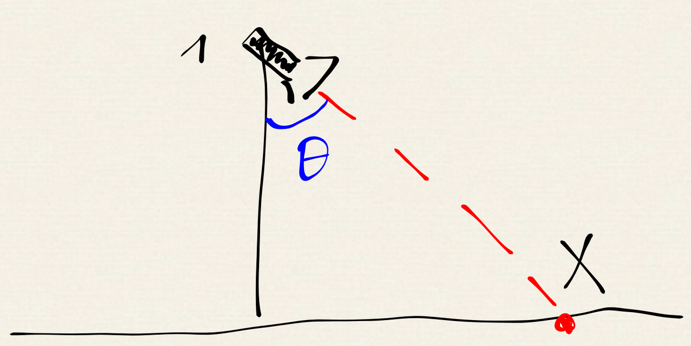
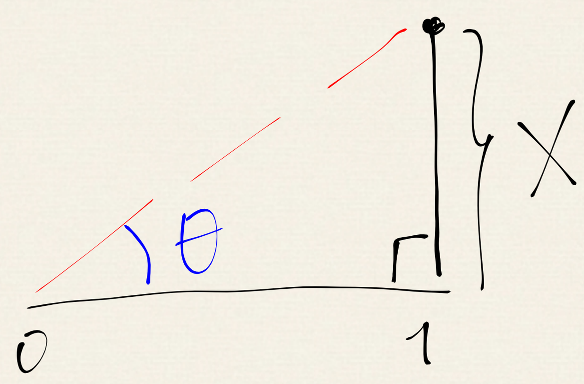
Exo 8.11



$\Theta \sim \mathrm{Unif}(]-\frac{\pi}{2}, \frac{\pi}{2}[)$

Question: Quelle est la densité de la loi de X ?



$\tan(\theta) = \frac{X}{1} = X$

Solution:

$$F_X(x) = \mathbb{P}(X \leq x)$$
$$= \mathbb{P}(\tan(\Theta) \leq x)$$
$$= \mathbb{P}(\Theta \leq \arctan(x))$$
$$= \int_{-\infty}^{\arctan(x)} f_\Theta(t)\, dt$$
$$= \int_{-\infty}^{\arctan(x)} \frac{1}{\pi} \mathbb{1}_{]-\frac{\pi}{2}, \frac{\pi}{2}[}(t)\, dt$$
$$= \frac{\arctan(x) - (-\frac{\pi}{2})}{\pi} = \frac{1}{\pi}\arctan(x) - \frac{1}{2}.$$

Rappel: Si $Y \sim \mathrm{Unif}(]a,b[)$ alors $f_Y(y) = \frac{1}{b-a} \mathbb{1}_{]a,b[}(y)$ $\forall y \in \mathbb{R}$.

F_X différentiable $\Rightarrow$ $f_X(x) = \frac{d}{dx}\left(\frac{1}{\pi}\arctan(x) - \frac{1}{2}\right)$

$$= \frac{1}{\pi} \frac{1}{1+x^2}$$

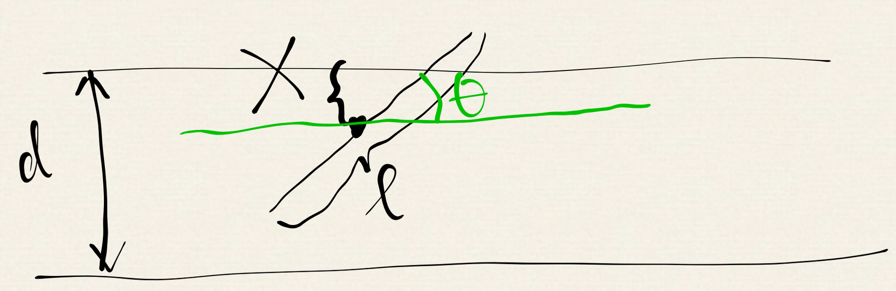
Donc X est une variable de Cauchy.

À 9h40 : l'exo 9.1.

Exo 9.1 (l'aiguille de Buffon)

On lance une aiguille sur un parquet de planches parallèles.

Q'n : quelle est la proba que l'aiguille intersecte une des rainures du parquet.



* $X \sim \mathrm{Unif}(]0, d/2[)$
* $\Theta \sim \mathrm{Unif}(]0, \frac{\pi}{2}[)$
* X et Θ indép.

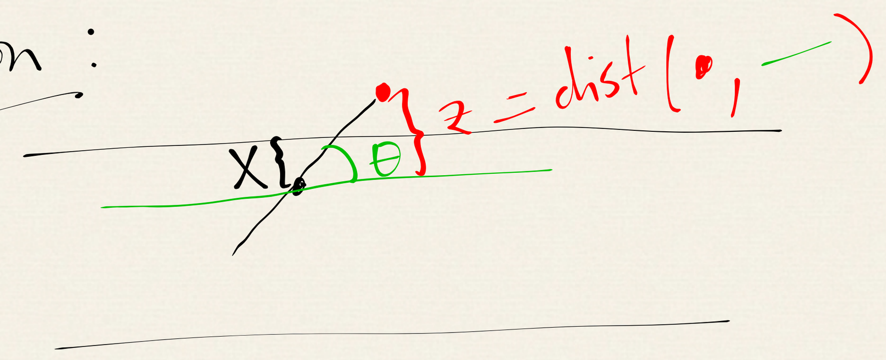
a) La densité jointe de (X,Θ).

