TD 8

Rappel : $X : \Omega \longrightarrow \mathbb{N}$ var. aléa.

Densité discrète : $p_X(k) = \mathbb{P}(X = k)$, $k \in \mathbb{N}$

Espérance : $\mathbb{E}(X) = \sum_{k=0}^{\infty} p_X(k) \cdot k$

Variance : $Var(X) = \mathbb{E}\left((X - \mathbb{E}(X))^2\right) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 \geq 0$

Formule de transfert : $g : \mathbb{N} \to \mathbb{R}$

$$\mathbb{E}(g(X)) = \sum_{k=0}^{\infty} p_X(k) \cdot g(k)$$

Exemple : si $g : \mathbb{N} \to \mathbb{N}$ déf pour $g(x) = x^2$, $\forall x \in \mathbb{N}$

on obtient : $\mathbb{E}(X^2) \overset{FdT}{=} \sum_{k=0}^{\infty} p_X(k) \cdot k^2$

Indication : (a) $\sum_{k=1}^{n} k = \binom{n+1}{2} = \frac{(n+1) \cdot n}{2}$

(b) $\sum_{k=1}^{n} k^2 = \frac{n \cdot (2n+1) \cdot (n+1)}{6}$

Preuve : (a) $\sum_{k=1}^{n} k^2 - (k-1)^2 = \underbrace{1^2 - 0^2}_{1^{ere}\text{ terme}} + \underbrace{2^2 - 1^2}_{2^{ième}} + \underbrace{3^2 - 2^2}_{3^{ième}} + \dots + \underbrace{n^2 - (n-1)^2}_{n^{ième}}$

$= n^2$ somme télescopique

Donc : $n^2 = \sum_{k=1}^{n} k^2 - (k^2 - 2k + 1) = \sum_{k=1}^{n} 2k - 1 = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1$

$= 2 \sum_{k=1}^{n} k - n$

donc $\sum_{k=1}^{n} k = \frac{n^2 + n}{2} = \frac{(n+1) \cdot n}{2}$.

(b) $n^3 = \sum_{k=1}^{n} k^3 - (k-1)^3 = \sum_{k=1}^{n} k^3 - (k^3 - 3k^2 + 3k - 1) = \sum_{k=1}^{n} 3k^2 - 3k + 1$

$= 3\sum_{k=1}^{n} k^2 - 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 \overset{(a)}{=} 3 \sum_{k=1}^{n} k^2 - 3 \frac{(n+1) \cdot n}{2} + n$

$\sum_{k=1}^{n} k^2 = \frac{1}{3}\left(n^3 + 3\frac{(n+1) \cdot n}{2} - n\right) = \frac{n(2n^2 + 3(n+1) - 2)}{6} = \frac{n(2n^2 + 3n + 1)}{6}$

$= \frac{n(2n+1)(n+1)}{6}$ $\square$

À g'20 : 5.1 (a).

Exo 5.1 (a) $X \sim Unif\{1, \dots, n\}$ c-à-d $\mathbb{P}(X = i) = \frac{1}{n}$ $\forall i = 1, \dots, n$.

Solution : $\mathbb{E}(X) = \sum_{k=1}^{n} p_X(k) \cdot k = \sum_{k=1}^{n} \frac{1}{n} \cdot k = \frac{1}{n} \sum_{k=1}^{n} k \overset{\text{Indication}}{=} \frac{1}{n} \cdot \frac{(n+1) \cdot n}{2} = \frac{n+1}{2}$

$\mathbb{E}(X^2) \overset{FdT}{=} \sum_{k=1}^{n} p_X(k) \cdot k^2 = \sum_{k=1}^{n} \frac{1}{n} k^2 = \frac{1}{n} \sum_{k=1}^{n} k^2 \overset{\text{Indication}}{=} \frac{1}{n} \frac{n(2n+1)(n+1)}{6} = \frac{(2n+1)(n+1)}{6}$

$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{(2n+1) \cdot (n+1)}{6} - \frac{(n+1)^2}{4} = \frac{2(2n+1)(n+1) - 3(n+1)^2}{12}$

$= \frac{(4n+2 - 3n - 3)(n+1)}{12} = \frac{(n-1)(n+1)}{12} = \frac{n^2 - 1}{12}$.

