C}\setminus\{0\}$, we have just seen that the composition

$$\mathbb{C} \to \mathbb{C}/(z \sim z + 1) \simeq \mathbb{C} \setminus \{0\}$$

is the universal covering map. Finally, in the proposition, the tori are given as quotients of $\mathbb{C}$. □

LECTURE 2

Quotients, metrics, conformal structures

2.1. More on quotients

2.1.1. Quotients of the complex plane, continued. We saw last time that any quotient Riemann surface of $\mathbb{C}$ is either $\mathbb{C}$, $\mathbb{C} - \{0\}$ or a torus. It turns out that moreover every Riemann surface structure on the torus comes from the complex plane. We have seen above that the universal cover cannot be the Riemann sphere, which means that (using the uniformization theorem) all we need to prove is that it cannot be the upper half plane either.

The fundamental group of the torus is isomorphic to $\mathbb{Z}^2$, so what we need to prove is that there is no subgroup of $\text{Aut}(\mathbb{H}^2) = \text{PSL}(2, \mathbb{R})$ that acts properly discontinuously and freely on $\mathbb{H}^2$ and is isomorphic to $\mathbb{Z}^2$. We will state this as a lemma (in which we don't unnecessarily assume that the action is free, even if in our context that would suffice):

LEMMA 2.1.1. *Suppose $G < \text{PSL}(2, \mathbb{R})$ acts properly on $\mathbb{H}^2$ and suppose furthermore that G is abelian. Then either $G \simeq \mathbb{Z}$ or G is finite and cyclic.*

PROOF. We will use the classification of isometries of $\mathbb{H}^2$ that we shall prove in the exercises: an element $g \in \text{PSL}(2, \mathbb{R})$ has either

- a single fixed point in $\mathbb{H}^2$, in which case it's called elliptic and can be conjugated into $\text{SO}(2)$
- a single fixed point on $\mathbb{R} \cup \{\infty\}$, in which case it's called parabolic and can be conjugated into $\left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}$
- or two fixed points on $\mathbb{R} \cup \{\infty\}$, in which case it's called hyperbolic (or loxodromic) and can be conjugated into $\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} : \lambda > 0 \right\}$.

If $g_1, g_2 \in \text{PSL}(2, \mathbb{R})$ commute and $p \in \mathbb{H}^2 \cup \mathbb{R} \cup \{\infty\}$ is a fixed point of g_1, then

$$g_1(g_2(p)) = g_2 \circ g_1(p) = g_2(p).$$

That is, $g_2(p)$ is also a fixed point of g_1.

So if G contains an elliptic element g, then all other $g' \in G \setminus \{e\}$ are elliptic as well, with the same fixed point. Moreover, by proper discontinuity (and compactness of $\text{SO}(2)$), the angles of rotation of all elements in G must be rationally related rational multiples of π. This means that G is a finite cyclic group.

Now suppose G contains a parabolic element g. Then all other $g' \in G \setminus \{e\}$ are parabolic as well, with the same fixed point (which we may assume to be ∞). If

$$\begin{pmatrix} 1 & t_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & t_2 \\ 0 & 1 \end{pmatrix} \in G$$

for some $t_1, t_2 \in \mathbb{R}$ that are not rationally related, then G is not discrete, which contradicts proper discontinuity (see the exercises). So $G \simeq \mathbb{Z}$.

The argument in the hyperbolic case is essentially the same as in the parabolic case. □

Combining this lemma with the uniformization theorem, we obtain:

COROLLARY 2.1.2. *Let X be a Riemann surface that is diffeomorphic to a torus. Then the universal cover of X is biholomorphic to $\mathbb{C}$.*

2.1.2. Quotients of the upper half plane. It will turn out that the richest family of Riemann surfaces is that of quotients of $\mathbb{H}^2$. Indeed, looking at the clasification of closed orientable surfaces, we note that we have so far only seen the sphere and the torus. It turns out that all the other closed orientable surfaces also admit the structure of a Riemann surface. In fact, they all admit lots of different such structures. The two propositions above imply that they must all arise as quotients of $\mathbb{H}^2$.

We will not yet discuss how to construct all these surfaces but instead discuss an example (partially taken from [GGD12, Example 1.7]). Fix some distinct complex numbers $a_1, \ldots, a_{2g+1}$ and consider the following subset of $\mathbb{C}^2$:

$$\mathring{X} = \left\{ (z,w) \in \mathbb{C}^2 \;:\; w^2 = (z-a_1)(z-a_2)\cdots(z-a_{2g+1}) \right\}.$$

Let X denote the one point compactification of $\mathring{X}$ obtained by adjoining the point (∞, ∞).

As opposed to charts, we will describe inverse charts, or *parametrizations* around every $p \in \mathring{X}$:

- Suppose $p = (z_0, w_0) \in \mathring{X}$ is so that $z_0 \neq a_i$ for all $i = 1, \ldots, 2g+1$. Set

  $$\varepsilon := \min_{i=1,\ldots,2g+1} \{|z_0 - a_i|/2\}$$

  Then define the map $\varphi^{-1} : \{ \zeta \in \mathbb{C} \,:\, |\zeta| < \varepsilon \} \to \mathring{X}$ by

  $$\varphi^{-1}(\zeta) = \left(\zeta + z_0, \quad \sqrt{(\zeta + z_0 - a_1)\cdots(\zeta + z_0 - a_{2g+1})} \right),$$

  where the branch of the square root is chosen so that $\varphi^{-1}(0) = (z_0, w_0)$, gives a parametrization.

- For $p = (a_j, 0)$, we set

  $$\varepsilon := \min_{i \neq j} \{ \sqrt{|a_j - a_i|/2} \}$$

Then define the map $\varphi^{-1} : \{\, \zeta \in \mathbb{C} \,:\, |\zeta| < \varepsilon \,\} \to \mathring{X}$ by

$$\varphi^{-1}(\zeta) = \left(\zeta^2 + a_j, \quad \zeta \sqrt{\prod_{i \neq j} (\zeta^2 + a_j - a_i)} \right).$$

The reason that we need to take different charts around these points is that

$$\sqrt{z - a_j}$$

is not a well defined holomorphic function near $z = a_j$.

Also note that the choice of the branch of the root does not matter. By changing the branch we would obtain a new parametrization $\widetilde{\varphi}^{-1}$ that satisfies $\widetilde{\varphi}^{-1}(\zeta) = \varphi^{-1}(-\zeta)$.

It's not hard to see that $\mathring{X}$ is not bounded as a subset of $\mathbb{C}^2$. This means in particular that it's not compact. We can however compactify it in a similar fashion to how we compactified $\mathbb{C}$ in order to obtain the Riemann sphere. That is, we add a point (∞, ∞) and around this point define a parametrization:

$$\varphi_\infty^{-1}(\zeta) = \begin{cases} \left(\zeta^{-2}, \quad \zeta^{-(2g+1)} \sqrt{(1 - a_1\zeta^2) \cdots (1 - a_{2g+1}\zeta^2)} \right) & \text{if } \zeta \neq 0 \\ (\infty, \infty) & \text{if } \zeta = 0, \end{cases}$$

for all $\zeta \in \{|\zeta| < \varepsilon\}$ and some appropriate $\varepsilon > 0$.

The reason that the resulting surface X is compact is that we can write it as the union of the sets

$$\left\{ (z, w) \in \mathring{X} \,:\, |z| \leq 1/\varepsilon^2 \right\} \cup \left(\left\{ (z, w) \in \mathring{X} \,:\, |z| \geq 1/\varepsilon^2 \right\} \cup \{(\infty, \infty)\} \right),$$

for some small $\varepsilon > 0$. The first set is compact because it's a bounded subset of $\mathbb{C}^2$. The second set is compact because it's $\varphi_\infty^{-1}(\{|\zeta| \leq \varepsilon\})$.

To see that X is connected, we could proceed using charts as well. We would have to find a collection of charts that are all connected, overlap and cover X. However, it's easier to use complex analysis. Suppose $z_0 \neq a_i$ for all $i = 1, \ldots, a_{2g+1}$ and $z_0 \neq \infty$. In that case, we can define a path

$$z(t) \mapsto \left(z(t), \quad \sqrt{\prod_{i=1}^{2g+1} (z(t) - a_i)} \right)$$

where $z(t)$ is some continuous path in $\mathbb{C}$ between z_0 and a_i and we pick a continuous branch of the square root, thus connecting any point $(z_0, w_0) \in X$ to $(0, a_i)$.

To figure out the genus of X, note that there is a map $\pi : X \to \widehat{\mathbb{C}}$ given by

$$\pi(z, w) = z \quad \text{for all } (z, w) \in X.$$

This map is two-to-one almost everywhere. Only the points $z = a_i$, $i = 1, \ldots, 2g + 1$ and the point $z = \infty$ have only one pre-image.

Now triangulate $\widehat{\mathbb{C}}$ so that the vertices of the triangulation coincide with the points $a_1, \ldots, a_{2g+1}, \infty$. If we lift the triangulation to X using π, we can compute the Euler characteristic of X. Every face and every edge in the triangulation of $\widehat{\mathbb{C}}$ has two pre-images, whereas each vertex has only one. This means that:

$$\chi(X) = 2\chi(\widehat{\mathbb{C}}) - (2g+2) = 2 - 2g.$$

Because X is an orientable closed surface, we see that it must have genus g (Lemma 1.1.5). In particular, if $g \geq 2$, these surfaces are quotients of $\mathbb{H}^2$. Note that this also implies that for $g \geq 1$, the Riemann surface $\mathring{X}$ is also a quotient of $\mathbb{H}^2$. Note that we could have also used the Riemann–Hurwitz formula for this calculation. Incidentally, this formula can be proved using a similar argument to what we just did above.

