Lecture 9

Models of random surfaces

9.1 Pants decompositions

A direct consequence of Proposition 8.6 is that every closed hyperbolic surface of genus $g \geq 2$ can be obtained from iterating Example 8.4. That is, every closed hyperbolic surface can be built by gluing together pairs of pants. Indeed, just take a system of homotopy classes of closed curves that cut the surface into pairs of pants. Proposition 8.6 tells us that each of these homotopy classes contains a unique geodesic

Let us formalize the notion of a pants decomposition:

Definition 9.1. Let X be a closed surface. A *pants decomposition* of X is a set of pairwise disjoint simple closed curves $\{\gamma_1, \ldots, \gamma_n\}$ so that

$$X \setminus (\bigcup_{i=1}^{n} \gamma_i)$$

is a disjoint union of pairs of pants.

An Euler characteristic argument shows that if X is a closed surface of genus g, then the number of curves in a pants decomposition is necessarily equal to $3g - 3$.

Figure 9.1 gives some examples:

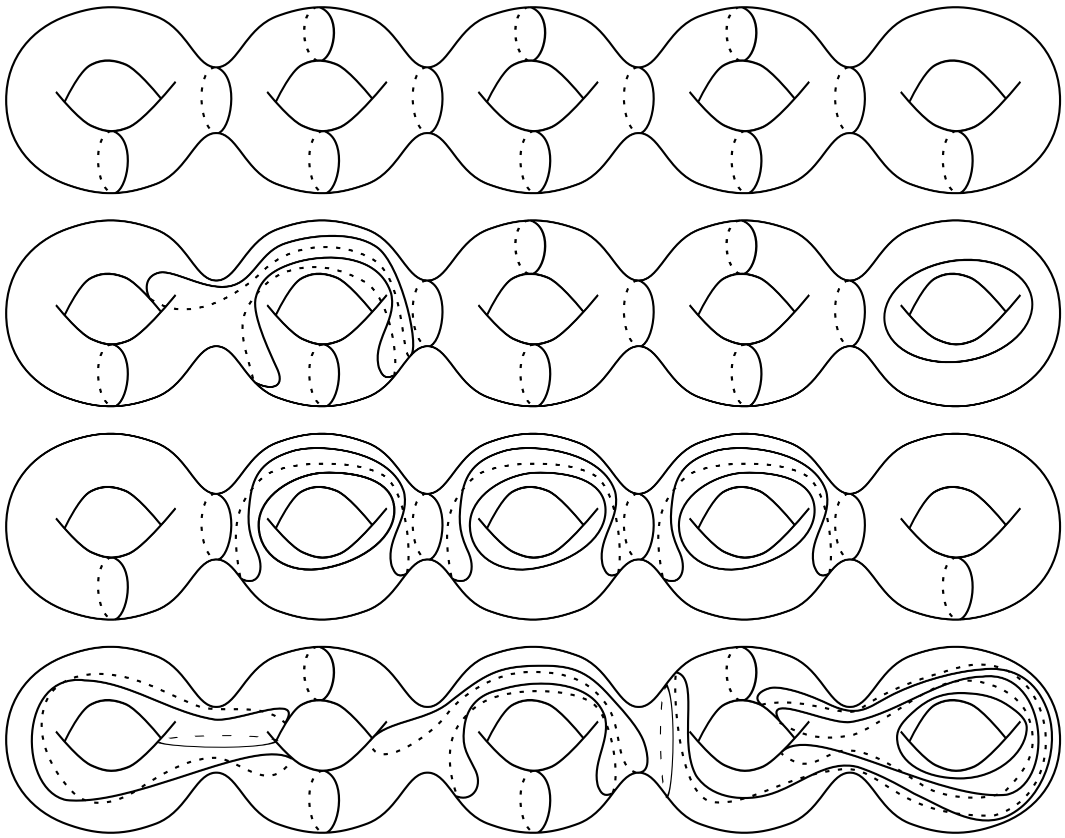

Figure 9.1: Four pants decompositions of a closed surface of genus 5.

