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BIRATIONAL MOTIVES AND THE NORM RESIDUE ISOMORPHISM THEOREM

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Abstract

We point out a relationship between the norm residue isomorphism theorem of Suslin-Voevodsky-Rost and the theory of birational motives, as well as its generalisation to "higher jets".

Let k be a perfect field, and let $\mathrm{DM}^{\mathrm{eff}}$ denote Voevodsky's (unbounded) triangulated category of effective Nisnevich motivic complexes over k. For $D \in \mathrm{DM}^{\mathrm{eff}}$ and $n \geqslant 0$, we have an adjunction morphism

$$\underline{\mathrm{Hom}}(\mathbf{Z}(n), D)(n) \to D \tag{1}$$

where $\underline{\mathrm{Hom}}$ is the internal Hom of $\mathrm{DM}^{\mathrm{eff}}$ and, as usual, we abbreviate the notation $\otimes\mathbf{Z}(n)$ to (n). The following lemma is well-known:

Lemma 1. *The morphism* (1) *is an isomorphism if and only if D is divisible by $\mathbf{Z}(n)$, i.e. if D is of the form $E(n)$.*

Proof. By Voevodsky's cancellation theorem [11], twisting by n is fully faithful on $\mathrm{DM}^{\mathrm{eff}}$. □

Recall that $\mathrm{DM}^{\mathrm{eff}}$ carries a "homotopy" t-structure. Let now $\mathrm{DM}^{\mathrm{eff}}_{\text{ét}}$ be the triangulated category of Ã©tale motivic complexes. It also carries a homotopy t-structure for which the "change of topology" functor

$$\mathrm{DM}^{\mathrm{eff}} \xrightarrow{\alpha^*} \mathrm{DM}^{\mathrm{eff}}_{\text{ét}} \tag{2}$$

is t-exact; the functor α^* has a right adjoint $R\alpha_*$ [5, C.4]. The main result of this note is:

Theorem 1. *Suppose that $D = R\alpha_* C$ with $C \in \mathrm{DM}^{\mathrm{eff}}_{\text{ét}}$ bounded and torsion (i.e. that $C \otimes \mathbf{Q} = 0$). Then* (1) *is an isomorphism.*

This provides a large quantity of objects of $\mathrm{DM}^{\mathrm{eff}}$ which are infinitely divisible by $\mathbf{Z}(1)$, of a quite different nature from those of [2, Rem. 1.10].

Corollary 1. *Suppose that $D = R\alpha_* C$ with $C \in \mathrm{DM}^{\mathrm{eff}}_{\text{ét}}$ bounded. Then the cone of* (1)Â *is uniquely divisible (multiplication by m is an isomorphism for all $m \neq 0$).*

Proof. Consider the exact triangles $C \xrightarrow{m} C \to C \otimes \mathbf{Z}/m \xrightarrow{+1}$. □

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Remarks 1. a) Theorem 1 is false for D torsion in general, as the example $D = \mathbf{Z}/l$ shows. Similarly, the torsion hypothesis on C is necessary, as the example $C = \alpha^*\mathbf{Q}$ shows.
b) For $C \in \mathrm{DM}^{\mathrm{eff}}_{\text{ét}}$, an adjunction game provides an isomorphism in $\mathrm{DM}^{\mathrm{eff}}$

$$R\alpha_* \underline{\mathrm{Hom}}_{\text{ét}}(\alpha^*\mathbf{Z}(n), C) \xrightarrow{\sim} \underline{\mathrm{Hom}}(\mathbf{Z}(n), R\alpha_* C)$$

where $\underline{\mathrm{Hom}}_{\text{ét}}$ is the internal Hom of $\mathrm{DM}^{\mathrm{eff}}_{\text{ét}}$. Let $m > 0$ be an integer invertible in k. The change of coefficients functor

$$\mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(k) \to \mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(k, \mathbf{Z}/m)$$

has a right adjoint i_m which induces a natural isomorphism

$$\underline{\mathrm{Hom}}_{\text{ét}}(C', i_m C) \simeq i_m \underline{\mathrm{Hom}}^m_{\text{ét}}(C' \otimes \mathbf{Z}/m, C)$$