To get a picture of what X looks like, draw a closed arc α_1 between a_1 and a_2 on $\widehat{\mathbb{C}}$, an arc α_2 between a_3 and a_4 that does not intersect the first arc and so on, and so forth. The last arc α_{g+1} goes between a_{2g+1} and ∞. Figure 1 shows a picture of what these arcs might look like.

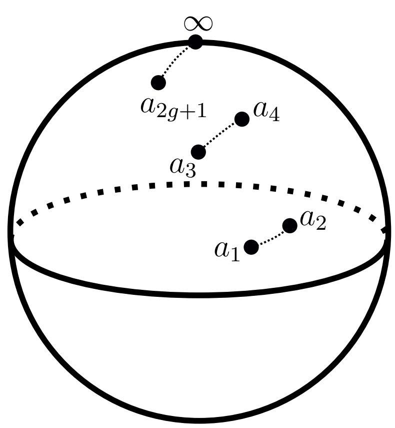

FIGURE 1. $\widehat{\mathbb{C}}$ with some intervals removed.

Let

$$D = \widehat{\mathbb{C}} \setminus \left(\bigcup_{i=1}^{g+1} \alpha_i \right).$$

The map

$$\pi|_{\pi^{-1}(D)} : \pi^{-1}(D) \to D$$

is now a two-to-one map. Moreover on the arcs, it's two-to-one on the interior and one-to-one on the boundary. Because it's also smooth, this means that the pre-image of the arcs is a circle. So, X may be obtained (topologically) by cutting $\widehat{\mathbb{C}}$ open along the arcs, taking two copies of that, and gluing these along their boundary. Figure 2 depicts this process.

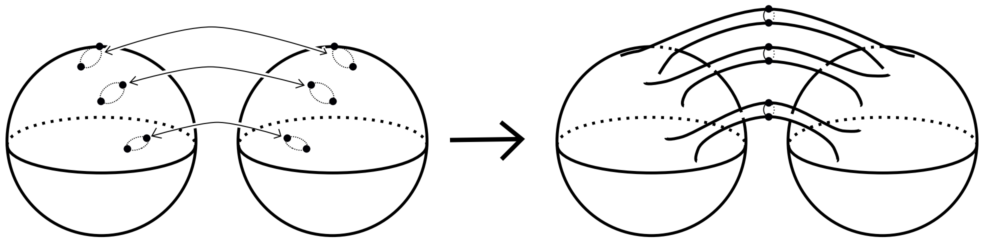

FIGURE 2. Gluing X out of two Riemann spheres.

Finally, we note that our Riemann surfaces come with an involution $\imath : X \to X$, given by

$$\imath(w) = \begin{cases} -w & \text{if } w \neq \infty \\ \infty & \text{if } w = \infty. \end{cases}$$

This map is called the *hyperelliptic involution* and the surfaces we described are hence called *hyperelliptic surfaces*. Note that $\pi : X \to \widehat{\mathbb{C}}$ is the quotient map $X \to X/\imath$.

2.2. Riemannian metrics and Riemann surfaces

We already noted that every Riemann surface comes with a natural Riemannian metric. Indeed the Riemann sphere has the usual round metric of constant curvature $+1$. Likewise, $\mathbb{C}$ has a flat metric, its usual Euclidean metric $\mathrm{Aut}(\mathbb{C})$ does not act by isometries. However, in the proof of Proposition 1.4.2, we saw that all the quotients are obtained by quotienting by a group that does act by Euclidean isometries. This means that the Euclidean metric descends. Finally, we proved in the exercises that $\mathrm{Aut}(\mathbb{H}^2)$ also acts by isometries of the hyperbolic metric defined in Section 1.2.5. So every quotient of $\mathbb{H}^2$ comes with a natural metric of constant curvature -1.

It turns out that we can also go the other way around. That is: Riemann surface structures on a given surface are in one-to-one correpsondence with complete metrics of constant curvature.

One way to see this uses the Killing-Hopf theorem. In the special case of surfaces, this states that every oriented surface equipped with a Riemannian metric of constant curvature $+1$, 0 or -1 can be obtained as the quotient by a group of orientation preserving isometries acting properly discontinuously and freely on $\mathbb{S}^2$ equipped with the round metric, $\mathbb{R}^2$ equipped with the Euclidean metric or $\mathbb{H}^2$ equipped with the hyperbolic metric respectively (see [CE08, Theorem 1.37] for a proof). For a Riemannian manifold M, let us write

$$\mathrm{Isom}^+(M) = \{ \varphi : M \to M \ : \ \varphi \text{ is an orientation preserving isometry} \}.$$

So, we need the fact that

(1) $\mathrm{Isom}^+(\mathbb{S}^2) = \mathrm{SO}(2,\mathbb{R})$ and this has no non-trivial subgroups that act properly discontinuously on $\mathbb{S}^2$.

(2) $\mathrm{Isom}^+(\mathbb{R}^2) = \mathrm{SO}(2,\mathbb{R}) \ltimes \mathbb{R}^2$, where $\mathbb{R}^2$ acts by translations. The only subgroups of this group that act properly discontinuously and freely are the fundamental groups of tori and cylinders.

(3) $\mathrm{Isom}^+(\mathbb{H}^2) = \mathrm{PSL}(2,\mathbb{R})$.

Given the above, we get our one-to-one correspondence:

PROPOSITION 2.2.1. *Given an orientable surface Σ of finite type with $\partial\Sigma = \emptyset$, the identification described above gives a one-to-one correspondence of sets*

$$\left\{ \begin{array}{c} \textit{Riemann surface} \\ \textit{structures on } \Sigma \end{array} \right\} \Big/ \sim \quad \leftrightarrow \quad \left\{ \begin{array}{c} \textit{Complete Riemannian} \\ \textit{metrics of constant} \\ \textit{curvature } \{-1,0,+1\} \\ \textit{on } \Sigma \end{array} \right\} \Big/ \sim,$$

where the equivalence on the left is biholomorphism and the equivalence on the right is isometry (and homothety in the Euclidean case).

PROOF SKETCH. From the above we see that a Riemann surface structure on Σ yields a metric of constant curvature and vice versa. We only need to check that biholomorphic Riemann surfaces yield isometric/homothetic metrics and vice versa.

Suppose $h : X \to Y$ is a biholomorphism. We may lift this to a biholomorphism $\widetilde{h} : \widetilde{X} \to \widetilde{Y}$ of the universal covers $\widetilde{X}$ and $\widetilde{Y}$ of X and Y respectively. There are three cases to treat: $\widetilde{X} \simeq \widetilde{Y} \simeq \mathbb{C}, \widehat{\mathbb{C}}, \mathbb{H}^2$. Because it's the most interesting case, we will treat the first, i.e. $\widetilde{X} \simeq \widetilde{Y} \simeq \mathbb{C}$. We will also assume X and Y are tori. If we write

$$X \simeq \mathbb{C}/\Lambda_1 \quad \text{and} \quad Y \simeq \mathbb{C}/\Lambda_2,$$

then we get that $\widetilde{h} \in \mathrm{Aut}(\mathbb{C})$ is such that $\widetilde{h}(\Lambda_1) = \Lambda_2$. Since all automorphisms of $\mathbb{C}$ are of the form $z \mapsto az + b$ for $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$, Λ_2 is obtained from Λ_1 by translating, scaling and rotating. This means that the quotient metrics are homothetic.

The proof of the reverse direction and both directions of all the remaining cases are similar. □

Whether the curvature is 0, +1 or −1 is determined by the topology of the underlying surface. This for instance follows from the discussion above. It can also be seen from the Gauss-Bonnet theorem. Recall that in the case of a closed Riemannian surface X, this states that

$$\int_X K \, dA = 2\pi \, \chi(\Sigma),$$

where K is the Gaussian curvature on X and dA the area measure. For constant curvature κ, this means that

$$\kappa \cdot \mathrm{area}(X) = 2\pi \, \chi(X)$$

So $\chi(X) = 0$ if and only of $\kappa = 0$ and otherwise $\chi(X)$ needs to have the same sign as κ. This last equality generalizes to finite type surfaces and we obtain:

LEMMA 2.2.2. *Let X be a hyperbolic surface homeomorphic to $\Sigma_{g,b,n}$ then*

$$\mathrm{area}(X) = 2\pi(2g + n + b - 2).$$

2.3. Conformal structures

There is another type of structures on a surface that is in one-to-one correspondence with Riemann surface structures, namely conformal structures.

We say that two Riemannian metrics ds_1^2 and ds_2^2 on a surface X are *conformally equivalent* is there exists a positive function $\rho : X \to \mathbb{R}_+$ so that

$$ds_1^2 = \rho \cdot ds_2^2.$$

So a conformal equivalence class of Riemannian metrics can be seen as a notion of angles on the surface.

We have already seen that a Riemann surface structure induces a Riemannian metric on the surface, so it certainly also induces a conformal class of metrics.