(b) $X \sim \text{Bern}(p)$, $p \in]0,1[$, c-à-d: $X : \Omega \to \{0,1\}$ $\mathbb{P}(X=1)=p$ et $\mathbb{P}(X=0)=1-p$.

À 9h40

Solution: $\mathbb{E}(X) = p_X(0)\cdot 0 + p_X(1)\cdot 1$

$= (1-p)\cdot 0 + p\cdot 1 = p.$

$\mathbb{E}(X^2) \overset{FdT}{=} p_X(0)\cdot 0^2 + p_X(1)\cdot 1^2$

$= (1-p)\cdot 0^2 + p \cdot 1^2 = p$

Donc: $Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = p - p^2 = p(1-p)$.

Exo 5.1 (c): À 10h15

Indication:

Méthode 1: Calcul direct.

Rappel: Newton: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

Donc: $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \cdot k = x \frac{\partial}{\partial x}\left(\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \right)$

$\overset{\text{Newton}}{=} x \frac{\partial}{\partial x}\left((x+y)^n\right)$

$= n\, x\, (x+y)^{n-1}$ $\quad *$

Et: $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} k^2 = x \frac{\partial}{\partial x}\left(\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} k \right)$

$\overset{*}{=} x \frac{\partial}{\partial x}\left(n\, x\, (x+y)^{n-1}\right)$

$= x \left(n\cdot (x+y)^{n-1} + n x \cdot (n-1) \cdot (x+y)^{n-2} \right)$

$= n\, x\, (x+y)^{n-1} + n(n-1)\, x^2 \cdot (x+y)^{n-2}$.

Méthode 2:

du cours: Si $X_1, \dots, X_n$ sont des v.a. indéps. et $X_i \sim \text{Bern}(p)$ $\forall\, i=1,\dots,n$ alors $X = X_1 + \dots + X_n \sim \text{Bin}(n,p)$.

De plus: Si $Y_1, \dots, Y_m$ sont des v.a. indéps alors

$Var(Y_1 + \dots + Y_n) = Var(Y_1) + \dots + Var(Y_n)$

Solution:

Méthode 1: $X \sim \text{Bin}(n,p)$ $X : \Omega \to \{0,\dots,n\}$ $\mathbb{P}(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$ $k=0,\dots,n$.

Donc: $\mathbb{E}(X) = \sum_{k=0}^{n} \binom{n}{k} \cdot p^k (1-p)^{n-k} \cdot k \underset{x=p,\, y=1-p}{\overset{\text{Indic. avec}}{=}} n\cdot p \cdot (p + 1 - p)^{n-1} = n \cdot p$

$\mathbb{E}(X^2) \overset{FdT}{=} \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \cdot k^2 \overset{\text{indic.}}{=} n\cdot p + n\cdot(n-1)\, p^2$

$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = n \cdot p + \underbrace{n\cdot(n-1)p^2}_{n^2p^2 - np^2} - n^2 p^2 = np - np^2 = np(1-p)$

Méthode 2: $\mathbb{E}(X) = \mathbb{E}(X_1 + \dots + X_n) = \mathbb{E}(X_1) + \dots + \mathbb{E}(X_n) \overset{(b)}{=} n \cdot p$
($X_1 \sim \text{Bern}(p)$, $X_n \sim \text{Bern}(p)$)

$Var(X) = Var(X_1 + \dots + X_n) \overset{\text{indép}}{=} Var(X_1) + \dots + Var(X_n) \overset{(b)}{=} n\, p\, (1-p)$.

où $X_1, \dots, X_n$ sont des $\text{Bern}(p)$ indéps.

