

# ON THE BÉNABOU-ROUBAUD THEOREM

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**ABSTRACT.** We give a detailed proof of the Bénabou-Roubaud theorem. As a byproduct, it yields a weakening of its hypotheses: the base category does not need fibre products and the Beck-Chevalley condition, in the form of a natural transformation, can be weakened by only requiring the latter to be epi.

To the memory of Jacques Roubaud.

**Introduction.** The Bénabou-Roubaud theorem [2] establishes, under certain conditions, an equivalence of categories between a category of descent data and a category of algebras over a monad. This result is widely cited, but [2] is a note “without proofs” and the ones I know in the literature are a bit terse ([7, pp. 50/51], [8, proof of Lemma 4.1], [11, Th. 8.5]), [9, 3.7]; moreover, [8] and [11] are formulated in more general contexts.

The aim of this note is to provide a detailed proof of this theorem in its original context. This exegesis has the advantage of showing that the original hypotheses can be weakened: it is not necessary to suppose that the base category admits fibred products<sup>1</sup>, and the Chevalley property of [2], formulated as an exchange condition, can also be weakened by requiring that the base change morphisms be only epi. I hope this will be useful to some readers. I also provided a proof of the equivalence between Chevalley’s property and the exchange condition (attributed to Beck, but see remark 1.1): this result is part of the folklore but, here again, I had difficulty finding a published proof. In Corollary 5.2, I give a condition (probably too strong) for the Eilenberg-Moore comparison functor to be essentially surjective. Finally, I give cases in Proposition 6.1 where the exchange isomorphism holds; this is certainly classical, but it recovers conceptually Mackey’s formula for the induced representations of a group (Example 6.3).

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*Date:* January 17, 2025.

*2020 Mathematics Subject Classification.* 18D30, 18C15, 18F20.

*Key words and phrases.* Descent, monad, Beck-Chevalley condition.

<sup>1</sup>As was pointed out by the referee, the corresponding arguments are related to Street’s notion of descent object relative to a truncated (co)simplicial category as in the beginning of [16]; but a “truncated cyclic category” à la Connes is also lurking in Proposition 4.6 b).

*Notation and conventions.* I keep that of [2]: thus  $P : \mathbf{M} \rightarrow \mathbf{A}$  is a bifibrant functor in the sense of [5, §10]. If  $A \in \mathbf{A}$ , we denote by  $\mathbf{M}(A)$  the fibre of  $P$  above  $A$ . For an arrow  $a : A_1 \rightarrow A_0$  of  $\mathbf{A}$ , we write  $a^* : \mathbf{M}(A_0) \rightarrow \mathbf{M}(A_1)$  and  $a_* : \mathbf{M}(A_1) \rightarrow \mathbf{M}(A_0)$  for the associated inverse and direct image functors ( $a_*$  is *left* adjoint to  $a^*$ ) and  $\eta^a$ ,  $\varepsilon^a$  for the associated unit and counit. We also write  $T^a = a^*a_*$  for the associated monad, equipped with its unit  $\eta^a$  and its multiplication  $\mu^a = a^*\varepsilon^aa_*$ . *We do not assume the existence of fibre products in  $\mathbf{A}$ .*

In order to simplify calculations, we shall assume that the pseudo-functor  $a \mapsto a^*$  is a functor. This can be justified by the fact that it can be rectified; more precisely, the morphism of pseudofunctors  $i \mapsto F_i$  of [10, §3, p. 141] is clearly faithful, hence any parallel arrows in its source which become equal in its target are already equal. (One could also use [3, I, Th. 2.4.2 or 2.4.4].) Then one can also choose the left adjoints  $a \mapsto a_*$  to form a functor [12, IV.8, Th. 1], which we do.

**1. Adjoint chases.** To elucidate certain statements and proofs, I start by doing two things: 1) “deploy” the single object  $M_1$  of [2] into several, which will allow us to remove the quotation marks from “natural” at the bottom of [2, p. 96], 2) not assume the Beck-Chevalley condition to begin with, which will allow us to clarify the functoriality in the first lemma of the note and to weaken hypotheses.