So, we need to explain how to go back. We will also only sketch this. First of all, suppose we are given a surface X with oriented charts $(U_j, (u_j, v_j))_j$ equipped with a Riemannian metric that in all local coordinates (u_j, v_j) is of the form

$$ds^2 = \rho(u_j, v_j) \cdot (du_j^2 + dv_j^2),$$

where $\rho : X \to \mathbb{R}_+$ is some smooth function. Consider the complex-valued coordinate

$$w_j = u_j + i\, v_j.$$

We claim that this is holomorphic. Indeed, applying a coordinate change on $U_j \cap U_k$, we have

$$ds^2 = \rho(u_k, v_k) \cdot \left[\left(\left(\frac{\partial u_j}{\partial u_k}\right)^2 + \left(\frac{\partial v_j}{\partial u_k}\right)^2 \right) du_k^2 + \left(\left(\frac{\partial u_j}{\partial v_k}\right)^2 + \left(\frac{\partial v_j}{\partial v_k}\right)^2 \right) dv_k^2 \right.$$
$$\left. + 2 \left(\frac{\partial u_j}{\partial u_k}\frac{\partial u_j}{\partial v_k} + \frac{\partial v_j}{\partial u_k}\frac{\partial v_j}{\partial v_k} \right) du_k dv_k \right].$$

Our assumption implies that

$$\left(\frac{\partial u_j}{\partial u_k}\right)^2 + \left(\frac{\partial v_j}{\partial u_k}\right)^2 = \left(\frac{\partial u_j}{\partial v_k}\right)^2 + \left(\frac{\partial v_j}{\partial v_k}\right)^2 \tag{2.3.1}$$

and

$$\frac{\partial u_j}{\partial u_k}\frac{\partial u_j}{\partial v_k} + \frac{\partial v_j}{\partial u_k}\frac{\partial v_j}{\partial v_k} = 0. \tag{2.3.2}$$

This last line can be written as

$$\det \begin{pmatrix} \partial u_j/\partial u_k & \partial v_j/\partial v_k \\ -\partial v_j/\partial u_k & \partial u_j/\partial v_k \end{pmatrix} = 0.$$

So this implies that

$$\begin{pmatrix} \partial u_j/\partial u_k \\ -\partial v_j/\partial u_k \end{pmatrix} = \lambda \cdot \begin{pmatrix} \partial v_j/\partial v_k \\ \partial u_j/\partial v_k \end{pmatrix}$$

for some $\lambda \in \mathbb{R}$. Filling this into (2.3.1), we obtain

$$\lambda^2 \cdot \left(\left(\frac{\partial u_j}{\partial v_k}\right)^2 + \left(\frac{\partial v_j}{\partial v_k}\right)^2 \right) = \left(\frac{\partial u_j}{\partial v_k}\right)^2 + \left(\frac{\partial v_j}{\partial v_k}\right)^2.$$

So $\lambda \in \{\pm 1\}$. Now using that our surface is oriented, i.e. that the determinant of the Jacobian of the chart transition is positive, we obtain that $\lambda = 1$. And hence

$$\frac{\partial u_j}{\partial u_k} = \frac{\partial v_j}{\partial v_k} \quad \text{and} \quad -\frac{\partial v_j}{\partial u_k} = \frac{\partial u_j}{\partial v_k},$$

the Cauchy-Riemann equations for the chart transition $w_k \circ w_j^{-1}$, which means that these coordinates are indeed holomorphic. The coordinates (U_j, w_j) are usually called *isothermal coordinates.*

Also note that we have not used the factor ρ, so any metric that is conformal to our metric will give us the same structure. Moreover, our usual coordinate 'z' on the three simply connected Riemann surfaces is an example of an isothermal coordinate, so if we apply the procedure above to the metric we obtain from our quotients, we find the same complex structure back.

This means that what we need to show is that for each Riemannian metric (that is not necessarily given to us in the form above), we can find a set of coordinates so that our metric takes this form. So, suppose our metric is given by

$$ds^2 = A\ dx^2 + 2B\ dx\ dy + C\ dy^2$$

in some local coordinates (x, y).

Writing $z = x + iy$, we get that

$$ds^2 = \lambda \left| dz + \mu d\overline{z} \right|^2 := \lambda (dz + \mu d\overline{z})(d\overline{z} + \overline{\mu} dz),$$

where

$$\lambda = \frac{1}{4}\left(A + C + 2\sqrt{AC - B^2}\right) \quad \text{and} \quad \mu = \frac{A - C + 2i\ B}{A + C + 2\sqrt{AC - B^2}}.$$

We are looking for a coordinate $w = u + iv$ so that

$$ds^2 = \rho(du^2 + dv^2) = \rho \left| dw \right|^2 = \rho \cdot \left| \frac{\partial w}{\partial z} \right|^2 \cdot \left| dz + \frac{\partial w / \partial \overline{z}}{\partial w / \partial z} d\overline{z} \right|^2,$$

where

$$\frac{\partial w}{\partial z} = \frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y} \quad \text{and} \quad \frac{\partial w}{\partial \overline{z}} = \frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y}$$

are called *Wirtinger derivatives*, we will discuss these in slightly more detail in Section **??**.

This means that isothermal coordinates exist if there is a solution to the partial differential equation

$$\frac{\partial w}{\partial \overline{z}} = \mu \cdot \frac{\partial w}{\partial z}.$$

It turns out this solution does indeed exist on a surface, which means that we obtain a Riemann surface structure. Moreover, it turns out this map is one-to-one. In particular, holmorphic maps are conformal. So we obtain

Proposition 2.3.1. *Given an orientable surface Σ of finite type with $\partial\Sigma = \emptyset$, the identification described above gives a one-to-one correspondence of sets*

$$\left\{ \begin{array}{c} \textit{Riemann surface} \\ \textit{structures on } \Sigma \end{array} \right\} \Big/ \textit{biholom.} \leftrightarrow \left\{ \begin{array}{c} \textit{Conformal classes} \\ \textit{of Riemannian} \\ \textit{metrics on } \Sigma \end{array} \right\} \Big/ \textit{diffeomorphism.}$$

Combined with Proposition 2.2.1, the proposition above also implies that in every conformal class of metrics there is a metric of constant curvature that is unique (up to scaling if the metric is flat). This can also be proved without passing through the uniformization theorem, which comes down to solving a non-linear PDE on the surface. This was treated in Olivier Biquard's course *Introduction à l'analyse géométrique*.

LECTURE 3

The Teichmüller space of the torus

3.1. Riemann surface structures on the torus

The goal of the rest of this course is to understand the deformation spaces associated to Riemann surfaces: Teichmüller and moduli spaces.

In general, the Teichmüller space associated to a surface will be a space of *marked* Riemann surface structures on that surface and the corresponding moduli space will be a space of isomorphism classes of Riemann surface structures. As such, the moduli space associated to a surface will be a quotient of the corresponding Teichmüller space.

First of all, note that the uniformization theorem tells us that there is only one Riemann surface structure on the sphere. This means that the corresponding moduli space will be a point. It turns out that the same holds for its Teichmüller space. This means that the lowest genus closed surface for which we can expect an intersting deformation space is the torus.

So, let us parametrize Riemann surface structures on the torus. Recall from Proposition 1.4.2 that every Riemann surface structure on the torus is of the form

$$\mathbb{C}/\left\langle \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix} \right\rangle$$

for some $\lambda, \mu \in \mathbb{C} \setminus \{0\}$ that are linearly independent over $\mathbb{R}$.

First of all note that every such torus is biholomorphic to a torus of the form

$$R_\tau := \mathbb{C}/\Lambda_\tau,$$

for some $\tau \in \mathbb{H}^2$, where

$$\Lambda_\tau = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} \right\rangle.$$

Indeed, rotating and rescaling the lattice induce biholomorphisms on the level of Riemann surfaces (as we have already noted in the proof sketch of Proposition 2.2.1)

However, there are still distinct $\tau, \tau' \in \mathbb{H}^2$ that lead to holomorphic tori. We have:

PROPOSITION 3.1.1. *Let $\tau, \tau' \in \mathbb{H}^2$. The two tori R_τ and $R_{\tau'}$ are biholomorphic if and only if*

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

for some $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$.

PROOF. First assume R_τ and $R_{\tau'}$ are biholomorphic and let $f : R_{\tau'} \to R_\tau$ be a biholomorphism. Lift f to a biholomorphism $\widetilde{f} : \mathbb{C} \to \mathbb{C}$. This means that

$$\widetilde{f}(z) = \alpha z + \beta$$

for some $\alpha, \beta \in \mathbb{C}$. By postcomposing with a biholomorphism of $\mathbb{C}$, we may assume that $\widetilde{f}(0) = 0$.

Because $\widetilde{f}$ is a lift, we know that both $\widetilde{f}(1)$ and $\widetilde{f}(\tau')$ are equivalent to 0 under Λ_τ. So

$$\widetilde{f}(\tau') = \alpha\tau' = a\tau + b$$

$$\widetilde{f}(1) = \alpha = c\tau + d$$

for some $a, b, c, d \in \mathbb{Z}$. So

$$\tau' = \frac{a\tau + b}{c\tau + d}.$$

So we only need to show that $ad - bc = 1$. Moreover, since $\widetilde{f}(\Lambda_{\tau'}) = \Lambda_\tau$, $f(\tau') = a\tau + b$ and $f(1) = c\tau + d$ generate Λ_τ. This means that the map

$$m\tau + n \mapsto m \cdot (a\tau + b) + n \cdot (c\tau + d)$$

is an automorphism of Λ, and hence $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z})$. So, we obtain $ad - bc = \pm 1$. Since

$$\mathrm{Im}(\tau') = \frac{ad - bc}{|c\tau + d|^2} > 0,$$

we get $ad - bc = 1$.

