

BIRATIONAL MOTIVES AND THE NORM RESIDUE ISOMORPHISM THEOREM

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ABSTRACT. We point out a relationship between the norm residue isomorphism theorem of Suslin-Voevodsky-Rost and the theory of birational motives, as well as its generalisation to “higher jets”.

Let k be a perfect field, and let $\mathrm{DM}^{\mathrm{eff}}$ denote Voevodsky’s (unbounded) triangulated category of effective Nisnevich motivic complexes over k . For $D \in \mathrm{DM}^{\mathrm{eff}}$ and $n \geq 0$, we have an adjunction morphism

$$(1) \quad \underline{\mathrm{Hom}}(\mathbf{Z}(n), D)(n) \rightarrow D$$

where $\underline{\mathrm{Hom}}$ is the internal Hom of $\mathrm{DM}^{\mathrm{eff}}$ and, as usual, we abbreviate the notation $\otimes \mathbf{Z}(n)$ to (n) . The following lemma is well-known:

Lemma 1. *The morphism (1) is an isomorphism if and only if D is divisible by $\mathbf{Z}(n)$, i.e. if D is of the form $E(n)$.*

Proof. By Voevodsky’s cancellation theorem [11], twisting by n is fully faithful on $\mathrm{DM}^{\mathrm{eff}}$. □

Recall that $\mathrm{DM}^{\mathrm{eff}}$ carries a “homotopy” t -structure. Let now $\mathrm{DM}_{\acute{\mathrm{e}}\mathrm{t}}^{\mathrm{eff}}$ be the triangulated category of étale motivic complexes. It also carries a homotopy t -structure for which the “change of topology” functor

$$(2) \quad \mathrm{DM}^{\mathrm{eff}} \xrightarrow{\alpha^*} \mathrm{DM}_{\acute{\mathrm{e}}\mathrm{t}}^{\mathrm{eff}}$$

is t -exact; the functor α^* has a right adjoint $R\alpha_*$ [5, C.4]. The main result of this note is:

Theorem 1. *Suppose that $D = R\alpha_* C$ with $C \in \mathrm{DM}_{\acute{\mathrm{e}}\mathrm{t}}^{\mathrm{eff}}$ bounded and torsion (i.e. that $C \otimes \mathbf{Q} = 0$). Then (1) is an isomorphism.*

This provides a large quantity of objects of $\mathrm{DM}^{\mathrm{eff}}$ which are infinitely divisible by $\mathbf{Z}(1)$, of a quite different nature from those of [2, Rem. 1.10].

Corollary 1. *Suppose that $D = R\alpha_* C$ with $C \in \mathrm{DM}_{\acute{\mathrm{e}}\mathrm{t}}^{\mathrm{eff}}$ bounded. Then the cone of (1) is uniquely divisible (multiplication by m is an isomorphism for all $m \neq 0$).*

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Proof. Consider the exact triangles $C \xrightarrow{m} C \rightarrow C \otimes \mathbf{Z}/m \xrightarrow{+1}$. \square

Remarks 1. a) Theorem 1 is false for D torsion in general, as the example $D = \mathbf{Z}/l$ shows. Similarly, the torsion hypothesis on C is necessary, as the example $C = \alpha^* \mathbf{Q}$ shows.

b) For $C \in \mathrm{DM}_{\acute{e}t}^{\mathrm{eff}}$, an adjunction game provides an isomorphism in $\mathrm{DM}^{\mathrm{eff}}$

$$R\alpha_* \underline{\mathrm{Hom}}_{\acute{e}t}(\alpha^* \mathbf{Z}(n), C) \xrightarrow{\sim} \underline{\mathrm{Hom}}(\mathbf{Z}(n), R\alpha_* C)$$

where $\underline{\mathrm{Hom}}_{\acute{e}t}$ is the internal Hom of $\mathrm{DM}_{\acute{e}t}^{\mathrm{eff}}$. Let $m > 0$ be an integer invertible in k . The change of coefficients functor

$$\mathrm{DM}_{\acute{e}t}^{\mathrm{eff}}(k) \rightarrow \mathrm{DM}_{\acute{e}t}^{\mathrm{eff}}(k, \mathbf{Z}/m)$$