1.1. Let  $a$  be as above; still following the notation of [2], we give ourselves a commutative square

$$(1.1) \quad \begin{array}{ccc} A_2 & \xrightarrow{a_2} & A_1 \\ a_1 \downarrow & & \downarrow a \\ A_1 & \xrightarrow{a} & A_0. \end{array}$$

except that we don’t require it to be Cartesian. The equality  $a_1^*a^* = a_2^*a^*$  yields a base change morphism

$$(1.2) \quad \chi : (a_2)_*a_1^* \Rightarrow T^a$$

equal to the composition  $\varepsilon^{a_2}T^a \circ (a_2)_*a_1^*\eta^a$ . Hence a map

$$(1.3) \quad \xi_{M,N} = \xi : \mathbf{M}(A_1)(T^a M, N) \xrightarrow{\chi_M^*} \mathbf{M}(A_1)((a_2)_*a_1^* M, N) \\ \xrightarrow[\sim]{\text{adj}} \mathbf{M}(A_2)(a_1^* M, a_2^* N)$$

for  $M, N \in \mathbf{M}(A_1)$ . It goes in the *opposite* direction to the map  $K^a$  of [2], which we will find back in (4.2). (See also Remark 4.4 in that section.)

*Remark 1.1.* The morphism (1.2) is sometimes called “Beck transformation”. However, it already appears in SGA4 (1963/64) to formulate the proper base change and smooth base change theorems [1, §4]. I have adopted the terminology “base change morphism” in reference to this seminar.

**Lemma 1.2** (key lemma). *For any  $\varphi \in \mathbf{M}(A_1)(T^a M, N)$ , one has*

$$\xi(\varphi) = a_2^* \varphi \circ a_1^* \eta_M^a.$$

*Proof.* For  $\psi \in \mathbf{M}(A_1)(a_2)_* a_1^* M, N)$  one has  $\text{adj}(\psi) = a_2^* \psi \circ \eta_{a_1^* M}^{a_2}$ , hence

$$\begin{aligned} \xi(\varphi) &= \text{adj}(\varphi \circ \chi_M) = a_2^*(\varphi \circ \chi_M) \circ \eta_{a_1^* M}^{a_2} \\ &= a_2^*(\varphi \circ (\varepsilon^{a_2} T^a \circ (a_2)_* a_1^* \eta_M^a)) \circ \eta_{a_1^* M}^{a_2} \\ &= a_2^* \varphi \circ a_2^* \varepsilon_{T^a M}^{a_2} \circ a_2^*(a_2)_* a_1^* \eta_M^a \circ \eta_{a_1^* M}^{a_2} \\ &= a_2^* \varphi \circ a_2^* \varepsilon_{T^a M}^{a_2} \circ \eta_{a_1^* T^a M}^{a_2} \circ a_1^* \eta_M^a \\ &= a_2^* \varphi \circ a_1^* \eta_M^a \end{aligned}$$

where we successively used the naturality of  $\eta^{a_2}$  and an adjunction identity.  $\square$

1.2. Let  $A_3 \in \mathbf{A}$  be equipped with “projections”  $p_1, p_2, p_3 : A_3 \rightarrow A_2$ . We assume that the “face identities”  $a_1 p_2 = a_1 p_3$ ,  $a_1 p_1 = a_2 p_3$ ,  $a_2 p_1 = a_2 p_2$  are satisfied; we call these morphisms respectively  $b_1, b_2, b_3$ .

*Canonical example 1.3.*  $A_2 = A_1 \times_{A_0} A_1$ ,  $A_3 = A_1 \times_{A_0} A_1 \times_{A_0} A_1$ , all morphisms given by the natural projections.