To decide whether or not pants decompositions are the same up to diffeomorphism, the following graph is very useful:

Definition 9.2. Let X be a closed surface and let $\mathcal{P} = \{\gamma_1, \ldots, \gamma_n\}$ be a pants decomposition of X. The *dual graph* $G_{\mathcal{P}}$ to $\mathcal{P}$ is the graph obtained by setting

- $V(G_{\mathcal{P}})$ to be the set of connected components of $X \smallsetminus \mathcal{P}$,
- $E(G_{\mathcal{P}}) = \mathcal{P}$ and
- γ_i is incident to a conneceted component $C \in V(G_{\mathcal{P}})$ if and only if it is a boundary component of C.

Note that $G_{\mathcal{P}}$ is a connected 3-regular graph with $3g-3$ edges and hence $2g-2$ vertices.

Proposition 9.3. *Let X be a closed surface and let $\mathcal{P}_1$ and $\mathcal{P}_2$ be pants decompositions. There exists a diffeomorphism $\varphi : X \to X$ so that $\varphi(\mathcal{P}_1) = \varphi(\mathcal{P}_2)$ if and only if $G_{\mathcal{P}_1}$ and $G_{\mathcal{P}_2}$ are isomorphic graphs.*

Proof. Exercise 9.1. □

Note that this proposition implies that there are finitely many pants decompositions of a given surface up to diffeomorphism.

We already noted that every closed hyperbolic surface can be obtained by gluing pairs of pants together. In fact, only pairs of pants with boundary of a bounded length (in terms of the genus) are needed, this is a theorem by Bers. The best bound is due to Parlier [Par14].

Theorem 9.4. *Every closed hyperbolic surface of genus g has a pants decomposition in which every curve has length at most*

$$20 \cdot g.$$

9.2 Teichmüller space

For applications later on, we will need a nice space to parameterize our hyperbolic surfaces. This role will be played by Teichmüller space. The definition we give is not the usual definition and in a course on Teichmüller theory would be a theorem (originally proven by Fenchel and Nielsen)

Definition/Theorem 9.5. Let $g \geq 2$. *Teichmüller space* is the manifold

$$\mathcal{T}_g = (0, \infty)^{3g-3} \times \mathbb{R}^{3g-3}.$$

The coordinates ℓ_i are called the *length coordinates* and the coordinates τ_i are called the *twist coordinates*.

Given a closed surface X with a pants decomposition $\mathcal{P} = \{\gamma_1, \dots, \gamma_{3g-3}\}$, we define a hyperbolic surface for every point $(\ell, \tau) \in \mathcal{T}_g$ as follows.

First of all, assign the length ℓ_i to γ_i for $i = 1, \dots, 3g-3$. Because of Proposition 8.3, this completely determines the geometry of the pair of pants $P_1, \dots P_{2g-2}$ in the decomposition, we only need to decide how to glue them together (we need to pick diffeomorphisms φ_{i,τ_i} between the corresponding boundary components). Let P and P' be the (not necessarily distinct) pairs of pants that meet at γ_i and use the standard parameterization described earlier to parameterize the corresponding boundary components $\delta_{i,1} : \mathbb{S}^1 \to P$ and $\delta_{i,2} : \mathbb{S}^2 \to P'$. Now define

$$\varphi_i : \delta_{i,1} \to \delta_{i,2}$$

by

$$\varphi_i(\delta_{i,1}(t)) = \delta_{i,2}(\tau_i - t).$$

The picture to have in mind is the following.

