ON THE BÉNABOU-ROUBAUD THEOREM

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ABSTRACT. We give a detailed proof of the Bénabou-Roubaud theorem. As a byproduct it yields a weakening of its hypotheses: the base category does not need fibre products and the Beck-Chevalley condition, in the form of a natural transformation, can be weakened by only requiring the latter to be epi.

Introduction. The Bénabou-Roubaud theorem [2] establishes, under certain conditions, an equivalence of categories between a category of descent data and a category of algebras over a monad. This result is widely cited, but [2] is a note "without proofs" and the ones I know in the literature are a bit terse ([5, pp. 50/51], [6, proof of Lemma 4.1], [7, Th. 8.5]); moreover, [6] and [7] are formulated in more general contexts.

The aim of this note is to provide a detailed proof of this theorem in its original context. This exegesis has the advantage of showing that the original hypotheses can be weakened: it is not necessary to suppose that the base category admits fibred products, and the Chevalley property of [2], formulated as an exchange condition, can also be weakened by requiring that the base change morphisms be only epi. I hope this will be useful to some readers. I also provided a proof of the equivalence between Chevalley's property and the exchange condition (attributed to Beck, but see remark 1.1): this result is part of the folklore but, here again, I had difficulty finding a published proof. In Corollary 5.2, I give a condition (probably too strong) for the Eilenberg-Moore comparison functor to be essentially surjective. Finally, I give in Proposition 6.1 cases where the exchange isomorphism holds; this is certainly classical, but it recovers conceptually Mackey's formula for the induced representations of a group (example 6.3).

Notation. I keep that of [2]: thus $P : \mathbf{M} \to \mathbf{A}$ is a bifibrant functor in the sense of [4, §10]. If $A \in \mathbf{A}$, we denote $\mathbf{M}(A)$ the fibre of P above A. For an arrow $a : A_1 \to A_0$ of $\mathbf{A}$, we write $a^* : \mathbf{M}(A_0) \to \mathbf{M}(A_1)$ and $a_* : \mathbf{M}(A_1) \to \mathbf{M}(A_0)$ for the associated inverse and direct image

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functors (a_* is *left* adjoint to a^*) and η^a, ε^a for the associated unit and counit. We also write $T^a = a^*a_*$ for the associated monad, equipped with its unit η^a and its multiplication $\mu^a = a^*\varepsilon^a a_*$. *We do not assume the existence of fibre products in* $\mathbf{A}$.

1. **Adjoint chases.** To elucidate certain statements and proofs, I start by doing two things: 1) "deploy" the notion of monad above, which will allow us to remove the quotation marks from "natural" at the bottom of [2, p. 96], 2) not assume the Beck-Chevalley condition to begin with, which will allow us to clarify the functoriality in the first lemma of the note and to weaken hypotheses.

Let $a_i : A_i \to A_0$ $(i = 1, 2, 3)$ be three morphisms in $\mathbf{A}$. For $i < j$, consider a commutative square

$$\begin{array}{ccc} A_{ij} & \xrightarrow{a_j^{ij}} & A_j \\ {\scriptstyle a_i^{ij}}\downarrow & & \downarrow{\scriptstyle a_j} \\ A_i & \xrightarrow{a_i} & A_0. \end{array} \tag{1.1}$$

The natural isomorphism

$$u : (a_i^{ij})^* a_i^* \overset{\sim}{\Rightarrow} (a_j^{ij})^* a_j^* \tag{1.2}$$

yields a base change morphism

$$\chi : (a_j^{ij})_*(a_i^{ij})^* \Rightarrow a_j^*(a_i)_* \tag{1.3}$$

equal to the composition $\varepsilon^{a_j^{ij}} a_j^*(a_i)_* \circ (a_j^{ij})_* u (a_i)_* \circ (a_j^{ij})_*(a_i^{ij})^* \eta^{a_i}$. Hence a map