We then have maps, for  $i < j$

$$(1.4) \quad \alpha_{ij}(M, N) = \alpha_{ij} : \mathbf{M}(A_2)(a_1^* M, a_2^* N) \rightarrow \mathbf{M}(A_3)(b_i^* M, b_j^* N)$$

given by

$$\alpha_{12} = p_3^*, \quad \alpha_{13} = p_2^*, \quad \alpha_{23} = p_1^*$$

hence composite maps

$$(1.5) \quad \theta_{ij} = \alpha_{ij} \circ \xi : \mathbf{M}(A_1)(T^a M, N) \rightarrow \mathbf{M}(A)(b_i^* M, b_j^* N).$$

In addition, we have the multiplication of  $T^a$  mentioned in the notations:

$$(1.6) \quad \mu^a = a^* \varepsilon^a a_* : T^a T^a \Rightarrow T^a.$$

The commutative square<sup>2</sup>

$$(1.7) \quad \begin{array}{ccc} A_3 & \xrightarrow{p_3} & A_2 \\ p_1 \downarrow & & \downarrow a_2 \\ A_2 & \xrightarrow{a_1} & A_1 \end{array}$$

yields another base change morphism  $\lambda : (p_1)_* p_3^* \Rightarrow a_1^*(a_2)_*$ , hence a composition

$$(1.8) \quad (b_3)_* b_1^* = (a_2)_* (p_1)_* p_3^* a_1^* \xrightarrow{(a_2)_* \lambda a_1^*} (a_2)_* a_1^* (a_2)_* a_1^* \xrightarrow{\chi^* \chi} T^a T^a$$

which, together with adjunction, induces a map

$$(1.9) \quad \rho : \mathbf{M}(A_1)(T^a T^a M, N) \rightarrow \mathbf{M}(A_2)(b_1^* M, b_3^* N).$$

**Lemma 1.4.** *a) The diagram of natural transformations*

$$\begin{array}{ccc} (a_2)_* (p_1)_* p_3^* a_1^* & \xlongequal{\quad} & (b_3)_* b_1^* \xlongequal{\quad} (a_2)_* (p_2)_* p_2^* a_1^* \\ (a_2)_* \lambda a_1^* \downarrow & & \downarrow (a_2)_* \varepsilon^{p_2} a_1^* \\ (a_2)_* a_1^* (a_2)_* a_1^* & & (a_2)_* a_1^* \\ \chi^* \chi \downarrow & & \downarrow \chi \\ T^a T^a & \xlongequal{\quad \mu^a \quad} & T^a \end{array}$$

is commutative.

b) One has  $\theta_{13} = \rho \circ \mu_a^*$  (see (1.5), (1.6) and (1.9)).

*Proof.* a) is a matter of developing the base change morphisms as done for  $\chi$  just below (1.2) (see proof of Lemma 1.2). This yields a commutative diagram

$$\begin{array}{ccc} \mathbf{M}(A_1)((b_3)_* b_1^* M, N) & \xleftarrow{((a_2)_* \varepsilon^{p_2} a_1^*)^*} & \mathbf{M}(A_1)((a_2)_* a_1^* M, N) \\ (1.8)^* \uparrow & & \uparrow \chi^* \\ \mathbf{M}(A_1)(T^a T^a M, N) & \xleftarrow{(\mu^a)^*} & \mathbf{M}(A_1)(T^a M, N) \end{array}$$

from which we get b) by developing the adjunction isomorphism for  $((b_3)_*, b_3^*)$ .  $\square$

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<sup>2</sup>Note that it is Cartesian in the canonical example.

Let now  $M_1, M_2, M_3 \in \mathbf{M}(A_1)$  and  $\varphi_{ij} \in \mathbf{M}(A_1)(T^a M_i, M_j)$  be three morphisms. We have a not necessarily commutative square:

$$(1.10) \quad \begin{array}{ccc} T^a T^a M_1 & \xrightarrow{T^a \varphi_{12}} & T^a M_2 \\ (\mu_a)_{M_1} \downarrow & & \varphi_{23} \downarrow \\ T^a M_1 & \xrightarrow{\varphi_{13}} & M_3. \end{array}$$

Write  $\hat{\varphi}_{ij} = \theta_{ij}(\varphi_{ij}) : b_i^* M_i \rightarrow b_j^* M_j$ .