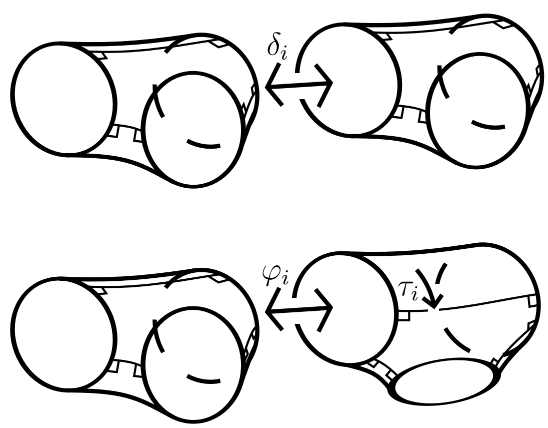

Figure 9.2: Twist.

The surface corresponding to $(\ell, \tau) \in \mathcal{T}_g$ is now given by

$$X(\ell, \tau) = \bigsqcup_{i=1}^{2g-2} P_i / \sim$$

where $\varphi_i(x) \sim x$ for all $x \in \delta_{i,1}$ and all $i = 1, \ldots, 3g-3$. Because this surface depends on the pants decomposition $\mathcal{P}$, we will sometimes denote it by $X_{\mathcal{P}}(\ell, \tau)$.

We note however that different points in $\mathcal{T}_g$ can give rise to isometric hyperbolic surfaces. For instance, there is an isometry

$$X(\ell_1, \ldots, \ell_{3g-3}, \tau_1, \ldots, \tau_{3g-3}) \to X(\ell_1, \ldots, \ell_{3g-3}, \tau_1 + \ell_1, \tau_2 \ldots, \tau_{3g-3}).$$

The quotient space $\mathcal{M}_g$ obtained by identifying points in Teichmüller space that define isometric surfaces is called the *moduli space* of closed hyperbolic surfaces of genus g. Note that from our definitions it is not clear that $\mathcal{M}_g$ is independent of the pants decomposition we use to define it. We will not prove the fact that it is indeed independent of the pants decomposition.

We also note that, even though we will not pursue this issue in this course, whereas the topology of Teichmüller space is very well understood, the toplogy of moduli space is a lot more complicated, so much so that many questions on it remain open.

In order to do probability theory later on, we need a volume form. To this end we will define the Weil-Petersson volume form. Again, we will not use the standard definition but rely on a theorem by Wolpert [Wol82] to define it.

Definition/Theorem 9.6. Let $A \subset \mathcal{T}_g$ be measurable. The *Weil-Petersson volume* of A is given by

$$\mathrm{vol}_{\mathrm{WP}}(A) = \int_A d\ell_1 \cdots d\ell_{3g-3} \cdot d\tau_1 \cdots d\tau_{3g-3}.$$

This measure descends to $\mathcal{M}_g$.

It is easy to see from the definition that the Weil-Petersson volume of $\mathcal{T}_g$ is infinite. From work by Wolpert [Wol82], it turns out that the Weil-Petersson volume of $\mathcal{M}_g$ is finite. The explicit bounds we need are due to Schumacher and Trappani [ST01], based on work of Penner [Pen92] and Grushevsky [Gru01].

Theorem 9.7. *There exist constants $a_1, a_2 > 0$ so that*

$$a_1^g \cdot g^{2g} \leq \mathrm{vol}_{\mathrm{WP}}(\mathcal{M}_g) \leq a_2^g \cdot g^{2g}.$$

The fact that $\mathrm{vol}_{\mathrm{WP}}(\mathcal{M}_g) < \infty$ leads to the following notion of random surfaces:

Definition 9.8. Let $g \in \mathbb{N}_{\geq 2}$ and let $\mathcal{B}(\mathcal{M}_g)$ be the Borel algebra of $\mathcal{M}_g$. We define the probability measure $\mathbb{P}_{\mathrm{WP}} : \mathcal{B}(\mathcal{M}_g) \to [0,1]$ by

$$\mathbb{P}_{\mathrm{WP}}[B] = \frac{\mathrm{vol}_{\mathrm{WP}}(B)}{\mathrm{vol}_{\mathrm{WP}}(\mathcal{M}_g)},$$

for all $B \in \mathcal{B}(\mathcal{M}_g)$.

