Introduction to moduli spaces of Riemann surfaces

Lecture notes

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Preface

These are the lecture notes for a course called *Introduction to moduli spaces of Riemann surfaces*, taught in January and February 2026 in the master's program *M2 de Mathématiques fondamentales* at Sorbonne University.

There are many references on various aspects of moduli spaces and Teichmüller spaces, like [**IT92**, **Bus10**, **GL00**, **Zor06**, **Hub06**, **FM12**, **Baa21**, **Wri15**]. All of these treat a lot more material than what we will have time for in the course, whence the present notes. Most of the material presented here is adapted from these references.

Thanks to Yiran Cheng, Greta Di Vincenzo, Leo Graf, Luka Hadji Jordanov, Paul Lemarchand, Sil Liskens, Qiaochu Ma, Ismaele Vanni, Shayan Zahedi and Christopher Zhang for catching mistakes and typos in earlier versions of these notes.

LECTURE 1

Reminder on surfaces

Riemann surfaces are objects that appear everywhere in mathematics. Of course, they play an important role in complex analysis and in geometry but also for example in dynamics, number theory and combinatorics.

Their moduli spaces - the spaces that parameterize Riemann surface structures on a fixed surface - are also studied from many different points of view. The goal of this course is to understand the geometry and topology of these moduli spaces.

Before we get to any of this, we need to talk about surfaces themselves. So, today we will recall some of the basics on surfaces.

1.1. Preliminaries on surface topology

1.1.1. Examples and classification. A **surface** is a smooth two-dimensional manifold. We call a surface **closed** if it is compact and has no boundary. A surface is said to be of **finite type** if it can be obtained from a closed surface by removing a finite number of points and (smooth) open disks with disjoint closures. In what follows, we will always assume our surfaces to be orientable.

EXAMPLE 1.1.1. To properly define a manifold, one needs to not only describe the set but also give smooth charts. In what follows we will content ourselves with the sets.

(a) The 2-**sphere** is the surface

$$\mathbb{S}^2 = \left\{ (x,y,z) \in \mathbb{R}^3 \;:\; x^2+y^2+z^2 = 1 \right\}.$$

(b) Let $\mathbb{S}^1$ denote the circle. The 2-**torus** is the surface

$$\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$$

(c) Given two (oriented) surfaces S_1, S_2, their **connected sum** $S_1 \# S_2$ is defined as follows. Take two closed sets $D_1 \subset S_1$ and $D_2 \subset S_2$ that are both diffeomorphic to closed disks, via diffeomorphisms

$$\varphi_i : \left\{ (x,y) \in \mathbb{R}^2 \;:\; x^2+y^2 \leq 1 \right\} \to D_i, \quad i = 1,2,$$

so that φ_1 is orientation preserving and φ_2 is orientation reversing.

Then

$$S_1 \# S_2 = \left(S_1 \setminus \mathring{D}_1 \sqcup S_2 \setminus \mathring{D}_2 \right) / \sim$$

where $\mathring{D}_i$ denotes the interior of D_i for $i = 1, 2$ and the equivalence relation $\sim$ is defined by

$$\varphi_1(x, y) \sim \varphi_2(x, y) \quad \text{for all } (x, y) \in \mathbb{R}^2 \text{ with } x^2 + y^2 = 1.$$

The figure below gives an example.

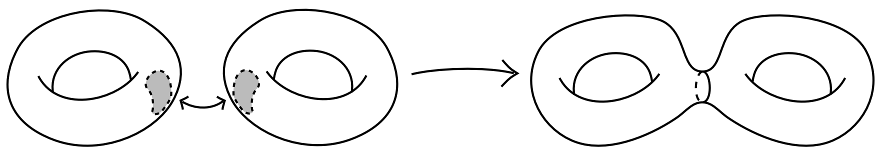

FIGURE 1. A connected sum of two tori.

Like our notation suggests, the manifold $S_1 \# S_2$ is independent (up to diffeomorphism) of the choices we make (the disks and diffeomorphisms φ_i). This is a non-trivial statement, the proof of which we will skip. Likewise, we will also not prove that the connected sum of surfaces is an associative operation and that $\mathbb{S}^2 \# S$ is diffeomorphic to S for all surfaces S.

A classical result from the 19^{th} century tells us that the three simple examples above are enough to understand all finite type surfaces up to diffeomorphism.

THEOREM 1.1.2 (Classification of closed surfaces). *Every closed orientable surface is diffeomorphic to the connected sum of a 2-sphere with a finite number of tori.*

Indeed, because the diffeormorphism type of a finite type surface does not depend on where we remove the points and open disks (another claim we will not prove), the theorem above tells us that an orientable finite type surface is (up to diffeomorphism) determined by a triple of positive integers (g, b, n), where

- g is the number of tori in the connected sum and is called the **genus** of the surface.
- b is the number of disks removed and is called the number of **boundary components** of the surface.
- n is the number of points removed and is called the number of **punctures** of the surface.

DEFINITION 1.1.3. The triple (g, b, n) defined above will be called the **signature** of the surface. We will denote the corresponding surface by $\Sigma_{g,b,n}$ and will write $\Sigma_g = \Sigma_{g,0,0}$.

1.1.2. Euler characteristic. The Euler characteristic is a useful topological invariant of a surface. There are multiple ways to define it. We will use triangulations. A **triangulation** $\mathcal{T} = (V, E, F)$ of a closed surface S will be the data of a finite set of points $V = \{v_1, \dots . v_k\} \subset S$ (called **vertices**), a finite set of arcs $E = \{e_1, \dots, e_l\}$ with endpoints in the vertices (called **edges**) so that the complement $S \setminus (\cup v_i \cup e_j)$ consists of a collection of disks $F = \{f_1, \dots, f_m\}$ (called **faces**) that are all incident to exactly 3 edges.

Note that a triangulation $\mathcal{T}$ here is a slightly more general notion than that of a simplicial complex (it's an example of what Hatcher calls a Δ-complex [Hat02, Page 102]). Figure 2 below gives an example of a triangulation of a torus that is not a simplicial complex.

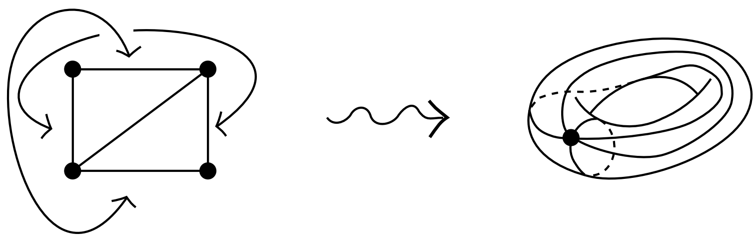

FIGURE 2. A torus with a triangulation

DEFINITION 1.1.4. S be a closed surface with a triangulation $\mathcal{T} = (V, E, F)$. The **Euler characteristic** of S is given by

$$\chi(S) = |V| - |E| + |F|.$$

Because $\chi(S)$ can be defined entirely in terms of singular homology (see [Hat02, Theorem 2.4] for details), it is a homotopy invariant. In particular this implies it should only depend on the genus of our surface S. Indeed, we have

LEMMA 1.1.5. *Let S be a closed connected and oriented surface of genus g. We have*

$$\chi(S) = 2 - 2g.$$

PROOF. Exercise: prove this using your favorite triangulation. □

For surfaces that are not closed, we can define

$$\chi(\Sigma_{g,b,n}) = 2 - 2g - b - n.$$

This can be computed with a triangulation as well. For surfaces with only boundary components, the usual definition still works. For surfaces with punctures there no longer is a finite triangulation, so the definition above no longer makes sense. There are multiple ways out. The most natural is to use the homological definition, which gives the formula above. Another option is to allow some vertices to be missing, that is, to allow edges to run between vertices and punctures. Both give the formula above.

1.2. Riemann surfaces

For the basics on Riemann surfaces, we refer to the lecture notes from the course by Elisha Falbel [Fal23] or any of the many books on them, like [Bea84, FK92]. For a text on complex functions of a single variable, we refer to [SS03].

1.2.1. Definition and first examples. A Riemann surface is a one-dimensional complex manifold. That is,

DEFINITION 1.2.1. A **Riemann surface** X is a connected Hausdorff topological space X, equipped with an open cover $\{U_\alpha\}_{\alpha\in A}$ of open sets and maps $\varphi_\alpha : U_\alpha \to \mathbb{C}$ so that

(1) $\varphi_\alpha(U_\alpha)$ is open and φ_α is a homeomorphism onto its image.

(2) For all $\alpha, \beta \in A$ so that $U_\alpha \cap U_\beta \neq \emptyset$ the map
$$\varphi_\alpha \circ (\varphi_\beta)^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$
is holomorphic.

The pairs $(U_\alpha, \varphi_\alpha)$ are usually called **charts** and the collection $((U_\alpha, \varphi_\alpha))_{\alpha\in A}$ is usually called an **atlas**.

Note that we do not a priori assume a Riemann surface X to be a second countable space. It is however a theorem by Radó that every Riemann surface is second countable (for a proof, see [**Hub06**, Section 1.3]). Moreover every Riemann surface is automatically orientable (see for instance [**GH94**, Page 18]).

EXAMPLE 1.2.2. (a) The simplest example is of course $X = \mathbb{C}$ equipped with one chart: the identity map.

(b) We set $X = \mathbb{C} \cup \{\infty\} = \widehat{\mathbb{C}}$ and give it the topology of the one point compactification of $\mathbb{C}$, which is homeomorphic to the sphere $\mathbb{S}^2$. The charts are
$$U_0 = \mathbb{C}, \quad \varphi_0(z) = z$$
and
$$U_\infty = X \setminus \{0\}, \quad \varphi_\infty(z) = 1/z.$$
So $U_0 \cap U_\infty = \mathbb{C} \setminus \{0\}$ and
$$\varphi_0 \circ (\varphi_\infty)^{-1}(z) = 1/z \quad \text{for all } z \in \mathbb{C} \setminus \{0\}$$
which is indeed holomorphic on $\mathbb{C} \setminus \{0\}$. $\widehat{\mathbb{C}}$ is usually called the **Riemann sphere**.

(c) $\widehat{\mathbb{C}}$ can also be identified with the projective line
$$\mathbb{P}^1(\mathbb{C}) = \left(\mathbb{C}^2 \setminus \{(0,0)\}\right) \Big/ \mathbb{C}^*,$$
where $\mathbb{C}^* \curvearrowright \mathbb{C}^2 \setminus \{(0,0)\}$ by $\lambda\cdot(z,w) = (\lambda\cdot z, \lambda\cdot w)$, for $\lambda \in \mathbb{C}^*$, $(z,w) \in \mathbb{C}^2 \setminus \{(0,0)\}$. Indeed, we may equip $\mathbb{P}^1(\mathbb{C})$ with two charts
$$U_0 = \{\, [z:w] \;:\; w \neq 0 \,\}, \quad \varphi_0([z:w]) = z/w$$
and
$$U_1 = \{\, [z:w] \;:\; z \neq 0 \,\}, \quad \varphi_1([z:w]) = w/z.$$
The map
$$[z:w] \mapsto \begin{cases} z/w & \text{if } w \neq 0 \\ \infty & \text{if } w = 0 \end{cases}$$
then defines a biholomorphism $\mathbb{P}^1(\mathbb{C}) \to \widehat{\mathbb{C}}$.

(d) Recall that a **domain** $D \subset \widehat{\mathbb{C}}$ is any connected and open set in $\widehat{\mathbb{C}}$. Any such domain inherits the structure of a Riemann surface from $\widehat{\mathbb{C}}$.

1.2.2. Automorphisms. To get a larger set of examples, we will consider quotients. First of all, we need the notion of a holomorphic map:

DEFINITION 1.2.3. Let X and Y be Riemann surfaces, equipped with atlasses $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ and $\{(V_\beta, \psi_\beta)\}_{\beta\in B}$ respectively. A function $f : X \to Y$ is called **holomorphic** if

$$\psi_\beta \circ f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap f^{-1}(V_\beta)) \to \psi_\beta(f(U_\alpha) \cap V_\beta)$$

is holomorphic for all $\alpha \in A, \beta \in B$ so that $f(U_\alpha) \cap V_\beta \neq \emptyset$. A bijective holomorphism is called a **biholomorphism** or **conformal**. $\mathrm{Aut}(X)$ will denote the **automorphism group** of X, the set of biholomorphisms $X \to X$.

The automorphism group of the Riemann sphere is

$$\mathrm{Aut}(\mathbb{P}^1(\mathbb{C})) = \mathrm{PGL}(2,\mathbb{C}) = \mathrm{GL}(2,\mathbb{C}) \Big/ \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \neq 0 \right\}.$$

It acts on $\mathbb{P}^1(\mathbb{C})$ through the projectivization of the linear action of $\mathrm{GL}(2,\mathbb{C})$ on $\mathbb{C}^2\backslash\{(0,0)\}$. We can also descibe the action on $\widehat{\mathbb{C}}$. We have:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \begin{cases} \frac{az+b}{cz+d} & \text{if } z \neq -d/c \\ \infty & \text{if } z = -d/c \end{cases} \tag{1.2.1}$$

and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \infty = \begin{cases} \frac{a}{c} & \text{if } c \neq 0 \\ \infty & \text{if } c = 0. \end{cases}$$

These maps are called Möbius transformations.

Finally, we observe that

$$\mathrm{PGL}(2,\mathbb{C}) \simeq \mathrm{PSL}(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{C}, \quad ad - bc = 1 \right\} \Big/ \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

1.2.3. Quotients. Many subgroups of $\mathrm{Aut}(\mathbb{P}^1(\mathbb{C}))$ give rise to Riemann surfaces:

THEOREM 1.2.4. *Let $D \subset \widehat{\mathbb{C}}$ be a domain and let $G < \mathrm{PSL}(2,\mathbb{C})$ such that*

(1) $g(D) = D$ for all $g \in G$

(2) If $g \in G \setminus \{e\}$ then the fixed points of g lie ourside of D.

(3) For each compact subset $K \subset D$, the set

$$\{ g \in G : g(K) \cap K \neq \emptyset \}$$

is finite.

Then the quotient space

$$D/G$$

has the structure of a Riemann surface.

A group that satisfies the second condition is said to act **freely** on D and a group that satisfies the third condition is said to act **properly discontinuously** on D. The proof of this theorem will be part of the exercises.

1.2.4. Tori. The theorem from the previous section gives us a lot of new examples. The first is that of tori. Consider the elements

$$g_1 := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, g_\tau := \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} \in \mathrm{PSL}(2, \mathbb{C}),$$

for some $\tau \in \mathbb{C}$ with $\mathrm{Im}(\tau) > 0$, acting on the domain $\mathbb{C} \subset \widehat{\mathbb{C}}$ by

$$g_1(z) = z + 1 \quad \text{and} \quad g_\tau(z) = z + \tau$$

for all $z \in \mathbb{C}$.

We define the group

$$\Lambda_\tau = \langle g_1, g_\tau \rangle < \mathrm{PSL}(2, \mathbb{C}).$$

A direct computation shows that

$$\begin{bmatrix} 1 & p + q\tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & r + s\tau \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & p + q + (r + s)\tau \\ 0 & 1 \end{bmatrix},$$

for all $p, q, r, s \in \mathbb{Z}$, from which it follows that

$$\Lambda_\tau = \left\{ \begin{bmatrix} 1 & n + m\tau \\ 0 & 1 \end{bmatrix} : m, n \in \mathbb{Z} \right\} \simeq \mathbb{Z}^2.$$

Let us consider the conditions from Theorem 1.2.4. (1) is trivially satisfied: Λ_τ preserves $\mathbb{C}$. Any non-trivial element in Λ_τ is of the form

$$\begin{bmatrix} 1 & n + m\tau \\ 0 & 1 \end{bmatrix}$$

and hence only has the point $\infty \in \widehat{\mathbb{C}}$ as a fixed point, which gives us condition (2). To check condition (3), suppose $K \subset \mathbb{C}$ is compact. Write $d_K = \sup\{\, |z - w| \,:\, z, w \in K \,\} < \infty$. Given $g \in \Lambda_\tau$, write

$$T_g = \inf\{\, |gz - z| \,:\, z \in \mathbb{C} \,\}$$

for the **translation length** of g. Note that $T_g = |gz - z|$ for all $z \in \mathbb{C}$ (this is quite special to quotients of $\mathbb{C}$). We have

$$\{\, g \in \Lambda_\tau \,:\, g(K) \cap K \neq \emptyset \,\} \subset \{\, g \in \Lambda_\tau \,:\, T_g \leq 2 d_K \,\}$$

and the latter is finite. So $\mathbb{C}/\Lambda_z$ is indeed a Riemann surface.

We claim that this is a torus. One way to see this is to note that the quotient map $\pi : \mathbb{C} \to \mathbb{C}/\Lambda_\tau$ restricted to the convex hull

$$\begin{aligned} \mathcal{F} &= \mathrm{conv}(\{0, 1, \tau, 1 + \tau) \\ &:= \{\, \lambda_1 + \lambda_2\tau + \lambda_3(1 + \tau) \,:\, \lambda_1, \lambda_1, \lambda_3 \in [0, 1], \lambda_1 + \lambda_2 + \lambda_3 \leq 1 \,\} \end{aligned}$$

is surjective. Figure 3 shows a picture of $\mathcal{F}$. On $\mathring{\mathcal{F}}$, π is also injective. So to understand what the quotient looks like, we only need to understand what happens to the sides of $\mathcal{F}$.

Since the quotient map identifies the left hand side of $\mathcal{F}$ with the right hand side and the top with the bottom, the quotient is a torus.

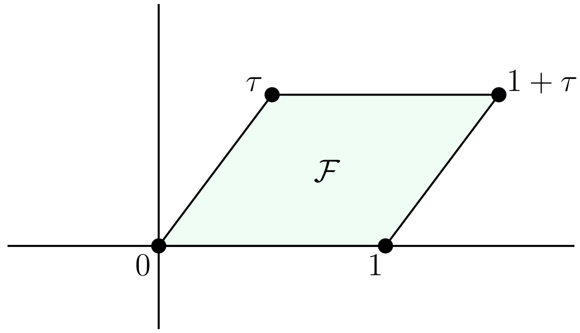

FIGURE 3. A fundamental domain for the action $\Lambda_\tau \curvearrowright \mathbb{C}$.

We can also prove that $\mathbb{C}/\Lambda_\tau$ is a torus by using the fact that for all $z \in \mathbb{C}$ there exist unique $x, y \in \mathbb{R}$ so that

$$z = x + y\tau.$$

The map $\mathbb{C}/\Lambda_\tau \to \mathbb{S}^1 \times \mathbb{S}^1$ given by

$$[x + y\tau] \mapsto (e^{2\pi i x}, e^{2\pi i y})$$

is a homeomorphism.

Note that we have not yet proven whether all these tori are distinct as Riemann surfaces. But it will turn out later that many of them are.

1.2.5. Hyperbolic surfaces. Set $\mathbb{H}^2 = \{ z \in \mathbb{C} : \mathrm{Im}(z) > 0 \}$, the upper half plane. It turns out that the automorphism group of $\mathbb{H}^2$ is $\mathrm{PSL}(2, \mathbb{R})$. We will see a lot more about this later during the course, but for now we will just note that there are many subgroups of $\mathrm{PSL}(2, \mathbb{R})$ that satisfy the conditions of Theorem 1.2.4.

It also turns out that $\mathrm{PSL}(2, \mathbb{R})$ is exactly the group of orientation preserving isometries of the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

This is a complete metric of constant curvature -1. So, this means that all these Riemann surfaces naturally come equipped with a complete metric of constant curvature -1. We will prove some of these statements and treat a first example in the first problem sheet.

1.3. The uniformization theorem and automorphism groups

The Riemann mapping theorem tells us that any pair of simply connected domains in $\mathbb{C}$ that are both not all of $\mathbb{C}$ are biholomorphic. In the early 20^{th} century this was generalized by Koebe and Poincaré to a classification of **all** simply connected Riemann surfaces:

THEOREM 1.3.1 (Uniformization theorem). *Let X be a simply connected Riemann surface. Then X is biholomorphic to exactly one of*

$$\widehat{\mathbb{C}}, \quad \mathbb{C} \quad \text{or} \quad \mathbb{H}^2.$$

PROOF. See for instance [**FK92**, Chapter IV]. □

This theorem implies that we can see obtain every Riemann surface as a quotient of one of three Riemann surfaces. Before we formally state this, we record the following fact:

PROPOSITION 1.3.2.
- $\mathrm{Aut}(\widehat{\mathbb{C}}) = \mathrm{PSL}(2,\mathbb{C})$ *acting by Möbius transformations,*
- $\mathrm{Aut}(\mathbb{C}) = \{\, \varphi : z \mapsto az + b \, : \, a \in \mathbb{C} \setminus \{0\}, b \in \mathbb{C} \,\} \simeq \mathbb{C} \rtimes \mathbb{C}^*$,
- $\mathrm{Aut}(\mathbb{H}^2) = \mathrm{PSL}(2,\mathbb{R})$ *acting by Möbius transformations.*

PROOF. See for instance [**Bea84**, Chapter 5] or [**IT92**, Section 2.3]. □

Note that in all three cases, we have

$$\mathrm{Aut}(X) = \left\{ g \in \mathrm{Aut}(\widehat{\mathbb{C}}) \, : \, g(X) = X \right\},$$

that is, all the automorphisms of $\mathbb{C}$ and $\mathbb{H}^2$ extend to $\widehat{\mathbb{C}}$. However, not all automorphisms of $\mathbb{H}^2$ extend to $\mathbb{C}$.

COROLLARY 1.3.3. *Let X be a Riemann surface. Then there exists a group $G < \mathrm{Aut}(D)$, where D is exactly one of $\mathbb{C}$, $\widehat{\mathbb{C}}$ or $\mathbb{H}^2$ so that*

- *G acts freely and properly discontinuously on D and*
- *$X = D/G$ as a Riemann surface.*

PROOF. Let $\widetilde{X}$ denote the universal cover of X and $\pi_1(X)$ its fundamental group. The fact that X is a Riemann surface, implies that $\widetilde{X}$ can be given the structure of a Riemann surface too, so that $\pi_1(X)$ acts freely and properly discontinuously on $\widetilde{X}$ by biholomorphisms (see for instance [**IT92**, Lemma 2.6]) and such that

$$\widetilde{X}/\pi_1(X) = X.$$

Since $\widetilde{X}$ is simply connected, it must be biholomorphic to exactly one of $\mathbb{C}$, $\widehat{\mathbb{C}}$ or $\mathbb{H}^2$. □

LECTURE 2

Quotients, metrics, conformal structures

2.1. Quotients of the three simply connected Riemann surfaces

Now that we know that we can obtain all Riemann surfaces as quotients of one of three simply connected Riemann surfaces, we should start looking for interesting quotients.

2.1.1. Quotients of the Rieman sphere. It turns out that for the Riemann sphere there are none:

PROPOSITION 2.1.1. *Let X be a Riemann surface. The universal cover of X is biholomorphic to $\widehat{\mathbb{C}}$ if and only if X is biholomorphic to $\widehat{\mathbb{C}}$.*

PROOF. The "if" part is clear. For the "only if" part, note that every element in $\mathrm{PSL}(2,\mathbb{C})$ has at least one fixed point on $\widehat{\mathbb{C}}$ (this either follows by direct computation or from the fact that orientation-preserving self maps of the sphere have at least one fixed point, by the Brouwer fixed point theorem [Mil65, Problem 6]). Since, by assumption

$$X = \widehat{\mathbb{C}}/G,$$

where G acts properly discontinuously and freely, we must have $G = \{e\}$. □

2.1.2. Quotients of the plane. In Section 1.2.4, we have already seen that in the case of the complex plane, the list of quotients is a lot more interesting: there are tori. This however turns out to be almost everything:

PROPOSITION 2.1.2. *Let X be a Riemann surface. The universal cover of X is biholomorphic to $\mathbb{C}$ if and only if X is biholomorphic to either $\mathbb{C}$, $\mathbb{C}\setminus\{0\}$ or*

$$\mathbb{C}/\left\langle \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix} \right\rangle$$

for some $\lambda, \mu \in \mathbb{C}\setminus\{0\}$ that are linearly independent over $\mathbb{R}$.

PROOF. First suppose $X = \mathbb{C}/G$. We claim that, since G acts properly discontinuously, G is one of the following three forms:

(1) $G = \{e\}$

(2) $G = \langle \varphi_b \rangle$, where $\varphi_b(z) = z + b$ for some $b \in \mathbb{C}\setminus\{0\}$

(3) $G = \langle \varphi_{b_1}, \varphi_{b_2} \rangle$ where $b_1, b_2 \in \mathbb{C}$ are independent over $\mathbb{R}$.

To see this, we first prove that G cannot contain any automorphism $z \mapsto az+b$ for $a \neq 1$. Indeed, if $a \neq 1$ then $b/(1-a)$ is a fixed point for this map, which would contradict freeness of the action. Moreover, since $z \mapsto z+b_1$ and $z \mapsto z+b_2$ commute for all $b_1, b_2 \in \mathbb{C}$, G is a free abelian group and

$$G \cdot z = \{\, z+b \,:\, \varphi_b \in G \,\}.$$

In particular, if G contains $\{z \mapsto z+b_1, z \mapsto z+b_2, z \mapsto z+b_3\}$ for $b_1, b_2, b_3 \in \mathbb{C}$ that are independent over $\mathbb{Q}$, then $\mathrm{span}_{\mathbb{Z}}(b_1, b_2, b_3)$ is dense in $\mathbb{C}$. This means that we can find a sequence $((k_i, l_i, m_i))_i$ such that

$$\varphi_{b_1}^{k_i} \circ \varphi_{b_2}^{l_i} \circ \varphi_{b_3}^{m_i}(z) \to z \quad \text{as } i \to \infty,$$

thus contradicting proper discontinuity. On a side note, we could have also used the classification of surfaces (of potentially infinite type) in the last step: there is no surface that has $\mathbb{Z}^k$ for $k \geq 3$ as a fundamental group.