for any $(C', C) \in \mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(k) \times \mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(k, \mathbf{Z}/m)$ where $\underline{\mathrm{Hom}}^m_{\text{ét}}$ is the internal Hom of $\mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(k, \mathbf{Z}/m)$. Take $C' = \alpha^*\mathbf{Z}(n)$; then $C' \otimes \mathbf{Z}/m = \mu_m^{\otimes n}$. Thus, if $C \in \mathrm{DM}^{\mathrm{eff}}_{\text{ét}}$ is bounded and m-torsion, the isomorphism (1) of Theorem 1 for $D = R\alpha_* C$ takes the form

$$R\alpha_* \underline{\mathrm{Hom}}^m_{\text{ét}}(\mu_m^{\otimes n}, C)(n) \xrightarrow{\sim} R\alpha_* C.$$

For $C = \mu_m^{\otimes i}$, this gives as a special case an isomorphism

$$R\alpha_* \mu_m^{\otimes i-n}(n) \xrightarrow{\sim} R\alpha_* \mu_m^{\otimes i}. \tag{3}$$

Let $\Gamma_m = Gal(k(\mu_m)/k)$: this is a subgroup of $(\mathbf{Z}/m)^*$. Since $\mu_m^{\otimes n} \simeq \mathbf{Z}/m$ when n is divisible by $\gamma_m = |\Gamma_m|$, this also gives the following corollary.

Corollary 2. *For $C \in \mathrm{DM}^{\mathrm{eff}}_{\text{ét}}$ bounded and of exponent m, the function $n \mapsto \underline{\mathrm{Hom}}(\mathbf{Z}(n), R\alpha_* C)$ is periodic of period γ_m.* □

The following reformulation of Theorem 1 will be useful. Recall that, in [6], we introduced and studied a triangulated category of birational motivic complexes DM^{o}; by *loc. cit.*, Prop. 4.2.5., one has

$$\mathrm{DM}^{\mathrm{o}} = \mathrm{DM}^{\mathrm{eff}} / \mathrm{DM}^{\mathrm{eff}}(1).$$

Higher versions of DM^{o} were introduced in [4, Def. 3.4] (they are also implicit in [2]):

$$\mathrm{DM}^{\mathrm{eff}}_{<n} = \mathrm{DM}^{\mathrm{eff}} / \mathrm{DM}^{\mathrm{eff}}(n)$$

so that $\mathrm{DM}_{<1} = \mathrm{DM}^{\mathrm{o}}$.

By *loc. cit.*, Prop. 3.5, the localisation functor $\nu_{<n} : \mathrm{DM}^{\mathrm{eff}} \to \mathrm{DM}^{\mathrm{eff}}_{<n}$ has a right adjoint ι_n; moreover, the homotopy t-structure of $\mathrm{DM}^{\mathrm{eff}}$ induces a t-structure on $\mathrm{DM}^{\mathrm{eff}}_{<n}$ via ι_n (*loc. cit.*, Prop. 3.6). By Lemma 1, *Theorem 1 is then equivalent to saying that $\nu_{<n} R\alpha_* C = 0$ for any bounded torsion $C \in \mathrm{DM}^{\mathrm{eff}}_{\text{ét}}$.*

Proof of Theorem 1. We start with a Grothendieckian dÃ©vissage, first reducing to the case where $C = \mathcal{F}[0]$ for a torsion sheaf $\mathcal{F}$. Such $\mathcal{F}$ comes from the *small* Ã©tale site of $\operatorname{Spec} k$ by the Suslin-Voevodsky rigidity theorem [8], i.e. is a Galois module. The functor $\nu_{<n}$ commutes with infinite direct sums as a left adjoint; since Ã©tale cohomology of sheaves has the same property, this reduces us to the case where the stalk(s) of $\mathcal{F}$ are finite, killed by some $m > 0$ that we may further assume to be a power of a prime number l different from the characteristic (since $\mathrm{DM}^{\mathrm{eff}}_{\text{ét}}$ is $\mathbf{Z}[1/p]$-linear, where p is the exponential characteristic of k [10, Prop. 3.3.3 2)]). A standard transfer/l-Sylow argument now allows us to assume that

$\mathcal{F}$ becomes constant after a finite Galois extension of k whose Galois group G has order a power of l. Since the statement is stable under extensions of sheaves and the l-group G acts unipotently on $\mathcal{F}$, we may finally assume that $\mathcal{F} = \mathbf{Z}/l$. By a standard argument due to Tate ($[k(\mu_l) : k]$ is prime to l), we further reduce to the case where $\mu_l \subset k$.