9.3 Minimal total pants length

The goal of this section is to apply the probabilistic method to study the lengths of pants decompositions. Concretely, we will present a proof, due to Guth, Parlier and Young [GPY11], that there are surfaces that do not allow short pants decompositions.

Let us start with the definition of a random variable, which we will call *minimal total pants length*, $PL : \mathcal{M}_g \to \mathbb{R}$ by

$$PL(X) = \min \left\{ \sum_{i=1}^{3g-3} \ell(c_i);\ \{c_i\}_{i=1}^{3g-3} \text{ forms a pants decomposition of } X \right\}.$$

As a direct corollary of Theorem 9.4, we obtain:

Corollary 9.9. *Let $X \in \mathcal{M}_g$, then*

$$PL(X) \leq 60 \cdot g^2 - 60 \cdot g.$$

The main question in this section is how sharp this upper bound is. To this end, define $MPL : \mathbb{N}_{\geq 2} \to \mathbb{R}$ by

$$MPL(g) = \sup \left\{ PL(X);\ X \in \mathcal{M}_g \right\}.$$

We will prove the following theorem due to Guth, Parlier and Young [GPY11].

Theorem 9.10. *For all $\varepsilon > 0$ we have*

$$\lim_{g\to\infty} \mathbb{P}_{\mathrm{WP}}\left[X \in \mathcal{M}_g;\ PL(X) \leq g^{7/6-\varepsilon}\right] = 0.$$

Proof. The upper bound follows directly from Corollary 9.9.

The main part of the proof consists of controlling the Weil-Petersson volume of sets of the form

$$\left\{X \in \mathcal{M}_g;\ PL(X) \leq x\right\}.$$

Like we noted above, $\mathcal{M}_g$ is well-defined. The projection

$$\pi_{\mathcal{P}} : \mathcal{T}_g \to \mathcal{M}_g$$

does however depend on the pants decomposition $\mathcal{P}$. Note however that if for pants decompositions $\mathcal{P}$ and $\mathcal{P}'$ there exists a diffeomorphism $\varphi : \Sigma_g \to \Sigma_g$ such that

$$\varphi(\mathcal{P}) = \mathcal{P}'$$

then

$$\pi_{\mathcal{P}}(\ell,\tau) = \pi_{\mathcal{P}'}(\ell,\tau),$$

because $X_{\mathcal{P}}(\ell,\tau)$ and $X_{\mathcal{P}'}(\ell,\tau)$ are isometric. This means that if we let $\mathcal{I}_g$ denote the (finite) set of diffeomorphism types of pants decompositions of Σ_g, we have

$$\left\{X \in \mathcal{M}_g;\ PL(X) \leq x\right\} \subset \bigcup_{\mathcal{P}\in\mathcal{I}_g} \pi_{\mathcal{P}}\left(\left\{ (\ell,\tau)\in\mathcal{T}_g;\quad \begin{array}{c} \sum_{i=1}^{3g-3} \ell_i \leq x \\ \text{and } 0\leq \tau_i \leq \ell_i \end{array} \right\}\right).$$

Write

$$A_{g,x} = \left\{(\ell,\tau)\in\mathcal{T}_g;\ \sum_{i=1}^{3g-3}\ell_i \leq x \text{ and } 0 \leq \tau_i \leq \ell_i\right\}.$$

Our observations above imply that

$$\mathrm{vol}_{\mathrm{WP}}(\left\{X \in \mathcal{M}_g;\ PL(X) \leq x\right\}) \leq |\mathcal{I}_g| \cdot \mathrm{vol}_{\mathrm{WP}}\left(A_{g,x}\right).$$

The rest of the proof consists of two steps: bounding $|\mathcal{I}_g|$ and bounding the volume of the set $A_{g,x}$.