We have already seen that the third case gives rise to tori. In the second case, the surface is biholomorphic to $\mathbb{C} \setminus \{0\}$. Indeed, the map

$$[z] \in \mathbb{C}/\langle \varphi_b \rangle \quad \mapsto \quad e^{2\pi i z/b} \in \mathbb{C} \setminus \{0\}$$

is a biholomorphism.

Now let us prove the converse. For $X = \mathbb{C}$ the statement is clear. Likewise, for $X = \mathbb{C}\setminus\{0\}$, we have just seen that the composition

$$\mathbb{C} \to \mathbb{C}/(z \sim z+1) \simeq \mathbb{C} \setminus \{0\}$$

is the universal covering map. Finally, in the proposition, the tori are given as quotients of $\mathbb{C}$. □

2.2. More on quotients

2.2.1. Quotients of the complex plane, continued. We saw last time that any quotient Riemann surface of $\mathbb{C}$ is either $\mathbb{C}$, $\mathbb{C} - \{0\}$ or a torus. It turns out that moreover every Riemann surface structure on the torus comes from the complex plane. We have seen above that the universal cover cannot be the Riemann sphere, which means that (using the uniformization theorem) all we need to prove is that it cannot be the upper half plane either.

The fundamental group of the torus is isomorphic to $\mathbb{Z}^2$, so what we need to prove is that there is no subgroup of $\mathrm{Aut}(\mathbb{H}^2) = \mathrm{PSL}(2,\mathbb{R})$ that acts properly discontinuously and freely on $\mathbb{H}^2$ and is isomorphic to $\mathbb{Z}^2$. We will state this as a lemma (in which we don't unnecessarily assume that the action is free, even if in our context that would suffice):

LEMMA 2.2.1. *Suppose $G < \mathrm{PSL}(2,\mathbb{R})$ acts properly on $\mathbb{H}^2$ and suppose furthermore that G is abelian. Then either $G \simeq \mathbb{Z}$ or G is finite and cyclic.*

PROOF. We will use the classification of isometries of $\mathbb{H}^2$ that we shall prove in the exercises: an element $g \in \mathrm{PSL}(2,\mathbb{R})$ has either

- a single fixed point in $\mathbb{H}^2$, in which case it's called elliptic and can be conjugated into $\mathrm{SO}(2)$

- a single fixed point on $\mathbb{R} \cup \{\infty\}$, in which case it's called parabolic and can be conjugated into $\left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}$
- or two fixed points on $\mathbb{R}\cup\{\infty\}$, in which case it's called hyperbolic (or loxodromic) and can be conjugated into $\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} : \lambda > 0 \right\}$.

If $g_1, g_2 \in \mathrm{PSL}(2,\mathbb{R})$ commute and $p \in \mathbb{H}^2 \cup \mathbb{R} \cup \{\infty\}$ is a fixed point of g_1, then

$$g_1(g_2(p)) = g_2 \circ g_1(p) = g_2(p).$$

That is, $g_2(p)$ is also a fixed point of g_1.

So if G contains an elliptic element g, then all other $g' \in G \setminus \{e\}$ are elliptic as well, with the same fixed point. Moreover, by proper discontinuity (and compactness of $\mathrm{SO}(2)$), the angles of rotation of all elements in G must be rationally related rational multiples of π. This means that G is a finite cyclic group.

Now suppose G contains a parabolic element g. Then all other $g' \in G \setminus \{e\}$ are parabolic as well, with the same fixed point (which we may assume to be ∞). If

$$\begin{pmatrix} 1 & t_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & t_2 \\ 0 & 1 \end{pmatrix} \in G$$

for some $t_1, t_2 \in \mathbb{R}$ that are not rationally related, then G is not discrete, which contradicts proper discontinuity (see the exercises). So $G \simeq \mathbb{Z}$.

The argument in the hyperbolic case is essentially the same as in the parabolic case. □

Combining this lemma with the uniformization theorem, we obtain:

COROLLARY 2.2.2. *Let X be a Riemann surface that is diffeomorphic to a torus. Then the universal cover of X is biholomorphic to $\mathbb{C}$.*

2.2.2. Quotients of the upper half plane. It will turn out that the richest family of Riemann surfaces is that of quotients of $\mathbb{H}^2$. Indeed, looking at the clasification of closed orientable surfaces, we note that we have so far only seen the sphere and the torus. It turns out that all the other closed orientable surfaces also admit the structure of a Riemann surface. In fact, they all admit lots of different such structures. The two propositions above imply that they must all arise as quotients of $\mathbb{H}^2$.

We will not yet discuss how to construct all these surfaces but instead discuss an example (partially taken from [GGD12, Example 1.7]). Fix some distinct complex numbers $a_1, \ldots, a_{2g+1}$ and consider the following subset of $\mathbb{C}^2$:

$$\mathring{X} = \left\{ (z, w) \in \mathbb{C}^2 \,:\, w^2 = (z - a_1)(z - a_2) \cdots (z - a_{2g+1}) \right\}.$$

Let X denote the one point compactification of $\mathring{X}$ obtained by adjoining the point (∞, ∞).

As opposed to charts, we will describe inverse charts, or **parametrizations** around every $p \in \mathring{X}$:

- Suppose $p = (z_0, w_0) \in \mathring{X}$ is so that $z_0 \neq a_i$ for all $i = 1, \ldots, 2g+1$. Set
$$\varepsilon := \min_{i=1,\ldots,2g+1} \{|z_0 - a_i|/2\}$$
Then define the map $\varphi^{-1} : \{\, \zeta \in \mathbb{C} \,:\, |\zeta| < \varepsilon \,\} \to \mathring{X}$ by
$$\varphi^{-1}(\zeta) = \left(\zeta + z_0, \quad \sqrt{(\zeta + z_0 - a_1)\cdots(\zeta + z_0 - a_{2g+1})}\right),$$
where the branch of the square root is chosen so that $\varphi^{-1}(0) = (z_0, w_0)$, gives a parametrization.

- For $p = (a_j, 0)$, we set
$$\varepsilon := \min_{i\neq j}\{\sqrt{|a_j - a_i|/2}\}$$
Then define the map $\varphi^{-1} : \{\, \zeta \in \mathbb{C} \,:\, |\zeta| < \varepsilon \,\} \to \mathring{X}$ by
$$\varphi^{-1}(\zeta) = \left(\zeta^2 + a_j, \quad \zeta \sqrt{\prod_{i\neq j}(\zeta^2 + a_j - a_i)}\right).$$
The reason that we need to take different charts around these points is that
$$\sqrt{z - a_j}$$
is not a well defined holomorphic function near $z = a_j$.

  Also note that the choice of the branch of the root does not matter. By changing the branch we would obtain a new parametrization $\widetilde{\varphi}^{-1}$ that satisfies $\widetilde{\varphi}^{-1}(\zeta) = \varphi^{-1}(-\zeta)$.

It's not hard to see that $\mathring{X}$ is not bounded as a subset of $\mathbb{C}^2$. This means in particular that it's not compact. We can however compactify it in a similar fashion to how we compactified $\mathbb{C}$ in order to obtain the Riemann sphere. That is, we add a point (∞, ∞) and around this point define a parametrization:
$$\varphi_\infty^{-1}(\zeta) = \begin{cases} \left(\zeta^{-2}, \quad \zeta^{-(2g+1)}\sqrt{(1 - a_1\zeta^2)\cdots(1 - a_{2g+1}\zeta^2)}\right) & \text{if } \zeta \neq 0 \\ (\infty, \infty) & \text{if } \zeta = 0, \end{cases}$$
for all $\zeta \in \{|\zeta| < \varepsilon\}$ and some appropriate $\varepsilon > 0$.

The reason that the resulting surface X is compact is that we can write it as the union of the sets
$$\left\{\, (z, w) \in \mathring{X} \,:\, |z| \leq 1/\varepsilon^2 \,\right\} \cup \left(\left\{\, (z, w) \in \mathring{X} \,:\, |z| \geq 1/\varepsilon^2 \,\right\} \cup \{(\infty, \infty)\}\right),$$
for some small $\varepsilon > 0$. The first set is compact because it's a bounded subset of $\mathbb{C}^2$. The second set is compact because it's $\varphi_\infty^{-1}(\{|\zeta| \leq \varepsilon\})$.

To see that X is connected, we could proceed using charts as well. We would have to find a collection of charts that are all connected, overlap and cover X. However, it's easier to use complex analysis. Suppose $z_0 \neq a_i$ for all $i = 1, \ldots, a_{2g+1}$ and $z_0 \neq \infty$. In that case, we can define a path

$$z(t) \mapsto \left(z(t), \quad \sqrt{\prod_{i=1}^{2g+1} (z(t) - a_i)} \right)$$

where $z(t)$ is some continuous path in $\mathbb{C}$ between z_0 and a_i and we pick a continuous branch of the square root, thus connecting any point $(z_0, w_0) \in X$ to $(0, a_i)$.

To figure out the genus of X, note that there is a map $\pi : X \to \widehat{\mathbb{C}}$ given by

$$\pi(z, w) = z \quad \text{for all } (z, w) \in X.$$

This map is two-to-one almost everywhere. Only the points $z = a_i$, $i = 1, \ldots, 2g + 1$ and the point $z = \infty$ have only one pre-image.

Now triangulate $\widehat{\mathbb{C}}$ so that the vertices of the triangulation coincide with the points $a_1, \ldots, a_{2g+1}, \infty$. If we lift the triangulation to X using π, we can compute the Euler characteristic of X. Every face and every edge in the triangulation of $\widehat{\mathbb{C}}$ has two pre-images, whereas each vertex has only one. This means that:

$$\chi(X) = 2\chi(\widehat{\mathbb{C}}) - (2g + 2) = 2 - 2g.$$

Because X is an orientable closed surface, we see that it must have genus g (Lemma 1.1.5). In particular, if $g \geq 2$, these surfaces are quotients of $\mathbb{H}^2$. Note that this also implies that for $g \geq 1$, the Riemann surface $\mathring{X}$ is also a quotient of $\mathbb{H}^2$. Note that we could have also used the Riemann–Hurwitz formula for this calculation. Incidentally, this formula can be proved using a similar argument to what we just did above.

To get a picture of what X looks like, draw a closed arc α_1 between a_1 and a_2 on $\widehat{\mathbb{C}}$, an arc α_2 between a_3 and a_4 that does not intersect the first arc and so on, and so forth. The last arc α_{g+1} goes between a_{2g+1} and ∞. Figure 1 shows a picture of what these arcs might look like.

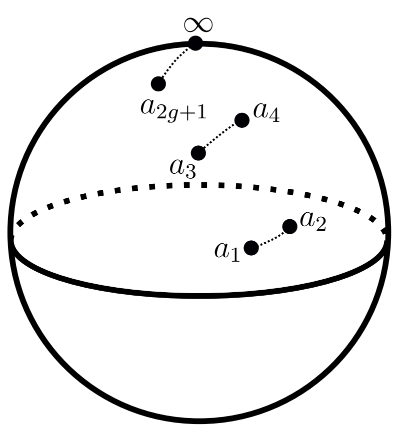

FIGURE 1. $\widehat{\mathbb{C}}$ with some intervals removed.

Let

$$D = \widehat{\mathbb{C}} \setminus \left(\bigcup_{i=1}^{g+1} \alpha_i \right).$$

The map

$$\pi|_{\pi^{-1}(D)} : \pi^{-1}(D) \to D$$

is now a two-to-one map. Moreover on the arcs, it's two-to-one on the interior and one-to-one on the boundary. Because it's also smooth, this means that the pre-image of the arcs is a circle. So, X may be obtained (topologically) by cutting $\widehat{\mathbb{C}}$ open along the arcs, taking two copies of that, and gluing these along their boundary. Figure 2 depicts this process.

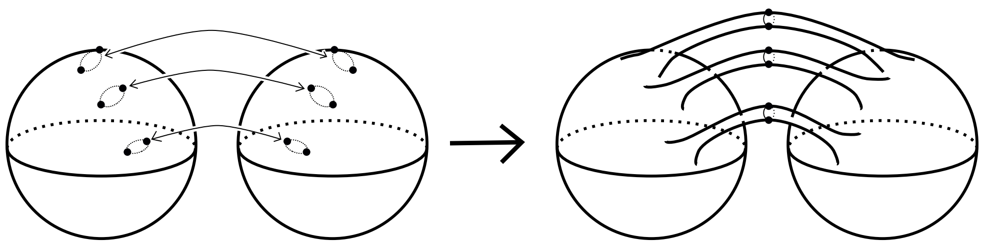

FIGURE 2. Gluing X out of two Riemann spheres.

Finally, we note that our Riemann surfaces come with an involution $\imath : X \to X$, given by

$$\imath(w) = \begin{cases} -w & \text{if } w \neq \infty \\ \infty & \text{if } w = \infty. \end{cases}$$

This map is called the **hyperelliptic involution** and the surfaces we described are hence called **hyperelliptic surfaces**. Note that $\pi : X \to \widehat{\mathbb{C}}$ is the quotient map $X \to X/\imath$.

LECTURE 3

Conformal structures and the moduli space of the torus

3.1. Riemannian metrics and Riemann surfaces

We already noted that every Riemann surface comes with a natural Riemannian metric. Indeed the Riemann sphere has the usual round metric of constant curvature $+1$. Likewise, $\mathbb{C}$ has a flat metric, its usual Euclidean metric $\mathrm{Aut}(\mathbb{C})$ does not act by isometries. However, in the proof of Proposition 2.1.2, we saw that all the quotients are obtained by quotienting by a group that does act by Euclidean isometries. This means that the Euclidean metric descends. Finally, we proved in the exercises that $\mathrm{Aut}(\mathbb{H}^2)$ also acts by isometries of the hyperbolic metric defined in Section 1.2.5. So every quotient of $\mathbb{H}^2$ comes with a natural metric of constant curvature -1.

It turns out that we can also go the other way around. That is: Riemann surface structures on a given surface are in one-to-one correpsondence with complete metrics of constant curvature.

One way to see this uses the Killing-Hopf theorem. In the special case of surfaces, this states that every oriented surface equipped with a Riemannian metric of constant curvature $+1$, 0 or -1 can be obtained as the quotient by a group of orientation preserving isometries acting properly discontinuously and freely on $\mathbb{S}^2$ equipped with the round metric, $\mathbb{R}^2$ equipped with the Euclidean metric or $\mathbb{H}^2$ equipped with the hyperbolic metric respectively (see [CE08, Theorem 1.37] for a proof). For a Riemannian manifold M, let us write

$$\mathrm{Isom}^+(M) = \{ \varphi : M \to M \ : \varphi \text{ is an orientation preserving isometry} \} .$$

So, we need the fact that

(1) $\mathrm{Isom}^+(\mathbb{S}^2) = \mathrm{SO}(2,\mathbb{R})$ and this has no non-trivial subgroups that act properly discontinuously on $\mathbb{S}^2$.

(2) $\mathrm{Isom}^+(\mathbb{R}^2) = \mathrm{SO}(2,\mathbb{R}) \ltimes \mathbb{R}^2$, where $\mathbb{R}^2$ acts by translations. The only subgroups of this group that act properly discontinuously and freely are the fundamental groups of tori and cylinders.

(3) $\mathrm{Isom}^+(\mathbb{H}^2) = \mathrm{PSL}(2,\mathbb{R})$.

Given the above, we get our one-to-one correspondence:

PROPOSITION 3.1.1. *Given an orientable surface Σ of finite type with $\partial\Sigma = \emptyset$, the identification described above gives a one-to-one correspondence of sets*

$$\left\{ \begin{array}{c} \text{Riemann surface} \\ \text{structures on } \Sigma \end{array} \right\} \Big/ \sim \quad \leftrightarrow \quad \left\{ \begin{array}{c} \text{Complete Riemannian} \\ \text{metrics of constant} \\ \text{curvature } \{-1,0,+1\} \\ \text{on } \Sigma \end{array} \right\} \Big/ \sim,$$

where the equivalence on the left is biholomorphism and the equivalence on the right is isometry (and homothety in the Euclidean case).

PROOF SKETCH. From the above we see that a Riemann surface structure on Σ yields a metric of constant curvature and vice versa. We only need to check that biholomorphic Riemann surfaces yield isometric/homothetic metrics and vice versa.

Suppose $h : X \to Y$ is a biholomorphism. We may lift this to a biholomorphism $\widetilde{h} : \widetilde{X} \to \widetilde{Y}$ of the universal covers $\widetilde{X}$ and $\widetilde{Y}$ of X and Y respectively. There are three cases to treat: $\widetilde{X} \simeq \widetilde{Y} \simeq \mathbb{C}, \widehat{\mathbb{C}}, \mathbb{H}^2$. Because it's the most interesting case, we will treat the first, i.e. $\widetilde{X} \simeq \widetilde{Y} \simeq \mathbb{C}$. We will also assume X and Y are tori. If we write

$$X \simeq \mathbb{C}/\Lambda_1 \quad \text{and} \quad Y \simeq \mathbb{C}/\Lambda_2,$$

then we get that $\widetilde{h} \in \text{Aut}(\mathbb{C})$ is such that $\widetilde{h}(\Lambda_1) = \Lambda_2$. Since all automorphisms of $\mathbb{C}$ are of the form $z \mapsto az + b$ for $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$, Λ_2 is obtained from Λ_1 by translating, scaling and rotating. This means that the quotient metrics are homothetic.

The proof of the reverse direction and both directions of all the remaining cases are similar. □

Whether the curvature is 0, +1 or −1 is determined by the topology of the underlying surface. This for instance follows from the discussion above. It can also be seen from the Gauss-Bonnet theorem. Recall that in the case of a closed Riemannian surface X, this states that

$$\int_X K \, dA = 2\pi \, \chi(\Sigma),$$

where K is the Gaussian curvature on X and dA the area measure. For constant curvature κ, this means that

$$\kappa \cdot \text{area}(X) = 2\pi \, \chi(X)$$

So $\chi(X) = 0$ if and only of $\kappa = 0$ and otherwise $\chi(X)$ needs to have the same sign as κ. This last equality generalizes to finite type surfaces and we obtain:

LEMMA 3.1.2. *Let X be a hyperbolic surface homeomorphic to $\Sigma_{g,b,n}$ then*

$$\text{area}(X) = 2\pi(2g + n + b - 2).$$

3.2. Conformal structures

There is another type of structures on a surface that is in one-to-one correspondence with Riemann surface structures, namely conformal structures.

We say that two Riemannian metrics ds_1^2 and ds_2^2 on a surface X are **conformally equivalent** is there exists a positive function $\rho : X \to \mathbb{R}_+$ so that

$$ds_1^2 = \rho \cdot ds_2^2.$$

So a conformal equivalence class of Riemannian metrics can be seen as a notion of angles on the surface.

We have already seen that a Riemann surface structure induces a Riemannian metric on the surface, so it certainly also induces a conformal class of metrics.

Now we need to explain how to go back.This consists of two parts, the second of which we will only sketch. We start with the first part. Suppose we are given a surface X with oriented charts $(U_j, (u_j, v_j))_j$ equipped with a Riemannian metric that in all local coordinates (u_j, v_j) is of the form

$$ds^2 = \rho(u_j, v_j) \cdot (du_j^2 + dv_j^2),$$

where $\rho : X \to \mathbb{R}_+$ is some smooth function. Consider the complex-valued coordinate

$$w_j = u_j + i\, v_j.$$

We claim that this is holomorphic. Indeed, applying a coordinate change on $U_j \cap U_k$, we have

$$ds^2 = \rho(u_k, v_k) \cdot \Bigg[\left(\left(\frac{\partial u_j}{\partial u_k}\right)^2 + \left(\frac{\partial v_j}{\partial u_k}\right)^2 \right) du_k^2 + \left(\left(\frac{\partial u_j}{\partial v_k}\right)^2 + \left(\frac{\partial v_j}{\partial v_k}\right)^2 \right) dv_k^2 + 2 \left(\frac{\partial u_j}{\partial u_k}\frac{\partial u_j}{\partial v_k} + \frac{\partial v_j}{\partial u_k}\frac{\partial v_j}{\partial v_k} \right) du_k dv_k \Bigg].$$

Our assumption implies that

$$\left(\frac{\partial u_j}{\partial u_k}\right)^2 + \left(\frac{\partial v_j}{\partial u_k}\right)^2 = \left(\frac{\partial u_j}{\partial v_k}\right)^2 + \left(\frac{\partial v_j}{\partial v_k}\right)^2 \tag{3.2.1}$$

and

$$\frac{\partial u_j}{\partial u_k}\frac{\partial u_j}{\partial v_k} + \frac{\partial v_j}{\partial u_k}\frac{\partial v_j}{\partial v_k} = 0. \tag{3.2.2}$$

This last line can be written as

$$\det \begin{pmatrix} \partial u_j/\partial u_k & \partial v_j/\partial v_k \\ -\partial v_j/\partial u_k & \partial u_j/\partial v_k \end{pmatrix} = 0.$$

So this implies that

$$\begin{pmatrix} \partial u_j/\partial u_k \\ -\partial v_j/\partial u_k \end{pmatrix} = \lambda \cdot \begin{pmatrix} \partial v_j/\partial v_k \\ \partial u_j/\partial v_k \end{pmatrix}$$

for some $\lambda \in \mathbb{R}$. Filling this into (3.2.1), we obtain

$$\lambda^2 \cdot \left(\left(\frac{\partial u_j}{\partial v_k} \right)^2 + \left(\frac{\partial v_j}{\partial v_k} \right)^2 \right) = \left(\frac{\partial u_j}{\partial v_k} \right)^2 + \left(\frac{\partial v_j}{\partial v_k} \right)^2 .$$

So $\lambda \in \{\pm 1\}$. Now using that our surface is oriented, i.e. that the determinant of the Jacobian of the chart transition is positive, we obtain that $\lambda = 1$. And hence

$$\frac{\partial u_j}{\partial u_k} = \frac{\partial v_j}{\partial v_k} \quad \text{and} \quad -\frac{\partial v_j}{\partial u_k} = \frac{\partial u_j}{\partial v_k},$$

the Cauchy-Riemann equations for the chart transition $w_k \circ w_j^{-1}$, which means that these coordinates are indeed holomorphic. The coordinates (U_j, w_j) are usually called **isothermal coordinates**.

Also note that we have not used the factor ρ, so any metric that is conformal to our metric will give us the same structure. Moreover, our usual coordinate 'z' on the three simply connected Riemann surfaces is an example of an isothermal coordinate, so if we apply the procedure above to the metric we obtain from our quotients, we find the same complex structure back.

This means that what we need to show is that for each Riemannian metric (that is not necessarily given to us in the form above), we can find a set of coordinates so that our metric takes this form.

3.2.1. Wirtinger derivatives. Before we do so, we briefly discuss some useful terminology from complex analysis, namely **Wirtinger derivatives**, i.e. differentiation with respect to complex coordinates. If (U, z) is a complex coordinate chart on a surface S, we write $z = x + iy$ and

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Moreover, the equation $\frac{\partial f(z)}{\partial \overline{z}} = 0$ is equivalent to the Cauchy-Riemann equations for $f(z)$, i.e. this equation is equivalent to f being holomorphic. One readily verifies that in these coordinates, the product rule takes the form

$$\frac{\partial}{\partial z}(f \cdot g) = \frac{\partial f}{\partial z} \cdot g + f \cdot \frac{\partial g}{\partial z}.$$

Moreover, the chain rules read

$$\frac{\partial (f \circ g)}{\partial z}(z_0) = \frac{\partial f}{\partial z}(g(z_0)) \cdot \frac{\partial g}{\partial z}(z_0) + \frac{\partial f}{\partial \overline{z}}(g(z_0)) \cdot \frac{\partial \overline{g}}{\partial z}(z_0)$$

and

$$\frac{\partial (f \circ g)}{\partial \overline{z}}(z_0) = \frac{\partial f}{\partial z}(g(z_0)) \cdot \frac{\partial g}{\partial \overline{z}}(z_0) + \frac{\partial f}{\partial \overline{z}}(g(z_0)) \cdot \frac{\partial \overline{g}}{\partial \overline{z}}(z_0).$$

3.2.2. Finding isothermal coordinates. Now, suppose our metric is given by

$$ds^2 = A\,dx^2 + 2B\,dx\,dy + C\,dy^2$$

in some local coordinates (x, y).

Writing $z = x + iy$, we get that

$$ds^2 = \lambda\,|dz + \mu d\overline{z}|^2 := \lambda(dz + \mu d\overline{z})(d\overline{z} + \overline{\mu} dz),$$

where

$$\lambda = \frac{1}{4}\left(A + C + 2\sqrt{AC - B^2}\right) \quad \text{and} \quad \mu = \frac{A - C + 2i\,B}{A + C + 2\sqrt{AC - B^2}}.$$

We are looking for a coordinate $w = u + iv$ so that

$$ds^2 = \rho(du^2 + dv^2) = \rho\,|dw|^2 = \rho \cdot \left|\frac{\partial w}{\partial z}\right|^2 \cdot \left|dz + \frac{\partial w/\partial \overline{z}}{\partial w/\partial z} d\overline{z}\right|^2.$$

This means that isothermal coordinates exist if there is a solution to the partial differential equation

$$\frac{\partial w}{\partial \overline{z}} = \mu \cdot \frac{\partial w}{\partial z}.$$

This equation is called the **Beltrami equation** and it turns out this solution does indeed exist on a surface, which means that we obtain a Riemann surface structure. Moreover, it turns out this map is one-to-one. In particular, holmorphic maps are conformal. So we obtain

PROPOSITION 3.2.1. *Given an orientable surface Σ of finite type with $\partial\Sigma = \emptyset$, the identification described above gives a one-to-one correspondence of sets*

$$\left\{ \begin{array}{c} \textit{Riemann surface} \\ \textit{structures on } \Sigma \end{array} \right\} \Big/ \textit{biholom.} \leftrightarrow \left\{ \begin{array}{c} \textit{Conformal classes} \\ \textit{of Riemannian} \\ \textit{metrics on } \Sigma \end{array} \right\} \Big/ \textit{diffeomorphism.}$$

Combined with Proposition 3.1.1, the proposition above also implies that in every conformal class of metrics there is a metric of constant curvature that is unique (up to scaling if the metric is flat). This can also be proved without passing through the uniformization theorem, which comes down to solving a non-linear PDE on the surface.