$$\begin{aligned} \xi_{ij} : \mathbf{M}(A_j)(a_j^*(a_i)_*M_i, M_j) &\xrightarrow{\chi^*_{M_i}} \mathbf{M}(A_j)((a_j^{ij})_*(a_i^{ij})^*M_i, M_j) \\ &\overset{\text{adj}}{\underset{\sim}{\longrightarrow}} \mathbf{M}(A_{ij})((a_i^{ij})^*M_i, (a_j^{ij})^*M_j) \end{aligned} \tag{1.4}$$

for $(M_i, M_j) \in \mathbf{M}(A_i) \times \mathbf{M}(A_j)$. It goes in the opposite direction to the map K^a of [2] (which we will find back in (4.1)).

Remark 1.1. The morphism (1.3) is sometimes called "Beck transformation". However, it already appears in SGA4 (1963/64) to formulate the proper base change and smooth base change theorems [1, §4]. I have adopted the terminology "base change morphism" in reference to this seminar.

Lemma 1.2 (key lemma)**.** *For any* $\varphi \in \mathbf{M}(A_j)(a_j^*(a_i)_*M_i, M_j)$*, one has*

$$\xi_{ij}(\varphi) = u_{(a_j)_*M_j} \circ (a_i^{ij})^*(\varphi \circ \eta^{a_i}_{M_i})$$

where u is the natural isomorphism of (1.2).

Proof. For $\psi \in \mathbf{M}(A_j)(a_j^{ij})_*(a_i^{ij})^*M_i, M_j)$ one has $\mathrm{adj}(\psi) = (a_j^{ij})^*\psi \circ \eta^{a_j^{ij}}_{(a_i^{ij})^*M_i}$, hence

$$
\begin{aligned}
\xi_{ij}(\varphi) &= \mathrm{adj}(\varphi \circ \chi_{M_i}) = (a_j^{ij})^*(\varphi \circ \chi_{M_i}) \circ \eta^{a_j^{ij}}_{(a_i^{ij})^*M_i}\\
&= (a_j^{ij})^*(\varphi \circ (\varepsilon^{a_j^{ij}} a_j^*(a_i)_* \circ (a_j^{ij})_* u (a_i)_* \circ (a_j^{ij})_*(a_i^{ij})^*\eta^{a_i})_{M_i}) \circ \eta^{a_j^{ij}}_{(a_i^{ij})^*M_i}\\
&= (a_j^{ij})^*\varphi \circ (a_j^{ij})^*\varepsilon^{a_j^{ij}}_{a_j^*(a_i)_*M_i} \circ (a_j^{ij})^*(a_j^{ij})_* u_{(a_i)_*M_i} \circ (a_j^{ij})^*(a_j^{ij})_*(a_i^{ij})^*\eta^{a_i}_{M_i} \circ \eta^{a_j^{ij}}_{(a_i^{ij})^*M_i}\\
&= (a_j^{ij})^*\varphi \circ (a_j^{ij})^*\varepsilon^{a_j^{ij}}_{a_j^*(a_i)_*M_i} \circ (a_j^{ij})^*(a_j^{ij})_*(u_{(a_i)_*M_i} \circ (a_i^{ij})^*\eta^{a_i}_{M_i}) \circ \eta^{a_j^{ij}}_{(a_i^{ij})^*M_i}\\
&= (a_j^{ij})^*\varphi \circ (a_j^{ij})^*\varepsilon^{a_j^{ij}}_{a_j^*(a_i)_*M_i} \circ \eta^{a_j^{ij}}_{(a_i^{ij})^*a_j^*(a_i)_*M_i} \circ u_{(a_i)_*M_i} \circ (a_i^{ij})^*\eta^{a_i}_{M_i}\\
&= (a_j^{ij})^*\varphi \circ u_{(a_i)_*M_i} \circ (a_i^{ij})^*\eta^{a_i}_{M_i} = u_{(a_j)_*M_j} \circ (a_i^{ij})^*(\varphi \circ \eta^{a_i}_{M_i})
\end{aligned}
$$

where we successively used the naturality of $\eta^{a_j^{ij}}$, an adjunction identity and the naturality of u. $\square$

Let $A \in \mathbf{A}$ be equipped with "projections" $c_{ij} : A \to A_{ij}$; we assume that $b_i = a_i^{ij} \circ c_{ij} : A \to A_i$ only depends on i.