$$\mathbb{E}(X) = \mathbb{E}(X_1 + \dots + X_n)$$

$X_i : \Omega \to \{0,1\}$

donc : $X_1 + \dots + X_n : \Omega \to \{0,1,\dots,n\}$

$$\left(= \sum_{k=0}^{n} p_{X_1+\dots+X_n}(k) \cdot k \right)$$

$$p_{X_1+\dots+X_n}(k) = \binom{n}{k} \cdot p^k (1-p)^{n-k}$$

$$= \mathbb{E}(X_1) + \dots + \mathbb{E}(X_n) = n \cdot p.$$

$$\mathbb{E}(X_1) = \sum_{k=0}^{1} p_{X_1}(k) \cdot k = (1-p) \cdot 0 + p \cdot 1 = p.$$

$$\mathbb{E}(X_2) = \mathbb{E}(X_3) = \dots = \mathbb{E}(X_n)$$

(d) $X \sim \text{Géom}(p)$ c-à-d $X : \Omega \to \mathbb{N}^*$ t.q. $p_X(k) = (1-p)^{k-1} \cdot p$, $k = 1, 2, \dots$

Indication:

Rappel: si $x \in]-1, 1[$ alors

$$\sum_{k=1}^{\infty} x^k = \frac{x}{1-x}$$

Donc: $$\sum_{k=1}^{\infty} x^k \cdot k = x \frac{d}{dx}\left(\sum_{k=1}^{\infty} x^k\right) = x \frac{d}{dx}\left(\frac{x}{1-x}\right) = x \frac{1\cdot(1-x) - (-1)\cdot x}{(1-x)^2} = x \frac{1-x+x}{(1-x)^2} = \frac{x}{(1-x)^2}$$

Et: $$\sum_{k=1}^{\infty} x^k k^2 = x \frac{d}{dx}\left(\sum_{k=1}^{\infty} x^k k\right) = x \frac{d}{dx}\left(\frac{x}{(1-x)^2}\right) = x \frac{1\cdot(1-x)^2 - 2(1-x)\cdot(-1)\cdot x}{(1-x)^4} = x \frac{1-x+2x}{(1-x)^3}$$

$$= \frac{x^2+x}{(1-x)^3}$$

À 11h20

Solution: $$\mathbb{E}(X) = \sum_{k=1}^{\infty} p_X(k) \cdot k = \sum_{k=1}^{\infty} (1-p)^{k-1} p \cdot k = \frac{p}{1-p} \sum_{k=1}^{\infty} (1-p)^k \cdot k \overset{\text{Indic.}}{\underset{x=1-p}{=}} \frac{p}{1-p} \frac{1-p}{(1-(1-p))^2}$$

$$= p \cdot \frac{1}{p^2} = \frac{1}{p}.$$

$$\mathbb{E}(X^2) \overset{\text{FdT}}{=} \sum_{k=1}^{\infty} p_X(k) \cdot k^2 = \sum_{k=1}^{\infty} (1-p)^{k-1} \cdot p \cdot k^2 = \frac{p}{1-p} \sum_{k=1}^{\infty} (1-p)^k \cdot k^2 = \frac{(1-p)^2 + (1-p)}{(1-(1-p))^3} \cdot \frac{p}{1-p}$$

$$= \frac{1-p+1}{p^2} = \frac{2-p}{p^2}$$

$$\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}.$$

5.1 (e): $X \sim \text{Poi}(\lambda)$, $\lambda > 0$. c-à-d: $X : \Omega \to \mathbb{N}$ et $\mathbb{P}(X=k) = \frac{\lambda^k e^{-\lambda}}{k!}$.