**Lemma 1.5.** *Let  $\psi$  (resp.  $\psi'$ ) be the composition of (1.10) passing through  $T^a M_2$  (resp. through  $T^a M_1$ ). Then  $\rho(\psi) = \hat{\varphi}_{23} \circ \hat{\varphi}_{12}$  and  $\rho(\psi') = \hat{\varphi}_{13}$ .*

*Proof.* The first point follows from a standard adjunction calculation similar to the previous ones, and the second follows from lemma 1.4.  $\square$

**Proposition 1.6.** *If (1.10) commutes, we have  $\hat{\varphi}_{13} = \hat{\varphi}_{23} \circ \hat{\varphi}_{12}$ ; the converse is true if  $\rho$  is injective in (1.9).*

*Proof.* This is obvious in view of Lemma 1.5.  $\square$

In (1.3), assume that  $M = N$  is of the form  $a^* M_0$  and write  $p = aa_1 = aa_2 : A_2 \rightarrow A_0$ . We have a composition

$$(1.11) \quad \begin{aligned} \mathbf{M}(A_1)(M, a^* M_0) &\xrightarrow{\sim} \mathbf{M}(A_0)(a_* M, M_0) \\ &\xrightarrow{a^*} \mathbf{M}(A_1)(T^a M, a^* M_0) \xrightarrow{\xi} \mathbf{M}(A_2)(a_1^* M, p^* M_0) \end{aligned}$$

where the first arrow is the adjunction isomorphism. A new adjoint chase gives:

**Lemma 1.7.** *The composition (1.11) is induced by  $a_1^*$ .*  $\square$

**2. Exchange condition and weak exchange condition.** Now we introduce the

**Definition 2.1.** A commutative square (1.1) is said to satisfy the *exchange condition* if the base change morphism (1.2) is an isomorphism; we say that (1.1) satisfies the *weak exchange condition* if (1.2) is epi.

**Lemma 2.2** (cf. [13, Prop. 11] and [14, II.3]). *The exchange condition of Definition 2.1 is equivalent to the Chevalley condition (C) of [2].*

*Proof.* Recall this condition: given a commutative square

$$(2.1) \quad \begin{array}{ccc} M'_1 & \xrightarrow{k_1} & M_1 \\ \chi' \downarrow & & \chi \downarrow \\ M'_0 & \xrightarrow{k_0} & M_0, \end{array}$$

above (1.1) (where we take  $(i, j) = (1, 2)$  to fix ideas), if  $\chi$  and  $\chi'$  are Cartesian and  $k_0$  is co-Cartesian, then  $k_1$  is co-Cartesian.

I will show that the exchange condition is equivalent to each of the following two conditions: (C) and

(C'): if  $k_0$  and  $k_1$  are co-Cartesian and  $\chi'$  is Cartesian, then  $\chi$  is Cartesian.

Let us translate the commutativity of (2.1) in terms of the square

$$(2.2) \quad \begin{array}{ccc} \mathbf{M}(A_2) & \xrightarrow{(a_2)_*} & \mathbf{M}(A_1) \\ a_1^* \uparrow & & a^* \uparrow \\ \mathbf{M}(A_1) & \xrightarrow{a_*} & \mathbf{M}(A_0). \end{array}$$

The morphisms of (2.1) correspond to morphisms  $\tilde{k}_0 : a_* M'_0 \rightarrow M_0$ ,  $\tilde{k}_1 : (a_1)_* M'_1 \rightarrow M_1$ ,  $\tilde{\chi} : M_1 \rightarrow a^* M_0$  and  $\tilde{\chi}' : M'_1 \rightarrow a_2^* M'_0$ , which fit in a commutative diagram of  $\mathbf{M}(A_1)$ :

$$\begin{array}{ccc} (a_2)_* a_1^* M'_0 & \xrightarrow{c} & T^a M'_0 \\ (a_2)_* \tilde{\chi}' \uparrow & & \downarrow a^* \tilde{k}_0 \\ (a_2)_* M'_1 & \xrightarrow{\tilde{k}_1} M_1 \xrightarrow{\tilde{\chi}} & a^* M_0 \end{array}$$

where  $c$  is the base change morphism of (1.2). The cartesianity conditions on  $\chi$  and  $\chi'$  (resp. co-cartesianity conditions on  $k_0$  and  $k_1$ ) amount to requesting the corresponding morphisms decorated with a  $\sim$  to be isomorphisms.