We now use Levine's "inverting the motivic Bott element" theorem [7]. The following corrects the presentation in [3, §3]. From the isomorphism

$$\mathbf{Z}(1) \simeq \mathbb{G}_m[-1] \tag{4}$$

we get an exact triangle

$$\mu_l[0] \to \mathbf{Z}/l(1) \to \mathbb{G}_m/l[-1] \xrightarrow{+1} . \tag{5}$$

Adding the isomorphism $\mathbf{Z}(n) \otimes \mathbf{Z}/l(1) \xrightarrow{\sim} \mathbf{Z}/l(n+1)$, we get a map in $\mathrm{DM}^{\mathrm{eff}}$:

$$\mathbf{Z}(n) \otimes \mu_l \to \mathbf{Z}/l(n+1). \tag{6}$$

For clarity, write $C \mapsto C\{n\}$ for tensoring an object $C \in \mathrm{DM}^{\mathrm{eff}}$ such that $lC = 0$ with the one-dimensional $\mathbf{Z}/l$-vector space $\mu_l^{\otimes n}$. Thus $\mathbf{Z}(n) \otimes \mu_l \simeq \mathbf{Z}/l(n)\{1\}$, hence we get from (6) another map

$$\mathbf{Z}/l(n) \to \mathbf{Z}/l(n+1)\{-1\}$$

which becomes an isomorphism after sheafifying for the étale topology. Iterating, we get a commutative diagram

$$\begin{array}{ccccccc} \mathbf{Z}/l(0) & \longrightarrow & \mathbf{Z}/l(1)\{-1\} & \longrightarrow & \mathbf{Z}/l(2)\{-2\} & \longrightarrow & \dots \\ & \searrow & \searrow & & \downarrow & \swarrow & \\ & & & & R\alpha_*\mathbf{Z}/l & & \end{array}$$

which induces a morphism

$$\operatorname{hocolim}_r \mathbf{Z}/l(r)\{-r\} \to R\alpha_*\mathbf{Z}/l \tag{7}$$

where "the" homotopy colimit is the one of BÃ¶kstedt-Neeman [1]; the main theorem of [7] is that (7) is an isomorphism when $l > 2$, or when $l = 2$ and either char $k > 0$ or -1 is a square in k.

This concludes the proof except for $l = 2$ in the exceptional case; we complete this case with the following proposition, which gives an "unstable" version of the previous divisibility at a higher cost. □

Proposition 1. *One has $\nu_{<n} R^q \alpha_* \mathbf{Z}/l = 0$ for $q > n$, and $\nu_{<n} R\alpha_*\mathbf{Z}/l = 0$. (See comment before the proof of Theorem 1 for $\nu_{<n}$.)*

Proof. By the Beilinson-Lichtenbaum conjecture [9, 12], we have an exact triangle for any $q \geqslant 0$:

$$\mathbf{Z}/l(q)\{-q\} \to R\alpha_*\mathbf{Z}/l \to \tau_{>q} R\alpha_*\mathbf{Z}/l \xrightarrow{+1} \tag{8}$$

Suppose that $q \geqslant n$. Applying $\nu_{<n}$ to (8), we get an isomorphism

$$\nu_{<n} R\alpha_*\mathbf{Z}/l \xrightarrow{\sim} \nu_{<n}\tau_{>q} R\alpha_*\mathbf{Z}/l.$$

Comparing this with the same isomorphism for $q+1$, we get the first statement. Therefore, $\nu_{<n}\tau_{>q} R\alpha_*\mathbf{Z}/l = 0$ for any $q \geqslant n$ (for example for $q = n$) and we conclude. □