Canonical example 1.3. $A_{ij} = A_i \times_{A_0} A_j$, $A = A_1 \times_{A_0} A_2 \times_{A_0} A_3$, all morphisms given by the natural projections.

We then have maps

$$\mathbf{M}(A_{ij})((a_i^{ij})^*M_i, (a_j^{ij})^*M_j) \to \mathbf{M}(A)(b_i^*M_i, b_j^*M_j)$$

induced by c_{ij}^*, hence composite maps

$$\theta_{ij} : \mathbf{M}(A_j)(a_j^*(a_i)_*M_i, M_j) \to \mathbf{M}(A)(b_i^*M_i, b_j^*M_j) \quad (i < j). \tag{1.5}$$

In addition, we have a natural transformation

$$\lambda = a_3^*\varepsilon^{a_2}(a_1)_* : a_3^*(a_2)_*a_2^*(a_1)_* \Rightarrow a_3^*(a_1)_*. \tag{1.6}$$

The commutative square (A, A_{12}, A_{23}, A_2)[1] yields another base change morphism $(c_{23})_*c_{12}^* \Rightarrow (a_2^{23})^*(a_2^{12})_*$, hence a composition

$$
\begin{aligned}
(b_3)_*b_1^* = (a_3^{23})_*(c_{23})_*c_{12}^*(a_1^{12})^* &\Rightarrow (a_3^{23})_*(a_2^{23})^*(a_2^{12})_*(a_1^{12})^*\\
&\Rightarrow a_3^*(a_2)_*a_2^*(a_1)_*
\end{aligned} \tag{1.7}
$$

which induces a map

$$\rho : \mathbf{M}(A_3)(a_3^*(a_2)_*a_2^*(a_1)_*M_1, M_3) \to \mathbf{M}(A)(b_1^*M_1, b_3^*M_3). \tag{1.8}$$

[1]Note that it is Cartesian in the canonical example.

An adjoint chase gives:

Lemma 1.4. *One has $\theta_{13} = \rho \circ \lambda^*$ (see (1.5), (1.6) and (1.8)).* □

Let $\varphi_{ij} \in \mathbf{M}(A_j)(a_j^*(a_i)_* M_i, M_j)$ now be three morphisms. We have a not necessarily commutative square:

$$
\begin{tikzcd}
a_3^*(a_2)_* a_2^*(a_1)_* M_1 \arrow[r, "a_3^*(a_2)_*\varphi_{12}"] \arrow[d, "\lambda_{M_1}"'] & a_3^*(a_2)_* M_2 \arrow[d, "\varphi_{23}"'] \\
a_3^*(a_1)_* M_1 \arrow[r, "\varphi_{13}"] & M_3.
\end{tikzcd}
\tag{1.9}
$$

Write $\hat\varphi_{ij} = \theta_{ij}(\varphi_{ij}) : b_i^* M_i \to b_j^* M_j$.