has a right adjoint i_m which induces a natural isomorphism

$$\underline{\mathrm{Hom}}_{\acute{e}t}(C', i_m C) \simeq i_m \underline{\mathrm{Hom}}_{\acute{e}t}^m(C' \otimes \mathbf{Z}/m, C)$$

for any $(C', C) \in \mathrm{DM}_{\acute{e}t}^{\mathrm{eff}}(k) \times \mathrm{DM}_{\acute{e}t}^{\mathrm{eff}}(k, \mathbf{Z}/m)$ where $\underline{\mathrm{Hom}}_{\acute{e}t}^m$ is the internal Hom of $\mathrm{DM}_{\acute{e}t}^{\mathrm{eff}}(k, \mathbf{Z}/m)$. Take $C' = \alpha^* \mathbf{Z}(n)$; then $C' \otimes \mathbf{Z}/m = \mu_m^{\otimes n}$. Thus, if $C \in \mathrm{DM}_{\acute{e}t}^{\mathrm{eff}}$ is bounded and m -torsion, the isomorphism (1) of Theorem 1 for $D = R\alpha_* C$ takes the form

$$R\alpha_* \underline{\mathrm{Hom}}_{\acute{e}t}^m(\mu_m^{\otimes n}, C)(n) \xrightarrow{\sim} R\alpha_* C.$$

For $C = \mu_m^{\otimes i}$, this gives as a special case an isomorphism

$$(3) \quad R\alpha_* \mu_m^{\otimes i-n}(n) \xrightarrow{\sim} R\alpha_* \mu_m^{\otimes i}.$$

Let $\Gamma_m = \mathrm{Gal}(k(\mu_m)/k)$: this is a subgroup of $(\mathbf{Z}/m)^*$. Since $\mu_m^{\otimes n} \simeq \mathbf{Z}/m$ when n is divisible by $\gamma_m = |\Gamma_m|$, this also gives the following corollary.

Corollary 2. *For $C \in \mathrm{DM}_{\acute{e}t}^{\mathrm{eff}}$ bounded and of exponent m , the function $n \mapsto \underline{\mathrm{Hom}}(\mathbf{Z}(n), R\alpha_* C)$ is periodic of period γ_m .* \square

The following reformulation of Theorem 1 will be useful. Recall that, in [6], we introduced and studied a triangulated category of birational motivic complexes DM° ; by *loc. cit.*, Prop. 4.2.5., one has

$$\mathrm{DM}^\circ = \mathrm{DM}^{\mathrm{eff}} / \mathrm{DM}^{\mathrm{eff}}(1).$$

Higher versions of DM° were introduced in [4, Def. 3.4] (they are also implicit in [2]):

$$\mathrm{DM}_{<n}^{\mathrm{eff}} = \mathrm{DM}^{\mathrm{eff}} / \mathrm{DM}^{\mathrm{eff}}(n)$$

so that $\mathrm{DM}_{<1} = \mathrm{DM}^\circ$.

By *loc. cit.*, Prop. 3.5, the localisation functor $\nu_{<n} : \mathrm{DM}^{\mathrm{eff}} \rightarrow \mathrm{DM}_{<n}^{\mathrm{eff}}$ has a right adjoint ι_n ; moreover, the homotopy t -structure of $\mathrm{DM}^{\mathrm{eff}}$ induces a t -structure on $\mathrm{DM}_{<n}^{\mathrm{eff}}$ via ι_n (*loc. cit.*, Prop. 3.6). By Lemma 1, Theorem 1 is then equivalent to saying that $\nu_{<n} R\alpha_* C = 0$ for any bounded torsion $C \in \mathrm{DM}_{\acute{e}t}^{\mathrm{eff}}$.