À 11h50

Solution: $$\mathbb{E}(X) = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \cdot k = \sum_{k=1}^{\infty} \frac{\lambda^k e^{-\lambda}}{(k-1)!} = \sum_{k'=0}^{\infty} \frac{\lambda^{k'+1} e^{-\lambda}}{k'!} = e^{-\lambda} \lambda \sum_{k'=0}^{\infty} \frac{\lambda^{k'}}{k'!}$$

$$\frac{k}{k!} = \frac{k}{k(k-1)\cdots 1} = \frac{1}{(k-1)\cdots 1} = \frac{1}{(k-1)!}$$

$$= e^{-\lambda} \cdot \lambda \cdot e^{\lambda} = \lambda.$$

Alternative:

$$= \lambda \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} = \lambda \sum_{k=0}^{\infty} \mathbb{P}(X=k) = \lambda \cdot \mathbb{P}(\Omega) = \lambda.$$

$$\frac{k(k-1)}{k!} = \frac{k(k-1)}{k(k-1)(k-2)\cdots 1} = \frac{1}{(k-2)!}$$

$$\mathbb{E}(X^2) = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} k^2 = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \cdot (k(k-1) + k) = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} k(k-1) + \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} k$$

$$= \sum_{k=2}^{\infty} \frac{\lambda^k e^{-\lambda}}{(k-2)!} + \lambda = \sum_{k'=0}^{\infty} \frac{\lambda^{k'+2} e^{-\lambda}}{k'!} + \lambda = e^{-\lambda} \lambda^2 \underbrace{\left(\sum_{k'=0}^{\infty} \frac{\lambda^{k'}}{k'!}\right)}_{e^{\lambda}} + \lambda = \lambda^2 + \lambda$$

Donc $\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$

Exo 5.3 : la prochaine fois.

$Y : \Omega \to \{2, \dots, 12\}$ $\quad Z : \Omega \to \{-5, \dots, 5\}$ $\quad$ v.a.

sont indép. ssi

$$\mathbb{P}(Y \in A \text{ et } Z \in B) = \mathbb{P}(Y \in A)\, \mathbb{P}(Z \in B) \quad \forall A \subset \{2, \dots, 12\} \quad B \subset \{-5, \dots, 5\}$$

ou (équivalent pour des v.a. discrètes) :

$$\mathbb{P}(Y = k \text{ et } Z = \ell) = \mathbb{P}(Y = k) \cdot \mathbb{P}(Z = \ell) \quad \forall k \in \{2, \dots, 12\} \text{ et } \ell \in \{-5, \dots, 5\}$$

$$\mathbb{P}(Y = k \text{ et } Z = \ell) = \mathbb{P}(\{w \in \Omega \,;\, Y(w) = k \text{ et } Z(w) = \ell\})$$

Pour nous : $\Omega = \{1, 2, 3, 4, 5, 6\}^2 = \{(i, j) \,;\, \begin{smallmatrix} 1 \le i \le 6 \\ 1 \le j \le 6 \end{smallmatrix}\}$

$\mathcal{F} = \mathcal{P}(\Omega)$

$\mathbb{P}$ = la proba. unif.

$X_1((i,j)) = i \qquad X_2((i,j)) = j$.

$$\mathbb{P}(X_1 = 2 \text{ et } X_2 = 3) = \mathbb{P}(\{w \in \Omega \,;\, X_1(w) = 2 \text{ et } X_2(w) = 3\})$$

$$= \mathbb{P}(\{(2,3)\}) = \frac{|\{(2,3)\}|}{|\Omega|} = \frac{1}{36}$$

$$\mathrm{Var}(X) = \mathbb{E}\left((X - \mathbb{E}(X))^2 \right)$$

Exemple : $X : \Omega \to \mathbb{N}$ où $\mathbb{P}(X = 27) = 1$ $\quad$ presque surement égal à 27

$\mathbb{E}(X) = \sum_{k \geq 0} p_X(k) \cdot k = 1 \cdot 27 = 27$.

$\mathbb{E}(X^2) = 1 \cdot 27^2 = 27^2$.

$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = 27^2 - 27^2 = 0$.

Obs : $\mathbb{E}(X - \mathbb{E}X) = \mathbb{E}(X) - \mathbb{E}(\mathbb{E}(X)) = \mathbb{E}(X) - \mathbb{E}(X) = 0$.