3.3. Riemann surface structures on the torus

The goal of the rest of this course is to understand the deformation spaces associated to Riemann surfaces: Teichmüller and moduli spaces.

In general, the Teichmüller space associated to a surface will be a space of **marked** Riemann surface structures on that surface and the corresponding moduli space will be a space of isomorphism classes of Riemann surface structures. As such, the moduli space associated to a surface will be a quotient of the corresponding Teichmüller space.

First of all, note that the uniformization theorem tells us that there is only one Riemann surface structure on the sphere. This means that the corresponding moduli space will be a point. It turns out that the same holds for its Teichmüller space. This means that the lowest genus closed surface for which we can expect an intersting deformation space is the torus.

So, let us parametrize Riemann surface structures on the torus. Recall from Proposition 2.1.2 that every Riemann surface structure on the torus is of the form

$$\mathbb{C}/\left\langle \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix} \right\rangle$$

for some $\lambda, \mu \in \mathbb{C} \setminus \{0\}$ that are linearly independent over $\mathbb{R}$.

First of all note that every such torus is biholomorphic to a torus of the form

$$R_\tau := \mathbb{C}/\Lambda_\tau,$$

for some $\tau \in \mathbb{H}^2$, where

$$\Lambda_\tau = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} \right\rangle.$$

Indeed, rotating and rescaling the lattice induce biholomorphisms on the level of Riemann surfaces (as we have already noted in the proof sketch of Proposition 3.1.1)

However, there are still distinct $\tau, \tau' \in \mathbb{H}^2$ that lead to holomorphic tori. We have:

PROPOSITION 3.3.1. *Let $\tau, \tau' \in \mathbb{H}^2$. The two tori R_τ and $R_{\tau'}$ are biholomorphic if and only if*

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

for some $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$.

PROOF. First assume R_τ and $R_{\tau'}$ are biholomorphic and let $f : R_{\tau'} \to R_\tau$ be a biholomorphism. Lift f to a biholomorphism $\widetilde{f} : \mathbb{C} \to \mathbb{C}$. This means that

$$\widetilde{f}(z) = \alpha z + \beta$$

for some $\alpha, \beta \in \mathbb{C}$. By postcomposing with a biholomorphism of $\mathbb{C}$, we may assume that $\widetilde{f}(0) = 0$.

Because $\widetilde{f}$ is a lift, we know that both $\widetilde{f}(1)$ and $\widetilde{f}(\tau')$ are equivalent to 0 under Λ_τ. So

$$\widetilde{f}(\tau') = \alpha\tau' = a\tau + b$$

$$\widetilde{f}(1) = \alpha = c\tau + d$$

for some $a, b, c, d \in \mathbb{Z}$. So

$$\tau' = \frac{a\tau + b}{c\tau + d}.$$

So we only need to show that $ad - bc = 1$. Moreover, since $\widetilde{f}(\Lambda_{\tau'}) = \Lambda_\tau$, $f(\tau') = a\tau + b$ and $f(1) = c\tau + d$ generate Λ_τ. This means that the map

$$m\tau + n \mapsto m \cdot (a\tau + b) + n \cdot (c\tau + d)$$

is an automorphism of Λ, and hence $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z})$. So, we obtain $ad - bc = \pm 1$. Since

$$\mathrm{Im}(\tau') = \frac{ad - bc}{|c\tau + d|^2} > 0,$$

we get $ad - bc = 1$.

Conversely, if

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

Then

$$f([z]) = [(c\tau + d)z]$$

gives a biholomorphic map $f : R_{\tau'} \to R_{\tau}$. $\square$

LECTURE 4

The moduli space of the torus

4.1. The Teichmüller and moduli spaces of tori

Looking at Proposition 3.3.1, we see that we can parametrize all complex structures on the torus with the set

$$\mathcal{M}_1 = \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}^2 = \mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}^2.$$

Moreover this set is the quotient of the hyperbolic place by a group $(\mathrm{PSL}(2,\mathbb{Z}))$ of isometries that acts properly discontinuously on it. However, the group doesn't quite act freely, so it's not directly a hyperbolic surface.

So, let us investigate the structure of this quotient. One way of doing this is to find a fundamental domain for the action of $\mathrm{PSL}(2,\mathbb{Z})$ on $\mathbb{H}^2$. Set

$$\mathcal{F} = \left\{ z \in \mathbb{H}^2 \; : \; |z| \geq 1 \text{ and } -\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2} \right\}.$$

Figure 1 shows a picture of $\mathcal{F}$.

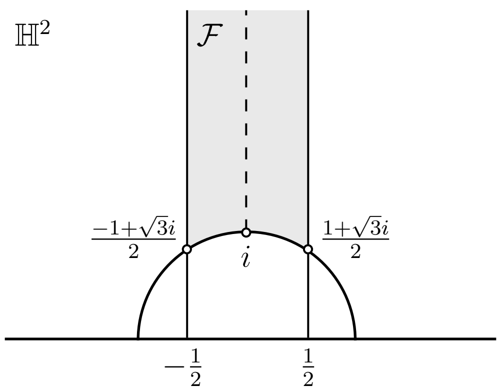

FIGURE 1. A fundamental domain for the action of $\mathrm{PSL}(2,\mathbb{Z})$ on $\mathbb{H}^2$.

We claim

PROPOSITION 4.1.1. *For all $\tau \in \mathbb{H}^2$ there exists an element $g \in \mathrm{PSL}(2,\mathbb{Z})$ so that $g\tau \in \mathcal{F}$. Moreover,*

- *if $\tau \in \mathring{\mathcal{F}}$ then*

$$\Big(\mathrm{PSL}(2,\mathbb{Z}) \cdot \tau\Big) \cap \mathcal{F} = \{\tau\},$$

- *if $\tau \in \mathcal{F}$ and $\mathrm{Re}(\tau) = -\frac{1}{2}$ then*

$$\Big(\mathrm{PSL}(2,\mathbb{Z}) \cdot \tau\Big) \cap \mathcal{F} = \{\tau, \tau+1\},$$

- *if* $\tau \in \mathcal{F}$ *and* $\mathrm{Re}(\tau) = \frac{1}{2}$ *then*
$$\Big(\mathrm{PSL}(2,\mathbb{Z}) \cdot \tau\Big) \cap \mathcal{F} = \{\tau, \tau - 1\}.$$
- *and if* $\tau \in \mathcal{F}$ *and* $|\tau| = 1$ *then*
$$\Big(\mathrm{PSL}(2,\mathbb{Z}) \cdot \tau\Big) \cap \mathcal{F} = \{\tau, -1/\tau\},$$

The proof of this proposition is part of this week's exercises.

In conclusion, $T : z \mapsto z + 1$ maps the line $\mathrm{Re}(z) = -1/2$ to the line $\mathrm{Re}(z) = 1/2$ and $S : z \mapsto -1/z$ fixes i and swaps $(-1 + \sqrt{3}i)/2$ and $(1 + \sqrt{3}i)/2$ (which are in turn the fixed points of ST). Moreover, these are the only side pairings and thus the quotient looks like Figure 2:

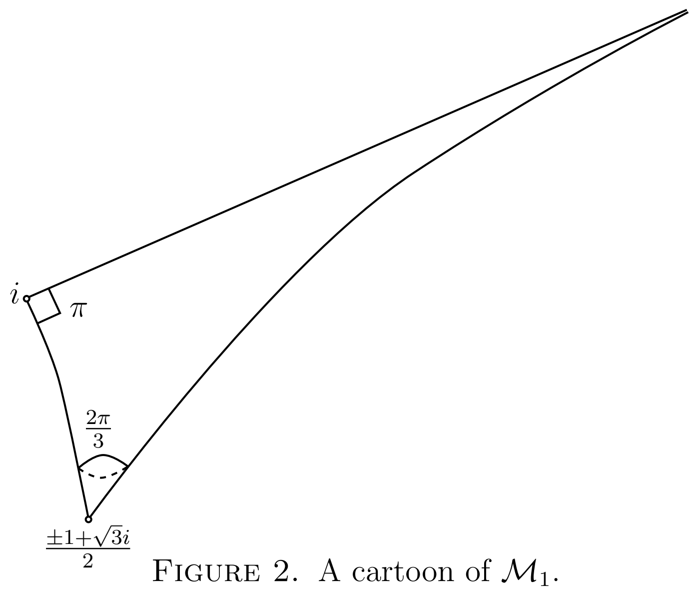

FIGURE 2. A cartoon of $\mathcal{M}_1$.

So $\mathcal{M}_1$ is a space that has the structure of a hyperbolic surface near almost every point. The only problematic points are the images of i and $(\pm 1 + \sqrt{3}i)/2$, where the $\mathcal{M}_1$ looks like a cone. The technical term for such a space is a hyperbolic **orbifold**.

$\mathcal{M}_1$ is called the **moduli space** of tori. $\mathcal{T}_1 = \mathbb{H}^2$ is called the **Teichmüller space** of tori.

Our next intermediate goal is to generalize this to all surfaces. To this end, we will introduce a different perspective on $\mathcal{T}_1$, that generalizes naturally to higher genus surfaces.

4.2. $\mathcal{T}_1$ as a space of marked structures

Our objective in this section is to understand what the information is that is parametrized by $\mathcal{T}_1$.

4.2.1. Markings as a choice of generators for $\pi_1(R)$. So, suppose $\tau \in \mathbb{H}^2$ and $\tau' = g\tau$ for some $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}(2, \mathbb{Z})$. Let $f : R_{\tau'} \to R_\tau$ denote the biholomorphism from the proof of Proposition 3.3.1. We saw that we can find a lift $\widetilde{f} : \mathbb{C} \to \mathbb{C}$ so that $\widetilde{f}(z) = (c\tau+d)z$. In particular, using the relation between τ and τ', we see that

$$\widetilde{f}(\{1,\ \tau'\}) = \{c\tau + d,\ a\tau + b\}.$$

So, the biholomorphism corresponds to a base change (i.e. the change of a choice of generators) for Λ_τ.

Let us formalize this idea of a base change. First we take a base point $p_0 = [0] \in R_\tau$ for the fundamental group $\pi_1(R_\tau, p_0)$. The segments between 0 and 1 and between 0 and τ project to simple closed curves on R_τ and determine generators

$$[A_\tau],\ [B_\tau] \in \pi_1(R_\tau, p_0).$$

This now also gives us a natural choice of isomorphism $\Lambda_\tau \simeq \pi_1(R_\tau, p_0)$, mapping

$$1 \mapsto [A_\tau] \quad \text{and} \quad \tau \mapsto [B_\tau].$$

Likewise, for $R_{\tau'}$ we also obtain a natural system of generators $[A_{\tau'}], [B_{\tau'}] \in \pi_1(R_{\tau'}, p_0)$. Moreover, if $f_* : \pi_1(R_{\tau'}, p_0) \to \pi_1(R_\tau, p_0)$ denotes the map f induces on the fundamental group, then

$$f_*([A_{\tau'}]) \neq [A_\tau] \quad \text{and} \quad f_*([B_{\tau'}]) \neq [B_\tau].$$

Let us package these choices of generators:

Definition 4.2.1. Let R be a Riemann surface homeomorphic to $\mathbb{T}^2$.

(1) A **marking** on R is a generating set $\Sigma_p \subset \pi_1(R, p)$ consisting of two elements.

(2) Two markings Σ_p and $\Sigma'_{p'}$ are called **equivalent** if there exists a continuous curve α from p to p' so that the corresponding isomorphism $T_\alpha : \pi_1(R, p) \to \pi_1(R, p')$ satisfies

$$T_\alpha(\Sigma_p) = \Sigma'_{p'}.$$

Two pairs (R, Σ) and (R', Σ') of marked Riemann surfaces homeomorphic to $\mathbb{T}^2$ are called **equivalent** if there exists a biholomorphic mapping $h : R \to R'$ such that

$$h_*(\Sigma) \simeq \Sigma'.$$

Note that above we have **not** proved that $(R_\tau, \{[A_\tau], [B_\tau]\})$ and $(R_{\tau'}, \{[A_{\tau'}], [B_{\tau'}]\})$ are equivalent as marked Riemann surfaces, because our map f_* did not send the generators to each other, and in fact, they are not equivalent:

Theorem 4.2.2. *Let $\tau, \tau' \in \mathcal{T}_1$. Then the marked Riemann surfaces*

$$(R_\tau, \{[A_\tau], [B_\tau]\}) \quad \textit{and} \quad (R_{\tau'}, \{[A_{\tau'}], [B_{\tau'}]\})$$

are equivalent if and only if $\tau' = \tau$. Moreover, we have an identification

$$\mathcal{T}_1 = \left\{ (R, \Sigma_p) \ : \ \begin{array}{c} R \text{ a Riemann surface homemorphic to } \mathbb{T}^2 \\ p \in R,\ \Sigma_p \text{ a marking on } R \end{array} \right\} \Big/ \sim .$$

Proof. We begin by proving part of the second claim: every marked complex torus is equivalent to a marked torus of the form $(R_\tau, \{[A_\tau], [B_\tau]\})$. So, suppose (R, Σ) is a marked torus. We know that R is biholomorphic to R_τ for some $\tau \in \mathcal{T}_1$. Moreover, since $\Sigma = \{[A], [B]\}$ is a minimal generating set for Λ_τ, we can find a lattice isomorphism $\varphi : \Lambda_\tau \to \Lambda_\tau$ so that

$$\varphi([A]) = 1.$$

Potentially switching the roles of $[A]$ and $[B]$, we can assume φ is an element of $\mathrm{SL}(2, \mathbb{Z})$ and hence that $\varphi([B])$ lies in $\mathbb{H}^2$. The torus $R_{\varphi([B])}$ is biholomorphic to R_τ. So (R, Σ) is equivalent to

$$(R_{\varphi([B])}, \{A_{\varphi([B])}, B_{\varphi([B])}\}).$$

So, to prove the theorem, we need to show that $(R_\tau, \{[A_\tau], [B_\tau]\})$ and $(R_{\tau'}, \{[A_{\tau'}], [B_{\tau'}]\})$ are equivalent if and only if $\tau = \tau'$. Of course, if $\tau = \tau'$ then the two corresponding marked surfaces are equivalent, so we need to show the other direction.

So let $h : R_{\tau'} \to R_\tau$ be a biholomorphism that induces the equivalence. We may assume that $h([0]) = [0]$ and take a lift $\widetilde{h} : \mathbb{C} \to \mathbb{C}$ so that

$$\widetilde{h}(0) = 0.$$

This means that $\widetilde{h}(z) = \alpha z$ for some $\alpha \in \mathbb{C} \setminus \{0\}$. Hence $1 = \widetilde{h}(1) = \alpha$, which implies that $\tau = \widetilde{h}(\tau') = \tau'$. $\square$

Note that so far, our alternate description of Teichmüller space only recovers the set $\mathcal{T}_1$ and not yet it topology. Of course we can use the bijection to define a topology. However, there is also an intrinsic defintion. We will discuss how to do this later.

4.3. Markings by diffeomorphisms

First, we give a third interpreation of $\mathcal{T}_1$. This goes through another (equivalent) way of marking Riemann surfaces.

To this end, once and for all fix a surface S diffeomorphic to $\mathbb{T}^2$. We define:

Definition 4.3.1. Let R and R' be Riemann surfaces and let

$$f : S \to R \quad \text{and} \quad f' : S \to R'$$

be orientation preserving diffeomorphisms. We say that the pairs (R, f) and (R', f') are **equivalent** if there exists a biholomorphism $h : R \to R'$ so that

$$(f')^{-1} \circ h \circ f : S \to S$$

is homotopic to the identity.

Note that if we pick a generating set $\{[A], [B]\}$ for the fundamental group $\pi_1(S, p)$ then every pair (R, f) as above defines a point

$$(R, \{f_*([A]), f_*([B])\}) \in \mathcal{T}_1 .$$

It turns out that this gives another description of the Teichmüller space of tori:

THEOREM 4.3.2. *Fix S and $[A],[B] \in \pi_1(S,p)$ as above. Then the map*

$$\left\{ (R,f) : \begin{array}{c} R \text{ a Riemann surface, } f : S \to R \\ \text{an orientation preserving diffeomorphism} \end{array} \right\} \Big/ \sim \quad \to \quad \mathcal{T}_1$$

given by

$$(R,f) \mapsto (R, \{f_*([A]).f_*([B])\}),$$

is a well-defined bijection.

PROOF. We start with well-definedness. Meaning, suppose (R,f) and (R',f') are equivalent. By definition, this means that there exists a biholomorphic map $h : R \to R'$ so that

$$h \circ f : S \to R' \quad \text{ and } f' : S \to R'$$

are homotopic. Now if α is a continuous arc between $f'(p)$ and $h(f(p))$, we see that T_α induces an equivalence between the markings

$$\{f'_*([A]),\ f'_*([B])\} \quad \text{and} \quad \{(h \circ f)_*([A]),\ (h \circ f)_*([B])\},$$

making $(R, \{f_*([A]),\ f_*[B]\})$ and $(R', \{f'_*([A]),\ f'_*([B])\})$ equivalent. This means that they correspond to the same point by the previous theorem. So, the map is well defined.

Moreover, the map is surjective. For any $\tau \in \mathcal{T}_1$ we can find an orientation preserving diffeomorphism $f : S \to R_\tau$. Indeed, we know that there exists some $\tau_0 \in \mathcal{T}_1$ such that $(S, \{[A],[B]\}) \sim (R_{\tau_0}, \{[A_{\tau_0}],[B_{\tau_0}]\})$ as marked surfaces. One checks that the map $f_\tau : \mathbb{C} \to \mathbb{C}$ given by

$$f_\tau(z) = \frac{(\tau - \overline{\tau}_0)z - (\tau - \tau_0)\overline{z}}{\tau_0 - \overline{\tau}_0}$$

descends to an orientation preserving diffeomorphism $R_{\tau_0} \to R_\tau$ that induces the marking $\{[A_\tau],[B_\tau]\}$ on R_τ.

For the injectivity, suppose that

$$\Big[(R, \{f_*([A]), f_*([B])\})\Big] = \Big[(R', \{f'_*([A]), f'_*([B])\})\Big].$$

Take $\tau_0 \in \mathcal{T}_1$ such that

$$\Big[(S, \{[A],[B]\})\Big] = \Big[(R_{\tau_0}, \{([A_{\tau_0}]), [B_{\tau_0}]\})\Big].$$

Moreover, let $h : R \to R'$ be a holomorphism such that

$$h_*\{f_*([A]), f_*([B])\} = \{f'_*([A]), f'_*([B])\}.$$

We choose lattices $\Lambda, \Lambda' \subset \mathbb{C}$, generated by $(1,a)$ and $(1,a')$ respectively such that

$$R = \mathbb{C}/\Lambda \quad \text{and} \quad R' = \mathbb{C}/\Lambda',$$

and the generators induce the bases $\{f_*([A]), f_*([B])\}$ and $\{f'_*([A]), f'_*([B])\}$ respectively.

Now, let $\widetilde{f}, \widetilde{f}', \widetilde{h} : \mathbb{C} \to \mathbb{C}$ be lifts. We may assume that

$$\widetilde{f}(0) = \widetilde{f}'(0) = \widetilde{h}(0) = 0, \quad \widetilde{f}(1) = \widetilde{f}'(1) = \widetilde{h}(1) = 1,$$

and

$$\widetilde{f}(\tau_0) = a, \quad \widetilde{f}'(\tau_0) = a' \quad \text{and } \widetilde{h}(a) = a'$$

So we obtain a homotopy $F_t : \mathbb{C} \to \mathbb{C}$ defined by

$$F_t(z) = (1-t)\,\widetilde{h} \circ \widetilde{f}(z) + t\,\widetilde{f}'(z)$$

between $\widetilde{h} \circ \widetilde{f}$ and $\widetilde{f}'$ that descends to a homotopy between $h \circ f : S \to R'$ and $f' : S \to R'$. □

4.4. The Teichmüller space of Riemann surfaces of a given type

The two description of the Teichmüller space of the torus above can be generalized to different Riemann surfaces. We will take the second one as a definition, as this is the most common definition in the literature. Moreover, it naturally leads to another key object in Teichmüller theory: the mapping class group.

Definition 4.4.1. Let S be a surface of finite type. Then the **Teichmüller space** of S is defined as

$$\mathcal{T}(S) = \left\{ (X,f) \; : \; \begin{array}{c} X \text{ a Riemann surface },\, f : S \to X \\ \text{an orientation preserving diffeomorphism} \end{array} \right\} \Big/ \sim,$$

where

$$(X,f) \sim (Y,g)$$

if and only if there exists a biholomorphism $h : X \to Y$ so that the map

$$g^{-1} \circ h \circ f : S \to S$$

is homotopic to the identity.

We will often write

$$\mathcal{T}(\Sigma_{g,n}) = \mathcal{T}_{g,n} \quad \text{and} \quad \mathcal{T}(\Sigma_g) = \mathcal{T}_g\,.$$

4.5. Teichmüller space in terms of markings

In order to get to the analogous definition to the space of marked tori, we need to single out particularly nice generating sets for the fundamental group, just like we did for tori. We will stick to closed surfaces. Recall that the fundamental group of a surface of genus g satisfies:

$$\pi_1(\Sigma_g, p) = \left\langle a_1, \ldots, a_g, b_1, \ldots, b_g \;\middle|\; \prod_{i=1}^{g} [a_i, b_i] = e \right\rangle.$$

In what follows, a generating set $A_1, \ldots, A_g, B_1, \ldots, B_g$ of $\pi_1(\Sigma_g, p)$ that satisfies

$$\prod_{i=1}^{g} [A_i, B_i] = e,$$

will be called a **canonical** generating set. Note that this includes the torus case.

Definition 4.5.1. Let R be a closed Riemann surface.

(1) A **marking** on R is a canonical generating set $\Sigma_p \subset \pi_1(R, p)$.

(2) Two markings Σ_p and $\Sigma'_{p'}$ are called **equivalent** if there exists a continuous curve α from p to p' so that the corresponding isomorphism $T_\alpha : \pi(R,p) \to \pi_1(R,p')$ satisfies

$$T_\alpha(\Sigma_p) = \Sigma'_{p'}.$$

Two pairs (R,Σ) and (R',Σ') of marked closed Riemann surfaces are called **equivalent** if there exists a biholomorphic mapping $h : R \to R'$ so that

$$h_*(\Sigma) \simeq \Sigma'.$$

Just like in the case of the torus, the space of marked Riemann surfaces turns out to be the same as Teichmüller space:

THEOREM 4.5.2. *Let S be a closed surface and Σ a marking on S. Then the map*

$$\mathcal{T}(S) \to \left\{ (R,\Sigma_p) \; : \; \begin{array}{c} R \text{ a closed Riemann surface diffeomorphic to } S \\ p \in R, \; \Sigma_p \text{ a marking on } R \end{array} \right\} \Big/ \sim .$$

given by

$$[(R,f)] \mapsto [(R, f_*(\Sigma)]$$

is a bijection.

Before we sketch the proof of this theorem, we state and prove a lemma that will be of use in the study of mapping class groups as well:

LEMMA 4.5.3 (Alexander Lemma). *Let D be a 2-dimensional closed disk and $\phi : D \to D$ a homeomorphism that restricts to the identity on ∂D. Then ϕ is isotopic to the identity $D \to D$*

PROOF OF THE ALEXANDER LEMMA. Identify D with the closed unit disk in $\mathbb{R}^2$ and define the map $F : D \times [0,1] \to D$ by

$$F_t(x) = \begin{cases} (1-t)\cdot\phi\left(\frac{x}{(1-t)}\right) & \text{if } ||x|| < 1-t \text{ and } t < 1 \\ x & \text{if } ||x|| > 1-t \text{ and } t < 1 \\ x & \text{if } t = 1. \end{cases}$$

This yields the isotopy we want. □

We can make this lemma work in the smooth category as well, but its proof is significantly less easy. It for instance follows from work by Smale [Sma59]. In this course we will generally gloss over the difference between homeomorphisms and diffeomorphisms.