The bound on $|\mathcal{I}|$ we need is

$$|\mathcal{I}_g| \leq a^g g^g$$

for some $a > 0$ independent of g. Proving this is Exercise 9.4.

We have

$$\begin{aligned} \mathrm{vol}_{\mathrm{WP}}(A_{g,x}) &= \int_{\sum_i \ell_i \leq x} \prod_{i=1}^{3g-3} \left(\int_0^{\ell_i} d\tau_i \right) d\ell_1 \cdots d\ell_{3g-3} \\ &= \int_{\sum_i \ell_i \leq x} \prod_{i=1}^{3g-3} \ell_i \, d\ell_1 \cdots d\ell_{3g-3} \end{aligned}$$

By the arithmetic-geometric mean inequality, we have

$$\prod_{i=1}^{3g-3} \ell_i \leq \left(\frac{\sum_{i=1}^{3g-3} \ell_i}{3g-3} \right)^{3g-3} \leq b^g \frac{x^{3g}}{g^{3g}}$$

for some $b > 0$ independent of g. So we obtain

$$\mathrm{vol}_{\mathrm{WP}}(A_{g,x}) \leq b^g \frac{x^{3g}}{g^{3g}} \int_{\sum_{i=1} \ell_i \leq x} d\ell_1 \cdots d\ell_{3g-3}.$$

It can be proved by induction that

$$\int_{\sum_{i=1} \ell_i \leq x} d\ell_1 \cdots d\ell_{3g-3} = \frac{x^{3g-3}}{(3g-3)!} \leq c^g \frac{x^{3g}}{g^{3g}},$$

for some $c > 0$ independent of g.

Putting all our estimates together, we obtain that

$$\mathrm{vol}_{\mathrm{WP}}(A_{g,x}) \leq d^g \frac{x^{6g}}{g^{5g}}.$$

Using Theorem 9.6, we obtain that

$$\mathbb{P}_{\mathrm{WP}} \left[X \in \mathcal{M}_g ; \; PL(X) \leq x \right] \leq r^g \frac{x^{6g}}{g^{7g}},$$

for some $r > 0$ independent of g. So, if $x \leq g^{7/6-\varepsilon}$ this probability tends to 0 as $g \to \infty$ and we are done. $\square$

As a consequence we obtain:

Corollary 9.11. *For all* $\varepsilon > 0$ *there exists a* $g_0 = g_0(\varepsilon) \in \mathbb{N}_{\geq 2}$ *so that*

$$g^{7/6-\varepsilon} \leq MPL(g) \leq 60 \cdot g^2 - 60 \cdot g$$

for all $g \geq g_0$.

9.4 Random triangulations

Another model for random surfaces is obtained from random triangulations. This is a model based on the configuration model of random 3-regular graphs on $2N$ vertices. Recall that the basis for the configuration model is a collection of disjoint sets $W_i(2N)$, $i = 1, \dots, 2N$. For convenience, we will just set

$$W_1(2N) = \{1, 2, 3\},\ W_2(2N) = \{4, 5, 6\}, \dots,$$
$$W_{2N}(2N) = \{6N-2, 6N-1, 6N\}.$$

We want to assign an oriented closed surface $S(C)$ without boundary to each 3-regular configuration C on $2N$ vertices. This goes as follows. Take $2N$ triangles (2-simplices) $\Delta_1, \dots, \Delta_{2N}$, and label the sides of the first triangle with the labels 1, 2 and 3, those of the second 4, 5 and 6 and so forth (see the figure below).

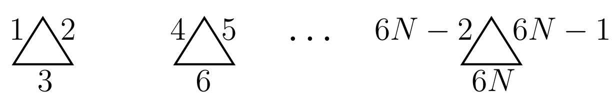

Figure 9.3: $2N$ labeled triangles.