Remarks 2. a) Since $R^0\alpha_*\mathbf{Z}/l = \mathbf{Z}/l$ is a birational sheaf, we have an isomorphism $\mathbf{Z}/l \xrightarrow{\sim} \nu_{<n}R^q\alpha_*\mathbf{Z}/l$ for $q = 0$. When $n = 1$, this allows us to compute $\nu_{<1}R^1\alpha_*\mathbf{Z}/l$ thanks to the isomorphisms

$$\tau_{\leqslant 1}\nu_{<1}R\alpha_*\mathbf{Z}/l \xrightarrow{\sim} \nu_{<1}R\alpha_*\mathbf{Z}/l = 0$$

where the first (*resp.* second) isomorphism follows from the first (*resp.* second) part of Proposition 1: this gives

$$\nu_{<1}R^1\alpha_*\mathbf{Z}/l \simeq \mathbf{Z}/l[2].$$

It is less clear how to compute $\nu_{<n}R^q\alpha_*\mathbf{Z}/l$ for $0 < q \leqslant n$ when $n \geqslant 2$.
b) In the spirit of [**3**], Proposition 1 for $n = 1$ conversely implies formally the Beilinson-Lichtenbaum conjecture: we neglect twists by powers of μ_l for simplicity, and argue by induction on q. Since $\otimes$ is right t-exact in $\mathrm{DM}^{\mathrm{eff}}$ and in view of (4), there is a natural map

$$\tau_{\leqslant q-1}R\alpha_*\mathbf{Z}/l \otimes \mathbf{Z}(1) \to \tau_{\leqslant q}R\alpha_*\mathbf{Z}/l \tag{9}$$

and we have to show that it is an isomorphism. The condition $\nu_{<1}R^q\alpha_*\mathbf{Z}/l = 0$ means that $R^q\alpha_*\mathbf{Z}/l$ is divisible by $\mathbf{Z}(1)$, which in turn implies that $\tau_{\leqslant q}R\alpha_*\mathbf{Z}/l$ is divisible by $\mathbf{Z}(1)$. But the computation of the Ã©tale cohomology of $X \times \mathbb{G}_m$ for smooth X shows that the adjoint of (9)

$$\tau_{\leqslant q-1}R\alpha_*\mathbf{Z}/l \to \underline{\mathrm{Hom}}(\mathbf{Z}(1), \tau_{\leqslant q}R\alpha_*\mathbf{Z}/l)$$

is an isomorphism, and we conclude with Lemma 1.
c) If k has virtually finite Ã©tale cohomological dimension (*e.g.* is finitely generated), we can relax the hypothesis "bounded" to "bounded below" in Theorem 1 and Corollary 1. Indeed, we reduce by a transfer argument to the case where k has finite cohomological dimension. As used before, $\nu_{<n}$ commutes with infinite direct sums, and so does $R\alpha_*$ by Lemma 2 below. Since, for any $C \in \mathrm{DM}^{\mathrm{eff}}_{\text{ét}}$, the natural map

$$\operatorname{hocolim} \tau_{\leqslant n}C \to C$$

is an isomorphism, this reduces us to the bounded case.

Lemma 2. *Suppose that k has finite Ã©tale cohomological dimension. Then $R\alpha_*$ commutes with infinite direct sums.*

Proof. Let $(C_i)_{i\in I}$ be a family of objects of $\mathrm{DM}^{\mathrm{eff}}_{\text{ét}}$. We want to show that the comparison map in $\mathrm{DM}^{\mathrm{eff}}$

$$\bigoplus_i R\alpha_*C_i \to R\alpha_*\bigoplus_i C_i$$

is an isomorphism. This can be tested against the generators $M(X)$, where X runs through smooth separated k-schemes of finite type. This yields the maps

$$H^n_{\mathrm{Nis}}(X, \bigoplus_i R\alpha_*C_i) \to H^n_{\text{ét}}(X, \bigoplus_i C_i), \quad n \in \mathbf{Z}$$

so we are reduced to showing that $H^n_{\mathrm{Nis}}(X, -)$ and $H^n_{\text{ét}}(X, -)$ commute with $\bigoplus_i$. Since hypercohomology spectral sequences are convergent (by the hypothesis on k for $H^n_{\text{ét}}(X, -)$), we are reduced to the case of direct sums of sheaves, and the result is true because the Nisnevich and Ã©tale sites are both coherent. □

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