PROOF SKETCH. Write $\Sigma = \{[A_1],\ldots,[A_g],[B_1],\ldots,[B_g]\}$, where A_i, B_i are simple closed curves based at a point $p_0 \in S$. Let us start with the injectivity. So, suppose

$$[(R, f_*(\Sigma)] = [(R', f'_*(\Sigma)].$$

This means that we can find a biholomorphic map $h : R \to R'$ and a self-diffeomorphism $g_0 : R' \to R'$ that is homotopic to the identity and such that

$$g_1 = g_0 \circ h \circ f$$

corresponds with f' on the curves $A_1, \ldots, A_g, B_1, \ldots, B_g$. The domain obtained by deleting these curves from S is a disk. This implies that f' and g_1 are homotopic (using the Alexander trick), which in turn means that

$$[(R, f)] = [(R', f')] \in \mathcal{T}(S).$$

For surjectivity, suppose we are given a marked Riemann surface (R, Σ_p). So we need to find an orientation preserving homeomorphism $f : S \to R$ so that $f_*(\Sigma) = \Sigma_p$. So, let us take simple closed smooth curves $A'_1, \ldots, A'_g, B'_1, \ldots, B'_g$ such that

$$\Sigma_p = \{[A'_1], \ldots, [A'_g], [B'_1], \ldots, [B'_g]\}.$$

Moreover, we will set

$$C = \bigcup_{j=1}^{g} (A_j \cup B_j), \quad C' = \bigcup_{j=1}^{g} (A'_j \cup B'_j), \quad S_0 = S \setminus C, \quad \text{and} \quad R_0 = R \setminus C'.$$

R_0 and S_0 are diffeomorphic to polygons with $4g$ sides. So we can find a diffeomorphism by extending a diffeomorphism R_0, S_0. For more details, see [**IT92**, Theorem 1.4]. □

4.5.1. Punctures and marked points. If $n \geq 1$, we can think of $\mathcal{T}(\Sigma_{g,n})$ as a space of surfaces with marked points (as opposed to punctures) as well:

PROPOSITION 4.5.4. *Let $n \geq 1$ and fix n distinct points $x_1, \ldots, x_n \in \Sigma_g$. There is a bijection*

$$\mathcal{T}(\Sigma_{g,n}) \longrightarrow \{\, (X, f) \,:\, f : \Sigma_g \to X \text{ an orientation preserving diffeomorphism} \} \Big/ \sim,$$

where $(X_1, f_1) \sim (X_2, f_2)$ if and only if there exists a biholomorphism $h : X_1 \to X_2$ such that

$$f_2^{-1} \circ h \circ f_1(x_i) = x_i \quad \text{for } i = 1, \ldots, n$$

and $f_2^{-1} \circ h \circ f_1 : \Sigma_g \to \Sigma_g$ is homotopic to the identity through maps fixing $x_1, \ldots x_n$.

We leave the proof to the reader.

LECTURE 5

Basic examples and mapping class groups

5.1. Basic examples

We have seen that the Teichmüller space of the torus can be identified with $\mathbb{H}^2$ (as a set for now). We will treat some further examples in this section.

PROPOSITION 5.1.1. *We have*

(a) Let S be diffeomorphic to Σ_0, $\Sigma_{0,1}$, $\Sigma_{0,2}$ or $\Sigma_{0,3}$, then $\mathcal{T}(S)$ is a point.

(b) $\mathcal{T}(\Sigma_{1,1})$ can be identified with $\mathcal{T}(\Sigma_1)$.

PROOF. For (a), suppose that $[X_1, f_1], [X_2, f_2] \in \mathcal{T}(\Sigma_{0,n})$ with $0 \leq n \leq 3$. We will think of these two as surfaces with marked points, coming from at most three marked points x_1, x_2, x_3 on $\mathbb{S}^2$. By the uniformization theorem, we can identify X_1 and X_2 with the Riemann sphere $\widehat{\mathbb{C}}$. Moreover (using that $n \leq 3$), there exists a Möbius transformation $\varphi : X_1 \to X_2$ such that

$$\varphi(f_1(x_i)) = f_2(x_i), \quad i = 1, \ldots, n.$$

As such the diffeomorphism $f_2^{-1} \circ \varphi \circ f_1 : \mathbb{S}^2 \to \mathbb{S}^2$ fixes $x_1, \ldots, x_n$. All we need to do, is show that this map is homotopic to the identity.

In fact this has nothing to do with the fact that φ is a Möbius transformation. We need to show that every orientation preserving diffeomorphism $f : \mathbb{S}^2 \to \mathbb{S}^2$ that fixes $x_1, \ldots, x_n$ is homotopic to the identity, through maps that fix $x_1, \ldots, x_n$. We will deal with the case $n \leq 2$ here, the other cases are covered by Proposition 5.4.1, that we will prove during the exercise class.

We start with $n = 0$. In this case, we need the fact that the degree is a complete invariant of homotopy classes of self maps of the sphere, which is due to Hopf. Indeed, since the degree of any orientation preserving homeomorphism is 1, this in particular holds for f and the identity map. We refer to [**Hat02**, Corollary 4.25] for a proof.

If $n = 1$, then we use that $\mathbb{S}^2 - \{x_1\} \simeq \mathbb{R}^2$. So we can think of our map as a map $f : \mathbb{R}^2 \to \mathbb{R}^2$. The map $F : \mathbb{R}^2 \times [0, 1] \to \mathbb{R}^2$, defined by

$$F_t(x) = (1 - t) \cdot f(x) + t \cdot x$$

is a homotopy between f and the identity $\mathbb{R}^2 \to \mathbb{R}^2$.

The proof for item (b) is similar. We again think in terms of surfaces with marked points. We have a surjective map

$$\mathcal{T}(\Sigma_{1,1}) \to \mathcal{T}(\Sigma_1),$$

mapping $[X, f] \in \mathcal{T}(\Sigma_{1,1})$ to $[X, f] \in \mathcal{T}(\Sigma_1)$. What we need to show is that this map is injective.

So, suppose $[X_1, f_1] = [X_2, f_2] \in \mathcal{T}(\Sigma_1)$. So there exists a biholomorphism $h : X_1 \to X_2$ such that $f_2^{-1} \circ h \circ f_1 : \Sigma_1 \to \Sigma_1$ is homotopic to the identity. We need to show that we can modify h in such a way that $f_2^{-1} \circ h \circ f_1$ remains homotopic to the identity and also fixes our favorite point $x_1 \in \Sigma_1$. To this end, let's write $X_2 = \mathbb{C}/\Lambda$ for some lattice Λ. Suppose $[p], [q] \in X_2$. Observe that $h_0 : X_2 \to X_2$, defined by

$$h_0([z]) = [z + q - p]$$

is a biholomorphic map $X_2 \to X_2$ that is homotopic to the identity and maps $[p]$ to $[q]$. So, if we set $[p] = h \circ f_1(x_1)$ and $[q] = f_2(x_1)$, then $h_0 \circ h : X_1 \to X_2$ is the biholomorphic map we're looking for. □

5.2. The mapping class group

5.2.1. Defintion. Just like in the case of the torus, we have a natural group action on the Teichmüller space of a surface, by a group called the mapping class group:

DEFINITION 5.2.1. Let S_0 be a compact surface of finite type and $\Sigma \subset S_0$ a finite set. Set $S = S_0 \setminus \Sigma$. The **mapping class group** of S is given by

$$\mathrm{MCG}(S) = \mathrm{Diff}^+(S, \partial S, \Sigma)/\mathrm{Diff}_0^+(S, \partial S, \Sigma)$$

where

$$\mathrm{Diff}^+(S, \partial S, \Sigma) = \left\{ f : S_0 \to S_0 \ : \begin{array}{c} f \text{ an orientation preserving diffeomorphism that} \\ \text{acts as the identity on the boundary components} \\ \text{of } S_0 \text{ and preserves the elements of } \Sigma \text{ pointwise} \end{array} \right\}$$

and

$$\mathrm{Diff}_0^+(S, \partial S, \Sigma) = \left\{ f \in \mathrm{Diff}^+(S, \partial S, \Sigma) \ : \begin{array}{c} f \text{ homotopic to the identity} \\ \text{through a homotopy preserving} \\ \text{the elements of } \Sigma \text{ pointwise} \end{array} \right\}.$$

The group operation is induced by composition of functions.

Some authors let go of the condition that $\mathrm{MCG}(S)$ fixes the punctures. The group we defined above is then often called the **pure mapping class group**.

5.3. Moduli space

Looking at Definition 4.4.1, we see there is a natural group action of the mapping class group of a surface on the corresponding Teichmüller space.

$$[g] \cdot [(R, f)] = [(R, f \circ g^{-1})].$$

The quotient is what will be called moduli space.

DEFINITION 5.3.1. Let S be a surface of finite type . The **moduli space** of S is the space

$$\mathcal{M}(S) = \mathcal{T}(S)/\,\mathrm{MCG}(S).$$

We will write

$$\mathcal{M}(\Sigma_{g,n}) = \mathcal{M}_{g,n} \quad \text{and} \quad \mathcal{M}(\Sigma_g) = \mathcal{M}_g\,.$$

Remark 5.3.2. Note that by using the convention that the mapping class group fixes boundary components and punctures, we leave these "marked", i.e. if two surfaces are isometric, but any isometry between them permutes the punctures, these surfaces represent different points in moduli space.

5.4. Elements and examples of mapping class groups

5.4.1. Basic examples. We have:

Proposition 5.4.1. *Let $n \leq 3$, then*

$$\mathrm{MCG}(\Sigma_{0,n}) = \{e\}.$$

Proof. We proved the case $n \leq 1$ during the proof of Proposition 5.1.1. Suppose $f : \Sigma_0 \to \Sigma_0$ is an orientation preserving diffeomorphism. The proof of the cases $n \in \{2, 3\}$ is part of this week's exercises. □

5.4.2. Dehn twists and the mapping class group of the annulus. Before we move on, let us describe some non-trivial elements of the mapping class group. First, consider an annulus

$$A := [0, 1] \times \mathbb{R}/\mathbb{Z}.$$

Define a map $T : A \to A$ by

$$T(t, [\theta]) = (t, [\theta + t])$$

for all $t \in [0, 1]$, $\theta \in \mathbb{R}$. This map is called a Dehn twist. Note that this map fixes ∂A pointwise. Figure 1 shows that this map does to a segment connecting the two boundary components of the annulus.

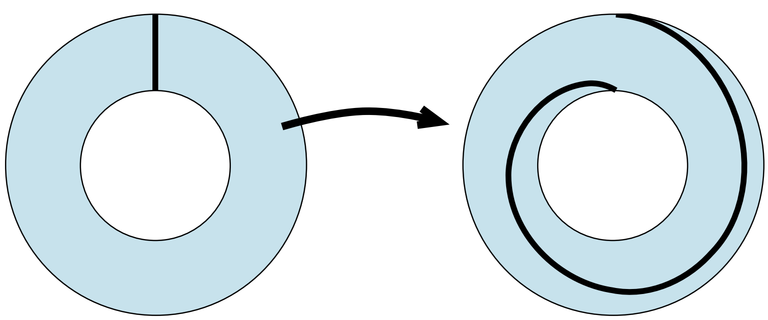

Figure 1. A Dehn twist on an annulus.

It turns out that T generates the mapping class group of the annulus:

Proposition 5.4.2. *Let $A = [0, 1] \times \mathbb{R}/\mathbb{Z}$. Then*

$$\mathrm{MCG}(A) \simeq \mathbb{Z} = \langle [T] \rangle.$$

Before we prove this, we will first show how to obtain mapping classes of more general surfaces using T.

Now let α be an essential (i.e. not homotopically trivial and not homotopic into a puncture or boundary component) simple closed curve on S. Let N be a closed regular neighborhood of α. Identifying N with A, we can define a map $T_\alpha : S \to S$ by

$$T_\alpha(p) = \begin{cases} T(p) & \text{if } p \in N \\ p & \text{if } p \in S \setminus N \end{cases}$$

Because $T|_{\partial A}$ is the identity map, this is a continuous map. To obtain an element in $\mathrm{MCG}(S)$, we need to start with a smooth map. There are multiple ways out at the moment. We could smoothen T. Or we could use surface topology to argue that T_α is homotopic to a smooth map. Since for the mapping class group, we only care about diffeomorphisms up to homotopy, the element we get in $\mathrm{MCG}(S)$ will not depend on how we do this.

Figure 2 shows an example of a Dehn twist.

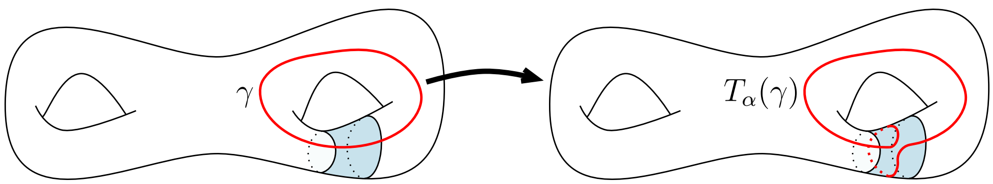

FIGURE 2. A Dehn twist on a surface of genus two.

We see that T_α maps a curve γ on the surface intersecting the defining curve α (of which we have only drawn the regular neighborhood) transversely to a curve that is not homotopic to γ. In particular, T_α is not homotopic to the identity and hence defines a non-trivial element in $\mathrm{MCG}(S)$.

PROOF OF PROPOSITION 5.4.2. We will first construct a homomorphism $\rho : \mathrm{MCG}(A) \to \mathbb{Z}$. Given an orientation preserving diffeomorphism $f : A \to A$ such that $f|_{\partial A} = \mathrm{Id}$, we can find a lift $\widetilde{f} : [0,1] \times \mathbb{R} \to [0,1] \times \mathbb{R}$ such that $\widetilde{f}(0,0) = (0,0)$. This means that

$$\widetilde{f}|_{\{0\}\times\mathbb{R}} = \mathrm{Id}.$$

Because $f|_{\partial A} = \mathrm{Id}$, the restriction $\widetilde{f}|_{\{1\}\times\mathbb{R}}$ is an integer translation. We let $\rho(f)$ be this integer.

ρ is surjective, because $\rho([T^n]) = n$. Now suppose $\rho([f]) = 0$. This means that $\widetilde{f}$ restricts to the identity on $\{0,1\} \times \mathbb{R}$. We have that

$$\widetilde{f}\Big(n \cdot (t,x)\Big) = f_*(n) \cdot \widetilde{f}(t,x), \quad n \in \mathbb{Z}, (t,x) \in [0,1] \times \mathbb{R},$$

where $f_* \in \mathrm{Aut}(\mathbb{Z}) = \{\pm \mathrm{Id}\}$. Because $\widetilde{f}|_{\{0,1\}\times\mathbb{R}} = \mathrm{Id}$, we need that $f_* = \mathrm{Id}$. Implying that

$$\widetilde{f}\Big(n \cdot (t,x)\Big) = n \cdot \widetilde{f}(t,x), \quad n \in \mathbb{Z}, (t,x) \in [0,1] \times \mathbb{R}$$

and thus that the straight line homotopy

$$F_s((t,x)) = (1-s) \cdot \widetilde{f}(x,t) + s \cdot (x,t), \quad s \in [0,1]$$

is a $\mathbb{Z}$-equivariant homotopy between $\widetilde{f}$ and the identity, that hence descends to A. This proves that ρ is injective and concludes the proof of the proposition. □

5.4.3. The mapping class group of the torus. We briefly return to the torus. The question is of course whether the general definition on the mapping class group really corresponds to what happens in the case of the torus. We recall that

$$\mathcal{M}_1 = \mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}^2.$$

This makes one wonder whether the mapping class group of the torus is maybe $\mathrm{PSL}(2,\mathbb{Z})$. This turns out to be almost correct. Indeed, we have the following theorem:

THEOREM 5.4.3. *We have*

$$\mathrm{MCG}(\mathbb{T}^2) \simeq \mathrm{SL}(2,\mathbb{Z}).$$

The action of $\mathrm{MCG}(\mathbb{T}^2)$ *on* $\mathcal{T}_1$ *is that given by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{d\tau - b}{-c\tau + a}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ *and* $\tau \in \mathcal{T}_1$.

PROOF. We will identify

$$\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$$

First observe that very element $A \in \mathrm{SL}(2,\mathbb{Z})$ induces a linear diffeomorphism $x \mapsto A \cdot x$ of $\mathbb{R}^2$. Moreover, since $\mathrm{SL}(2,\mathbb{Z})$ preserves $\mathbb{Z}^2 \subset \mathbb{R}^2$, the action on $\mathbb{R}^2$ descends to an action by diffeomorphisms

$$\mathbb{T}^2 \to \mathbb{T}^2$$

that are orientation preserving because $\det(A) > 0$.

Our goal is to show that every orientation preserving diffeomorphism $\phi : \mathbb{T}^2 \to \mathbb{T}^2$ is homotopic to such a map. To this end, we may homotope ϕ so that it fixes $[0] \in \mathbb{T}^2$ and we can take a lift $\widetilde{\phi} : \mathbb{R}^2 \to \mathbb{R}^2$ that fixes the origin of $\mathbb{R}^2$. We have

$$\widetilde{\phi}(x + (m,n)) = \widetilde{\phi}(x) + \phi_*(m,n),$$

for all $(m,n) \in \mathbb{Z}^2$ where $\phi_* : \mathbb{Z}^2 \to \mathbb{Z}^2$ is an isomorphism, i.e. an element of $\mathrm{GL}(2,\mathbb{Z})$. For a general surface S, the map $[\phi] \in \mathrm{MCG}(S) \mapsto \phi_* \in \mathrm{Aut}(\pi_1(S))$ does not yield a homomorphism: we have chosen a homotopy to make ϕ fix a base point. Changing this choice a priori changes ϕ_* by an inner automorphism of $\pi_1(S)$. So we only obtain a map to $\mathrm{Out}(\pi_1(S))$. However, because $\mathbb{Z}^2$ is abelian, we have $\mathrm{Out}(\mathbb{Z}^2) \simeq \mathrm{Aut}(\mathbb{Z}^2)$. So in the case of the torus, we obtain a homomorphism $\mathrm{MCG}(\mathbb{T}^2) \to \mathrm{GL}(2,\mathbb{Z})$.

Write A_ϕ for the $\mathrm{GL}(2,\mathbb{Z})$ matrix corresponding to ϕ. Observe that

$$F_t(x) = tA_\phi \cdot x + (1-t)\widetilde{\phi}(x), \quad t \in [0,1], x \in \mathbb{R}^2$$

gives a $\mathbb{Z}^2$-equivariant homotopy between $\widetilde{\phi}$ and the linear map $x \mapsto A_\phi \cdot x$. Since $\widetilde{\phi}$ is orientation preserving, $\det(A_\phi) > 0$, and hence $A_\phi \in \mathrm{SL}(2,\mathbb{Z})$. So we obtain a map $\mathrm{MCG}(\mathbb{T}^2) \to \mathrm{SL}(2,\mathbb{Z})$. The map is surjective, because ϕ_A maps to A. Moreover, the map is injective, because if A_ϕ is the identity matrix, F_t gives a homotopy of $\widetilde{\phi}$ to the identity.

Since the action of $[\phi] \in \mathrm{MCG}(\mathbb{T}^2)$ on $\mathcal{T}(\mathbb{T}^2)$ is by precomposition with ϕ^{-1}, the action is as described. □

REMARK 5.4.4. Note that the theorem above implies that the mapping class group action is not faithful. The kernel of the action is the center of $\mathrm{SL}(2,\mathbb{Z})$, i.e. the subgroup

$$\left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} < \mathrm{SL}(2,\mathbb{Z}).$$

On the other hand, we do have

$$\mathbb{H}^2 / \mathrm{PSL}(2,\mathbb{Z}) = \mathbb{H}^2 / \mathrm{SL}(2,\mathbb{Z}).$$

This means that the mapping class group action is indeed a generalization of the situation for the torus case.

5.4.4. Mapping class groups in higher genus. We proved in the exercises that $\mathrm{SL}(2,\mathbb{Z})$ can be generated by the matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We can also generate $\mathrm{SL}(2,\mathbb{Z})$ by the matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Indeed, a calculation shows that $S = T^{-1}RT^{-1}$.

Now identify $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ again and write α and β for the closed curves in $\mathbb{T}^2$ that are the images of the straight line segments between the origin and $(0,1)$ and $(1,0)$ respectively. Tracing the proof of Theorem 5.4.3, we see that $T = [T_\alpha]$ and $R = [T_\beta]$. That is, $\mathrm{MCG}(\mathbb{T}^2)$ can be generated by two Dehn twists.

It actually turns out that an analogous statement holds for all mapping class groups. In the following theorem, a non-separating curve will be a curve α so that $S \setminus \alpha$ is connected. Figure 3 shows an example.

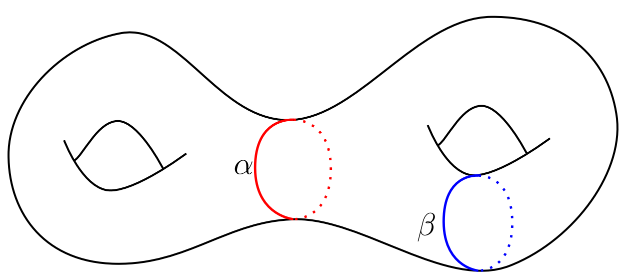

FIGURE 3. A separating curve (α) and a non-separating curve (β).

THEOREM 5.4.5 (Dehn - Lickorish theorem). *Let S be a surface of finite type, the mapping class group $\mathrm{MCG}(S)$ is generated by finitely many Dehn twists about nonseparating simple closed curves.*

LECTURE 6

Mapping class groups, quasiconformal mappings and Beltrami coefficients

6.1. More on mapping class groups

6.1.1. The action on homology. If S is a surface, then $\mathrm{MCG}(S)$ acts on its homology $H_1(S)$. Indeed every diffeomorphism $f : S \to S$ induces an automorphism $f_* : H_1(S,\mathbb{Z}) \to H_1(S,\mathbb{Z})$. In this section, we briefly descibe some aspects of this action. We will restrict to closed surfaces.

First of all, it turns out the action preserves some extra structure: the algebraic intersection number between oriented curves. In order to define it, let α and β be two oriented closed curves on an oriented surface S that intersect each other transversely at every intersection point. Then the **algebraic intersection number** between α and β is given by

$$i(\alpha,\beta) = \sum_{p\in\alpha\cap\beta} \mathrm{sgn}(\omega(v_p(\alpha), v_p(\beta))),$$

where $\mathrm{sgn} : \mathbb{R} \to \{\pm 1\}$ denotes the sign function, ω is any volume form that induces the orientation and $v_p(\alpha)$ and $v_p(\beta)$ denote the unit tangent vectors to α and β respectively at p. Note that

$$i(\beta,\alpha) = -i(\alpha,\beta).$$

Figure 1 shows an example of a positive contribution to the intersection number.

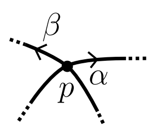

FIGURE 1. A positive contribution to $i(\alpha,\beta)$ if the orientation points out of the page.

We note that this form descends to homology. That is, it induces a form

$$i : H_1(S,\mathbb{Z}) \times H_1(S,\mathbb{Z}) \to \mathbb{Z}$$

called the **intersection form**, with the properties:

(1) i is bilinear.

(2) i is **alternating**, i.e.

$$i(a,b) = -i(b,a)$$

for all $a,b \in H_1(S,\mathbb{Z})$.

(3) i is **non-degenerate**, i.e. if $a \in H_1(S,\mathbb{Z})$ is such that
$$i(a,b) = 0 \quad \text{for all } b \in H_1(S,\mathbb{Z})$$
then $a = 0$.

(see [**FK92**, Section III.1] for more details). Such a form is called a **symplectic form**.

First of all note that the image preserves the intersection form. Moreover, isotopic maps give rise to the same automorphism. So this gives us a representation
$$\mathrm{MCG}(S) \to \mathrm{Aut}(H_1(S,\mathbb{Z}), i)$$
called the **homology representation** of the mapping class group. Recall that if S is a closed orientable surface of genus g, then $H_1(S,\mathbb{Z}) \simeq \mathbb{Z}^{2g}$. Choosing an identification, the homology representation becomes a map
$$\mathrm{MCG}(S) \to \mathrm{Sp}(2g,\mathbb{Z}) = \left\{ A \in \mathrm{Mat}_{2g}(\mathbb{Z}) \,:\, i(Av, Aw) = i(v,w),\ \forall v, w \in \mathbb{Z}^{2g} \right\}.$$
It turns out that this representation is surjective (this can be proved using a finite generating set for $\mathrm{Sp}(2g,\mathbb{Z})$ consisting of transvections, which can be realized by Dehn twists), but generally highly non-injective. A notable exception is the case of the torus, there is an isomorphism
$$\mathrm{Sp}(2,\mathbb{Z}) \simeq \mathrm{SL}(2,\mathbb{Z})$$
and indeed the the homology representation $\mathrm{MCG}(\mathbb{T}^2) \to \mathrm{Sp}(2,\mathbb{Z})$ is an isomorphism.

6.2. Finite subgroups of the mapping class group

We begin with a classcial theorem about automorphism groups of closed Riemann surfaces:

THEOREM 6.2.1 (Hurwitz). *Let X be a closed Riemann surface of genus $g \geq 2$ then*
$$|\mathrm{Aut}(X)| \leq 84(g-1).$$

The same holds for groups of orientation self isometries of a Riemann surface. In fact, using the uniformization theorem, one can prove that if X is a hyperbolic surface and its isometry group is denoted by
$$\mathrm{Isom}^+(X) = \{ \varphi : X \to X \,:\, \varphi \text{ an orientation preserving isometry} \}$$
then
$$\mathrm{Aut}(X) \simeq \mathrm{Isom}^+(X),$$
simply because every isometry is a complex automorphism and vice versa.

Because isometries/automorphisms are diffeomorphisms, we get a map $\mathrm{Aut}(X) \to \mathrm{MCG}(X)$. We now have the following proposition:

PROPOSITION 6.2.2. *Let S be a closed surface.*

(1) The map $\mathrm{Aut}(X) \to \mathrm{MCG}(X)$ is injective.

(2) A point $[X, f] \in \mathcal{T}(S)$ is fixed by $h \in \mathrm{MCG}(S)$ if and only if $f \circ h \circ f^{-1} : X \to X$ is homotopic to an automorphism.

We will postpone the proof of the first part until we have some more tools from hyperbolic geometry available. We'll prove the second part now.