Lemma 1.5. *Let ψ (resp. ψ') be the composition of (1.9) passing through $a_3^*(a_2)_* M_2$ (resp. through $a_3^*(a_1)_* M_1$). Then $\rho(\psi) = \hat\varphi_{23} \circ \hat\varphi_{12}$ and $\rho(\psi') = \hat\varphi_{13}$.*

Proof. The first point follows from a standard adjunction calculation, and the second follows from lemma 1.4. □

Proposition 1.6. *If (1.9) commutes, we have $\hat\varphi_{13} = \hat\varphi_{23} \circ \hat\varphi_{12}$; the converse is true if ρ is injective in (1.8).*

Proof. This is obvious in view of Lemma 1.5. □

In (1.4), assume M_j is of the form $a_j^* M_0$ and write $a^{ij} : A_{ij} \to A_0$ for the projection. We have a composition

$$
\begin{aligned}
\mathbf{M}(A_1)(M_1, (a_i)^* M_0) &\xrightarrow{\sim} \mathbf{M}(A_0)((a_i)_* M_i, M_0) \\
&\xrightarrow{a_j^*} \mathbf{M}(A_j)(a_j^*(a_i)_* M_i, a_j^* M_0) \xrightarrow{\xi_{ij}} \mathbf{M}(A_{ij})((a_i^{ij})^* M_i, (a^{ij})^* M_0)
\end{aligned}
\tag{1.10}
$$

where the first arrow is the adjunction isomorphism. A new adjoint chase gives:

Lemma 1.7. *The composition (1.10) is induced by $(a_i^{ij})^*$.* □

2. **Exchange condition and weak exchange condition.** Now we introduce the

Definition 2.1. A commutative square (1.1) is said to satisfy the *exchange condition* if the base change morphism (1.3) is invertible; we say that (1.1) satisfies the *weak exchange condition* if (1.3) is epi.

Lemma 2.2 (cf. [9, Prop. 11])**.** *The exchange condition of Definition 2.1 is equivalent to the Chevalley condition (C) of [2].*

Proof. Recall this condition: given a commutative square

$$\begin{tikzcd} M_1' \arrow[r, "k_1"] \arrow[d, "\chi'"'] & M_1 \arrow[d, "\chi"] \\ M_0' \arrow[r, "k_0"] & M_0, \end{tikzcd} \tag{2.1}$$

above (1.1) (where we take $(i,j)=(1,2)$ to fix ideas), if χ and χ' are Cartesian and k_0 is co-Cartesian, then k_1 is co-Cartesian.

I will show that the exchange condition is equivalent to the following two conditions: (C) and

(C'): if k_0 and k_1 are co-Cartesian and χ' is Cartesian, then χ is Cartesian.

Let us translate the commutativity of (2.1) in terms of the square

$$\begin{tikzcd} \mathbf{M}(A_{12}) \arrow[r, "(a_2^{12})_*"] & \mathbf{M}(A_2) \\ \mathbf{M}(A_1) \arrow[r, "(a_1)_*"] \arrow[u, "(a_1^{12})^*"] & \mathbf{M}(A_0). \arrow[u, "a_2^*"'] \end{tikzcd} \tag{2.2}$$

The morphisms of (2.1) correspond to morphisms $\tilde{k}_0 : (a_1)_*M_0' \to M_0$, $\tilde{k}_1 : (a_1^{12})_*M_1' toM_1$, $\tilde{\chi} : M_1 \to a_2^*M_0$ and $\tilde{\chi}' : M_1' \to (a_2^{12})^*M_0'$, which fit in a commutative diagram of $\mathbf{M}(A_1)$:

$$\begin{tikzcd} (a_2^{12})_*(a_1^{12})^*M_0' \arrow[rr, "c"] & & a_2^*(a_1)_*M_0' \arrow[d, "a_2^*\tilde{k}_0"] \\ (a_2^{12})_*M_1' \arrow[u, "(a_2^{12})_*\tilde{\chi}'"] \arrow[r, "\tilde{k}_1"] & M_1 \arrow[r, "\tilde{\chi}"] & a_2^*M_0 \end{tikzcd}$$

where c is the base change morphism of (1.3). The cartesianity conditions on χ and χ' (resp. co-cartesianity conditions on k_0 and k_1) amount to requesting the corresponding morphisms decorated with a $\tilde{}$ to be isomorphisms.