Proof of Theorem 1. We start with a Grothendieckian dévissage, first reducing to the case of a single torsion sheaf \mathcal{F} . Such \mathcal{F} comes from the *small* étale site of $\mathrm{Spec} k$ by the Suslin-Voevodsky rigidity theorem [8], i.e. is a Galois module. The functor $\nu_{<n}$ commutes with infinite direct sums as a left adjoint; since étale cohomology of sheaves has the same property, this reduces us to the case where the stalk(s) of \mathcal{F} are finite, killed by some $m > 0$ that we may further assume to be a power of a prime number l different from the characteristic (since $\mathrm{DM}_{\text{ét}}^{\text{eff}}$ is $\mathbf{Z}[1/p]$ -linear, where p is the exponential characteristic of k [10, Prop. 3.3.3 2])). A standard transfer/ l -Sylow argument now allows us to assume that \mathcal{F} becomes constant after a finite Galois extension of k whose Galois group G has order a power of l . Since the statement is stable under extensions of sheaves and the l -group G acts unipotently on \mathcal{F} , we may finally assume that $\mathcal{F} = \mathbf{Z}/l$. By a standard argument due to Tate ($[k(\mu_l) : k]$ is prime to l), we further reduce to the case where $\mu_l \subset k$.

We now use Levine’s “inverting the motivic Bott element” theorem [7]. The following corrects the presentation in [3, §3]. From the isomorphism

$$(4) \quad \mathbf{Z}(1) \simeq \mathbb{G}_m[-1]$$

we get an exact triangle

$$(5) \quad \mu_l[0] \rightarrow \mathbf{Z}/l(1) \rightarrow \mathbb{G}_m/l[-1] \xrightarrow{+1} .$$

Adding the isomorphism $\mathbf{Z}(n) \otimes \mathbf{Z}/l(1) \xrightarrow{\sim} \mathbf{Z}/l(n+1)$, we get a map in DM^{eff} :

$$(6) \quad \mathbf{Z}(n) \otimes \mu_l \rightarrow \mathbf{Z}/l(n+1).$$

For clarity, write $C \mapsto C\{n\}$ for tensoring with the (constant) sheaf $\mu_l^{\otimes n}$ an object $C \in \mathrm{DM}^{\text{eff}}$ such that $lC = 0$. Thus $\mathbf{Z}(n) \otimes \mu_l \simeq \mathbf{Z}/l(n)\{1\}$, hence we get from (6) another map

$$\mathbf{Z}/l(n) \rightarrow \mathbf{Z}/l(n+1)\{-1\}$$

which becomes an isomorphism after sheafifying for the étale topology. Iterating, we get a commutative diagram

$$\begin{array}{ccccccc} \mathbf{Z}/l(0) & \longrightarrow & \mathbf{Z}/l(1)\{-1\} & \longrightarrow & \mathbf{Z}/l(2)\{-2\} & \longrightarrow & \dots \\ & & \searrow & & \downarrow & & \swarrow \\ & & & & R\alpha_*\mathbf{Z}/l & & \end{array}$$

which induces a morphism

$$(7) \quad \mathrm{hocolim}_r \mathbf{Z}/l(r)\{-r\} \rightarrow R\alpha_*\mathbf{Z}/l$$

where “the” homotopy colimit is the one of Bökstedt-Neeman [1]; the main theorem of [7] is that (7) is an isomorphism when $l > 2$, or when $l = 2$ and either $\text{char} k > 0$ or -1 is a square in k .

This concludes the proof except for $l = 2$ in the exceptional case; we complete this case with the following proposition, which gives an “unstable” version of the previous divisibility at a higher cost. \square

Proposition 1. *One has $\nu_{<n} R^q \alpha_* \mathbf{Z}/l = 0$ for $q > n$, and $\nu_{<n} R \alpha_* \mathbf{Z}/l = 0$. (See comment before the proof of Theorem 1 for $\nu_{<n}$.)*

Proof. By the Beilinson-Lichtenbaum conjecture [9, 12], we have an exact triangle for any $q \geq 0$:

$$(8) \quad \mathbf{Z}/l(q)\{-q\} \rightarrow R \alpha_* \mathbf{Z}/l \rightarrow \tau_{>q} R \alpha_* \mathbf{Z}/l \xrightarrow{+1}$$

Suppose that $q \geq n$. Applying $\nu_{<n}$ to (8), we get an isomorphism

$$\nu_{<n} R \alpha_* \mathbf{Z}/l \xrightarrow{\sim} \nu_{<n} \tau_{>q} R \alpha_* \mathbf{Z}/l.$$

Comparing this with the same isomorphism for $q + 1$, we get the first statement. Therefore, $\nu_{<n} \tau_{>q} R \alpha_* \mathbf{Z}/l = 0$ for any $q \geq n$ (for example for $q = n$) and we conclude. \square