PROOF OF PROPOSITION 6.2.2(2). This is a matter of unwrapping the definitions. Indeed, h fixes $[X, f]$ if and only if $(X, f) \sim (X, f \circ h^{-1})$ which happens if and only if there is a biholomorphism $\varphi : X \to X$ such that $h \circ f^{-1} \circ \varphi \circ f : S \to S$ is homotopic to the identity $S \to S$ which is tur if and only if $f \circ h \circ f^{-1} X \to X$ is homotopic to φ. □

6.3. Beltrami coefficients

In order to topologize Teichmüller spaces and hence also moduli spaces, we need a notion of "closeness" of different Riemann surface structures on the same surface. We will use **quasiconformal mappings** to do this. We will start by developing this idea for orientation preserving smooth maps $f : D \to \mathbb{C}$, where $D \subset \mathbb{C}$ is some domain.

We have observed in Sections 3.2.1 and 3.2.2 that f is holomorphic at $z_0 \in D$ if and only if

$$\frac{\partial f}{\partial \overline{z}}(z_0) = 0$$

and that moreover

$$\mu_f = \frac{\partial f / \partial \overline{z}}{\partial f / \partial z}$$

shows up when trying to find isothermal coordinates. Indeed, it appears as an error term that measures how far a set of coordinates are from being isothermal. We will call $\mu_f : D \to \mathbb{C}$ the **Beltrami coefficient** of f.

Our next goal is to explain what μ_f describes geometrically. Before we get to this, we prove the following lemma:

LEMMA 6.3.1. *Suppose that $f : D \to \mathbb{C}$ is an orientation preserving smooth map. Then*

$$\frac{\partial f(z)}{\partial z} \neq 0 \quad \textit{and} \quad |\mu_f(z)| < 1 \quad \textit{for all } z \in D.$$

PROOF. We write $f(x + iy) = u(x, y) + iv(x, y)$ and we'll write Df for the Jacobian matrix of f. We then have, for all $z = x + iy \in D$:

$$0 < \det(Df(x,y)) = \det \begin{pmatrix} \partial u(x,y)/\partial x & \partial u(x,y)/\partial y \\ \partial v(x,y)/\partial x & \partial v(x,y)/\partial y \end{pmatrix}$$
$$= \frac{\partial u(x,y)}{\partial x}\frac{\partial v(x,y)}{\partial y} - \frac{\partial v(x,y)}{\partial x}\frac{\partial u(x,y)}{\partial y} = \left|\frac{\partial f}{\partial z}\right|^2 - \left|\frac{\partial f}{\partial \overline{z}}\right|^2,$$

which proves the lemma. □

This lemma in particular shows that μ_f is well defined everywhere, assuming that f is orientation preserving on D (or more generally that $\det(Df)$ has no zeroes).

6.3.1. Quasiconformal dilatation. Now we are ready to discuss the geometric interpretation of μ_f. Indeed, consider the derivative $Df : T_zD \to T_{f(z)}\mathbb{C}$ of f. We'll identify T_zD and $T_{f(z)}\mathbb{C}$ with $\mathbb{C}$ and use complex coordinates, which means that, above 0,

$$Df(0) \cdot z = \frac{\partial f(0)}{\partial z} \cdot z + \frac{\partial f(0)}{\partial \overline{z}} \cdot \overline{z}, \quad z \in T_0 D$$

And thus that, if we write $z = r \cdot e^{i\theta}$

$$|Df(0) \cdot z| = \left|\frac{\partial f(0)}{\partial z}\right| \cdot |r| \cdot \left|e^{i\theta} + \mu_f(0) \cdot e^{i\theta}\right| = \left|\frac{\partial f(0)}{\partial z}\right| \cdot |r| \cdot \left|1 + \mu_f(0) \cdot e^{-2i\theta}\right|.$$

In particular

$$\frac{\max\{\,|Df(0)\cdot z| \,:\, |z| = 1\,\}}{\min\{\,|Df(0)\cdot z| \,:\, |z| = 1\,\}} = \frac{1 + |\mu_f(0)|}{1 - |\mu_f(0)|}.$$

So $Df(z)$ sends the unit circle in T_zD to an ellipse in $T_{f(z)}\mathbb{C}$ and the ratio of the axes of this ellipse is given by

$$K_f(z) = \frac{1 + |\mu_f(z)|}{1 - |\mu_f(z)|}.$$

We call $K_f(z)$ the **quasiconformal dilatation** of f at z. We will set

$$K_f = \sup\{\, K_f(z) \,:\, z \in D\,\}.$$

A map $f : D \to \mathbb{C}$ is called a **quasiconformal mapping** if this number is finite.

6.3.2. Examples. It's high time for some examples:

Example 6.3.2. (1) First of all, we observe that a holomorphic (or conformal) map $f : D \to \mathbb{C}$ is quasiconformal with $\mu_f \equiv 0$ and $K_f = 1$.

(2) The affine map $f : \mathbb{C} \to \mathbb{C}$ given by

$$f(z) = az + b\overline{z} + c, \quad z \in \mathbb{C}$$

with $a, b, c \in \mathbb{C}$ and $|a| > |b|$ is quasiconformal with

$$K_f = \frac{|a| + |b|}{|a| - |b|}.$$

(3) The map $f : \{\, z \in \mathbb{C} \,:\, |z| < 1\,\} \to \mathbb{C}$ given by

$$f(z) = \frac{z}{1 - |z|^2}$$

satisfies

$$\frac{\partial f}{\partial z} = \frac{1}{(1 - |z|^2)^2} \quad \text{and} \quad \frac{\partial f}{\partial \overline{z}} = \frac{z^2}{(1 - |z|^2)^2}$$

which means that

$$\mu_f(z) = z^2$$

and in particular that $K_f = \infty$. In fact, it turns out that there are no quasiconformal mappings between the unit disk and $\mathbb{C}$.

LECTURE 7

Lower regularity

Next up, we need to generalize the notion of quasiconformal mappings to maps of lower regularity. This will make for a more flexible notion. In particular, we don't want to assume differentiability.

First we recall the definition of absolute continuity:

DEFINITION 7.0.1. A function $f : [a, b] \to \mathbb{R}$ is **absolutely continuous** on $[a, b]$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all finite sequences $a \leq x_1 < y_1 < \ldots x_N < y_N \leq b$ with

$$\sum_{k=1}^{N} (y_k - x_k) < \delta,$$

we have

$$\sum_{k=1}^{N} |f(y_k) - f(x_k)| < \varepsilon.$$

We have the following classical proposition:

PROPOSITION 7.0.2. *The following are equivalent:*

- $f : [a, b] \to \mathbb{R}$ *is absolutely continuous on* $[a, b]$
- f *is differentiable, with derivative* $f' : [a, b.] \to \mathbb{R}$*, almost everywhere on* $[a, b]$ *and*

$$f(x) = f(a) + \int_a^x f'(t)\, dt$$

  for all $x \in [a, b]$.
- *There exists a Lebesgue integrable function* $g : [a, b] \to \mathbb{R}$ *such that*

$$f(x) = f(a) + \int_a^x g(t)\, dt$$

  for all $x \in [a, b]$.

For functions on a domain in $\mathbb{C}$, we define:

DEFINITION 7.0.3. Let $D \subset \mathbb{C}$ be a domain, then $f : D \to \mathbb{C}$ is called **absolutely continuous on lines (ACL)** if for every rectangle $R = [a, b] \times [c, d] \subset D$:

- the function $x \in [a, b] \mapsto f(x + iy)$ is absolutely continuous for almost every $y \in [c, d]$, and

- the function $y \in [c,d] \mapsto f(x+iy)$ is absolutely continuous for almost every $x \in [a,b]$.

We note that the proposition above implies that if $f : D \to \mathbb{C}$ is ACL, then
$$\frac{\partial f}{\partial z}(z_0) \quad \text{and} \quad \frac{\partial f}{\partial \overline{z}}(z_0)$$
are well defined almost everywhere on D and moreover define measurable functions.

This notion now allows us to define a more general class of quasiconformal mappings:

Definition 7.0.4. Let $D \subset \mathbb{C}$ be a domain and let $f : D \to \mathbb{C}$ be an orientation preserving homeomorphism onto its image. We say that f is **quasiconformal on** D if

- f is ACL on D, and
- there exists a constant $k \in [0,1)$ such that
$$\left|\frac{\partial f}{\partial \overline{z}}\right| \leq k \left|\frac{\partial f}{\partial z}\right| \quad \text{almost everywhere on } D.$$
We also say that f is $\frac{1+k}{1-k}$-quasiconformal in this case.

If f is quasiconformal on D, we will write
$$K_f = \inf\left\{ \frac{1+k}{1-k} \;:\; k \in [0,1), \quad \left|\frac{\partial f}{\partial \overline{z}}\right| \leq k \left|\frac{\partial f}{\partial z}\right| \text{ almost everywhere on } D \right\}$$
for the **quasiconformal dilatation** of f.

We observe that, by defintion, the Beltrami coefficient of an orientation preserving homeomorphism onto its image $f : D \to \mathbb{C}$ is an element of
$$B(D)_1 = \{\, \mu : D \to \mathbb{C} \text{ a bounded measurable function } : \ ||\mu||_\infty < 1 \,\},$$
where we recall that
$$||\mu||_\infty = \text{ess.sup}_{z \in D}\{\mu(z)\}.$$

We now first note that post-composing a quasi-conformal mapping with a conformal mapping preserves quasiconformality:

Lemma 7.0.5. *Let $D, D' \subset \mathbb{C}$ be domains, $f : D \to D'$ a quasiconformal mapping and $g : D' \to D'$ a conformal mapping. Then $g \circ f : D \to D'$ is quasiconformal and*
$$K_{g\circ f} = K_f.$$

Proof. We recall from Section 3.2.1 that
$$\frac{\partial (g\circ f)}{\partial \overline{z}}(z_0) = \frac{\partial g}{\partial \overline{z}}(f(z_0)) \cdot \frac{\partial \overline{f}}{\partial \overline{z}}(z_0) + \frac{\partial g}{\partial z}(f(z_0)) \cdot \frac{\partial f}{\partial \overline{z}}(z_0) = \frac{\partial g}{\partial z}(f(z_0)) \cdot \frac{\partial f}{\partial \overline{z}}(z_0)$$
and likewise
$$\frac{\partial (g\circ f)}{\partial z}(z_0) = \frac{\partial g}{\partial \overline{z}}(f(z_0)) \cdot \frac{\partial \overline{f}}{\partial z}(z_0) + \frac{\partial g}{\partial z}(f(z_0)) \cdot \frac{\partial f}{\partial z}(z_0) = \frac{\partial g}{\partial z}(f(z_0)) \cdot \frac{\partial f}{\partial z}(z_0),$$

wherever these derivatives exist. So indeed

$$\left|\frac{\partial(g\circ f)}{\partial\overline{z}}(z_0)\right| \leq k\cdot\left|\frac{\partial(g\circ f)}{\partial z}(z_0)\right| \quad \text{if and only if} \quad \left|\frac{\partial f}{\partial\overline{z}}(z_0)\right| \leq k\cdot\left|\frac{\partial f}{\partial z}(z_0)\right|$$

at z_0. □

7.1. Equivalent definitions

Our next goal is to describe two equivalent definitions of quasiconformal mappings. We won't prove the equivalence in this course and refer to [**IT92**, Chapter 4] for these proofs.

We first recall the notion of distributional derivatives.

DEFINITION 7.1.1. We say that f_z and $f_{\overline{z}}$ are the **distributional derivatives** of $f : D \to \mathbb{C}$ with respect to z and $\overline{z}$ respectively if

$$\int_D f_z \cdot \varphi \, dx\, dy = -\int_D f \cdot \frac{\partial\varphi}{\partial z}\, dx\, dy$$

and

$$\int_D f_{\overline{z}} \cdot \varphi \, dx\, dy = -\int_D f \cdot \frac{\partial\varphi}{\partial \overline{z}}\, dx\, dy$$

respectively for all $\varphi \in C_0^\infty(D)$.

It turns out that if $f : D \to \mathbb{C}$ is quasiconformal, then its distributional derivatives coincide with the derivatives of f that exist almost everywhere.

We now arrive at the first equivalent definition of quasiconformal mappings:

THEOREM 7.1.2. *Suppose that $f : D \to \mathbb{C}$ is an orientation preserving homeomorphism onto its image. Then f is $\frac{1+k}{1-k}$-quasiconformal if and only if*

(1) The distributional derivatives of f with respect to z and $\overline{z}$ can be represented by functions f_z and $f_{\overline{z}}$ that are locally L^2, and

(2) we have that

$$|f_{\overline{z}}| \leq k \cdot |f_z| \ \text{ almost everywhere on } D$$

The next equivalent definition we will discuss is a more geometric definition. To this end we define a notion of quadrilaterals (see Figure 1 for a picture) :

DEFINITION 7.1.3. A **quadrilateral** is a pair $(Q; q_1, q_2, q_3, q_4)$, where $Q \subset \mathbb{C}$ is a Jordan closed domain (i.e. $Q \subset \mathbb{C}$ is a closed set that can be obtained as the closure of a domain and the boundary of Q is a Jordan curve) and $q_1, q_2, q_3, q_4 \in \partial Q$ are distinct and appear in this cyclic order, which is consistent with the orientation on $\mathbb{C}$.

We have the following fact about quadrilaterals:

PROPOSITION 7.1.4. *Given a quadrilateral $\mathbf{Q} = (Q; q_1, q_2, q_3, q_4)$, there exist $a, b > 0$ and a homeomorphism*

$$h : Q \to R = [0, a] \times [0, b]$$

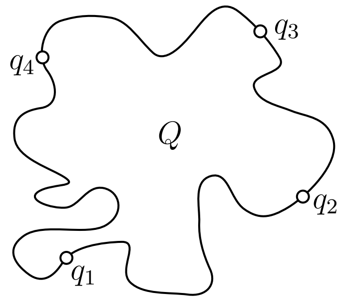

FIGURE 1. A quadrilateral.

that is conformal in the interior $\mathring{Q}$ of Q and such that

$$h(q_1) = 0, \quad h(q_2) = a, \quad h(q_3) = a + ib \quad \text{and} \quad h(q_4) = ib.$$

*Moreover, the number a/b is independent of h and is called the **module** $M(\mathbf{Q})$ of the quadrilateral.*

Before we prove it, we recall the following classical theorem from complex analysis:

THEOREM 7.1.5 (Carathéodory's theorem). *Let $D \subset \mathbb{C}$ be a domain and $f : D \to \mathbb{H}^2$ a conformal map, then f admits a continous one-to-one extension to $\overline{D}$ if and only if ∂D is a Jordan curve.*

PROOF OF PROPOSITION 7.1.4. Combining the Riemann mapping theorem with Carathéodory's theorem, we obtain a homeomorphism $h_1 : Q \to \overline{\mathbb{H}^2} = \mathbb{H}^2 \cup \mathbb{R} \cup \{\infty\}$ that is conformal on the interior $\mathring{Q}$ of Q.

We may assume, by post-composing with a Möbius transformation, that

$$h_1(q_1) = -1, \quad h_1(q_2) = 1 \quad \text{and} \quad h_1(q_3) = -h(q_4) > 1.$$

Indeed, we can map the unique orthogonal to two hyperbolic geodesics between $h_1(q_1)$ and $h_1(q_2)$ and between $h_1(q_3)$ and $h_1(q_4)$ respectively to the imaginary axis with a Möbius transformation. This achieves $h_1(q_1) = -h(q_2)$ and $h_1(q_3) = -h(q_4)$. After this, we scale by a Möbius transformation preserving the imaginary axis so that $h_1(q_1) = -1$.

Now we set $k = 1/h_1(q_3)$ and define $h_2 : \mathbb{H}^2 \to \mathbb{C}$ by

$$h_2(z) = \int_0^z \frac{1}{\sqrt{(1-w^2)(1-k^2w^2)}} dw.$$

Because $\overline{\mathbb{H}^2}$ is simply connected, this integral is well defined and does not on the path chosen from 0 to z. Moreover, because k is real, the integrand $w \mapsto \frac{1}{\sqrt{(1-w^2)(1-k^2w^2)}}$ is a holomorphic function on $\mathbb{H}^2$ that doesn't vanish (for any choice of branch of the square root). This means that h_2 is holomorphic with a non-vanishing derivative and hence conformal on $\mathbb{H}^2$.

The function h_2 is also real differentiable on $\partial\mathbb{H}^2$, except at the points -1, 1, $-1/k$ and $1/k$. At these points, the expression $(1-w^2)(1-k^2w^2)$ changes sign and hence its square root changes from real to purely imaginary, or vice versa. This means that $h_2(\mathbb{H}^2)$ is a rectangle. Moreover $h_2(0) = 0$ which implies it's of the form $[-K, K] \times [0, K']$.

Combining all of the above, we get that

$$z \mapsto h_2 \circ h_1(z) + K$$

is a map that does what we want.

Now we still need to prove that the ratio of the side lengths of the image rectangle does not depend on the choices we have made. Suppose $\widetilde{h} : Q \to [0, a'] \times [0, b']$ is another map that satisfies our requirements. Using the Schwartz reflection principle, we can extend $\widetilde{h} \circ h^{-1} : [0, a] \times [0, b] \to [0, a'] \times [0, b']$ to an automorphism of $\mathbb{C}$. As such

$$\widetilde{h} \circ h^{-1}(z) = \alpha z + \beta$$

for some $\alpha \neq 0$ and some β. Because this map send $[0, a] \times [0, b]$ to $[0, a'] \times [0, b']$ and preserves the vertices pointwise, we obtain that $\beta = 0$ and

$$\alpha = a'/a = b'/b,$$

which proves our second claim. □

The notion of module allows us to give another definition of quasiconformal mappings:

Theorem 7.1.6. *Let $D \subset \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ an orientation preserving homeomorphism. Then f is K-quasiconformal if and only if*

$$M(f(\mathbf{Q})) \leq K \cdot M(\mathbf{Q})$$

for all quadrilaterals $\mathbf{Q}$ in D.

7.2. Further properties

We record some further properties of quasiconformal mappings now.

Theorem 7.2.1. *(1) The inverse of a K-quasiconformal maping is also K-quasiconformal*

(2) If $f : D_1 \to D_2$ is K-quasiconformal and $h_1 : D_1 \to D_1$ and $h_2 : D_2 \to D_2$ are conformal, then $h_2 \circ f \circ : D_1 \to D_2$ is K-quasiconformal.

(3) If $f : D_1 \to D_2$ is K_1-quasiconformal and $g : D_2 \to D_3$ is K_2-quasiconformal, then $g \circ f : D_1 \to D_3$ is $K_1 \cdot K_2$ quasiconformal.

Proof. The proof of this theorem is an exercise. □

We also record the analogue of Lemma 6.3.1 in this setting:

Proposition 7.2.2. *If $f : D \to \mathbb{C}$ is quasiconformal, then $\frac{\partial f}{\partial z} \neq 0$ almost everywhere on D.*

For a proof, see [**IT92**, Proposition 4.11].

Finally, we note:

Proposition 7.2.3. *Let $\mu \in B(D)_1$ and suppose that there exists a quasiconformal mapping $f : D \to \mathbb{C}$ with $\mu_f = \mu$. Then for every conformal mapping $h : f(D) \to f(D)$, we have*

$$\mu_{h \circ f} = \mu_f.$$

Conversely, if $g : D \to \mathbb{C}$ is such that $\mu_g = \mu$ as well, then $g \circ f^{-1}$ is a conformal mapping of $f(D)$.

7.3. Existence

Next, we mention an important existence theorem for solutions to the Beltrami equation, that we already alluded to in Section 3.2.

The measurable Riemann mapping theorem now states:

Theorem 7.3.1 (Measurable Riemann mapping theorem). *For every Beltrami coefficient $\mu \in B(\mathbb{C})_1$, there exists a quasiconformal mapping $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\mu_f = \mu$. Moreover, there is a unique such f with*

$$f(0) = 0, \quad f(1) = 1 \quad \text{and} \quad f(\infty) = \infty.$$

We will denote the latter map by f^μ.

Also the proof of this theorem goes beyond what will fit in a six week course. We refer to [**IT92**, Theorem 4.30] instead.

We note the following consequence:

Proposition 7.3.2. *Let $\mu \in B(\mathbb{H}^2)_1$, then there exists a quasiconformal mapping $f : \mathbb{H}^2 \to \mathbb{H}^2$ with $\mu_f = \mu$. This mapping extends to a continuous map $f : \overline{\mathbb{H}^2} \to \overline{\mathbb{H}^2}$. Moreover, there is a unique such map such that*

$$f(0) = 0, \quad f(1) = 1, \quad \text{and} \quad f(\infty) = \infty.$$

Proof. Let's first prove that all quasiconformal maps extend to $\overline{\mathbb{H}^2}$. So, let $f : \mathbb{H}^2 \to \mathbb{H}^2$ be any quasiconformal map. We can extend μ_f to all of $\mathbb{C}$ by setting it equal to 0 on $\mathbb{C} - \mathbb{H}^2$. Theorem 7.3.1 then gives us a quasiconformal mapping $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with $\mu_g = \mu_f$. The composition $f \circ g^{-1}$ is then a quasiconformal mapping of $\mathbb{H}^2$, by Proposition 7.2.3. Carathéodory's theorem (Theorem 7.1.5) now tells us that $f \circ g^{-1} : \mathbb{H}^2 \to \mathbb{H}^2$ extends to a homeommorphism $\overline{\mathbb{H}^2} \to \overline{\mathbb{H}^2}$ and hence, so does $f = f \circ g^{-1} \circ g$.

This now also means that, by Proposition 7.2.3, if we can find a map, it's unique. So now we just need to construct it.

Extend μ to $\mathbb{C}$ by:

$$\widehat{\mu}(z) = \begin{cases} \mu(z) & \text{if } z \in \mathbb{H}^2 \\ 0 & \text{if } z \in \mathbb{R} \\ \overline{\mu(\overline{z})} & \text{if } \overline{z} \in \mathbb{H}^2 \end{cases}$$

By Theorem 7.3.1, there exists a unique quasiconformal mapping $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ that fixes 0, 1 and ∞ and such that $\mu_f = \widehat{\mu}$. By construction, the Beltrami differential associated to

$z \mapsto \overline{f(\overline{z})}$ equals $\widehat{\mu}$ as well, so by uniqueness

$$f(z) = \overline{f(\overline{z})}$$

This means that $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ preserves $\mathbb{R} \cup \{\infty\}$. f is also orientation preserving, so it must preserve $\mathbb{H}^2$. $\square$

LECTURE 8

Beltrami coefficients on a Riemann surface and hyperbolic geometry

8.1. Beltrami coefficients on Riemann surfaces

Now we're going back to Riemann surfaces. For the moment, suppose that $f : S \to R$ is a smooth orientation preserving homeomorphism between Riemann surfaces. Locally on S, we can define μ_f. Our first goal is to understand what kind of an object this yields on the Riemann surface S.

LEMMA 8.1.1. *Let S and R be Riemann surfaces and $f : S \to R$ a smooth map. Suppose that (U, z) is a holomorphic local coordinate on S and (V, w) one on R. Then the smooth function $\mu : z(U) \to \mathbb{C}$ defined by*

$$\mu = \left(\frac{\partial F}{\partial \overline{z}}\right) \Big/ \left(\frac{\partial F}{\partial z}\right),$$

where $F = w \circ f \circ z^{-1} : z(U) \to \mathbb{C}$ is independent of the choice of coordinate (V, w).

PROOF. Suppose (V', w') is a different holomorphic local coordinate with $f(U) \subset V'$. Write $F' = w' \circ f \circ z^{-1} : z(U) \to \mathbb{C}$. By the chain rule,

$$\frac{\partial F'}{\partial \overline{z}} = \frac{\partial (w' \circ w^{-1} \circ F)}{\partial \overline{z}} = \left(\frac{\partial (w' \circ w^{-1})}{\partial z} \circ F\right) \cdot \frac{\partial F}{\partial \overline{z}},$$

where the second term disappears because $w' \circ w^{-1}$ is holomorphic. Likewise,

$$\frac{\partial F'}{\partial z} = \frac{\partial (w' \circ w^{-1} \circ F)}{\partial z} = \left(\frac{\partial (w' \circ w^{-1})}{\partial z} \circ F\right) \cdot \frac{\partial F}{\partial z}.$$

So when we divide the two, we obtain the same μ. □

Observe that μ *does* depend on the local coordinate (U, z). Indeed, if (U', z') is a different holomorphic local coordinate and we write $F' = w \circ F \circ (z')^{-1}$, then

$$\begin{aligned}\frac{\partial F'}{\partial \overline{z'}} = \frac{\partial (F \circ z \circ (z')^{-1})}{\partial \overline{z'}} &= \left(\frac{\partial F}{\partial \overline{z}} \circ z \circ (z')^{-1}\right) \cdot \frac{\partial \overline{(z \circ (z')^{-1})}}{\partial \overline{z'}} \\ &= \left(\frac{\partial F}{\partial \overline{z}} \circ z \circ (z')^{-1}\right) \cdot \overline{\left(\frac{\partial (z \circ (z')^{-1})}{\partial z'}\right)},\end{aligned}$$

again using the fact that the coordinate change is holomorphic to conclude that the other term disappears, and

$$\frac{\partial F'}{\partial z'} = \frac{\partial (F \circ z \circ (z')^{-1})}{\partial z'} = \left(\frac{\partial F}{\partial z} \circ z \circ (z')^{-1}\right) \cdot \frac{\partial (z \circ (z')^{-1})}{\partial z'},$$

where we have used that

$$\frac{\partial\overline{(z\circ(z')^{-1})}}{\partial z'} = \overline{\left(\frac{\partial(z\circ(z')^{-1})}{\partial\overline{z'}}\right)} = 0.$$

In conclusion

$$\mu(z') = \mu(z) \cdot \overline{\left(\frac{\partial(z\circ(z')^{-1})}{\partial z'}\right)} \Big/ \frac{\partial(z\circ(z')^{-1})}{\partial z'}$$

and thus a smooth map $f : S \to R$ yields a differential μ_f of type $(-1,1)$ associated to f. That is, it makes sense to write

$$\mu_f = \mu(z)\cdot\frac{d\overline{z}}{dz}.$$

So, we have a well defined differential μ_f associated to f that satisfies

$$\mu_f = 0 \quad\Leftrightarrow\quad f \text{ is biholomorphic.}$$

Indeed, if $\mu_f = 0$ then $\partial f/\partial\overline{z} = 0$ everywhere, so f is holomorphic. Since f is also invertible (and the inverse of a bijective holomorphic function is holomorphic), f is biholomorphic. We will call μ_f the **Beltrami coefficient** of f.