Suppose c is an isomorphism. If $\tilde{\chi}'$ and $\tilde{k}_0$ are isomorphisms, $\tilde{\chi}$ is an isomorphism if and only if $\tilde{k}_1$ is. Thus, the exchange condition implies conditions (C) and (C'). Conversely, M_0' being given, let $\tilde{k}_0$, $\tilde{\chi}$ and $\tilde{\chi}'$ be identities, which successively defines M_0, M_1 and M_1'. The arrow c then defines an arrow $\tilde{k}_0$, which is an isomorphism if and only if so is c. This shows that the exchange condition implies (C), and we argue in the same way for (C') by taking $\tilde{k}_1$ to be the identity. □

Remarks 2.3. a) This proof did not use the hypothesis that (1.1) be Cartesian.

b) Under conservativity assumptions for $(a_2^{12})^*$ or a_2^*, we obtain converses to (C) and (C').

3. **Pre-descent data.** We now assume $A_1 = A_2 = A_3$, $M_1 = M_2 = M_3 =: M$ and that the three squares (1.1) are identical. We note a for a_i $(i = 1, 2, 3)$.

Definition 3.1. A *pre-descent datum* on M is a morphism $v \in \mathbf{M}(A_{12})((a_1^{12})^*M, (a_2^{12})^*M)$ which verifies the condition of Proposition 1.6. We write $\mathbf{D}^{\mathrm{pre}}$ for the category whose objects are pairs (M, v), where v is a pre-descent datum on M, and whose morphisms are those of $M(A_1)$ which commute with pre-descent data.

Let us introduce the

Hypothesis 3.2. The weak exchange condition is verified by the squares (A_{12}, A_1, A_2, A_0), (A_{23}, A_2, A_3, A_0) and (A, A_{12}, A_{23}, A_2).

(Of course, the first two squares coincide and $A_{12} = A_{23}$ in the third, but I keep these notations for clarity.)

Proposition 3.3 (cf. [2, lemme]). *In* (1.9)*, assume* $\varphi_{12} = \varphi_{23} = \varphi_{13} =: \varphi$*. If* φ *satisfies the associativity condition of a* T^a*-algebra, then* $\xi_{12}(\varphi)$ *in* (1.4) *is a weak descent datum; the converse is true under Hypothesis 3.2.*

Proof. In view of Proposition 1.6, it suffices to show that Hypothesis 3.2 implies the injectivity of ρ, which is induced by the composition of the two natural transformations of (1.7). The second is epi, therefore induces an injection on Hom's, and so does the first by adjunction. □

Corollary 3.4. *Let* $\mathbf{M}^a_{\mathrm{ass}}$ *denote the category of associative* T^a*-algebras which are not necessarily unital. Then Proposition 3.3 defines a faithful functor* $\xi : \mathbf{M}^a_{\mathrm{ass}} \to \mathbf{D}^{\mathrm{pre}}$ *commuting with the forgetful functors to* $\mathbf{M}(A_1)$*; under Hypothesis 3.2, it is an isomorphism of categories.*

Proof. Commutation of ξ with the forgetful functors is obvious. This already shows that it is faithful; under Hypothesis 3.2, it is essentially surjective by Proposition 3.3 and we see immediately that it is also full. □

4. **The unit condition.** We now introduce an additional ingredient: a "diagonal" morphism $\Delta : A_1 \to A_{12}$ such that $a_1^{12}\Delta = a_2^{12}\Delta = 1_{A_1}$.

Definition 4.1. A *descent datum* on M is a pre-descent datum v such that $\Delta^* v = 1_M$ modulo the isomorphisms $\Delta^*(a_i^{12})^* \xrightarrow{\sim} \mathrm{Id}_{\mathbf{M}(A_1)}$ for $i = 1.2$. We denote by $\mathbf{D}$ the full subcategory of $\mathbf{D}^{\mathrm{pre}}$ given by the descent data.

Remark 4.2. In the canonical example 1.3, a pre-descent datum v satisfies the condition of Definition 4.1 if and only if v is invertible (therefore is a descent datum in the classical sense): this follows from [3, A.1.d pp. 303–304].