Conversely, if

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

Then

$$f([z]) = [(c\tau + d)z]$$

gives a biholomorphic map $f : R_{\tau'} \to R_\tau$. □

3.2. The Teichmüller and moduli spaces of tori

Looking at Proposition 3.1.1, we see that we can parametrize all complex structures on the torus with the set

$$\mathcal{M}_1 = \mathrm{SL}(2, \mathbb{Z}) \backslash \mathbb{H}^2 = \mathrm{PSL}(2, \mathbb{Z}) \backslash \mathbb{H}^2.$$

Moreover this set is the quotient of the hyperbolic place by a group ($\mathrm{PSL}(2, \mathbb{Z})$) of isometries that acts properly discontinuously on it. However, the group doesn't quite act freely, so it's not directly a hyperbolic surface.

So, let us investigate the structure of this quotient. One way of doing this is to find a fundamental domain for the action of $\mathrm{PSL}(2, \mathbb{Z})$ on $\mathbb{H}^2$. Set

$$\mathcal{F} = \left\{ z \in \mathbb{H}^2 \,:\, |z| \geq 1 \text{ and } -\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2} \right\}.$$

Figure 1 shows a picture of $\mathcal{F}$.

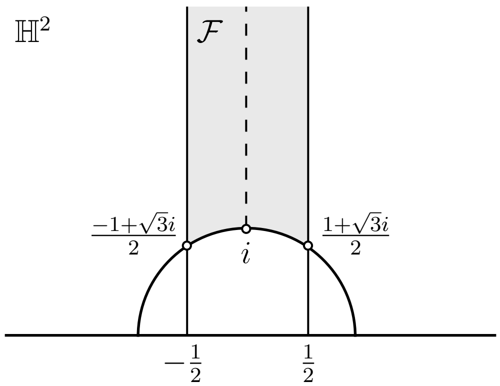

FIGURE 1. A fundamental domain for the action of $\mathrm{PSL}(2,\mathbb{Z})$ on $\mathbb{H}^2$.

We claim

PROPOSITION 3.2.1. *For all* $\tau \in \mathbb{H}^2$ *there exists an element* $g \in \mathrm{PSL}(2,\mathbb{Z})$ *so that* $g\tau \in \mathcal{F}$. *Moreover,*

- *if* $\tau \in \mathring{\mathcal{F}}$ *then*

$$\Big(\mathrm{PSL}(2,\mathbb{Z}) \cdot \tau\Big) \cap \mathcal{F} = \{\tau\},$$

- *if* $\tau \in \mathcal{F}$ *and* $\mathrm{Re}(\tau) = -\frac{1}{2}$ *then*

$$\Big(\mathrm{PSL}(2,\mathbb{Z}) \cdot \tau\Big) \cap \mathcal{F} = \{\tau, \tau+1\},$$

- *if* $\tau \in \mathcal{F}$ *and* $\mathrm{Re}(\tau) = \frac{1}{2}$ *then*

$$\Big(\mathrm{PSL}(2,\mathbb{Z}) \cdot \tau\Big) \cap \mathcal{F} = \{\tau, \tau-1\}.$$

- *and if* $\tau \in \mathcal{F}$ *and* $|\tau| = 1$ *then*

$$\Big(\mathrm{PSL}(2,\mathbb{Z}) \cdot \tau\Big) \cap \mathcal{F} = \{\tau, -1/\tau\},$$

The proof of this proposition is part of this week's exercises.

In conclusion, $T : z \mapsto z+1$ maps the line $\mathrm{Re}(z) = -1/2$ to the line $\mathrm{Re}(z) = 1/2$ and $S : z \mapsto -1/z$ fixes i and swaps $(-1+\sqrt{3}i)/2$ and $(1+\sqrt{3}i)/2$ (which are in turn the fixed points of ST). Moreover, these are the only side pairings and thus the quotient looks like Figure 2:

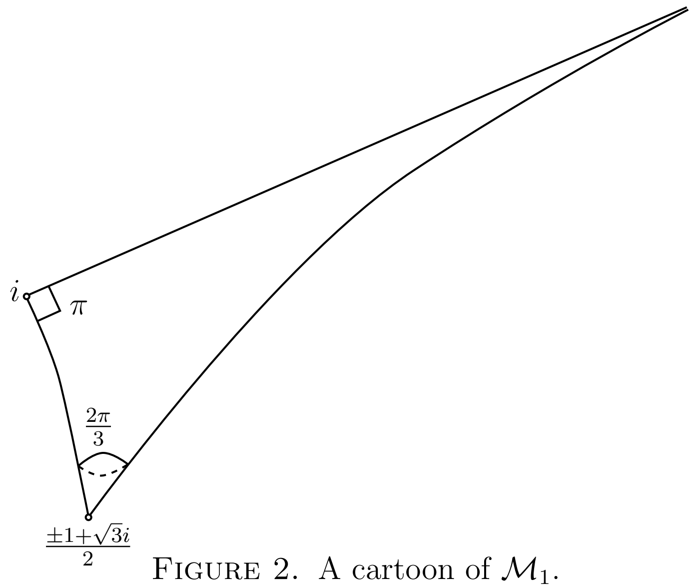

FIGURE 2. A cartoon of $\mathcal{M}_1$.

So $\mathcal{M}_1$ is a space that has the structure of a hyperbolic surface near almost every point. The only problematic points are the images of i and $(\pm 1 + \sqrt{3}i)/2$, where the $\mathcal{M}_1$ looks like a cone. The technical term for such a space is a hyperbolic *orbifold*.

$\mathcal{M}_1$ is called the *moduli space* of tori. $\mathcal{T}_1 = \mathbb{H}^2$ is called the *Teichmüller space* of tori.

Our next intermediate goal is to generalize this to all surfaces. To this end, we will introduce a different perspective on $\mathcal{T}_1$, that generalizes naturally to higher genus surfaces.

3.3. $\mathcal{T}_1$ as a space of marked structures

Our objective in this section is to understand what the information is that is parametrized by $\mathcal{T}_1$.

3.3.1. Markings as a choice of generators for $\pi_1(R)$. So, suppose $\tau \in \mathbb{H}^2$ and $\tau' = g\tau$ for some $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}(2, \mathbb{Z})$. Let $f : R_{\tau'} \to R_\tau$ denote the biholomorphism from the proof of Proposition 3.1.1. We saw that we can find a lift $\widetilde{f} : \mathbb{C} \to \mathbb{C}$ so that $\widetilde{f}(z) = (c\tau + d)z$. In particular, using the relation between τ and τ', we see that

$$\widetilde{f}(\{1,\ \tau'\}) = \{c\tau + d,\ a\tau + b\}.$$

So, the biholomorphism corresponds to a base change (i.e. the change of a choice of generators) for Λ_τ.

Let us formalize this idea of a base change. First we take a base point $p_0 = [0] \in R_\tau$ for the fundamental group $\pi_1(R_\tau, p_0)$. The segments between 0 and 1 and between 0 and τ project to simple closed curves on R_τ and determine generators

$$[A_\tau],\ [B_\tau] \in \pi_1(R_\tau, p_0).$$

This now also gives us a natural choice of isomorphism $\Lambda_\tau \simeq \pi_1(R_\tau, p_0)$, mapping

$$1 \mapsto [A_\tau] \quad \text{and} \quad \tau \mapsto [B_\tau].$$

Likewise, for $R_{\tau'}$ we also obtain a natural system of generators $[A_{\tau'}], [B_{\tau'}] \in \pi_1(R_{\tau'}, p_0)$. Moreover, if $f_* : \pi_1(R_{\tau'}, p_0) \to \pi_1(R_\tau, p_0)$ denotes the map f induces on the fundamental group, then

$$f_*([A_{\tau'}]) \neq [A_\tau] \quad \text{and} \quad f_*([B_{\tau'}]) \neq [B_\tau].$$

Let us package these choices of generators:

DEFINITION 3.3.1. Let R be a Riemann surface homeomorphic to $\mathbb{T}^2$.

(1) A *marking* on R is a generating set $\Sigma_p \subset \pi_1(R, p)$ consisting of two elements.