Remarks 2. a) Since $R^0 \alpha_* \mathbf{Z}/l = \mathbf{Z}/l$ is a birational sheaf, we have an isomorphism $\mathbf{Z}/l \xrightarrow{\sim} \nu_{<n} R^q \alpha_* \mathbf{Z}/l$ for $q = 0$. When $n = 1$, this allows us to compute $\nu_{<1} R^1 \alpha_* \mathbf{Z}/l$ thanks to the isomorphisms

$$\tau_{\leq 1} \nu_{<1} R \alpha_* \mathbf{Z}/l \xrightarrow{\sim} \nu_{<1} R \alpha_* \mathbf{Z}/l = 0$$

where the first (*resp.* second) isomorphism follows from the first (*resp.* second) part of Proposition 1: this gives

$$\nu_{<1} R^1 \alpha_* \mathbf{Z}/l \simeq \mathbf{Z}/l[2].$$

It is less clear how to compute $\nu_{<n} R^q \alpha_* \mathbf{Z}/l$ for $0 < q \leq n$ when $n \geq 2$.

b) In the spirit of [3], Proposition 1 for $n = 1$ conversely implies formally the Beilinson-Lichtenbaum conjecture: we neglect twists by powers of μ_l for simplicity, and argue by induction on q . Since \otimes is right t -exact in DM^{eff} and in view of (4), there is a natural map

$$(9) \quad \tau_{\leq q-1} R \alpha_* \mathbf{Z}/l \otimes \mathbf{Z}(1) \rightarrow \tau_{\leq q} R \alpha_* \mathbf{Z}/l$$

and we have to show that it is an isomorphism. The condition $\nu_{<1} R^q \alpha_* \mathbf{Z}/l = 0$ means that $R^q \alpha_* \mathbf{Z}/l$ is divisible by $\mathbf{Z}(1)$, which in turn implies that $\tau_{\leq q} R \alpha_* \mathbf{Z}/l$ is divisible by $\mathbf{Z}(1)$. But the computation of the étale cohomology of $X \times \mathbb{G}_m$ for smooth X shows that the adjoint of (9)

$$\tau_{\leq q-1} R \alpha_* \mathbf{Z}/l \rightarrow \underline{\text{Hom}}(\mathbf{Z}(1), \tau_{\leq q} R \alpha_* \mathbf{Z}/l)$$

is an isomorphism, and we conclude with Lemma 1.

c) If k has virtually finite étale cohomological dimension (*e.g.* is finitely

generated), we can relax the hypothesis “bounded” to “bounded below” in Theorem 1 and Corollary 1. Indeed, we reduce by a transfer argument to the case where k has finite cohomological dimension. As used before, $\nu_{<n}$ commutes with infinite direct sums, and so does $R\alpha_*$ by Lemma 2 below. Since, for any $C \in \mathrm{DM}_{\acute{e}t}^{\mathrm{eff}}$, the natural map

$$\mathrm{hocolim} \tau_{\leq n} C \rightarrow C$$

is an isomorphism, this reduces us to the bounded case.

Lemma 2. *Suppose that k has finite étale cohomological dimension. Then $R\alpha_*$ commutes with infinite direct sums.*

Proof. Let $(C_i)_{i \in I}$ be a family of objects of $\mathrm{DM}_{\acute{e}t}^{\mathrm{eff}}$. We want to show that the comparison map in $\mathrm{DM}^{\mathrm{eff}}$

$$\bigoplus_i R\alpha_* C_i \rightarrow R\alpha_* \bigoplus_i C_i$$

is an isomorphism. This can be tested against the generators $M(X)$, where X runs through smooth separated k -schemes of finite type. This yields the maps

$$H_{\mathrm{Nis}}^n(X, \bigoplus_i R\alpha_* C_i) \rightarrow H_{\acute{e}t}^n(X, \bigoplus_i C_i), \quad n \in \mathbf{Z}$$

so we are reduced to showing that $H_{\mathrm{Nis}}^n(X, -)$ and $H_{\acute{e}t}^n(X, -)$ commute with \bigoplus_i . Since hypercohomology spectral sequences are convergent (by the hypothesis on k for $H_{\acute{e}t}^n(X, -)$), we are reduced to the case of direct sums of sheaves, and the result is true because the Nisnevich and étale sites are both coherent. \square

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