Let $\mathbf{M}^a \subset \mathbf{M}^a_{\mathrm{ass}}$ be the category of T^a-algebras.

Theorem 4.3 (cf. [2, théorème]). *For all $\varphi \in \mathbf{M}(A_1)(a^*a_*M, M)$, we have*

$$\Delta^*\xi_{12}(\varphi) = \varphi \circ \eta^a_M$$

modulo the isomorphisms $\Delta^*(a_i^{12})^* \xrightarrow{\sim} \mathrm{Id}_{\mathbf{M}(A_1)}$. In particular, $\xi(\mathbf{M}^a) \subset \mathbf{D}$ and $\xi : \mathbf{M}^a \to \mathbf{D}$ is an isomorphism of categories under Hypothesis 3.2.

Proof. This follows from Lemma 1.2 and Corollary 3.4. □

As in [8, VI.3, Th. 1], we have the Eilenberg-Moore comparison functor

$$\begin{aligned} K^a : \mathbf{M}(A_0) &\to \mathbf{M}^a \\ M_0 &\mapsto (a^*M_0, a^*\varepsilon^a_{M_0}). \end{aligned} \tag{4.1}$$

On the other hand, we have the natural isomorphism of (1.2)

$$u_{M_0} : (a_1^{12})^*a^*M_0 \xrightarrow{\sim} (a_2^{12})^*a^*M_0$$

and Lemma 1.7 yields:

Proposition 4.4. *We have $u_{M_0} = \xi_{12}(a^*\varepsilon^a_{M_0})$. In other words, in the diagram*

$$\begin{tikzcd} \mathbf{M}(A_0) \arrow[r, "\Psi^a"] \arrow[dr, "K^a"'] & \mathbf{D} \arrow[r, "U^a"] & \mathbf{M}(A_1) \\ & \mathbf{M}^a \arrow[u, "\xi"] \arrow[ur, "U^{T^a}"'] & \end{tikzcd}$$

the left triangle commutes. □

5. **A complement.**

Proposition 5.1. *Let a^* be fully faithful and $\mathbf{M}(A_0)$ Karoubian. Let $\varphi : a^*a_*M \to M$, verify the identity $\varphi \circ \eta^a_M = 1_M$. Then there exists $M_0 \in \mathbf{M}(A_0)$ and an isomorphism $\nu : M_1 \xrightarrow{\sim} a^*M_0$ such that $\varphi = \nu^{-1}a^*\varepsilon^a_{M_0}a^*a_*\nu$.*

Proof. Let e denote the idempotent $\eta^a_M\varphi \in \mathrm{End}_{\mathbf{M}(A_1)}(a^*a_*M)$. By hypothesis, $e = a^*\tilde{e}$ where $\tilde{e}$ is an idempotent of $\mathrm{End}_{\mathbf{M}(A_0)}(a_*M)$, of image

M_0. Then a^*M_0 is isomorphic to the image M of e via a morphism ν as in the statement, such that

$$\nu \circ \varphi = a^*\pi, \quad a^*\iota \circ \nu = \eta^a_M$$

where $\iota\pi$ is the epi-mono factorization of $\tilde{e}$.

To finish, it is enough to see that $a^*\pi = a^*\varepsilon^a_{M_0} a^* a_* \nu$. But we also have

$$\eta^a_{a^*M_0} \circ \nu = a^* a_* \nu \circ \eta^a_M = a^* a_* \nu \circ a^* \iota \circ \nu$$

hence $\eta^a_{a^*M_0} = a^* a_* \nu \circ a^*\iota$. This concludes, since $\eta^a_{a^*M_0} a^* \varepsilon^a_{M_0}$ is the epi-mono factorisation of the idempotent of $\mathrm{End}(a^*a_*a^*M_0)$ of image a^*M_0. □