Each of these triangles naturally comes with an orientation (induced by the cyclic order of the labels on the sides). For each pair of labels $c = \{i, j\} \in C$ fix an orientation reversing simplicial map φ_c between the corresponding sides. We set

$$S(C) = \bigsqcup_{i=1}^{2N} \Delta_i / \sim$$

where the equivalence relation is given by the collection of maps $\{\varphi_c\}_{c\in C}$. From now on we will speak of configurations on $2N$ *triangles* instead of on $2N$ vertices.

Figure 9.4 gives some examples for $N = 1$.

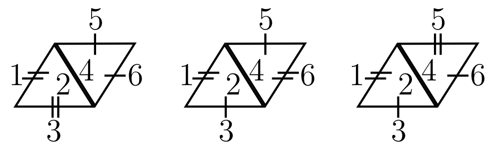

Figure 9.4: The surfaces corresponding to the configurations $\{\{1,3\},\{2,4\},\{5,6\}\}$, $\{\{1,6\},\{2,4\},\{3,5\}\}$ and $\{\{1,5\},\{2,4\},\{3,6\}\}$: a sphere, a torus and a sphere respectively.

Let us denote the set of all configurations on $2N$ triangles by Ω_N. We define a probability measure using the counting measure again:

Definition 9.12. Let $N \in \mathbb{N}$. We define the probability measure $\mathbb{P}_N : \mathcal{P}(\Omega_N) \to [0,1]$ by

$$\mathbb{P}_N[A] = \frac{|A|}{|\Omega_N|}$$

for all $A \subset \Omega_N$.

The main question we will work on in this course is the topology of these surfaces. This model can also be turned into a model for random hyperbolic surfaces, but we will not discuss the geometry of these surfaces in this course (see [BM04] for more details).

We first state, but will not prove, a theorem on the connectivity of these surfaces due to Bollobaás [Bol81] and Wormald [Wor81]:

Theorem 9.13. *We have*

$$\lim_{N\to\infty} \mathbb{P}_N[S \textit{ is connected}] = 1.$$

Because of the classification of surfaces, this theorem implies that in order to understand the topology of these surfaces, the only thing that remains to be understood is the distribution of their genus, which is the content of the following lecture.

9.5 Exercises

Exercise 9.1. Prove Proposition 9.3.

Hint: for one of the directions, find a way to use Proposition 8.3 and the fact that a surjective distance preserving map between closed hyperbolic surfaces is automatically a diffeomorphism (this is a special case of what is called the Myers-Steenrod theorem).

Exercise 9.2. Which of the pants decompositions in Figure 9.1 are diffeomorphic?

Exercise 9.3. It is known that

$$\mathbb{P}_{n,3}[\text{The graph is connected}] \to 1$$

as $n \to \infty$ for the configuration model for random regular graphs (Bonus exercise that will not be part of the exam: show this).

Show that the number of pants decompositions of a closed surface of genus g in which there are no two pairs of pants that share two boundary components and no curves incident to just one pair of pants is asymptotic to

$$\frac{e^{-2}(6g-6)!!}{6^{2g-2} \cdot (2g-2)!}$$

as $g \to \infty$.

Exercise 9.4. Let $\mathcal{I}_g$ denote the set of diffeomorphism classes of pants decompositions.

(a) Given a pants decomposition $\mathcal{P}$ of Σ_g, show that the number of automorphisms of the dual graph $G_{\mathcal{P}}$ can be bounded by

$$(2g-2) \cdot 6^{2g-2}.$$

Hint: suppose we know that an automorphism sends a vertex v in $G_{\mathcal{P}}$ to a vertex w, how many choices are left?

(b) Show that there exists a constant $a > 0$ so that

$$|\mathcal{I}_g| \leq a^g \cdot g^g$$

Exercise 9.5. Give an example of a sequence of configurations $(C_g)_{g=1}^{\infty}$ so that $S(C_g)$ is a connected surface of genus g for every g.

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