Suppose  $c$  is an isomorphism. If  $\tilde{\chi}'$  and  $\tilde{k}_0$  are isomorphisms,  $\tilde{\chi}$  is an isomorphism if and only if  $\tilde{k}_1$  is. Thus, the exchange condition implies conditions (C) and (C'). Conversely,  $M'_0$  being given, let  $\tilde{k}_0$ ,  $\tilde{\chi}$  and  $\tilde{\chi}'$  be identities, which successively defines  $M_0$ ,  $M_1$  and  $M'_1$ . The arrow  $c$  then defines an arrow  $\tilde{k}_1$ , which is an isomorphism if and only if so is  $c$ . This shows that the exchange condition is implied by (C), and we argue symmetrically for (C') by taking  $\tilde{\chi}'$ ,  $\tilde{k}_1$  and  $\tilde{k}_0$  to be identities.  $\square$

*Remarks 2.3.* a) This proof did not use the hypothesis that (1.1) be Cartesian.

b) Under conservativity assumptions for  $a_2^*$  or  $a^*$ , we obtain converses to (C) and (C').

**3. Pre-descent data.** Here we come back to the set-up of Section 1: namely, we give ourselves a commutative diagram (1.1) as in §1.1 and a system  $(A_3, p_1, p_2, p_3)$  as in the beginning of §1.2 satisfying the identities of *loc. cit.* In other words, we have a set of objects and morphisms of  $\mathbf{A}$

$$(A_0, A_1, A_2, A_3, a, a_1, a_2, p_1, p_2, p_3)$$

subject to the relations

$$aa_1 = aa_2, \quad a_1p_2 = a_1p_3, \quad a_1p_1 = a_2p_3, \quad a_2p_1 = a_2p_2.$$

Let  $M \in \mathbf{M}(A_1)$  and  $v \in \mathbf{M}(A_2)(a_1^*M, a_2^*M)$ . We associate to  $v$  three morphisms

$$\hat{\varphi}_{ij} = \alpha_{ij}(v) : b_i^*M \rightarrow b_j^*M \quad (i < j)$$

where  $\alpha_{ij}$  are the maps of (1.4).

**Definition 3.1.** We say that  $v$  is a *pre-descent datum* on  $M$  if the  $\hat{\varphi}_{ij}$  satisfy the condition  $\hat{\varphi}_{13} = \hat{\varphi}_{23} \circ \hat{\varphi}_{12}$  of Proposition 1.6. We write  $\mathbf{D}^{\text{pre}}$  for the category whose objects are pairs  $(M, v)$ , where  $v$  is a pre-descent datum on  $M$ , and whose morphisms are those of  $\mathbf{M}(A_1)$  which commute with pre-descent data.

Let us introduce the

**Hypothesis 3.2.** The weak exchange condition is verified by the squares (1.1) and (1.7).

**Proposition 3.3** (cf. [2, lemme]). *In (1.10), assume  $\varphi_{12} = \varphi_{23} = \varphi_{13} =: \varphi$ . If  $\varphi$  satisfies the associativity condition of a  $T^a$ -algebra, then  $\xi(\varphi)$  in (1.3) is a pre-descent datum; the converse is true under Hypothesis 3.2.*

*Proof.* In view of Proposition 1.6, it suffices to show that Hypothesis 3.2 implies the injectivity of  $\rho$ , which is induced by the composition of the two natural transformations of (1.8). The second is epi, therefore induces an injection on Hom's, and so does the first by adjunction.  $\square$