(2) Two markings Σ_p and $\Sigma'_{p'}$ are called *equivalent* if there exists a continuous curve α from p to p' so that the corresponding isomorphism $T_\alpha : \pi_1(R, p) \to \pi_1(R, p')$ satisfies
$$T_\alpha(\Sigma_p) = \Sigma'_{p'}.$$

Two pairs (R, Σ) and (R', Σ') of marked Riemann surfaces homeomorphic to $\mathbb{T}^2$ are called *equivalent* if there exists a biholomorphic mapping $h : R \to R'$ such that

$$h_*(\Sigma) \simeq \Sigma'.$$

Note that above we have *not* proved that $(R_\tau, \{[A_\tau], [B_\tau]\})$ and $(R_{\tau'}, \{[A_{\tau'}], [B_{\tau'}]\})$ are equivalent as marked Riemann surfaces, because our map f_* did not send the generators to each other, and in fact, they are not equivalent:

THEOREM 3.3.2. *Let $\tau, \tau' \in \mathcal{T}_1$. Then the marked Riemann surfaces*

$$(R_\tau, \{[A_\tau], [B_\tau]\}) \quad \textit{and} \quad (R_{\tau'}, \{[A_{\tau'}], [B_{\tau'}]\})$$

are equivalent if and only if $\tau' = \tau$. Moreover, we have an identification

$$\mathcal{T}_1 = \left\{ (R, \Sigma_p) \ : \ \begin{array}{c} R \textit{ a Riemann surface homemorphic to } \mathbb{T}^2 \\ p \in R,\ \Sigma_p \textit{ a marking on } R \end{array} \right\} \Big/ \sim .$$

PROOF. We begin by proving part of the second claim: every marked complex torus is equivalent to a marked torus of the form $(R_\tau, \{[A_\tau], [B_\tau]\})$. So, suppose (R, Σ) is a marked torus. We know that R is biholomorphic to R_τ for some $\tau \in \mathcal{T}_1$. Moreover, since $\Sigma = \{[A], [B]\}$ is a minimal generating set for Λ_τ, we can find a lattice isomorphism $\varphi : \Lambda_\tau \to \Lambda_\tau$ so that

$$\varphi([A]) = 1.$$

Potentially switching the roles of $[A]$ and $[B]$, we can assume φ is an element of $\mathrm{SL}(2, \mathbb{Z})$ and hence that $\varphi([B])$ lies in $\mathbb{H}^2$. The torus $R_{\varphi([B])}$ is biholomorphic to R_τ. So (R, Σ) is equivalent to

$$(R_{\varphi([B])}, \{A_{\varphi([B])}, B_{\varphi([B])}\}).$$

So, to prove the theorem, we need to show that $(R_\tau, \{[A_\tau], [B_\tau]\})$ and $(R_{\tau'}, \{[A_{\tau'}], [B_{\tau'}]\})$ are equivalent if and only if $\tau = \tau'$. Of course, if $\tau = \tau'$ then the two corresponding marked surfaces are equivalent, so we need to show the other direction.

So let $h : R_{\tau'} \to R_\tau$ be a biholomorphism that induces the equivalence. We may assume that $h([0]) = [0]$ and take a lift $\widetilde{h} : \mathbb{C} \to \mathbb{C}$ so that

$$\widetilde{h}(0) = 0.$$

This means that $\widetilde{h}(z) = \alpha z$ for some $\alpha \in \mathbb{C} \setminus \{0\}$. Hence $1 = \widetilde{h}(1) = \alpha$, which implies that $\tau = \widetilde{h}(\tau') = \tau'$. □

Note that so far, our alternate description of Teichmüller space only recovers the set $\mathcal{T}_1$ and not yet it topology. Of course we can use the bijection to define a topology. However, there is also an intrinsic defintion. We will discuss how to do this later.

3.4. Markings by diffeomorphisms

First, we give a third interpreation of $\mathcal{T}_1$. This goes through another (equivalent) way of marking Riemann surfaces.

To this end, once and for all fix a surface S diffeomorphic to $\mathbb{T}^2$. We define:

DEFINITION 3.4.1. Let R and R' be Riemann surfaces and let

$$f : S \to R \quad \text{and} \quad f' : S \to R'$$

be orientation preserving diffeomorphisms. We say that the pairs (R, f) and (R', f') are *equivalent* if there exists a biholomorphism $h : R \to R'$ so that

$$(f')^{-1} \circ h \circ f : S \to S$$

is homotopic to the identity.

Note that if we pick a generating set $\{[A], [B]\}$ for the fundamental group $\pi_1(S, p)$ then every pair (R, f) as above defines a point

$$(R, \{f_*([A]), f_*([B])\}) \in \mathcal{T}_1 .$$

It turns out that this gives another description of the Teichmüller space of tori:

THEOREM 3.4.2. *Fix S and $[A], [B] \in \pi_1(S, p)$ as above. Then the map*

$$\left\{ (R, f) : \begin{array}{c} R \text{ a Riemann surface, } f : S \to R \\ \text{an orientation preserving diffeomorphism} \end{array} \right\} \Big/ \sim \quad \to \quad \mathcal{T}_1$$

given by

$$(R, f) \mapsto (R, \{f_*([A]).f_*([B])\}),$$

is a well-defined bijection.

PROOF. We start with well-definedness. Meaning, suppose (R, f) and (R', f') are equivalent. By definition, this means that there exists a biholomorphic map $h : R \to R'$ so that

$$h \circ f : S \to R' \quad \text{and } f' : S \to R'$$

are homotopic. Now if α is a continuous arc between $f'(p)$ and $h(f(p))$, we see that T_α induces an equivalence between the markings

$$\{f'_*([A]),\ f'_*([B])\} \quad \text{and} \quad \{(h \circ f)_*([A]),\ (h \circ f)_*([B])\},$$

making $(R, \{f_*([A]),\ f_*[B]\})$ and $(R', \{f'_*([A]),\ f'_*([B])\})$ equivalent. This means that they correspond to the same point by the previous theorem. So, the map is well defined.

Moreover, the map is surjective. For any $\tau \in \mathcal{T}_1$ we can find an orientation preserving diffeomorphism $f : S \to R_\tau$. Indeed, we know that there exists some $\tau_0 \in \mathcal{T}_1$ such that $(S, \{[A], [B]\}) \sim (R_{\tau_0}, \{[A_{\tau_0}], [B_{\tau_0}]\})$ as marked surfaces. One checks that the map $f_\tau : \mathbb{C} \to \mathbb{C}$ given by

$$f_\tau(z) = \frac{(\tau - \overline{\tau}_0)z - (\tau - \tau_0)\overline{z}}{\tau_0 - \overline{\tau}_0}$$

descends to an orientation preserving diffeomorphism $R_{\tau_0} \to R_\tau$ that induces the marking $\{[A_\tau], [B_\tau]\}$ on R_τ.

For the injectivity, suppose that

$$\Big[(R, \{f_*([A]), f_*([B])\})\Big] = \Big[(R', \{f'_*([A]), f'_*([B])\})\Big].$$

Take $\tau_0 \in \mathcal{T}_1$ such that

$$\Big[(S, \{[A], [B]\})\Big] = \Big[(R_{\tau_0}, \{([A_{\tau_0}]), [B_{\tau_0}]\})\Big].$$

Moreover, let $h : R \to R'$ be a holomorphism such that

$$h_*\{f_*([A]), f_*([B])\} = \{f'_*([A]), f'_*([B])\}.$$

We choose lattices $\Lambda, \Lambda' \subset \mathbb{C}$, generated by $(1, a)$ and $(1, a')$ respectively such that

$$R = \mathbb{C}/\Lambda \quad \text{and} \quad R' = \mathbb{C}/\Lambda',$$

and the generators induce the bases $\{f_*([A]), f_*([B])\}$ and $\{f'_*([A]), f'_*([B])\}$ respectively.

Now, let $\widetilde{f}, \widetilde{f}', \widetilde{h} : \mathbb{C} \to \mathbb{C}$ be lifts. We may assume that

$$\widetilde{f}(0) = \widetilde{f}'(0) = \widetilde{h}(0) = 0, \quad \widetilde{f}(1) = \widetilde{f}'(1) = \widetilde{h}(1) = 1,$$

and

$$\widetilde{f}(\tau_0) = a, \quad \widetilde{f}'(\tau_0) = a' \quad \text{and } \widetilde{h}(a) = a'$$

So we obtain a homotopy $F_t : \mathbb{C} \to \mathbb{C}$ defined by

$$F_t(z) = (1 - t)\, \widetilde{h} \circ \widetilde{f}(z) + t\, \widetilde{f}'(z)$$

between $\widetilde{h} \circ \widetilde{f}$ and $\widetilde{f}'$ that descends to a homotopy between $h \circ f : S \to R'$ and $f' : S \to R'$. □

3.5. The Teichmüller space of Riemann surfaces of a given type

The two description of the Teichmüller space of the torus above can be generalized to different Riemann surfaces. We will take the second one as a definition, as this is the most common definition in the literature. Moreover, it naturally leads to another key object in Teichmüller theory: the mapping class group.

DEFINITION 3.5.1. Let S be a surface of finite type. Then the *Teichmüller space* of S is defined as

$$\mathcal{T}(S) = \left\{ (X, f) \ : \ \begin{array}{c} X \text{ a Riemann surface }, f : S \to X \\ \text{an orientation preserving diffeomorphism} \end{array} \right\} \Big/ \sim,$$

where

$$(X, f) \sim (Y, g)$$

if and only if there exists a biholomorphism $h : X \to Y$ so that the map

$$g^{-1} \circ h \circ f : S \to S$$

is homotopic to the identity.

We will often write

$$\mathcal{T}(\Sigma_{g,n}) = \mathcal{T}_{g,n} \quad \text{and} \quad \mathcal{T}(\Sigma_g) = \mathcal{T}_g\,.$$

LECTURE 4

Markings, mapping class groups and moduli spaces

4.1. Teichmüller space in terms of markings

In order to get to the analogous definition to the space of marked tori, we need to single out particularly nice generating sets for the fundamental group, just like we did for tori. We will stick to closed surfaces. Recall that the fundamental group of a surface of genus g satisfies:

$$\pi_1(\Sigma_g, p) = \left\langle a_1, \ldots, a_g, b_1, \ldots, b_g \;\middle|\; \prod_{i=1}^{g} [a_i, b_i] = e \right\rangle.$$

In what follows, a generating set $A_1, \ldots, A_g, B_1, \ldots, B_g$ of $\pi_1(\Sigma_g, p)$ that satisfies

$$\prod_{i=1}^{g} [A_i, B_i] = e,$$

will be called a *canonical* generating set. Note that this includes the torus case.