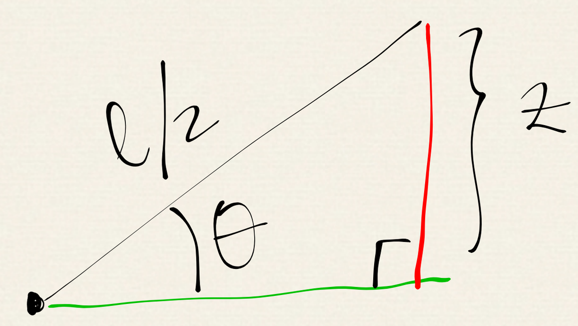
Solution : $f_{(X,\Theta)}(x,t) \overset{\text{indép.}}{=} f_X(x) \cdot f_\Theta(t) = \frac{2}{d} \mathbb{1}_{]0,d/2[}(x) \cdot \frac{2}{\pi} \mathbb{1}_{]0,\pi/2[}(t)$

$$= \frac{4}{d \cdot \pi} \mathbb{1}_{]0,d/2[}(x) \cdot \mathbb{1}_{]0,\pi/2[}(t)$$

b) "l'aiguille soit à cheval sur la rainure du parquet" $= \left\{ X < \frac{\ell}{2} \cdot \sin(\theta) \right\}$

Solution:





$$\sin(\theta) = \frac{z}{\ell/2}$$

$$z = \frac{\ell}{2} \cdot \sin(\theta)$$

Donc:

$$\{X < z\} = \left\{ X < \frac{\ell}{2} \sin(\theta) \right\}$$

$=$ "l'aiguille soit à cheval sur la rainure du parquet"

(c) On suppose $\ell \leq d$. Prouver que $\mathbb{P}\left(X < \frac{\ell}{2} \sin(\theta) \right) = \frac{2\ell}{\pi d}$.

Solution:

$$\mathbb{P}\left(X < \frac{\ell}{2} \cdot \sin(\theta) \right) = \iint_{\substack{x < \frac{\ell}{2}\sin(t) \\ 0 \leq x \leq d/2 \\ 0 \leq t \leq \pi/2}} f_{(X,\theta)}(x,t) \, dx \, dt$$

$$= \int_0^{\pi/2} \int_0^{\overbrace{\min\left\{ \frac{d}{2}, \frac{\ell}{2}\sin(t) \right\}}^{\ell \leq d \ \Rightarrow\ = \frac{\ell}{2}\sin(t)}} \frac{4}{\pi d} \, dx \, dt$$

$$= \int_0^{\pi/2} \int_0^{(\ell/2)\sin(t)} \frac{4}{\pi d} \, dx \, dt$$

$$= \int_0^{\pi/2} \frac{4}{\pi d} \frac{\ell}{2} \sin(t) \, dt$$

$$= \frac{2\ell}{\pi d} \left[-\cos(t) \right]_0^{\pi/2} = \frac{2\ell}{\pi d} \left[0 - -1 \right] = \frac{2\ell}{\pi d} \qquad \square$$

À $10^h 15$: l'exo 9.5.

Attention: $f_{S_n}(x) = \frac{\lambda^n}{(n-1)!} x^{n-1} e^{-\lambda x} \cdot \mathbb{1}_{]0,+\infty[}(x)$.

Exo 9.5 $(X_n)_{n \geq 1}$ suite de v.a. indéps. de loi $\mathcal{E}(\lambda)$ chacune.
On définit: $S_n = \sum_{k=1}^{n} X_k$. Prouver que $f_{S_n}(x) = \frac{\lambda^n}{(n-1)!} x^{n-1} e^{-\lambda x} \cdot \mathbb{1}_{]0,+\infty[}(x)$.

Solution: Pour $n=1$: $S_1 = X_1 \sim \mathcal{E}(\lambda)$ donc $f_{S_1}(x) = \lambda e^{-\lambda x} \mathbb{1}_{]0,+\infty[}(x)$.

Hérédité: Rappel (prop. 5.30): X, Y v.a. à densité indép. Alors $f_{X+Y}(z) = \int_{-\infty}^{+\infty} f_X(x) \cdot f_Y(z-x) \, dx$.

Donc: $f_{S_{n+1}}(z) = f_{S_n + X_{n+1}}(z) = \int_{-\infty}^{\infty} f_{S_n}(x) \cdot f_{X_{n+1}}(z-x) \, dx \overset{\text{hyp.}}{=} \int_0^{\infty} \underbrace{\frac{\lambda^n}{(n-1)!} x^{n-1} e^{-\lambda x}}_{f_{S_n}(x)} \cdot \underbrace{\lambda e^{-\lambda(z-x)} \cdot \mathbb{1}_{]0,+\infty[}(z-x)}_{f_{X_{n+1}}(z-x)} dx$

$= \frac{\lambda^{n+1}}{(n-1)!} e^{-\lambda z} \int_0^{z} x^{n-1} \, dx$ (z ← vient de $\mathbb{1}_{]0,+\infty[}(z-x)$) $= \frac{\lambda^{n+1}}{(n-1)!} e^{-\lambda z} \mathbb{1}_{]0,+\infty[}(z) \cdot \left[\frac{x^n}{n} \right]_0^z = \frac{\lambda^{n+1}}{n!} z^n e^{-\lambda z} \cdot \mathbb{1}_{]0,+\infty[}(z)$ $\square$

<u>La</u> prochaine fois: l'exo 9.7.