**Corollary 3.4.** *Let  $\mathbf{M}_{\text{ass}}^a$  denote the category of associative  $T^a$ -algebras which are not necessarily unital. Then Proposition 3.3 defines a faithful functor  $\xi : \mathbf{M}_{\text{ass}}^a \rightarrow \mathbf{D}^{\text{pre}}$  commuting with the forgetful functors to  $\mathbf{M}(A_1)$ ; under Hypothesis 3.2, it is an isomorphism of categories.*

*Proof.* Commutation of  $\xi$  with the forgetful functors is obvious. This already shows that it is faithful; under Hypothesis 3.2, it is essentially surjective by Proposition 3.3 and we see immediately that it is also full.  $\square$

**4. The unit condition.** We keep the hypotheses and notation of Section 3, and introduce an additional ingredient: a “diagonal” morphism  $\Delta : A_1 \rightarrow A_2$  such that  $a_1\Delta = a_2\Delta = 1_{A_1}$ .

**Definition 4.1.** A *descent datum* on  $M$  is a pre-descent datum  $v$  such that  $\Delta^*v = 1_M$ . We denote by  $\mathbf{D}$  the full subcategory of  $\mathbf{D}^{\text{pre}}$  given by the descent data.

Let  $\mathbf{M}^a \subset \mathbf{M}_{\text{ass}}^a$  be the category of  $T^a$ -algebras.

**Theorem 4.2** (cf. [2, théorème]). *For all  $\varphi \in \mathbf{M}(A_1)(T^a M, M)$ , we have*

$$(4.1) \quad \Delta^* \xi(\varphi) = \varphi \circ \eta_M^a.$$

*In particular,  $\xi(\mathbf{M}^a) \subset \mathbf{D}$  and  $\xi : \mathbf{M}^a \rightarrow \mathbf{D}$  is an isomorphism of categories under Hypothesis 3.2.*

*Proof.* Suppose that  $M = N$  in Lemma 1.2. Applying  $\Delta^*$  to its identity, we get (4.1). In particular, if  $\varphi$  is the action of a  $T^a$ -algebra then  $v = \xi(\varphi)$  verifies  $\Delta^* v = 1_M$ . We conclude with Corollary 3.4.  $\square$

As in [12, VI.3, Th. 1], we have the Eilenberg-Moore comparison functor

$$(4.2) \quad \begin{aligned} K^a : \mathbf{M}(A_0) &\rightarrow \mathbf{M}^a \\ M_0 &\mapsto (a^* M_0, a^* \varepsilon_{M_0}^a). \end{aligned}$$

Lemma 1.7 yields:

**Proposition 4.3.** *We have  $\xi(a^* \varepsilon_{M_0}^a) = 1_{M_0}$ . In other words, in the diagram*

$$\begin{array}{ccccc} \mathbf{M}(A_0) & \xrightarrow{\Psi^a} & \mathbf{D} & \xrightarrow{U^a} & \mathbf{M}(A_1) \\ & \searrow K^a & \uparrow \xi & \nearrow U^{T^a} & \\ & & \mathbf{M}^a & & \end{array}$$

*the left triangle commutes (as well as the right one, trivially).*  $\square$

**Remark 4.4.** In [9, 3.7], Janelidze and Tholen construct a functor from  $\mathbf{D}$  to  $\mathbf{M}$  (same direction as in [2]) by using the *inverses* of the base change morphisms (1.2).

**Remark 4.5.** In the canonical example 1.3, a pre-descent datum  $v$  satisfies the condition of Definition 4.1 if and only if it is invertible (therefore is a descent datum in the classical sense): this follows from [4, A.1.d pp. 303–304]. In *loc. cit.*, Grothendieck uses an elegant Yoneda argument. It is an issue to see how this result extends to our more general situation: this is done in the next proposition. I am indebted to the referee for prodding me to investigate this.

Note that I merely looked for what is necessary to translate Grothendieck's arguments, and not for the greatest generality.