DEFINITION 4.1.1. Let R be a closed Riemann surface.

(1) A *marking* on R is a canonical generating set $\Sigma_p \subset \pi_1(R, p)$.
(2) Two markings Σ_p and $\Sigma'_{p'}$ are called *equivalent* if there exists a continuous curve α from p to p' so that the corresponding isomorphism $T_\alpha : \pi(R, p) \to \pi_1(R, p')$ satisfies

$$T_\alpha(\Sigma_p) = \Sigma'_{p'}.$$

Two pairs (R, Σ) and (R', Σ') of marked closed Riemann surfaces are called *equivalent* if there exists a biholomorphic mapping $h : R \to R'$ so that

$$h_*(\Sigma) \simeq \Sigma'.$$

Just like in the case of the torus, the space of marked Riemann surfaces turns out to be the same as Teichmüller space:

THEOREM 4.1.2. *Let S be a closed surface and Σ a marking on S. Then the map*

$$\mathcal{T}(S) \to \left\{ (R, \Sigma_p) \; : \; \begin{array}{c} R \text{ a closed Riemann surface diffeomorphic to } S \\ p \in R, \; \Sigma_p \text{ a marking on } R \end{array} \right\} \Big/ \sim .$$

given by

$$[(R, f)] \mapsto [(R, f_*(\Sigma)]$$

is a bijection.

Before we sketch the proof of this theorem, we state and prove a lemma that will be of use in the study of mapping class groups as well:

LEMMA 4.1.3 (Alexander Lemma). *Let D be a 2-dimensional closed disk and $\phi : D \to D$ a homeomorphism that restricts to the identity on ∂D. Then ϕ is isotopic to the identity $D \to D$*

PROOF OF THE ALEXANDER LEMMA. Identify D with the closed unit disk in $\mathbb{R}^2$ and define the map $F : D \times [0,1] \to D$ by

$$F_t(x) = \begin{cases} (1-t)\cdot \phi\left(\frac{x}{(1-t)}\right) & \text{if } ||x|| < 1-t \text{ and } t<1 \\ x & \text{if } ||x|| > 1-t \text{ and } t<1 \\ x & \text{if } t=1. \end{cases}$$

This yields the isotopy we want. □

We can make this lemma work in the smooth category as well, but its proof is significantly less easy. It for instance follows from work by Smale [**Sma59**]. In this course we will generally gloss over the difference between homeomorphisms and diffeomorphisms.

PROOF SKETCH. Write $\Sigma = \{[A_1],\ldots,[A_g],[B_1],\ldots,[B_g]\}$, where A_i, B_i are simple closed curves based at a point $p_0 \in S$. Let us start with the injectivity. So, suppose

$$[(R, f_*(\Sigma)] = [(R', f'_*(\Sigma)].$$

This means that we can find a biholomorphic map $h : R \to R'$ and a self-diffeomorphism $g_0 : R' \to R'$ that is homotopic to the identity and such that

$$g_1 = g_0 \circ h \circ f$$

corresponds with f' on the curves $A_1,\ldots,A_g,B_1,\ldots,B_g$. The domain obtained by deleting these curves from S is a disk. This implies that f' and g_1 are homotopic (using the Alexander trick), which in turn means that

$$[(R,f)] = [(R',f')] \in \mathcal{T}(S).$$

For surjectivity, suppose we are given a marked Riemann surface (R,Σ_p). So we need to find an orientation preserving homeomorphism $f : S \to R$ so that $f_*(\Sigma) = \Sigma_p$. So, let us take simple closed smooth curves $A'_1,\ldots,A'_g,B'_1,\ldots,B'_g$ such that

$$\Sigma_p = \{[A'_1],\ldots,[A'_g],[B'_1],\ldots,[B'_g]\}.$$

Moreover, we will set

$$C = \bigcup_{j=1}^{g}(A_j \cup B_j), \quad C' = \bigcup_{j=1}^{g}(A'_j \cup B'_j), \quad S_0 = S \setminus C, \quad \text{and} \quad R_0 = R \setminus C'.$$

R_0 and S_0 are diffeomorphic to polygons with $4g$ sides. So we can find a diffeomorphism by extending a diffeomorphism R_0, S_0. For more details, see [**IT92**, Theorem 1.4]. □

4.1.1. Punctures and marked points. If $n \geq 1$, we can think of $\mathcal{T}(\Sigma_{g,n})$ as a space of surfaces with marked points (as opposed to punctures) as well:

Proposition 4.1.4. *Let $n \geq 1$ and fix n distinct points $x_1, \ldots, x_n \in \Sigma_g$. There is a bijection*

$$\mathcal{T}(\Sigma_{g,n}) \longrightarrow \left\{ (X, f) \; : \; f : \Sigma_g \to X \text{ an orientation preserving diffeomorphism} \right\} \Big/ \sim,$$

where $(X_1, f_1) \sim (X_2, f_2)$ if and only if there exists a biholomorphism $h : X_1 \to X_2$ such that

$$f_2^{-1} \circ h \circ f_1(x_i) = x_i \quad \text{for } i = 1, \ldots, n$$

and $f_2^{-1} \circ h \circ f_1 : \Sigma_g \to \Sigma_g$ is homotopic to the identity through maps fixing $x_1, \ldots x_n$.

We leave the proof to the reader.

4.1.2. Basic examples. We have seen that the Teichmüller space of the torus can be identified with $\mathbb{H}^2$ (as a set for now). We will treat some further examples in this section.

Proposition 4.1.5. *We have*

(a) *Let S be diffeomorphic to Σ_0, $\Sigma_{0,1}$, $\Sigma_{0,2}$ or $\Sigma_{0,3}$, then $\mathcal{T}(S)$ is a point.*

(b) *$\mathcal{T}(\Sigma_{1,1})$ can be identified with $\mathcal{T}(\Sigma_1)$.*

Proof. For (a), suppose that $[X_1, f_1], [X_2, f_2] \in \mathcal{T}(\Sigma_{0,n})$ with $0 \leq n \leq 3$. We will think of these two as surfaces with marked points, coming from at most three marked points x_1, x_2, x_3 on $\mathbb{S}^2$. By the uniformization theorem, we can identify X_1 and X_2 with the Riemann sphere $\widehat{\mathbb{C}}$. Moreover (using that $n \leq 3$), there exists a Möbius transformation $\varphi : X_1 \to X_2$ such that

$$\varphi(f_1(x_i)) = f_2(x_i), \quad i = 1, \ldots, n.$$

As such the diffeomorphism $f_2^{-1} \circ \varphi \circ f_1 : \mathbb{S}^2 \to \mathbb{S}^2$ fixes $x_1, \ldots, x_n$. All we need to do, is show that this map is homotopic to the identity. If $n = 0$, we can perform a homotopy such that $f_2^{-1} \circ \varphi \circ f_1 : \mathbb{S}^2 \to \mathbb{S}^2$ fixes a point, which we shall call x_1. This means that $f_2^{-1} \circ \varphi \circ f_1 : \mathbb{S}^2 \to \mathbb{S}^2$ can be restricted to a self homeomorphism of $\mathbb{S}^2 - \{x_1\} \simeq \mathbb{R}^2$, that we call $f : \mathbb{R}^2 \to \mathbb{R}^2$. The map $F : \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$, defined by

$$F_t(x) = (1-t) \cdot f(x) + t \cdot x$$

is a homotopy between f and the identity $\mathbb{R}^2 \to \mathbb{R}^2$. Because both $f_2^{-1} \circ \varphi \circ f_1$ and the identity fix $x_1 \in \mathbb{S}^2$, the homotopy above can be extended to $\mathbb{S}^2$.

The proof for item (b) is similar. We again think in terms of surfaces with marked points. We have a surjective map

$$\mathcal{T}(\Sigma_{1,1}) \to \mathcal{T}(\Sigma_1),$$

mapping $[X, f] \in \mathcal{T}(\Sigma_{1,1})$ to $[X, f] \in \mathcal{T}(\Sigma_1)$. What we need to show is that this map is injective.