We thus obtain the following complement:

Corollary 5.2. *Assume Hypothesis 3.2, and also that a^* is fully faithful and $\mathbf{M}(A_0)$ Karoubian. Then*
a) every unital T^a-algebra is associative;
b) K^a is essentially surjective. □

Can one weaken the full faithfulness assumption in this corollary? The following lemma does not seem sufficient:

Lemma 5.3. *Let $M, N \in \mathbf{M}(A_1)$. Then the map*

$$a^* : \mathbf{M}(A_0)(a_*M, a_*N) \to \mathbf{M}(A_1)(a^*a_*M, a^*a_*N)$$

has a retraction r given by $r(f) = \varepsilon^a_{a_*N} \circ a_*f \circ a_*\eta^a_M$. More generally, we have an identity of the form $r(a^*g \circ f) = g \circ r(f)$.

Proof. For $f : a^*a_*M \to a^*a_*N$ et $g : a_*N \to a_*P$, we have

$$r(a^*g \circ f) = \varepsilon^a_{a_*P} \circ a_*a^*g \circ a_*f \circ a_*\eta^a_M = g \circ \varepsilon^a_{a_*N} \circ a_*f \circ a_*\eta^a_M = g \circ r(f).$$

Taking $f = 1_{a^*a_*M}$, we obtain that r is a retraction. □

6. **Appendix: a case where the exchange condition is verified.**

Let $\mathcal{A}$ be a category. Take for $\mathbf{A}$ the category of presheaves of sets on $\mathcal{A}$. Write $\int A$ for the category associated to $A \in \mathbf{A}$ by the Grothendieck construction [4, §8]. Let $\mathcal{C}$ be another category. We take for $\mathbf{M}$ the fibred category of representations of $\mathbf{A}$ in $\mathcal{C}$: for $A \in \mathbf{A}$, an object of $\mathbf{M}(A)$ is a functor from $\int A$ to $\mathcal{C}$. For all $a \in \mathbf{A}(A_1, A_0)$ we have an obvious pull-back functor $a^* : \mathbf{M}(A_0) \to \mathbf{M}(A_1)$, which has a left adjoint a_* (direct image) given by the usual colimit if $\mathcal{C}$ is cocomplete. We can then ask whether the exchange condition is true for Cartesian squares of $\mathcal{A}$.

Proposition 6.1. *This is the case if $\mathcal{C}$ is the category of sets* **Set***, and more generally if $\mathcal{C}$ admits a forgetful functor $\Omega : \mathcal{C} \to \mathbf{Set}$ with a left adjoint L such that (L, Ω) satisfies the conditions of Beck's theorem* [8, VI.7, Th. 1].

Proof. First suppose $\mathcal{C} = \mathbf{Set}$; to verify that (1.3) is a natural isomorphism, it is enough to test it on representable functors. Consider Diagram (2.2) again. For $(c, \gamma) \in \int A_1$ and $(d, \delta) \in \int A_2$ (with $c, d \in \mathcal{A}$ and $\gamma \in A_1(c)$, $\delta \in A_2(d)$), we have

$$\begin{aligned} a_2^*(a_1)_* y(c,\gamma)(d,\delta) = a_2^* y(c, a_1(\gamma))(d,\delta) = y(c, a_1(\gamma))(d, a_2(\delta)) \\ = \{\varphi \in \mathcal{A}(d,c) \mid \varphi^* a_1(\gamma) = a_2(\delta)\} \end{aligned}$$

and

$$(a_2^{12})_*(a_1^{12})^* y(c,\gamma)(d,\delta) = \varinjlim_{(e,\eta)\in(d,\delta)\downarrow a_2^{12}} (a_1^{12})^* y(c,\gamma)(e,\eta)$$

$$= \varinjlim_{(e,\eta)\in(d,\delta)\downarrow a_2^{12}} y(c,\gamma)(e, a_1^{12}(\eta)) = \varinjlim_{(e,\eta)\in(d,\delta)\downarrow a_2^{12}} \{\psi \in \mathcal{A}(e,c) \mid \psi^*\gamma = a_1^{12}(\eta)\}.$$