**Proposition 4.6.** *Let  $(A_0, A_1, A_2, A_3, a, a_1, a_2, p_1, p_2, p_3)$  be as in Section 3. Let  $M \in \mathbf{M}(A_1)$  and let  $v \in \mathbf{M}(A_2)(a_1^* M, a_2^* M)$  be a pre-descent datum as in Definition 3.1. Further, let  $\Delta$  be as in the beginning of the present section. Consider the following conditions:*



- (i)  $\Delta^*v = 1_M$  (i.e.  $v$  is a descent datum).
- (ii)  $v$  is invertible.

Then:

a) (ii)  $\Rightarrow$  (i) under one of the following conditions: there exists a morphism  $s_1$  (resp.  $s_2$ ) from  $A_2$  to  $A_3$  such that

$$p_1s_1 = \Delta a_2, \quad p_2s_1 = p_3s_1 = 1$$

(resp.

$$p_1s_2 = p_2s_2 = 1, \quad p_3s_2 = \Delta a_2).$$

b) (i)  $\Rightarrow$  (ii) under the following condition: there exists an involution  $\sigma$  of  $A_2$  and a morphism  $\Gamma : A_2 \rightarrow A_3$  such that

$$p_1\Gamma = \sigma, \quad p_2\Gamma = \Delta a_1, \quad p_3\Gamma = 1_{A_2}.$$

(In the case of the canonical example 1.3, we may take for  $s_1$  and  $s_2$  the partial diagonals, for  $\sigma$  the exchange of factors and for  $\Gamma$  the graph of  $a_1$ , given in formula by  $(\alpha_1, \alpha_2) \mapsto (\alpha_1, \alpha_2, \alpha_1)$ .)

*Proof.* The predescent condition on  $v$  is

$$(4.3) \quad p_2^*v = p_1^*v \circ p_3^*v.$$

a) Applying  $s_1^*$  to (4.3), we get

$$v = a_2^*\Delta^*v \circ v$$

hence  $a_2^*\Delta^*v = 1_{A_2}$  and

$$\Delta^*v = \Delta^*a_2^*\Delta^*v = 1_{A_1}.$$

Same reasoning with  $s_2$ , *mutatis mutandis*. Note that with  $s_1$  (resp.  $s_2$ ), it suffices to assume that  $v$  is right (resp. left) cancellable.

b) Applying  $\Gamma^*$  to (4.3), we get

$$1_{A_2} = a_1^*\Delta^*v = \sigma^*v \circ v.$$

Applying now  $\sigma^*$ , we also get  $v \circ \sigma^*v = 1_{A_2}$ . □

**5. A supplement.** Recall [6, Ex. 8.7.8] that a category is called *Karoubian* if any idempotent endomorphism has an image.

**Proposition 5.1.** *Let  $a^*$  be fully faithful and  $\mathbf{M}(A_0)$  Karoubian. Let  $\varphi : T^aM \rightarrow M$  satisfy the identity  $\varphi \circ \eta_M^a = 1_M$ . Then there exists  $M_0 \in \mathbf{M}(A_0)$  and an isomorphism  $\nu : M \xrightarrow{\sim} a^*M_0$  such that  $\varphi = \nu^{-1} \circ a^*\varepsilon_{M_0}^a \circ T^a\nu$ .*

*Proof.* Let  $e$  denote the idempotent  $\eta_M^a \varphi \in \text{End}_{\mathbf{M}(A_1)}(T^a M)$ . By hypothesis,  $e = a^* \tilde{e}$  where  $\tilde{e}$  is an idempotent of  $\text{End}_{\mathbf{M}(A_0)}(a_* M)$ , with image  $M_0$ . Then  $a^* M_0$  is isomorphic to the image  $M$  of  $e$  via a morphism  $\nu$  as in the statement, such that

$$\nu \circ \varphi = a^* \pi, \quad a^* \iota \circ \nu = \eta_M^a$$

where  $\iota \pi$  is the epi-mono factorization of  $\tilde{e}$ .