So, suppose $[X_1, f_1] = [X_2, f_2] \in \mathcal{T}(\Sigma_1)$. So there exists a biholomorphism $h : X_1 \to X_2$ such that $f_2^{-1} \circ h \circ f_1 : \Sigma_1 \to \Sigma_1$ is homotopic to the identity. We need to show that we can modify h in such a way that $f_2^{-1} \circ h \circ f_1$ remains homotopic to the identity and also fixes our favorite point $x_1 \in \Sigma_1$. To this end, let's write $X_2 = \mathbb{C}/\Lambda$ for some lattice Λ. Suppose $[p], [q] \in X_2$. Observe that $h_0 : X_2 \to X_2$, defined by

$$h_0([z]) = [z + q - p]$$

is a biholomorphic map $X_2 \to X_2$ that is homotopic to the identity and maps $[p]$ to $[q]$. So, if we set $[p] = h \circ f_1(x_1)$ and $[q] = f_2(x_1)$, then $h_0 \circ h : X_1 \to X_2$ is the biholomorphic map we're looking for. □

4.2. The mapping class group

4.2.1. Defintion. Just like in the case of the torus, we have a natural group action on the Teichmüller space of a surface, by a group called the mapping class group:

DEFINITION 4.2.1. Let S_0 be a compact surface of finite type and $\Sigma \subset S_0$ a finite set. Set $S = S_0 \setminus \Sigma$. The *mapping class group* of S is given by

$$\mathrm{MCG}(S) = \mathrm{Diff}^+(S, \partial S, \Sigma)/\mathrm{Diff}^+_0(S, \partial S, \Sigma)$$

where

$$\mathrm{Diff}^+(S, \partial S, \Sigma) = \left\{ f : S_0 \to S_0 \ : \begin{array}{c} f \text{ an orientation preserving diffeomorphism that} \\ \text{acts as the identity on the boundary components} \\ \text{of } S_0 \text{ and preserves the elements of } \Sigma \text{ pointwise} \end{array} \right\}$$

and

$$\mathrm{Diff}^+_0(S, \partial S, \Sigma) = \left\{ f \in \mathrm{Diff}^+(S, \partial S, \Sigma) \ : \begin{array}{c} f \text{ homotopic to the identity} \\ \text{through a homotopy preserving} \\ \text{the elements of } \Sigma \text{ pointwise} \end{array} \right\}.$$

The group operation is induced by composition of functions.

Some authors let go of the condition that $\mathrm{MCG}(S)$ fixes the punctures. The group we defined above is then often called the *pure mapping class group*.

4.3. Moduli space

Looking at Definition 3.5.1, we see there is a natural group action of the mapping class group of a surface on the corresponding Teichmüller space.

$$[g] \cdot [(R, f)] = [(R, f \circ g^{-1})].$$

The quotient is what will be called moduli space.

DEFINITION 4.3.1. Let S be a surface of finite type . The *moduli space* of S is the space

$$\mathcal{M}(S) = \mathcal{T}(S)/\,\mathrm{MCG}(S).$$

We will write

$$\mathcal{M}(\Sigma_{g,n}) = \mathcal{M}_{g,n} \quad \text{and} \quad \mathcal{M}(\Sigma_g) = \mathcal{M}_g\,.$$

REMARK 4.3.2. Note that by using the convention that the mapping class group fixes boundary components and punctures, we leave these "marked", i.e. if two surfaces are isometric, but any isometry between them permutes the punctures, these surfaces represent different points in moduli space.

4.4. Elements and examples of mapping class groups

4.4.1. Basic examples. We have:

PROPOSITION 4.4.1. *Let* $n \leq 3$*, then*

$$\mathrm{MCG}(\Sigma_{0,n}) = \{e\}.$$

PROOF. We start with the case $n \leq 1$. This is very similar to some of what we did in the proof of Proposition 4.1.5. Suppose $f : \Sigma_0 \to \Sigma_0$ is an orientation preserving diffeomorphism. We can (up to homotopy if $n = 0$) assume that f fixes a point $x \in \Sigma_0$. This allows us to restrict f to $\Sigma_0 - \{x\} \simeq \mathbb{R}^2$ and use a straight line homotopy to homotope $f|_{\mathbb{R}^2}$ to the identity. This extends to a homotopy between f and the identity on Σ_0, because both fix x.

The proof of the cases $n \in \{2, 3\}$ is part of this week's exercises. □

4.4.2. Dehn twists and the mapping class group of the annulus. Before we move on, let us describe some non-trivial elements of the mapping class group. First, consider an annulus

$$A := [0, 1] \times \mathbb{R}/\mathbb{Z}.$$

Define a map $T : A \to A$ by

$$T(t, [\theta]) = (t, [\theta + t])$$

for all $t \in [0, 1]$, $\theta \in \mathbb{R}$. This map is called a Dehn twist. Note that this map fixes ∂A pointwise. Figure 1 shows that this map does to a segment connecting the two boundary components of the annulus.

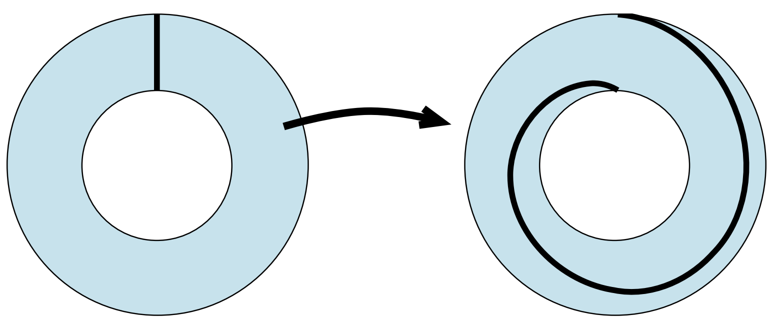

FIGURE 1. A Dehn twist on an annulus.

Before we show how to turn T into a non-trivial element of a mapping class group of a different surface, we mention that T generates the mapping class group of the annulus:

PROPOSITION 4.4.2. *Let* $A = [0, 1] \times \mathbb{R}/\mathbb{Z}$*. Then*

$$\mathrm{MCG}(A) \simeq \mathbb{Z} = \langle [T] \rangle.$$

We will prove below..

Now let α be an essential (i.e. not homotopically trivial and not homotopic into a puncture or boundary component) simple closed curve on S. Let N be a closed regular neighborhood of α. Identifying N with A, we can define a map $T_\alpha : S \to S$ by

$$T_\alpha(p) = \begin{cases} T(p) & \text{if } p \in N \\ p & \text{if } p \in S \setminus N \end{cases}$$

Because $T|_{\partial A}$ is the identity map, this is a continuous map. To obtain an element in $\mathrm{MCG}(S)$, we need to start with a smooth map. There are multiple ways out at the moment. We could smoothen T. Or we could use surface topology to argue that T_α is homotopic to a smooth map. Since for the mapping class group, we only care about diffeomorphisms up to homotopy, the element we get in $\mathrm{MCG}(S)$ will not depend on how we do this.

Figure 2 shows an example of a Dehn twist.

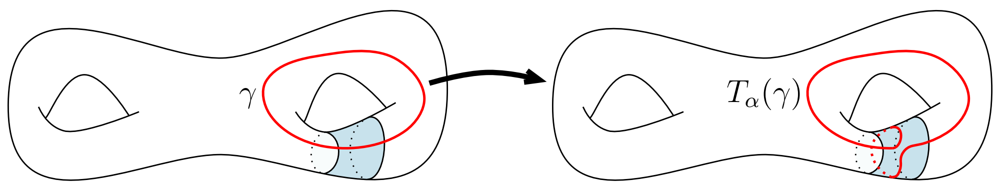

FIGURE 2. A Dehn twist on a surface of genus two.

We see that T_α maps a curve γ on the surface intersecting the defining curve α (of which we have only drawn the regular neighborhood) transversely to a curve that is not homotopic to γ. In particular, T_α is not homotopic to the identity and hence defines a non-trivial element in $\mathrm{MCG}(S)$.

PROOF OF PROPOSITION 4.4.2. We will first construct a homomorphism $\rho : \mathrm{MCG}(A) \to \mathbb{Z}$. Given an orientation preserving diffeomorphism $f : A \to A$ such that $f|_{\partial A} = \mathrm{Id}$, we can find a lift $\widetilde{f} : [0,1] \times \mathbb{R} \to [0,1] \times \mathbb{R}$ such that $\widetilde{f}(0,0) = (0,0)$. This means that

$$\widetilde{f}|_{\{0\}\times\mathbb{R}} = \mathrm{Id}.$$

Because $f|_{\partial A} = \mathrm{Id}$, the restriction $\widetilde{f}|_{\{1\}\times\mathbb{R}}$ is an integer translation. We let $\rho(f)$ be this integer.

ρ is surjective, because $\rho([T^n]) = n$. Now suppose $\rho([f]) = 0$. This means that $\widetilde{f}$ restricts to the identity on $\{0,1\} \times \mathbb{R}$. We have that

$$\widetilde{f}\Big(n \cdot (t,x)\Big) = f_*(n) \cdot \widetilde{f}(t,x), \quad n \in \mathbb{Z}, (t,x) \in [0,1] \times \mathbb{R},$$

where $f_* \in \mathrm{Aut}(\mathbb{Z}) = \{\pm \mathrm{Id}\}$. Because $\widetilde{f}|_{\{0,1\}\times\mathbb{R}} = \mathrm{Id}$, we need that $f_* = \mathrm{Id}$. Implying that

$$\widetilde{f}\Big(n \cdot (t,x)\Big) = n \cdot \widetilde{f}(t,x), \quad n \in \mathbb{Z}, (t,x) \in [0,1] \times \mathbb{R}$$

and thus that the straight line homotopy

$$F_s((t,x)) = (1-s) \cdot \widetilde{f}(x,t) + s \cdot (x,t), \quad s \in [0,1]$$

is a $\mathbb{Z}$-equivariant homotopy between $\widetilde{f}$ and the identity, that hence descends to A. This proves that ρ is injective and concludes the proof of the proposition. $\square$

4.4.3. The mapping class group of the torus. We briefly return to the torus. The question is of course whether the general definition on the mapping class group really corresponds to what happens in the case of the torus. We recall that

$$\mathcal{M}_1 = \mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}^2.$$

This makes one wonder whether the mapping class group of the torus is maybe $\mathrm{PSL}(2,\mathbb{Z})$. This turns out to be almost correct. Indeed, we have the following theorem:

THEOREM 4.4.3. *We have*

$$\mathrm{MCG}(\mathbb{T}^2) \simeq \mathrm{SL}(2,\mathbb{Z}).$$

The action of $\mathrm{MCG}(\mathbb{T}^2)$ *on* $\mathcal{T}_1$ *is that given by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{f\tau - b}{-c\tau + a}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ *and* $\tau \in \mathcal{T}_1$.