We also observe that the coordinate transition above implies that $z \mapsto |\mu_f(z)|$ is a well defined function on S. Moreover, it is uniformly bounded by 1 for all orientation preserving diffeomorphisms, by Lemma 6.3.1.

A differential μ of type $(-1,1)$ on a Riemann surface X whose L^∞-norm satisfies

$$||\mu||_\infty = \sup_{z\in X}\{|\mu(z)|\} < 1$$

is called a *Beltrami differential*.

The following transformation formula for Beltrami coefficients will also be useful to us:

LEMMA 8.1.2. *Let S, X_1 and X_2 be Riemann surfaces and let*

$$S \xrightarrow{f} X_1 \xrightarrow{g} X_2$$

be orientation preserving diffeomorphisms. Then

$$\mu_g\circ f = \left(\frac{\partial f}{\partial z} \Big/ \overline{\left(\frac{\partial f}{\partial z}\right)}\right)\cdot\frac{\mu_{g\circ f} - \mu_f}{1 - \overline{\mu}_f\cdot\mu_{g\circ f}}.$$

PROOF. This is part of this week's exercises. □

Indeed, we can derive from it that Beltrami coefficients can recognize biholomorphisms:

LEMMA 8.1.3. *Let X_1 and X_2 be Riemann surfaces and let*

$$f_1 : R \to X_1 \quad \textit{and} \quad f_2 : R \to X_2$$

be orientation preserving diffeomorphisms. Then the map $f_2\circ f_1^{-1} : X_1 \to X_2$ is biholomorphic if and only if

$$\mu_{f_1} = \mu_{f_2}.$$

PROOF. $f_2 \circ f_1^{-1}$ is biholomorphic if and only if $\mu_{f_2 \circ f_1^{-1}} = 0$ as a Beltrami differential on X_1. This is true if and only if, as a Beltrami differential on S,

$$0 = \mu_{f_2 \circ f_1^{-1}} \circ f_1 = \frac{\overline{\partial f_1/\partial z}}{\partial f_1/\partial z} \cdot \frac{\mu_{f_2} - \mu_{f_1}}{1 - \overline{\mu_{f_1}} \cdot \mu_{f_2}},$$

where we have used the previous lemma. Because neither the first factor on the right, nor the denominator (because $|\mu_f| < 1$ for an orientation preserving diffeomorphism) can be 0, we obtain that the equation $\mu_{f_2} = \mu_{f_1}$ is equivalent to $\mu_{f_2 \circ f_1^{-1}} \circ f_1$ being 0. □

8.2. Topologizing Teichmüller space

We are going the use Beltrami coefficients to topologize Teichmüller space. First of all, the following theorem implies that Riemann surface structures up to homotopy can be recognized using Beltrami differentials:

THEOREM 8.2.1. *Let S, X_1 and X_2 be Riemann surfaces and*

$$f_1 : S \to X_1 \quad \textit{and} \quad f_2 : S \to X_2$$

be orientation preserving diffeomorphisms. Then there exists a biholomorphic mapping

$$h : X_1 \to X_2$$

if and only if

$$\mu_{f_1} = \mu_{f_2 \circ \varphi^{-1}}$$

for some $\varphi \in \mathrm{Diff}^+(S)$. Moreover, the map

$$(f_2)^{-1} \circ h \circ f_1 : S \to S$$

is homotopic to the identity if and only if $\varphi \in \mathrm{Diff}^+_0(S, \Sigma)$.

PROOF. First suppose that there exists a biholormorphic map $h : X_1 \to X_2$. Then we set

$$\varphi = (f_2)^{-1} \circ h \circ f_1 : S \to S.$$

Then

$$\mu_{f_2} = \mu_{h \circ f_1 \circ \varphi^{-1}} = \mu_{f_1 \circ \varphi^{-1}},$$

where we have used Lemma 8.1.2 for the second equality. Since $\varphi = (f_2)^{-1} \circ h \circ f_1$, the second claim is immediate.

Conversely, suppose there exists some $\varphi \in \mathrm{Diff}^+(S)$ such that

$$\mu_{f_1} = \mu_{f_2 \circ \varphi^{-1}}$$

Lemma 8.1.2 then shows that $h = f_2 \circ \varphi \circ f_1^{-1} : X_1 \to X_2$ is biholomorphic. Again, the second claim follows from the form of φ. Indeed $f_2^{-1} \circ h \circ f_1 = \varphi$. □

So, given a Riemann surface S, we can define the space

$$B(S)_1 := \left\{ \mu_f \ : \ \begin{array}{c} f : S \to R \text{ an orientation} \\ \text{preserving diffeomorphism,} \\ R \text{ a Riemann surface} \end{array} \right\}$$

That we equip with the L^∞ topology. By the measurable Riemann mapping theorem (Theorem [?]), we can also think of this as a space of bounded differentials of type $(-1,1)$ whose essential supremum is at most 1. This space admits an action of the group of orientation preserving diffeomorphisms $\mathrm{Diff}^+(S)$ by

$$\varphi \cdot \mu_f = \mu_{f\circ\varphi^{-1}}, \quad \varphi \in \mathrm{Diff}^+(S), \mu_f \in B(S)_1.$$

A direct consequence of the above is the following:

COROLLARY 8.2.2. *The map from the set of marked Riemann surfaces defined by*

$$(R,f) \mapsto \mu_f$$

induces bijections

$$\mathcal{T}(S) \to B(S)_1/\mathrm{Diff}_0^+(S,\Sigma)$$

and

$$\mathcal{M}(S) \to B(S)_1/\mathrm{Diff}^+(S,\Sigma).$$

In particular, since $B(S)_1$ is a topological space, these bijections equip $\mathcal{T}(S)$ and $\mathcal{M}(S)$ with a topology. It is not so hard to see that the choice of Riemann surface structure on S does not influence the topology on Teichmüller space. Indeed, if S and S' are two Riemann surfaces and $g : S' \to S$ is any orientation preserving diffeomorphism between them, then

$$[X,f] \in \mathcal{T}(S) \mapsto [X, f\circ g] \in \mathcal{T}(S')$$

is a homeomorphism.

8.3. The Teichmüller metric

Looking at Theorem 7.2.1, it seems likely that the following is a good candidate for a metric on Teichmüller space:

DEFINITION 8.3.1. Let S be a Riemann surface. Then the *Teichmüller distance* between $[X_1, f_1], [X_2, f_2] \in \mathcal{T}(S)$ is

$$\mathrm{d_T}\Big([X_1,f_1],[X_2,f_2]\Big) = \frac{1}{2}\log\left(\inf\left\{ K_g \;:\; \begin{array}{c} g : X_1 \to X_2 \text{ an orientation} \\ \text{preserving diffeomorphism} \\ \text{homotopic to } f_2\circ f_1^{-1} \end{array} \right\}\right).$$

The factor $\frac{1}{2}$ is a convention. The first thing to observe is that the Teichmüller distance is indeed a metric:

LEMMA 8.3.2. *Let S be a Riemann surface. Then the Teichmüller distance $\mathrm{d_T} : \mathcal{T}(S) \times \mathcal{T}(S) \to [0,\infty)$ defines a metric.*

PROOF SKETCH. The fact that $\mathrm{d_T}$ is symmetric and satisfies the triangle inequality are direct from Theorem 7.2.1. In order to show non-degeneracy, suppose

$$\mathrm{d_T}\Big([X_1,f_1],[X_2,f_2]\Big) = 0.$$

There are two ways to show that this implies that $[X_1,f_1] = [X_2,f_2]$:

- We can use Teichmüller's theorem (that we will not prove later on in the course), which states that there is a quasiconformal map that realizes $\mathrm{d_T}$. This map must have $K_g = 1$ and hence is a biholomorphism.
- We can run an approximation argument. Suppose $g_n : X_1 \to X_2$ is a sequence of maps in the homotopy class of $f_2 \circ f_1^{-1}$ such that
$$\frac{1}{2}\log(K_{g_n}) \overset{n\to\infty}{\longrightarrow} \mathrm{d_T}\Big([X_1, f_1], [X_2, f_2]\Big) = 0.$$
This means that
$$K_{g_n} \overset{n\to\infty}{\longrightarrow} 1,$$
which in turn implies that g_n locally uniformly converges to a holomorphic map $X_1 \to X_2$ (see [IT92, Proposition 4.36] for details).

□

We could have used $\mathrm{d_T}$ to topologize Teichmüller space as well:

LEMMA 8.3.3. *The Teichmüller metric is compatible with the topology on $\mathcal{T}(S)$.*

PROOF. This is a matter of tracing the definitions. Two points in $[X_1, f_1], [X_2, f_2] \in \mathcal{T}(S)$ are close if we can make $\mu_{f_2 \circ f_1^{-1}}$ close to 0 in the L^∞ topology by precomposing f_1 with a homotopically trivial self-diffeomorphism $X_1 \to X_1$. This is the same as saying that K_g is small for g in the homotopy class of $f_2 \circ f_1^{-1}$. □

8.4. Hyperbolic surfaces

Hyperbolic geometry is a powerful tool in the study of Teichmüller and moduli spaces of surfaces of higher genus. Indeed, we will use it to prove that Teichmüller space is a ball. Before we get to that, we start by recalling some of the basics of the hyperbolic geometry of surfaces. We have already seen some of what follows in the first problem set, so we refer to this problem set for some of the proofs. Our next main goal will be to show how to use pants decompositions to build hyperbolic surfaces and to use those in turn to prove that Teichmüller space is a homeomorphic to a ball.

8.4.1. The hyperbolic plane. Hyperbolic geometry originally developed in the early 19^{th} century to prove that the parallel postulate in Euclidean geometry is independent of the other postulates. From this perspective, the hyperbolic plane can be seen as a geometric object satisfying a collection of axioms very similar to Euclid's axioms for Euclidean geometry, but with the parallel postulate replaced by something else. From a more modern perspective, hyperbolic geometry is the study of manifolds that admit a Riemannian metric of constant curvature -1.

8.4.2. The upper half plane model. From the classical point of view, any concrete description of the hyperbolic plane is a *model* for two-dimensional hyperbolic geometry, in the same way that $\mathbb{R}^2$ is a model for Euclidean geometry.

As we've already mentioned in Lecture 1, the hyperbolic plane can be defined as follows.

DEFINITION 8.4.1. The hyperbolic plane $\mathbb{H}^2$ is the complex domain

$$\mathbb{H}^2 = \{ z \in \mathbb{C} \, : \, \mathrm{Im}(z) > 0 \}$$

equipped with the Riemannian metric given by

$$ds^2 = \frac{1}{y^2}\left(dx^2 + dy^2\right)$$

at $x + iy \in \mathbb{H}^2$

The group of orientation preserving isometries of $(\mathbb{H}^2, ds^2)$ coincides with the group of complex automorphisms of $\mathbb{H}^2$. That is,

$$\mathrm{Isom}^+(\mathbb{H}^2) = \mathrm{PSL}(2, \mathbb{R}).$$

Moreover, we've already seen during the exercises that the associated distance function is given by

$$\mathrm{d}(z, w) = \cosh^{-1}\left(1 + \frac{|z-w|^2}{2 \cdot \mathrm{Im}(z) \cdot \mathrm{Im}(w)}\right).$$

Finally, we have also seen in the exercises what geodesics look like:

PROPOSITION 8.4.2. *The image of a geodesic $\gamma : \mathbb{R} \to \mathbb{H}^2$ is a vertical line or a half circle orthogonal to $\mathbb{R}$. Moreover, every vertical line and half circle orthogonal to the real line can be parameterized as a geodesic.*

We will often forget about the parametrization and call the image of a geodesic a geodesic as well. Note that it follows from the proposition above that given any two distinct points $z, w \in \mathbb{H}^2$ there exists a unique geodesic $\gamma \subset \mathbb{H}^2$ so that both $z \in \gamma$ and $w \in \gamma$. Furthermore, it also follows given a point $z \in \mathbb{H}^2$ and a geodesic γ that does not contain it, there is a unique perpendicular from z to γ (a geodesic γ' that intersects γ once perpendicularly and contains z)

The final fact we will need about the hyperbolic plane is:

PROPOSITION 8.4.3. *Let $z \in \mathbb{H}$ and let $\gamma \subset \mathbb{H}^2$ be a geodesic so that $z \notin \gamma$. Then*

$$\mathrm{d}(z, \gamma) := \inf \{ \mathrm{d}(z, w) \, : \, w \in \gamma \}$$

is realized by the intersection point of the perpendicular from z to γ.

Likewise, any two geodesics that don't intersect and are not asymptotic to the same point in $\mathbb{R} \cup \{\infty\}$ have a unique common perpendicular. Moreover, this perpendicular minimizes the distance between them.

PROOF. The first claim follows from Pythagoras' theorem for hyperbolic triangles. Indeed, given three points in $\mathbb{H}^2$ so that the three geodesics through them form a right angled hyperbolic triangle with sides of length a, b and c (where c is the side opposite the right angle), we have

$$\cosh(a)\cosh(b) = \cosh(c).$$

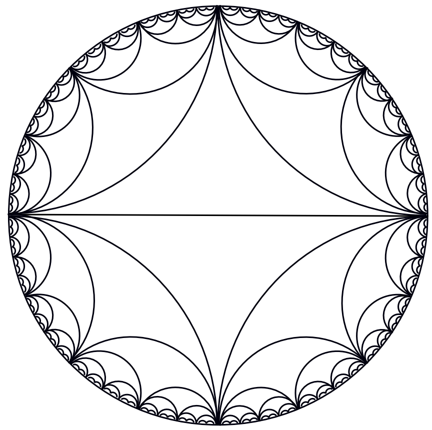

FIGURE 1. The Farey tesselation.

Indeed, this can be computed directly by putting the triangle in some standard position and then using the distance formula, a computation that we leave to the reader. This means in particular that $c > b$.

So, any other point on γ is further away from z than the point w realizing the perpendicular. Because that other point forms a right angled triangle with w and z.

The second claim follows from the first. □

8.4.3. The disk model. Another useful model, especially if one likes compact pictures, is the disk model of the hyperbolic plane. Set

$$\Delta = \{ z \in \mathbb{C} \; : \; |z| < 1 \} .$$

The map $f : \mathbb{H}^2 \to \Delta$ given by

$$f(z) = \frac{z - i}{z + i}$$

is a biholormorphism. We can also use it to push forward the hyperbolic metric to Δ. A direct computation tells us that the metric we obtain is given by

$$ds^2 = 4 \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}.$$

Since f is conformal, the angles in the disk model are still the same as Euclidean angles.

Using the fact that the map f above is a Möbius transformation and thus sends circles and lines to circles and lines, one can prove:

PROPOSITION 8.4.4. *The hyperbolic geodesics in Δ are*

- *straight diagonals through the origin $0 \in \Delta$*
- *$C \cap \Delta$ where $C \subset \mathbb{C}$ is a circle that intersects $\partial\Delta$ orthogonally.*

For example, Figure 1 shows a collection of geodesics in Δ, known as the Farey tesselation.

8.4.4. Hyperbolic surfaces. A hyperbolic surface will be a finite type surface equipped with a metric that locally makes it look like $\mathbb{H}^2$.

Because we will want to deal with surfaces with boundary later on, we need half spaces. Let $\gamma \subset \mathbb{H}^2$ be a geodesic. $\mathbb{H}^2 \setminus \gamma$ consists of two connected components C_1 and C_2. We will call $\mathcal{H}_i = C_i \cup \gamma$ a *closed half space* ($i = 1, 2$). So for example

$$\left\{ z \in \mathbb{H}^2 \ : \ \mathrm{Re}(z) \leq 0 \right\}$$

is a closed half space.

We formalize the notion of a hyperbolic surface as follows:

Definition 8.4.5. A finite type surface S with atlas $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ is called a *hyperbolic surface* if $\varphi_\alpha(U_\alpha) \subset \mathbb{H}^2$ for all $\alpha \in A$ and

1. for each $p \in S$ there exists an $\alpha \in A$ so that $p \in U_\alpha$ and
   - If $p \in \partial S$ then
     $$\varphi_\alpha(U_\alpha) = V \cap \mathcal{H}$$
     for some open set $V \subset \mathbb{H}^2$ and some closed half space $\mathcal{H} \subset \mathbb{H}^2$.
   - If $p \in \mathring{S}$ then $\varphi_\alpha(U_\alpha) \subset \mathbb{H}^2$ is open.
2. For every $\alpha, \beta \in A$ and for each connected component C of $U_\alpha \cap U_\beta$ we can find a Möbius transformation $A : \mathbb{H}^2 \to \mathbb{H}^2$ so that
   $$\varphi_\alpha \circ \varphi_\beta^{-1}(z) = A(z)$$
   for all $z \in \varphi_\beta(C) \subset \mathbb{H}^2$.

Note that every hyperbolic surface comes with a metric: every chart is identified with an open set of $\mathbb{H}^2$ which gives us a metric. Because the chart transitions are restrictions of isometries of $\mathbb{H}^2$, this metric does not depend on the choice of chart and hence is well defined.

Definition 8.4.6. A hyperbolic surface S is called *complete* if the induced metric is complete.

We have seen in Lecture 2 that complete hyperbolic surfaces without boundary (considered up to isometry) correspond one-to-one to Riemann surfaces (considered up to biholomorphism).

8.4.5. Right angled hexagons. Even though Definition 8.4.5 is a complete definition, it is not very descriptive. In what follows we will describe a concrete cutting and pasting construction for hyperbolic surfaces.

We start with right angled hexagons. A right angled hexagon $H \subset \mathbb{H}^2$ is a compact simply connected closed subset whose boundary consists of 6 geodesic segments, that meet each other orthogonally. The picture to have in mind is displayed in Figure 2.

It turns out that the lengths of three non-consecutive sides determine a right angled hexagon up to isometry.

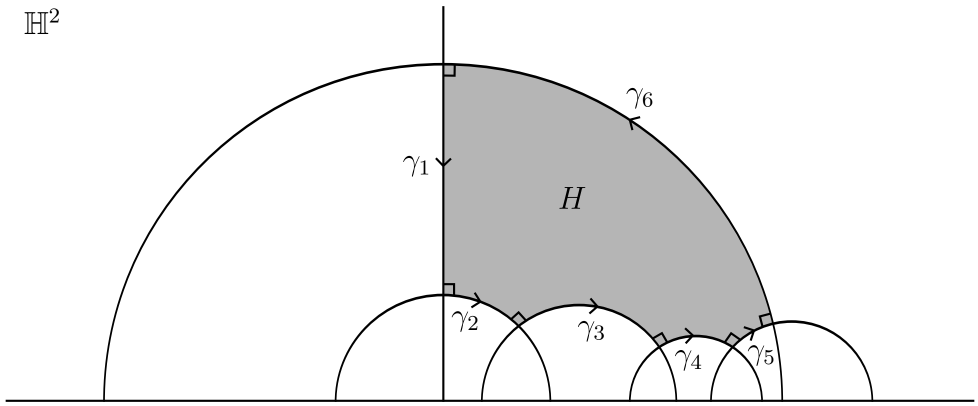

FIGURE 2. A right angled hexagon H.

PROPOSITION 8.4.7. *Let $a, b, c \in (0, \infty)$. Then there exists a right angled hexagon $H \subset \mathbb{H}^2$ with three non-consecutive sides of length a, b and c respectively. Moreover, if H' is another right angled hexagon with this property, then there exists a Möbius transformation $A : \mathbb{H}^2 \to \mathbb{H}^2$ so that*

$$A(H) = H'.$$

PROOF. Let us start with the existence. Let γ_{im} denote the positive imaginary axis and set

$$B = \left\{ z \in \mathbb{H}^2 \,:\, \mathrm{d}(z, \gamma_{im}) = c \right\}.$$

B is a one-dimensional submanifold of $\mathbb{H}^2$. Because the map $z \mapsto \lambda z$ is an isometry that preserves γ_{im} for every $\lambda > 0$, it must also preserve B. This means that B is a (straight Euclidean) line.

Now construct the following picture:

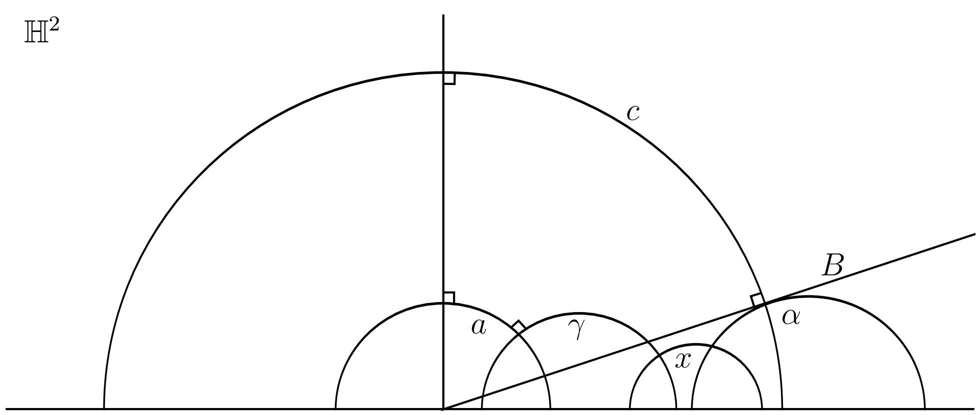

FIGURE 3. Constructing a right angled hexagon $H(a, b, c)$.

That is, we take the geodesic though the point $i \in \mathbb{H}^2$ perpendicular to γ_{im} and at distance a draw a perpendicular geodesic γ. furthermore, for any $p \in B$, we draw the geodesic α

that realizes a right angle with the perpendicular from p to γ_{im}. Now let

$$x = \mathrm{d}(\alpha, \gamma) = \inf \{ \mathrm{d}(z, w) \, : \, z \in \gamma, \; w \in \alpha \} .$$

Because of Proposition 8.4.3, x is realized by the common perpendicular to α and γ. By moving p over B, we can realize any positive value for x and hence obtain our hexagon $H(a, b, c)$.

We also obtain uniqueness from the picture above. Indeed, given any right angled hexagon H' with three non-consequtive sides of length a, b and c, apply a Möbius transformation $A : \mathbb{H}^2 \to \mathbb{H}^2$ so that the geodesic segment of length a starts at i and is orthogonal to the imaginary axis. This implies that the geodesic after a gets mapped to the geodesic γ. Furthermore, one of the endpoints of the geodesic segment of length c needs to lie on the line B. We now know that the the geodesic α before that point needs to be tangent to B. The tangency point of α to B determines the picture entirely. Because the function that assigns the length x of the common perpendicular to the tangency point is injective, we obtain that there is a unique solution. □

LECTURE 9

Pairs of pants and geodesics on hyperbolic surfaces

9.1. The universal cover of a hyperbolic surface with boundary

It will be useful to have a description of the Riemannian universal cover of a surface with boundary. To this end, we first prove:

PROPOSITION 9.1.1. *Let X be a hyperbolic surface with non-empty boundary that consists of closed geodesics. Then there exists a complete hyperbolic surface X^* without boundary in which X can be isometrically embedded so that X is a deformation retract of X^*.*

PROOF. For each $\ell \in (0, \infty)$, we define a hyperbolic surface

$$F_\ell = [0, \infty) \times \mathbb{R}/\{t \sim t + 1\},$$

equipped with the metric

$$ds^2 = d\rho^2 + \ell^2 \cosh^2(\rho) \cdot dt^2$$

for all $(\rho, t) \in F_\ell$. We will call such a surface a *funnel*. One can check that this is a metric of constant curvature -1, in which the boundary is totally geodesic. Alternatively, we can identify

$$F_\ell = \left\{ z \in \mathbb{H}^2 \, : \, \mathrm{Re}(z) \geq 0 \right\} \Big/ \left\langle \begin{bmatrix} e^{\ell/2} & 0 \\ 0 & e^{\ell/2} \end{bmatrix} \right\rangle .$$

We can glue funnels of the right length along the boundary components, in a similar way to Example 9.3.1. Figure 1 shows an example.

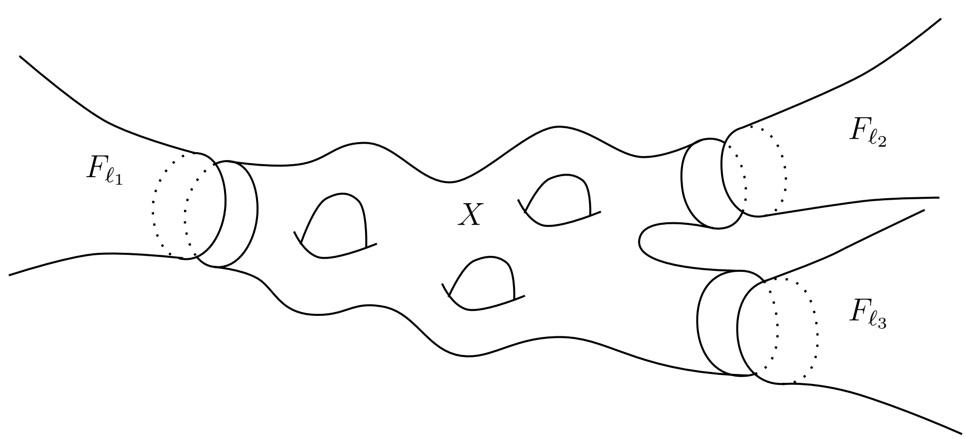

FIGURE 1. Attaching funnels

Since both F_ℓ and X are complete, the resulting surface X^* is complete.