We have

$$(d,\delta) \downarrow a_2^{12} = \{(e, \eta_1, \eta_2, \theta) \in \mathcal{A} \times A_1(e) \times_{A_0(e)} A_2(e) \times \mathcal{A}(d,e) \mid \theta^*\eta_2 = \delta\}.$$

This category has the initial set $\{(d, \eta_1, \delta, 1_d) \mid a_1(\eta_1) = a_2(\delta)\}$, so

$$\begin{aligned} (a_2^{12})_*(a_1^{12})^* y(c,\gamma)(d,\delta) = \coprod_{\{(\eta_1 \in A_1(d) \mid a_1(\eta_1) = a_2(\delta)\}} \{\varphi \in \mathcal{A}(d,c) \mid \varphi^*\gamma = \eta_1\} \\ = \{\varphi \in \mathcal{A}(d,c) \mid a_1(\varphi^*\gamma) = a_2(\delta)\} \end{aligned}$$

and the map $(a_2^{12})_*(a_1^{12})^* y(c,\gamma)(d,\delta) \to (a_2^{12})_*(a_1^{12})^* y(c,\gamma)(d,\delta)$ is clearly equal to the identity.

General case: let us write more precisely $\mathbf{M}^{\mathcal{C}}(A) = \mathbb{CAT}(\int A, \mathcal{C})$. The functors L and Ω induce pairs of adjoint functors (same notation)

$$L : \mathbf{M}^{\mathbf{Set}}(A) \leftrightarrows \mathbf{M}^{\mathcal{C}}(A) : \Omega.$$

These two functors commute with pull-backs; as L is a left adjoint, it also commutes with direct images. Therefore, in the above situation, the base change morphism $\chi_M : (a_2^{12})_*(a_1^{12})^* M \to a_2^*(a_1)_* M$ is an isomorphism when $M \in \mathbf{M}^{\mathcal{C}}(A_1)$ is of the form LX for $X \in \mathbf{M}^{\mathbf{Set}}(A_1)$. For any M, we have its canonical presentation [8, (5) p. 153]

$$(L\Omega)^2 M \rightrightarrows L\Omega M \to M \tag{6.1}$$

whose image by Ω is a split coequaliser (*loc. cit.*). Given the hypothesis that Ω creates such coequalisers, (6.1) is a coequaliser. Since pull-backs are cocontinuous, as well as direct images (again, as left adjoints), (6.1) remains a coequaliser after applying the functors $(a_2^{12})_*(a_1^{12})^*$ and $a_2^*(a_1)_*$. Finally, a coequaliser of isomorphisms is an isomorphism. $\square$

Examples 6.2 (for $\mathcal{C}$). Varieties (category of groups, abelian groups, rings...): [8, VI.8, Th. 1].

Example 6.3 (for $\mathcal{A}$). The category with one object $\underline{G}$ associated with a group G: then $\mathbf{A}$ is the category of G-sets. Let us take for $\mathcal{C}$ the category of R-modules where R is a commutative ring. If $A \in \mathbf{A}$ is G-transitive, $\int A$ is a connected groupoid, which is equivalent to $\underline{H}$ for the stabilizer H of any element of A; thus, $\mathbf{M}(A)$ is equivalent to $\mathbf{Rep}_R(H)$. If $a : A_1 \to A_0$ is the morphism of $\mathbf{A}$ defined by an inclusion $K \subset H \subset G$ ($A_1 = G/K$, $A_0 = G/H$), then a^* is restriction from H to K and a_* is induction $V \mapsto RH \otimes_{RK} V$. From Proposition 6.1, we thus recover conceptually the Mackey formula of [10, 7.3, Prop. 22], proven "by hand" in *loc. cit.*

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