PROOF. We will identify

$$\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$$

First observe that very element $A \in \mathrm{SL}(2,\mathbb{Z})$ induces a linear diffeomorphism $x \mapsto A \cdot x$ of $\mathbb{R}^2$. Moreover, since $\mathrm{SL}(2,\mathbb{Z})$ preserves $\mathbb{Z}^2 \subset \mathbb{R}^2$, the action on $\mathbb{R}^2$ descends to an action by diffeomorphisms

$$\mathbb{T}^2 \to \mathbb{T}^2$$

that are orientation preserving because $\det(A) > 0$.

Our goal is to show that every orientation preserving diffeomorphism $\phi : \mathbb{T}^2 \to \mathbb{T}^2$ is homotopic to such a map. To this end, we may homotope ϕ so that it fixes $[0] \in \mathbb{T}^2$ and we can take a lift $\widetilde{\phi} : \mathbb{R}^2 \to \mathbb{R}^2$ that fixes the origin of $\mathbb{R}^2$. We have

$$\widetilde{\phi}(x + (m,n)) = \widetilde{\phi}(x) + \phi_*(m,n),$$

for all $(m,n) \in \mathbb{Z}^2$ where $\phi_* : \mathbb{Z}^2 \to \mathbb{Z}^2$ is an isomorphism, i.e. an element of $\mathrm{GL}(2,\mathbb{Z})$. For a general surface S, the map $[\phi] \in \mathrm{MCG}(S) \mapsto \phi_* \in \mathrm{Aut}(\pi_1(S))$ does not yield a homomorphism: we have chosen a homotopy to make ϕ fix a base point. Changing this choice a priori changes ϕ_* by an inner automorphism of $\pi_1(S)$. So we only obtain a map to $\mathrm{Out}(\pi_1(S))$. However, because $\mathbb{Z}^2$ is abelian, we have $\mathrm{Out}(\mathbb{Z}^2) \simeq \mathrm{Aut}(\mathbb{Z}^2)$. So in the case of the torus, we obtain a homomorphism $\mathrm{MCG}(\mathbb{T}^2) \to \mathrm{GL}(2,\mathbb{Z})$.

Write A_ϕ for the $\mathrm{GL}(2,\mathbb{Z})$ matrix corresponding to ϕ. Observe that

$$F_t(x) = tA_\phi \cdot x + (1-t)\widetilde{\phi}(x), \quad t \in [0,1], x \in \mathbb{R}^2$$

gives a $\mathbb{Z}^2$-equivariant homotopy between $\widetilde{\phi}$ and the linear map $x \mapsto A_\phi \cdot x$. Since $\widetilde{\phi}$ is orientation preserving, $\det(A_\phi) > 0$, and hence $A_\phi \in \mathrm{SL}(2,\mathbb{Z})$. So we obtain a map $\mathrm{MCG}(\mathbb{T}^2) \to \mathrm{SL}(2,\mathbb{Z})$. The map is surjective, because ϕ_A maps to A. Moreover, the map is injective, because if A_ϕ is the identity matrix, F_t gives a homotopy of $\widetilde{\phi}$ to the identity.

Since the action of $[\phi] \in \mathrm{MCG}(\mathbb{T}^2)$ on $\mathcal{T}(\mathbb{T}^2)$ is by precomposition with ϕ^{-1}, the action is as described. □

REMARK 4.4.4. Note that the theorem above implies that the mapping class group action is not faithful. The kernel of the action is the center of $\mathrm{SL}(2,\mathbb{Z})$, i.e. the subgroup

$$\left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} < \mathrm{SL}(2,\mathbb{Z}).$$

On the other hand, we do have

$$\mathbb{H}^2 / \mathrm{PSL}(2,\mathbb{Z}) = \mathbb{H}^2 / \mathrm{SL}(2,\mathbb{Z}).$$

This means that the mapping class group action is indeed a generalization of the situation for the torus case.

4.4.4. Mapping class groups in higher genus. We proved in the exercises that $\mathrm{SL}(2,\mathbb{Z})$ can be generated by the matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We can also generate $\mathrm{SL}(2,\mathbb{Z})$ by the matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Indeed, a calculation shows that $S = T^{-1} R T^{-1}$.

Now identify $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ again and write α and β for the closed curves in $\mathbb{T}^2$ that are the images of the straight line segments between the origin and $(0,1)$ and $(1,0)$ respectively. Tracing the proof of Theorem 4.4.3, we see that $T = [T_\alpha]$ and $R = [T_\beta]$. That is, $\mathrm{MCG}(\mathbb{T}^2)$ can be generated by two Dehn twists.

It actually turns out that an analogous statement holds for all mapping class groups. In the following theorem, a non-separating curve will be a curve α so that $S \setminus \alpha$ is connected. Figure 3 shows an example.

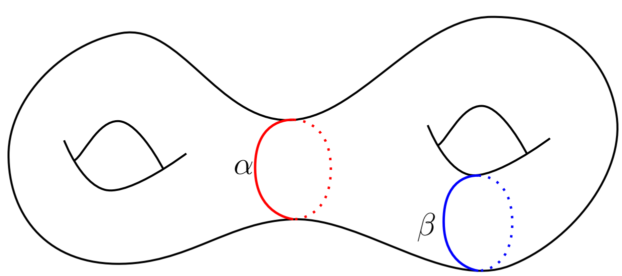

FIGURE 3. A separating curve (α) and a non-separating curve (β).

THEOREM 4.4.5 (Dehn - Lickorish theorem). *Let S be a surface of finite type, the mapping class group $\mathrm{MCG}(S)$ is generated by finitely many Dehn twists about nonseparating simple closed curves.*

4.4.5. The action on homology. If S is a surface, then $\mathrm{MCG}(S)$ acts on its homology $H_1(S)$. Indeed every diffeomorphism $f : S \to S$ induces an automorphism $f_* : H_1(S,\mathbb{Z}) \to H_1(S,\mathbb{Z})$. In this section, we briefly descibe some aspects of this action. We will restrict to closed surfaces.

First of all, it turns out the action preserves some extra structure: the algebraic intersection number between oriented curves. In order to define it, let α and β be two oriented closed curves on an oriented surface S that intersect each other transversely at every intersection point. Then the *algebraic intersection number* between α and β is given by

$$i(\alpha,\beta) = \sum_{p\in\alpha\cap\beta} \mathrm{sgn}(\omega(v_p(\alpha), v_p(\beta))),$$

where $\mathrm{sgn} : \mathbb{R} \to \{\pm 1\}$ denotes the sign function, ω is any volume form that induces the orientation and $v_p(\alpha)$ and $v_p(\beta)$ denote the unit tangent vectors to α and β respectively at p. Note that

$$i(\beta,\alpha) = -i(\alpha,\beta).$$

Figure 4 shows an example of a positive contribution to the intersection number.

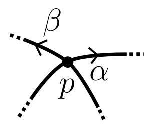

FIGURE 4. A positive contribution to $i(\alpha,\beta)$ if the orientation points out of the page.

We note that this form descends to homology. That is, it induces a form

$$i : H_1(S,\mathbb{Z}) \times H_1(S,\mathbb{Z}) \to \mathbb{Z}$$

called the *intersection form*, with the properties:

(1) i is bilinear.

(2) i is *alternating*, i.e.

$$i(a,b) = -i(b,a)$$

for all $a,b \in H_1(S,\mathbb{Z})$.

(3) i is *non-degenerate*, i.e. if $a \in H_1(S,\mathbb{Z})$ is such that

$$i(a,b) = 0 \quad \text{for all } b \in H_1(S,\mathbb{Z})$$

then $a = 0$.

(see [**FK92**, Section III.1] for more details). Such a form is called a *symplectic form*.

First of all note that the image preserves the intersection form. Moreover, isotopic maps give rise to the same automorphism. So this gives us a representation

$$\mathrm{MCG}(S) \to \mathrm{Aut}(H_1(S,\mathbb{Z}), i)$$

called the *homology representation* of the mapping class group. Recall that if S is a closed orientable surface of genus g, then $H_1(S,\mathbb{Z}) \simeq \mathbb{Z}^{2g}$. Choosing an identification, the homology representation becomes a map

$$\mathrm{MCG}(S) \to \mathrm{Sp}(2g,\mathbb{Z}) = \left\{ A \in \mathrm{Mat}_{2g}(\mathbb{Z}) \,:\, i(Av,Aw) = i(v,w),\ \forall v,w \in \mathbb{Z}^{2g} \right\}.$$

It turns out that this representation is surjective (this can be proved using a finite generating set for $\mathrm{Sp}(2g,\mathbb{Z})$ consisting of transvections, which can be realized by Dehn twists), but generally highly non-injective. A notable exception is the case of the torus, there is an isomorphism

$$\mathrm{Sp}(2,\mathbb{Z}) \simeq \mathrm{SL}(2,\mathbb{Z})$$

and indeed the the homology representation $\mathrm{MCG}(\mathbb{T}^2) \to \mathrm{Sp}(2,\mathbb{Z})$ is an isomorphism.

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