Moreover, since F_ℓ retracts onto its boundary component, X is a deformation retract of X^*. □

See [**Bus10**, Theorem 1.4.1] for a version of the above to surfaces with more general types of boundary components.

Recall that a subset $C \subset M$ of a Riemannian manifold M is called *convex* if for all $p, q \in C$ there exists a length minimizing geodesic $\gamma : [0, \mathrm{d}(p,q)] \to M$ such that

$$\gamma(0) = p, \quad \gamma(\mathrm{d}(p,q)) = q \quad \text{and} \quad \gamma(t) \in C \; \forall \, t \in [0, \mathrm{d}(p,q)].$$

As a result of this construction we obtain:

PROPOSITION 9.1.2. *Let X be a complete hyperbolic surface with non-empty boundary that consists of closed geodesics. Then the universal Riemannian cover $\widetilde{X}$ of X is isometric to a convex subset of $\mathbb{H}^2$ whose boundary consists of complete geodesics.*

PROOF. The Killing-Hopf theorem tells us that the universal cover of X^* is the hyperbolic plane $\mathbb{H}^2$. Here X^* is the surface given by Proposition 9.1.1.

Let us denote the covering map by $\pi : \mathbb{H}^2 \to X^*$. Now let C be a connected component of $\pi^{-1}(X)$. The boundary of C consists of the lifts of ∂X and hence of a countable collection of disjoint complete geodesics in $\mathbb{H}^2$. As such, it's a countable intersection of half spaces (which are convex) and hence convex. □

9.2. Pairs of pants and gluing

One of our main building blocks for hyperbolic surfaces is the following:

DEFINITION 9.2.1. Let $a, b, c \in (0, \infty)$. A *pair of pants* is a hyperbolic surface that is diffeomorphic to $\Sigma_{0,3,0}$ such that the boundary components have length a, b and c respectively.

PROPOSITION 9.2.2. *Let $a, b, c \in (0, \infty)$ and let P and P' be pairs of pants with boundary curves of lengths a, b and c. Then there exists an isometry $\varphi : P \to P'$.*

PROOF SKETCH. There exists a unique orthogonal geodesic (this essentially follows from Proposition 8.4.3 below, the proof of Proposition 9.5.1 that we will do in full during the exercises, is similar) between every pair of boundary components of P.

These three orthogonals decompose P into right-angled hexagons out of which three non-consecutive sides are determined. Proposition 8.4.7 now tells us that this determines the hexagons up to isometry and this implies that P is also determined up to isometry. □

Note that it also follows from the proof sketch above that the unique perpendiculars cut each boundary curve on P into two geodesic segments of equal length.

9.3. Non-compact pairs of pants

In order to deal with non-compact surfaces, we will need non-compact polygons. To this end, we note that, looking at Proposition 8.4.2, complete geodesics in $\mathbb{H}^2$ are parametrized by their *endpoints*: pairs of distinct point in

$$\partial\mathbb{H}^2 := \mathbb{R} \cup \{\infty\}$$

(or $\mathbb{S}^1$ if we use the disk model).

A (not necessarily compact) polygon now is a closed connected simply connected subset $P \subset \mathbb{H}^2$, whose boundary consists of geodesic segments.

If two consecutive segments "meet" at a point in $\partial\mathbb{H}^2$, this point will be called an *ideal* vertex of the boundary. Note that the angle at an ideal vertex is always 0. A polygon all of whose vertices are ideal is called an *ideal polygon*.

We can also make sense of a pair of pants where some of the boundary components have "length" 0. In this case, we obtain a complete hyperbolic structure on a surface with boundary and punctures such that

$$\#\text{punctures} + \#\text{boundary components} = 3.$$

Such pairs of pants can be obtained by gluing either

- two pentagons with one ideal vertex each and right angles at the other vertices,
- two quadrilaterals with two ideal vertices each right angles at the other vertices or
- two ideal triangles.

Along the sides of infinite length there however is a gluing condition. We will come back to this later (see Proposition 10.0.2). Moreover, we obtain a similar uniqueness statement to the proposition above. As always in the non-compact case, the adjective "complete" does need to be added.

Example 9.3.1. If P is a pair of pants and $\delta \subset \partial P$ is one of its boundary components, let us write $\ell(\delta)$ for the length of δ. Given two pairs of pants P_1 with boundary components δ_1, δ_2 and δ_3 and P_2 with boundary components γ_1, γ_2 and γ_3 so that

$$\ell(\delta_1) = \ell(\gamma_1),$$

we can choose an orientation reversing isometry $\varphi : \delta_1 \to \gamma_1$ and from that obtain a hyperbolic surface

$$S = P_1 \sqcup P_2 / \sim,$$

where $\varphi(x) \sim x$ for all $x \in \delta_1$. One way to see that this surface comes with a well defined hyperbolic structure, is that locally it's obtained by gluing two half spaces in $\mathbb{H}^2$ together along their defining geodesics. Note that S is diffeomorphic to $\Sigma_{0,4,0}$.

Repeating the construction above, we can build hyperbolic surfaces of any genus and any number of boundary components. In what follows we will prove that every hyperbolic surface can be obtained from this construction.

9.4. The geometry of isometries

Recall that we can classify isometries in $\mathrm{PSL}(2,\mathbb{R})$ into three different types:

DEFINITION 9.4.1. Let $g \in \mathrm{PSL}(2,\mathbb{R})$.

(1) If $\mathrm{tr}\,(g)^2 < 4$ then g is called *elliptic*.

(2) If $\mathrm{tr}\,(g)^2 = 4$ then g is called *parabolic*.

(3) If $\mathrm{tr}\,(g)^2 > 4$ then g is called *hyperbolic*.

Note that, since trace is conjugacy invariant, conjugate elements in $\mathrm{PSL}(2,\mathbb{R})$ are of the same type. It turns out (as we will see below) that closed geodesics correspond exactly to conjugacy classes of hyperbolic elements.

We've seen during the exercises that the classification above can equivalently be described as:

LEMMA 9.4.2. *Let $g \in \mathrm{PSL}(2,\mathbb{R})$. Then*

(1) g is elliptic if and only if g has a single fixed point inside $\mathbb{H}^2$.

(2) g is parabolic if and only if g has a single fixed point on $\mathbb{R}\cup\{\infty\}$.

(3) g is hyperbolic if and only if g has two distinct fixed points on $\mathbb{R}\cup\{\infty\}$.

Given a hyperbolic isometry, we can define its translation distance as follows:

DEFINITION 9.4.3. Let $g \in \mathrm{PSL}(2,\mathbb{R})$ be hyperbolic. Then its *translation distance* is given by

$$T_g := \inf\left\{ z \in \mathbb{H}^2 \,:\, \mathrm{d}(z, gz) \right\}.$$

Moreover, its *axis* is defined as

$$\alpha_g := \left\{ z \in \mathbb{H}^2 \,:\, \mathrm{d}(z, gz) = T_g \right\}.$$

We have:

LEMMA 9.4.4. *Let $g \in \mathrm{PSL}(2,\mathbb{R})$ be hyperbolic with fixed points $x_1, x_2 \in \partial\mathbb{H}^2$. Then its axis α_g is the unique geodesic between x_1 and x_2 and its translation length is given by*

$$T_g = 2\cosh^{-1}\left(\frac{|\mathrm{tr}\,(g)|}{2}\right).$$

PROOF. Since the claim is conjugacy invariant, we can conjugate g so that $x_1 = 0$ and $x_2 = \infty$. Which means that we can assume without loss of generality that

$$g = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$$

for some $\lambda \in (0,\infty)$. Using the fact that $2\cosh(\frac{1}{2}\cosh^{-1}(x)) = \sqrt{2x+2}$, We get that

$$2\cosh(d(z,gz)/2) = \sqrt{4 + \frac{(\lambda^2-1)^2 \cdot (\mathrm{Im}(z)^2 + \mathrm{Re}(z)^2)}{\lambda^2 \mathrm{Im}(z)^2}} \geq \sqrt{4 + \frac{(\lambda^2-1)^2}{\lambda^2}} = \lambda + \frac{1}{\lambda},$$

with equality if and only $\mathrm{Re}(z) = 0$, thus proving the lemma. □

9.5. Geodesics and conjugacy classes

Recall that there is a one-to-one correspondence

$$\left\{ \begin{array}{c} \text{Conjugacy classes of} \\ \text{non-trivial elements in } \pi_1(X) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{free homotopy classes of} \\ \text{non-trivial closed curves on } X \end{array} \right\}.$$

We will call a curve puncture parallel if it can be homotoped into a puncture.

It turns out that on a hyperbolic surface (or more generally a negatively curved Riemannian manifold), every free homotopy class of essential closed curves contains a unique geodesic:

Proposition 9.5.1. *Let X be a complete hyperbolic surface with totally geodesic boundary.*

(1) Then in every homotopy class of non-puncture parallel closed curves γ on X, there exists a unique geodesic that minimizes the length among all curves in the homotopy class.

(2) Moreover, if the free homotopy class contains a simple closed curve, then the corresponding geodesic is also simple.

(3) More generally, if γ and γ' are non-homotopic non-puncture parallel and non-trivial closed curves, then

- *The number of self-intersections of the unique geodesic $\overline{\gamma}$ homotopic to γ is mimimal among all closed curves homotopic to γ and*
- *$\#\overline{\gamma} \cap \overline{\gamma'}$ is minimal among all pairs of curves homotopic to γ and γ' respectively.*

Proof. The proof will be part of this week's exercises. □

LECTURE 10

Fenchel–Nielsen coordinates

We will use the proposition above to prove:

PROPOSITION 10.0.1. *Let X be a complete hyperbolic surface with totally geodesic boundary. Then there are one-to-one correspondences between the following three sets:*

$$\left\{ \begin{array}{c} \text{Non-trivial free homotopy classes of} \\ \text{non puncture-parallel closed curves on } X \end{array} \right\},$$

$$\left\{ \begin{array}{c} \text{Conjugacy classes of} \\ \text{hyperbolic elements in } \Gamma \end{array} \right\}$$

and

$$\left\{ \begin{array}{c} \text{Oriented, unparametrized} \\ \text{closed geodesics on } \mathbb{H}^2/\Gamma \end{array} \right\}.$$

PROOF. The correspondence between the last and the first set is given by the previous proposition, so we only need to show that conjugacy classes of hyperbolic elements correspond one-to-one to oriented, unparametrized geodesics.

In order to make our lives a little easier, we will assume X to be closed. The argument for the general case is similar. We will hence not worry about the assumption that the curve is non puncture parallel, nor about boundary components.

First of all consider a conjugacy class $K \subset \Gamma$ of hyperbolic elements. Let us pick an element $g \in K$, with axis $\alpha_g \subset \mathbb{H}^2$. The projection map $\pi : \mathbb{H}^2 \to X$ sends α_g to a closed geodesic of length T_g. Moreover, since

$$\pi\Big(\alpha_{hgh^{-1}}\Big) = \pi\Big(h\alpha_g\Big) = \pi\Big(\alpha_g\Big),$$

the resulting geodesic does not depend on the choice of g.

In the opposite direction, a closed geodesic on $\mathbb{H}^2/\Gamma$ lifts to a countable union of geodesics in $\mathbb{H}^2$ (the orbit of a single such geodesic under Γ), each invariant under a cyclic group of deck transformations. These transformations need to fix the endpoints of the given geodesic, so they are hyperbolic. The action of Γ on the geodesics corresponds to conjugation of these hyperbolic elements. □

Before we get to pants decompositions, we record what happens to curves that are parallel to a puncture.

PROPOSITION 10.0.2. *Let X be a complete hyperbolic surface and make an identification $X = C/\Gamma$, where C is a convex subset of $\mathbb{H}^2$, bounded by complete geodesics and $\Gamma <$*

$\mathrm{PSL}(2,\mathbb{R})$ *acts prorperly discontinuosly and freely on* C. *Then there is a one-to-one correspondences*

$$\left\{ \begin{array}{c} \textit{Conjugacy classes of} \\ \textit{parabolic elements in } \Gamma \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{Oriented, unparametrized} \\ \textit{puncture-parallel closed curves } \mathbb{H}^2/\Gamma \end{array} \right\}.$$

This proposition gives us the gluing condition we spoke about in Section 9.2: the gluing needs to be so that the resulting puncture parallel curves give rise to parabolic elements, this turns out to uniquely determine the gluing.

10.1. Automorphism groups as subgroups of the mapping class group

Now that we know that free homotopy classes of essential closed curves contain unique geodesics, we can also prove a proposition from a while ago:

Proof of Proposition 6.2.2(1). Suppose that an isometry φ is homotopic to the identity. We need to proved that it is the identity. To this end, let $\gamma_1, \gamma_2 \subset X$ be two simple closed geodesics that pairwise intersect once. Because φ is homotopic to the identity, $\varphi(\gamma_i)$ is homotopic to γ_i.

The fact that φ is an isometry and that there is a unique geodesic in the homotopy class of γ_i, means that φ maps γ_i to itself for $i = 1, 2$. Moreover, the intersection point $p \in \gamma_1 \cap \gamma_2$ also gets mapped to itself. This means that $\varphi|_{\gamma_i}$ is either the identity or of order two. However, if it's of order two, then it maps the curve γ_i to γ_i^{-1}. This is not a contradiction with the fact that φ is globally orientation preserving. It is however a contradiction with the fact that φ is homotopically trivial. So, we conclude that φ acts like the identity on all the curves γ_i, for $i = 1, 2$.

In particular, φ fixes the four on $\gamma_1 \cup \gamma_2$ at distance $\varepsilon > 0$ to p. This implies that φ is the identity. □

10.2. Every hyperbolic surface admits a pants decomposition

As an immediate consequence to Proposition 9.5.1 we get that hyperbolic surfaces admit pants decompositions.

Definition 10.2.1. Let X be a complete, orientable hyperbolic surface of finite area. A pants decomposition of X is a collection of pairwise disjoint simple closed geodesics $\mathcal{P} = \{\alpha_1, \ldots, \alpha_k\}$ in X so that each connected component of

$$X \setminus \left(\bigcup_{i=1}^{k} \alpha_i \right)$$

consists of hyperbolic pairs of pants whose boundary components have been removed.

We have the following:

Lemma 10.2.2. *Let* $\mathcal{P}$ *be a pants decomposition of a hyperbolic surface* X *that is homeomorphic to* $\Sigma_{g,b,n}$ *then*

- $\mathcal{P}$ *contains* $3g + n + b - 3$ *closed geodesics and*

- *$X \setminus \mathcal{P}$ consists of $2g+n+b-2$ pairs of pants.*

PROOF. This can be proved using the Euler characteristic. □

PROPOSITION 10.2.3. *Let X be a complete, orientable hyperbolic surface of finite area and totally geodesic boundary. Then X admits a pants decomposition.*

PROOF. Take any collection of simple closed curves on $\Sigma_{g,b,n}$ that decompose it into pairs of pants. Proposition 9.5.1 tells us that these curves can be realized by unique geodesics. □

Note that we actually get countably many such pants decompositions: given a pants decomposition we can apply a diffeomorphism not isotopic to the identity (of which we already know there are many) to obtain a new topological pants decomposition, that is realized by different geodesics.

Finally, we remark, that lengths alone are not enough to determine the hyperbolic metric:

EXAMPLE 10.2.4. φ in Example 9.3.1 is determined up to 'twist'. That is, if we parameterize δ_1 by a simple closed geodesic $x : \mathbb{R}/(\ell(\delta_1)\mathbb{Z}) \to \delta_1$ and $\varphi' : \delta_1 \to \gamma_1$ is a different orientation reversing isometry, then there exists some $t_0 \in \mathbb{R}$ such that

$$\varphi'(x(t)) = \varphi(x(t_0 + t))$$

for all $t \in \mathbb{R}/(\ell(\delta_1)\mathbb{Z})$.

Summarizing the discussion above, we get the following parametrization of all hyperbolic surfaces:

THEOREM 10.2.5. *Let (g, b, n) be so that*

$$\chi(\Sigma_{g,b,n}) < 0.$$

If we fix a pants decomposition $\mathcal{P}$ of $\Sigma_{g,b,n}$ and vary the lengths $\ell_i \in (0, \infty)$ and twist $\tau_i \in [0, \ell_i]$, we obtain all complete hyperbolic surfaces homeomorphic to $\Sigma_{g,b,n}$.

Note however that there is no guarantee that we don't obtain the same surface multiple times (and in fact we do).

10.3. Annuli

10.3.1. Hyperbolic annuli. Our goal is to use pants decompositions to define global coordinates on Teichmüller space. In order to prove continuity of the coordinates we obtain, we need to understand (to some degree) how the complex structure and the hyperbolic metric depend on each other. It turns out that understanding this for annuli will suffice. So, before we get to Fenchel–Nielsen coordinates, we will discuss annuli.

If $g \in \mathrm{PSL}(2, \mathbb{R})$ is a hyperbolic or parabolic isometry then the group $\langle g \rangle \simeq \mathbb{Z}$ acts on $\mathbb{H}^2$ properly disconinuously and freely. This means that

$$N_g = \mathbb{H}^2/\langle \gamma \rangle$$

is an orientable hyperbolic surface with fundamental group $\mathbb{Z}$ and hence an annulus. First we note that the geometry of the annulus only depends on the translation length of g. We record this as a lemma, the proof of which we leave to the reader.

LEMMA 10.3.1. *Let $g, h \in \mathrm{PSL}(2,\mathbb{R})$ be either both hyperbolic or both parabolic elements so that their translation lengths satisfy $T_g = T_h$. Then the annuli N_g and N_h are isometric. Moreover, every complete hyperbolic annulus is isometric to N_g for some parabolic or hyperbolic $g \in \mathrm{PSL}(2,\mathbb{R})$.*

Note that this includes the case where $T_g = T_h = 0$.

10.3.2. Complex annuli. The complex parametrization of annuli we will need is:

$$A_m := \{\, z \in \mathbb{C} \,:\, 0 < \mathrm{Im}(z) < m \,\} / \mathbb{Z}$$

for all $m > 0$. Here the $\mathbb{Z}$-action is given by $k \cdot z = z + k$ for all $k \in \mathbb{Z}$, $z \in \mathbb{C}$.

We also record a version of Grötzsch's theorem for these annuli (the proof of which is a variation of the proof we saw in the exercises).

THEOREM 10.3.2 (Grötzsch's theorem). *Let $f : A_m \to A_{m'}$ be a K-quasiconformal map. Then*

$$\frac{1}{K} \leq \frac{m}{m'} \leq K.$$

Moreover, equality is realized if and only if f can be lifted to a map

$$\widetilde{f} : \{\, z \in \mathbb{C} \,:\, 0 < \mathrm{Im}(z) < m \,\} \to \{\, z \in \mathbb{C} \,:\, 0 < \mathrm{Im}(z) < m' \,\}$$

given by

$$\widetilde{f}(x+iy) = b + x + i\frac{m'}{m}y$$

for some $b \in \mathbb{R}$.

We observe that this theorem also implies that A_m and $A_{m'}$ are biholomorphic if and only if $m = m'$. The number m is called the *modulus* of the annulus.

The question now becomes whether N_g is biholomorphic to A_m for some m and if so, to which. In order to solve this question, we introduce a new (somewhat uncommon) model for the hyperbolic plane the *band model*. Set

$$\mathbb{B} = \left\{\, z \in \mathbb{C} \,:\, |\mathrm{Im}(z)| < \frac{\pi}{2} \,\right\},$$

equipped with the metric

$$ds^2 = \frac{dx^2 + dy^2}{\cos^2(y)}.$$

This is another model for the hyperbolic plane, moreover the real line is a geodesic in $\mathbb{B}$. Maps of the from $\varphi_b : \mathbb{B} \to \mathbb{B}$ defined by

$$z \mapsto z + b$$

for some $b > 0$ are isometries for this metric. Moreover $\langle \varphi_b \rangle \simeq \mathbb{Z}$ acts on $\mathbb{B}$ properly discontinuously, which means that

$$M_b = \mathbb{B} / \langle \varphi_b \rangle$$

is a hyperbolic annulus. Moreover, the translation length of φ_b is b, so using Lemma 10.3.1, we see that

$$M_b \simeq N_g$$

as hyperbolic surfaces, where $g \in \mathrm{PSL}(2,\mathbb{R})$ is any hyperbolic element with translation length b.

We now claim that:

LEMMA 10.3.3. *Let $m > 0$. The annuli A_m and $M_{\pi/m}$ are biholomorphic.*

PROOF. The map $z \mapsto z - i\ m/2$ is a biholormophism of $\mathbb{C}$ that commutes with the $\mathbb{Z}$-action. As such, A_m is biholomorphic to

$$\left\{ z \in \mathbb{C} \, : \, |\mathrm{Im}(z)| < \frac{m}{2} \right\} / \mathbb{Z}.$$

The map $\left\{ z \in \mathbb{C} \, : \, |\mathrm{Im}(z)| < \frac{m}{2} \right\} \to \mathbb{B}$ given by $z \mapsto \frac{\pi}{m} z$ is a $\mathbb{Z}$-equivariant biholomorphism and hence descends to a biholomorphism

$$\left\{ z \in \mathbb{C} \, : \, |\mathrm{Im}(z)| < \frac{m}{2} \right\} / \mathbb{Z} \simeq M_{\pi/m}.$$

□

For the parabolic case we have:

LEMMA 10.3.4. *let $g \in \mathrm{PSL}(2,\mathbb{R})$ be parabolic. The annuli N_g and $\mathbb{D} \setminus \{0\}$ are biholomorphic.*

PROOF. Using Lemma 10.3.1, we may assume that

$$g = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The map $\mathbb{H}^2 \to \mathbb{D}$ given by

$$z \mapsto e^{-2\pi i z}$$

induces the biholomorphism. □

10.4. Fenchel-Nielsen coordinates

Now we're ready to introduce Fenchel–Nielsen coordinates on Teichmüller spaces of hyperbolic surfaces. In particular, in this section, we will assume that our base surface S admits a complete hyperbolic metric. Moreover, we will fix a (topological) pants decomposition $\mathcal{P}$ on S.

10.4.1. Lengths. Given any essential closed curve γ on S, we obtain a function

$$\ell_\gamma : \mathcal{T}(S) \to \mathbb{R}_+$$

called a *length function*, defined as follows. Each $[R, f] \in \mathcal{T}(S)$ can be seen as a marked hyperbolic surface. So, Proposition 9.5.1 implies that the homotopy class of $f(\gamma)$ on R contains a unique geodesic. $\ell_\gamma([R, f])$ is the length of this geodesic.

Hence, given S and $\mathcal{P}$ as above, we obtain a map

$$\ell_\mathcal{P} : \mathcal{T}(S) \to \mathbb{R}_+^{3g-3+n}$$

defined by

$$\ell_\mathcal{P}([R, f]) = \Big(\ell_\gamma([R, f])\Big)_{\gamma\in\mathcal{P}}.$$

We have:

Lemma 10.4.1 (Wolpert). *Let R and S be closed Riemann surfaces and $f : R \to S$ a K-quasiconformal map. Then*

$$\frac{1}{K} \cdot \ell_\gamma(R) \quad \leq \quad \ell_{f(\gamma)}(S) \quad \leq \quad K \cdot \ell_\gamma(R)$$

for any essential simple closed curve γ on R. Moreover, the function

$$\ell_\gamma : \mathcal{T}(S) \to \mathbb{R}$$

is 2-Lipschitz with respect to the Teichmüller metric, i.e.

$$|\log(\ell_\gamma([R, f])) - \log(\ell_\gamma([R', f']))| \leq 2\, \mathrm{d_T}([R, f], [R', f'])$$

for all $[R, f], [R', f'] \in \mathcal{T}(S)$.

Proof. Fix a basepoint $p \in R$ so that we can identify γ with an element of $\pi_1(R, p)$, that we will also denote by γ. The infinite cyclic subgroup of $\pi_1(R, p)$ generated by γ induces a $\mathbb{Z}$-cover

$$R_\gamma \to R.$$

We will write $S_{f(\gamma)}$ for the corresponding covering space of S. Just like in the proof of Proposition 9.5.1, these are annuli and by Lemma 10.3.3, they are biholomorphic to $A_{\pi/\ell_\gamma(R)}$ and $A_{\pi/\ell_{f(\gamma)}(S)}$ respectively. K-quasiconformal maps between R and S lift to K-quasiconformal maps between R_γ and $S_{f(\gamma)}$. The result follows from Götzsch's theorem for annuli that states the modulus of an annulus gets multiplied by at most the multiplicative constant (the proof of which is a variation of the proof for squares that we have done in the exercises). □

10.4.2. Twists. So, given S and $\mathcal{P}$ as above, we have a continuous map

$$\ell_\mathcal{P} : \mathcal{T}(S) \to \mathbb{R}_+^{3g-3+n}.$$

It's however not quite injective. The problem is that we can still rotate the hyperbolic metric along the curves in the pants decomposition. Twist coordinates will remedy this.

First we pick a collection of disjoint simple closed curves Γ so that for each pair of pants P in $S \setminus \mathcal{P}$, $\Gamma \cap \mathcal{P}$ consists of three arcs, each connecting a different pair of boundary components of P. Figure 1 shows an example.

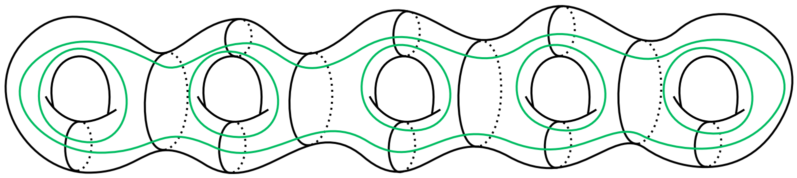

FIGURE 1. A pants decomposition $\mathcal{P}$ with a set of curves Γ.

Regardless of our choice of pants decomposition $\mathcal{P}$, such a system of curves Γ always exists.

Now let $\gamma \in \mathcal{P}$ be a pants curve. Then γ bounds either one P or two pairs of pants P_1 and P_2 in the decomposition. Let us assume the latter for simplicity, the other case is analogous. The left hand side of Figure 2 shows an example of such a curve γ.

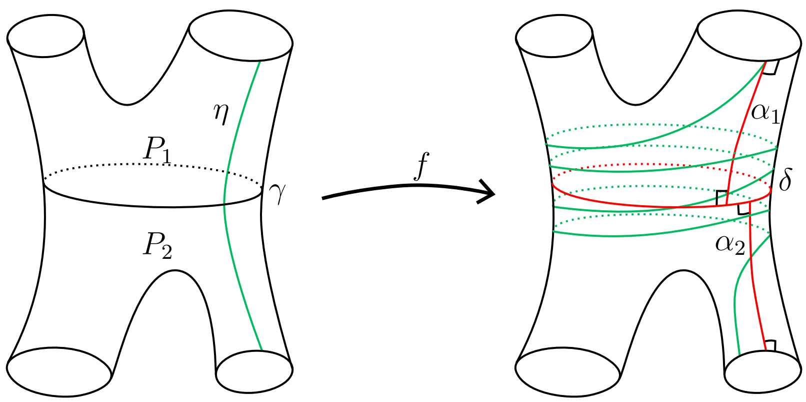

FIGURE 2. The image of an arc under a diffeomorphism.

If $f : S \to R$ is an orientation preserving diffeomorphism, then it maps $\mathcal{P}$ to some pants decomposition of R. Moreover, if η is one of the (two) components of $(P_1 \cup P_2) \cap \Gamma$ that intersects γ, then $f(\eta)$ is some arc between boundary components of $f(P_1)$ and $f(P_2)$ (like on the right hand side of Figure 2). Now

- δ will be the unique simple closed geodesic in the free homotopy class of $f(\gamma)$ on R.
- α_1 and α_2 the two unique perpendiculars between the boundary components between which $f(\eta)$ runs and δ (see Figure 2).

Then relative to the boundary of $f(P_1 \cup P_2)$, the arc $f(\eta)$ is freely homotopic to $\alpha_2 \cdot \delta^k \cdot \alpha_1$ for some $k \in \mathbb{Z}$.

The twist along γ is now

$$\tau_\gamma([R, f]) = k \cdot \ell_\gamma([R, f]) \pm \mathrm{d}(p_1, p_2) \in \mathbb{R}$$

where

- p_1 and p_2 are the points where α_1 and α_2 hit δ.

- The signs are determined by the orientation of R in the following way. The orientation of R gives a notion of "twisting to the left" along δ. Left twists are counted positively and right twists negatively.

Let us prove that twists are continuous:

LEMMA 10.4.2. *Let S and γ be as above. The function*

$$\tau_\gamma : \mathcal{T}(S) \to \mathbb{R}$$

is continuous.

PROOF SKETCH. Suppose that

$$d_T([R, f], [R', f'])$$

is small. This means that the map $f' \circ f^{-1} : R \to R'$ is close to an isometry. Since it maps the curves and arcs used to define $\tau_\gamma([R, f])$ to those used to define $\tau_\gamma([R', f'])$. So, this map lifts to a map $\widetilde{f' \circ f^{-1}} : \mathbb{H}^2 \to \mathbb{H}^2$ that is close to conformal and hence close to an isometry. This means that the numbers $\tau_\gamma([R, f])$ and $\tau_\gamma([R', f'])$ are close. ☐

Putting the above together, we obtain a continuous map

$$\mathrm{FN}_{\mathcal{P}} : \mathcal{T}(S) \to \mathbb{R}_+^{3g-3+n} \times \mathbb{R}^{3g-3+n}$$

defined by

$$\mathrm{FN}_{\mathcal{P}}([R, f]) = \Big(\ell_\gamma([R, f]), \tau_\gamma([R, f])\Big)_{\gamma \in \mathcal{P}}.$$

LECTURE 11

The geometry and topology of Teichmüller and moduli spaces

11.1. Fenchel–Nielssen coordinates yield a homeomorphism

It turns out that the Fenchel Nielssen map is a homeomorphism:

THEOREM 11.1.1. *Let S be a surface of finite type such that $\chi(S) < 0$ and let $\mathcal{P}$ be a pants decomposition of S. Then the map*

$$\mathrm{FN}_{\mathcal{P}} : \mathcal{T}(S) \to \mathbb{R}_+^{3g-3+n} \times \mathbb{R}^{3g-3+n},$$

is a homeomorphism.

PROOF. Since we have already proved that lengths and twists are continuous, we only need to provide a continuous inverse to the map $\mathrm{FN}_{\mathcal{P}}$.

Given a vector $(\ell_\gamma, \tau_\gamma)_{\gamma\in\mathcal{P}}$, we can use the gluing construction we discussed above in order to produce a hyperbolic surface R. The lengths give us the geometry of the pairs of pants and the gluing along a curve γ is determined by

$$\tau_\gamma^{(0)} = \tau_\gamma + k \cdot \ell_\gamma,$$

where k is such that $\tau_\gamma^{(0)} \in [0, \ell_\gamma)$. Call this surface $R\Big((\ell_\gamma, [\tau_\gamma])_\gamma\Big)$. In particular, by varying the twist τ_γ, we obtain the same surface countably many times.

The question however is what the marking, i.e. the map $f : S \to R\Big((\ell_\gamma, [\tau_\gamma])_\gamma\Big)$, should be. In order to do this, we fix open regular neighborhoods N_γ^S of the curves $\gamma \in \mathcal{P}$ on S so that

$$S \setminus \bigcup_{\gamma\in\mathcal{P}} N_\gamma$$

consists of disjoint pairs of pants $P_1^S, \ldots, P_k^S$. We will once and for all parametrize the annuli

$$N_\gamma^S = \Big(\mathbb{R}/\mathbb{Z}\Big) \times (-1, 1).$$

On $R(\ell_\gamma, [\tau_\gamma])$ we pick such neighborhoods too and obtain neighborhoods N_γ^R and pairs of pants P_i^R. We will assume that

$$N_\gamma^R = \Big\{ x \in R\Big((\ell_\gamma, [\tau_\gamma])_\gamma\Big) \ : \ \mathrm{d}(x, \gamma) < \varepsilon \Big\}$$

for some ε small enough. Moreover, we assume ε varies continuously as a function of $(\ell_\gamma, [\tau_\gamma])_\gamma$.

In order to build f, we now pick a parametrization

$$N_\gamma^R = \Big(\mathbb{R}/\ell_\gamma\mathbb{Z}\Big) \times (-1,1)$$

where the subset

$$\Big(\mathbb{R}/\ell_\gamma\mathbb{Z}\Big) \times \{t\} \subset N_\gamma^R$$

is one of the (one or two) components of

$$\Big\{ x \in R\Big((\ell_\gamma, [\tau_\gamma])_\gamma\Big) \,:\, \mathrm{d}(x,\gamma) = |t| \cdot \varepsilon \Big\},$$

parametrized by a constant multiple (depending on t) of arclength for all $t \in (-1,1)$.

The map $f_\gamma : N_\gamma^S \to N_\gamma^R$ is now given by

$$f_\gamma(\theta, t) = \left(\ell_\gamma \cdot \theta + \tau_\gamma \cdot \frac{t+1}{2}, t\right).$$

The awkward $(t+1)/2$ is an artifact of choosing the interval $(-1,1)$ instead of $(0,1)$ (the latter would have made some of the previous equations more awkward).

For the complements of the annuli we choose arbitrary homeomorphisms and $f_i^P : P_i^S \to P_i^R$ that smoothly extend the f_γ.

This map is clearly an inverse and since we can make everything depend on the input continuously, it's continuous. □

Remark 11.1.2. Looking at the proof above, it's a natural question to ask whether we maybe get a fundamental domain for moduli space by only considering $\tau_\gamma \in [0, \ell_\gamma)$.

However, this not the case. To see this, take any $f \in \mathrm{Diff}^+(S,\Sigma)$ (where $S = S_0 \setminus \Sigma$, S_0 is closed and Σ a finite set) that is not homotopic to the identity. Then we get a surface isometric to $R\Big((\ell_\gamma, [\tau_\gamma])_{\gamma\in\mathcal{P}}\Big)$ if we assign the lengths of the curves in $f(\mathcal{P})$ to the curves in $\mathcal{P}$ instead (the isometry will be induced by f).

11.2. Complex structures on Teichmüller and moduli spaces

Our next goal is to describe a complex structure on Teichmüller and moduli spaces of higher genus surfaces. We will use the **Bers embedding** to do this. This is an embedding of $\mathcal{T}(\Sigma_g)$ as a bounded domain in $\mathbb{C}^{3g-3}$. The mapping class group acts on this bounded domain by biholomorphic maps and acts properly discontinuously (but not freely, as we have discussed before, see Proposition 6.2.2), which gives $\mathcal{M}(\Sigma_g)$ the structure of a complex orbifold.

11.3. The Schwarzian derivative

First we discuss the Schwarzian derivative. Given a domain $D \subset \widehat{\mathbb{C}}$ and an analytic function $f : D \to \widehat{\mathbb{C}}$ with non vanishing derivatives, the Schwarzian $\mathcal{S}(f)$ measures how far away f is from a Möbius transformation.

11.3.1. Definition. Before defining the Schwartzian derivative, we start with a lemma that will make it appear more natural:

LEMMA 11.3.1. *Let $D \subset \widehat{\mathbb{C}}$ be a domain and $z_0 \in D$. For every analytic map $f : D \to \widehat{\mathbb{C}}$ with non-vanishing derivative there exists a unique element $g \in \mathrm{PSL}(2, \mathbb{C})$ such that*

$$f(z_0) = g(z_0), \quad \frac{df}{dz}(z_0) = \frac{dg}{dz}(z_0) \quad \text{and} \quad \frac{d^2 f}{dz^2}(z_0) = \frac{d^2 g}{dz^2}(z_0).$$

PROOF. Because translations are Möbius transformations whose derivatives vanish after the first order, we can post and precompose with translations such that $z_0 = 0 \in D$ and $f(z_0) = 0$. This means that around z_0,

$$f(z) = \sum_{k \geq 1} \frac{a_k}{k!} z^k \quad \text{with} \quad a_k = \frac{\partial^k}{\partial z^k} f(z_0).$$

If we require $g(0) = 0$, this means that g satisfies

$$g(z) = \frac{\alpha z}{1 + \beta z} = \alpha z \sum_{k \geq 0} (-\beta z)^k$$

for some $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$. So we get the equations

$$-\alpha \cdot \beta = a_1 \quad \text{and} \quad 2 \cdot \alpha \cdot \beta^2 = a_2$$

which means that

$$\alpha = a_1 \quad \text{and} \quad \beta = -\frac{a_2}{2a_1},$$

which indeed determines them uniquely. □

Looking at this lemma, we see that the third order term of the Taylor expansion of $f - g$ is a good measure of how far f is from a Möbius transformation. Looking at the proof above, this corresponds to computing the difference between $\alpha \cdot \beta^2 = a_2^2/4a_1$ and $a_3/6$. Expressing this in terms of derivatives of f and dividing by $f'/6$, we arrive at the following defintion:

DEFINITION 11.3.2. Let $D \subset \widehat{\mathbb{C}}$ be a domain and $z_0 \in D$. Moreover, let $f : D \to \widehat{\mathbb{C}}$ be an analytic map with non-vanishing derivative. Then we define its **Schwartzian derivative** as

$$\mathcal{S}(f)(z) = \left(\frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2 \right).$$

The notation $\{f, z\}$ is also quite common for the Schwartzian derivative.

11.3.2. The Schwartzian on a Riemann surface. The following properties of the Schwarzian derivative are immediate:

PROPOSITION 11.3.3. *Let $D \subset \widehat{\mathbb{C}}$ be a domain and let $f : D \to \widehat{\mathbb{C}}$ be an analytic map with non-vanishing first derivative and let $g \in \mathrm{Aut}(\widehat{\mathbb{C}})$, then*

$$\mathcal{S}(f \circ g)(z) = \mathcal{S}(f)(g(z)) \cdot g'(z)^2 \quad \text{for } z \in g^{-1}(D),$$
$$\mathcal{S}(g \circ f)(z) = \mathcal{S}(f)(z) \quad \text{for } z \in D$$

and

$$\mathcal{S}(g)(z) = 0 \quad \textit{for } z \in \widehat{\mathbb{C}}.$$

PROOF. The last claim is a consquence of Lemma 11.3.1. The other two follow from the chain rule. □

It follows from this proposition that if R and S are Riemann surfaces and $f : R \to S$ is an orientation preserving diffeomorphism (which by Lemma 6.3.1 implies that $f'(z) \neq 0$) then $\mathcal{S}(g)$ is naturally a **quadratic differential**. That is, the local expression

$$\mathcal{S}(f)(z) \cdot dz^2$$

is globally well-defined.

11.3.3. The Bers embedding. We have decribed above how to parametrize Teichmüller space using Beltrami differentials. That is, for a closed Riemann surface:

$$\mathcal{T}(S) = B(S)_1/\mathrm{Diff}_0^+(S)$$

The Bers embedding embeds this space into a finite dimensional vector space. We will describe how this works now. We will fix an identification $S = \Gamma\backslash\mathbb{H}^2$ and write

$$B(S)_1 \simeq B(\mathbb{H}^2,\Gamma)_1 = \left\{ \mu \text{ a Beltrami coefficient on } \mathbb{H}^2 \;:\; \begin{array}{c} \mu\circ\gamma = \mu\cdot\frac{\overline{\gamma'}}{\gamma} \text{ for all} \\ \gamma\in\Gamma \text{ and } ||\mu||_\infty < 1 \end{array} \right\}$$

Given $\mu \in B(\mathbb{H}^2,\Gamma)_1$, we may extend it to a Beltrami coefficient on $\mathbb{C}$ by

$$\widehat{\mu}(z) = \begin{cases} \mu(z) & \text{if } z \in \mathbb{H}^2 \\ 0 & \text{if } z \in \mathbb{C} - \mathbb{H}^2. \end{cases}$$

Theorem 7.3.1 provides us with a unique map

$$f_\mu : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}} \quad \text{such that} \quad f(0) = 0,\ f(1) = 1,\ f(\infty) = \infty \text{ and } \mu_{f_\mu} = \mu.$$

Note that f_μ is holomorphic on the lower half plane $\mathbb{H}^*$. This in particular means that, by Proposition 11.3.3, we obtain a map

$$[\mu] \in \mathcal{T}(\Gamma\backslash\mathbb{H}^2) \quad \overset{\Phi_{\mathrm{Bers}}}{\mapsto} \quad \mathcal{S}\left((f_\mu)|_{\mathbb{H}^*}\right)\cdot dz^2 \in \mathcal{Q}(\Gamma\backslash\mathbb{H}^*),$$

where $\mathcal{Q}(\Gamma\backslash\mathbb{H}^*)$ denotes the space of **holomorphic quadratic differentials** on the Riemann surface $\Gamma\backslash\mathbb{H}^*$. The map Φ_{Bers} is called the **Bers embedding**.

First we note the following consequence of the Riemann–Roch theorem:

THEOREM 11.3.4. *Let S be a closed Riemann surface of genus $g \geq 2$. Then*

$$\dim_{\mathbb{C}}(\mathcal{Q}(S)) = 3g - 3.$$

So the Bers embedding is a map into a finite dimensional complex vector space. As the name suggests, the map is in fact an embedding an thus equips Teichmüller space with a complex structure that also turns out not to depend on the choice of base surface. It will take us a while to prove this. We start with a proposition:

PROPOSITION 11.3.5. *The Bers embedding $\Phi_{\mathrm{Bers}} : \mathcal{T}(\Gamma\backslash\mathbb{H}^2) \to \mathcal{Q}(\Gamma\backslash\mathbb{H}^*)$ is continuous and injective.*

PROOF. Injectivity is part of the exercises. For continuity, we refer to [IT92, Proposition 6.5]. □

THEOREM 11.3.6. *The Bers embedding is a homeomorphism onto its image.*

PROOF. In Theorem 11.1.1, we proved that $\mathcal{T}(\Gamma\backslash\mathbb{H}^2)$ is in fact homeomorphic to $\left(\mathbb{R}_{>0}\times\mathbb{R}\right)^{3g-3}$. Combined with the proposition above and Brouwer's invariance of domain this implies that Φ_{Bers} is a homeomorphism onto its image. □

The aforementioned independence of base points is the following theorem:

THEOREM 11.3.7. *Let $[R, f] \in \mathcal{T}(S)$ then the map*

$$[X, h] \in \mathcal{T}(S) \mapsto [X, h \circ f] \in \mathcal{T}(R)$$

is a biholomorphism.

PROOF. See [IT92, Theorem 6.12]. □

Finally, we mention that the expression

$$||q||_\infty = \sup_{z\in\mathbb{H}^*} \operatorname{Im}(z)^2 \cdot |q(z)| \quad q \in \mathcal{Q}(\Gamma\backslash\mathbb{H}^*)$$

is a well defined norm on $\mathcal{Q}(\Gamma\backslash\mathbb{H}^*)$. It turns out that the image of the Bers embbeding is bounded:

THEOREM 11.3.8 (Nehari). *Let $R = \Gamma\backslash\mathbb{H}^2$ be a closed Riemann surface. Then*

$$\Phi_{\text{Bers}} \subset \left\{ q \in \mathcal{Q}(\Gamma\backslash\mathbb{H}^*) \;:\; ||q||_\infty < \frac{3}{2} \right\}.$$

PROOF. See [IT92, Theorem 6.6]. □

We also observe that this is consistent with the case of the torus, in which the complex structure is that of a bounded domain in $\mathbb{C}$.

11.4. Proper discontinuity of the action of the mapping class group

We have now almost recovered all the structure for Teichmüller spaces of higher genus surfaces that the Teichmüller space of the torus has. Now we want to proce that the corresponding moduli spaces also have a similar structure. To this end, we need to show that the moduli space of a closed Riemann surface of genus $g \geq 2$ has the structure of a complex $(3g-3)$-dimensional orbifold.

Before we prove this, we record a lemma that is useful in its own right. In this lemma, the **length spectrum** of a hyperbolic surface X is is the multi-set $\mathcal{L}(X)$ of all the lengths of closed geodesics on X, including their multiplicities (the numbers of geodesics realizing the lengths).

LEMMA 11.4.1. *Let X be a closed orientable hyperbolic surface. Then the length spectrum is discrete and the multiplicity of every length is finite.*

PROOF. We will write $X = \Gamma\backslash\mathbb{H}^2$ where Γ acts on $\mathbb{H}^2$ freely and properly discontinuously. We take a compact fundamental domain $\mathcal{F}$ for the Γ-action on $\mathbb{H}^2$.

Every closed geodesic $\gamma \subset X$ has a lift that intersects $\mathcal{F}$. This lift need not be unique, in particular, if γ is not simple, it never is. However, there are at most finitely many lifts. This lift is the axis of a unique (up to taking inverses) primitive hyperbolic element $g \in \Gamma$. This element has the property that

$$A = \mathrm{d}_{\mathbb{H}^2}\Big(g(\mathcal{F}), \mathcal{F}\Big) \leq \ell(\gamma)$$

In particular, the each closed geodesic of length $\leq R$ yields at least one distinct element of

$$\{\, g \in \Gamma \,:\, g \cdot N_{\leq R}(\mathcal{F}) \cap N_{\leq R}(\mathcal{F}) \neq \emptyset \,\},$$

where

$$N_{\leq R}(\mathcal{F}) = \left\{\, x \in \mathbb{H}^2 \,:\, \mathrm{d}_{\mathbb{H}^2}(x, \mathcal{F}) \leq R \,\right\},$$

which is compact. By proper discontinuity, this implies that A, and thus the set of geodesics of length $\leq R$ is finite. □

THEOREM 11.4.2. *The action of* $\mathrm{MCG}(\Sigma_g)$ *on* $\mathcal{T}(\Sigma_g)$ *is properly discontinuous.*

PROOF. Given a compact set $K \subset \mathcal{T}(\Sigma_g)$, we need to show that

$$\{\, \varphi \in \mathrm{MCG}(\Sigma_g) \,:\, \varphi K \cap K \neq \emptyset \,\}$$

is finite. Writing $\mathrm{diam}_{\mathrm{T}}(K)$ for the diameter of K with respect to the Teichmüller metric, we note that

$$\{\, \varphi \in \mathrm{MCG}(\Sigma_g) \,:\, \varphi K \cap K \neq \emptyset \,\} \subset \{\, \varphi \in \mathrm{MCG}(\Sigma_g) \,:\, \mathrm{d}_{\mathrm{T}}([X,f], \varphi([X,f])) \leq 2\mathrm{diam}_{\mathrm{T}}(K) \,\},$$

for some arbitrary point $[X,f] \in K$. Now let $\gamma_1, \gamma_2 \subset \Sigma_g$ be two simple closed curves that together fill Σ_g and write

$$L = \max\{\ell_{\gamma_1}([X,f]), \ell_{\gamma_2}([X,f])\}.$$

Suppose that $\mathrm{d}_{\mathrm{T}}([X,f], \varphi([X,f])) \leq 2\mathrm{diam}_{\mathrm{T}}(K)$, then by Wolpert's lemma (Lemma 10.4.1),

$$\ell_{\varphi^{-1}(\gamma_i)}([X,f]) = \ell_{\gamma_i}(\varphi \cdot [X,f]) \leq e^{4\mathrm{diam}_{\mathrm{T}}(K)} \cdot L$$

By Lemma 11.4.1, the number of distinct pairs $(\varphi^{-1}(\gamma_1), \varphi^{-1}(\gamma_2))$ that appear when we vary of all φ with $\mathrm{d}_{\mathrm{T}}([X,f], \varphi([X,f])) \leq 2\mathrm{diam}_{\mathrm{T}}(K)$ is finite. But, because the pair (γ_1, γ_2) is filling, the Alexander Lemma (Lemma 4.5.3) implies that $(\varphi^{-1}(\gamma_1), \varphi^{-1}(\gamma_2))$ determines φ up to homotopy and thus the number of distinct φ is finite. □

COROLLARY 11.4.3. *Let* $g \geq 2$. *The moduli space* $\mathcal{M}(\Sigma_g)$ *has the structure of a* $(3g-3)$-*dimensional complex orbifold.*

Bibliography

[Baa21] Sebastian Baader. *Geometry and topology of surfaces*. Zurich Lectures in Advanced Mathematics. EMS Press, Berlin, [2021] ©2021.

[Bea84] A. F. Beardon. *A primer on Riemann surfaces*, volume 78 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1984.

[Bus10] Peter Buser. *Geometry and spectra of compact Riemann surfaces*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2010. Reprint of the 1992 edition.

[CE08] Jeff Cheeger and David G. Ebin. *Comparison theorems in Riemannian geometry*. AMS Chelsea Publishing, Providence, RI, 2008. Revised reprint of the 1975 original.

[Fal23] Elisha Falbel. Introduction to Riemann surfaces. Lecture notes, available at: https://webusers.imj-prg.fr/~elisha.falbel/Surfaces/RS.html, 2023.

[FK92] H. M. Farkas and I. Kra. *Riemann surfaces*, volume 71 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1992.

[FM12] Benson Farb and Dan Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.

[GGD12] Ernesto Girondo and Gabino González-Diez. *Introduction to compact Riemann surfaces and dessins d'enfants*, volume 79 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2012.

[GH94] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994. Reprint of the 1978 original.

[GL00] Frederick P. Gardiner and Nikola Lakic. *Quasiconformal Teichmüller theory*, volume 76 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2000.

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[Hub06] John Hamal Hubbard. *Teichmüller theory and applications to geometry, topology, and dynamics. Vol. 1*. Matrix Editions, Ithaca, NY, 2006. Teichmüller theory, With contributions by Adrien Douady, William Dunbar, Roland Roeder, Sylvain Bonnot, David Brown, Allen Hatcher, Chris Hruska and Sudeb Mitra, With forewords by William Thurston and Clifford Earle.

[IT92] Y. Imayoshi and M. Taniguchi. *An introduction to Teichmüller spaces*. Springer-Verlag, Tokyo, 1992. Translated and revised from the Japanese by the authors.

[Mil65] John W. Milnor. *Topology from the differentiable viewpoint*. University Press of Virginia, Charlottesville, VA, 1965. Based on notes by David W. Weaver.

[Sma59] Stephen Smale. Diffeomorphisms of the 2-sphere. *Proc. Amer. Math. Soc.*, 10:621–626, 1959.

[SS03] Elias M. Stein and Rami Shakarchi. *Complex analysis*, volume 2 of *Princeton Lectures in Analysis*. Princeton University Press, Princeton, NJ, 2003.

[Wri15] Alex Wright. Translation surfaces and their orbit closures: an introduction for a broad audience. *EMS Surv. Math. Sci.*, 2(1):63–108, 2015.

[Zor06] Anton Zorich. Flat surfaces. In *Frontiers in number theory, physics, and geometry. I*, pages 437–583. Springer, Berlin, 2006.