To finish, it is enough to see that  $a^* \pi = a^* \varepsilon_{M_0}^a \circ T^a \nu$ . But we also have

$$\eta_{a^* M_0}^a \circ \nu = T^a \nu \circ \eta_M^a = T^a \nu \circ a^* \iota \circ \nu$$

hence  $\eta_{a^* M_0}^a = T^a \nu \circ a^* \iota$ . This concludes the proof, since  $\eta_{a^* M_0}^a \circ a^* \varepsilon_{M_0}^a$  is the epi-mono factorisation of the idempotent of  $\text{End}(T^a a^* M_0)$  with image  $a^* M_0$ .  $\square$

We thus obtain the following complement:

**Corollary 5.2.** *Assume Hypothesis 3.2, and also that  $a^*$  is fully faithful and  $\mathbf{M}(A_0)$  Karoubian. Then*

- a) *every unital  $T^a$ -algebra is associative;*
- b)  *$K^a$  is essentially surjective.*

$\square$

Can one weaken the full faithfulness assumption in this corollary? The following lemma does not seem sufficient:

**Lemma 5.3.** *Let  $M, N \in \mathbf{M}(A_1)$ . Then the map*

$$a^* : \mathbf{M}(A_0)(a_* M, a_* N) \rightarrow \mathbf{M}(A_1)(T^a M, T^a N)$$

*has a retraction  $r$  given by  $r(f) = \varepsilon_{a_* N}^a \circ a_* f \circ a_* \eta_M^a$ . More generally, we have an identity of the form  $r(a^* g \circ f) = g \circ r(f)$ .*

*Proof.* For  $f : T^a M \rightarrow T^a N$  and  $g : a_* N \rightarrow a_* P$ , we have

$$r(a^* g \circ f) = \varepsilon_{a_* P}^a \circ a_* a^* g \circ a_* f \circ a_* \eta_M^a = g \circ \varepsilon_{a_* N}^a \circ a_* f \circ a_* \eta_M^a = g \circ r(f).$$

Taking  $f = 1_{T^a M}$ , we obtain that  $r$  is a retraction.  $\square$

## 6. Appendix: a case where the exchange condition is verified.

Let  $\mathcal{A}$  be a category. Take for  $\mathbf{A}$  the category of presheaves of sets on  $\mathcal{A}$ . Write  $\int A$  for the category associated to  $A \in \mathbf{A}$  by the Grothendieck construction [5, §8]. Recall its definition in this simple case: the objects of  $\int A$  are pairs  $(X, a)$  where  $X \in \mathcal{A}$  and  $a \in A(X)$ , and a morphism from  $(X, a)$  to  $(Y, b)$  is a morphism  $f \in \mathcal{A}(X, Y)$  such that  $A(f)(x) = y$ .

Let  $\mathcal{C}$  be another category. We take for  $\mathbf{M}$  the fibred category of representations of  $\mathbf{A}$  in  $\mathcal{C}$ : for  $A \in \mathbf{A}$ , an object of  $\mathbf{M}(A)$  is a functor from  $\int A$  to  $\mathcal{C}$ . For all  $a \in \mathbf{A}(A_1, A_0)$  we have an obvious pull-back functor  $a^* : \mathbf{M}(A_0) \rightarrow \mathbf{M}(A_1)$ , which has a left adjoint  $a_*$  (direct image)

given by the usual colimit if  $\mathcal{C}$  is cocomplete. We can then ask whether the exchange condition is true for Cartesian squares of  $\mathcal{A}$ .

**Proposition 6.1.** *This is the case if  $\mathcal{C}$  is the category of sets **Set**, and more generally if  $\mathcal{C}$  admits a forgetful functor  $\Omega : \mathcal{C} \rightarrow \mathbf{Set}$  with a left adjoint  $L$  such that  $(L, \Omega)$  satisfies the conditions of Beck's theorem [12, VI.7, Th. 1].*

*Proof.* First suppose  $\mathcal{C} = \mathbf{Set}$ ; to verify that (1.2) is a natural isomorphism, it is enough to test it on representable functors. Consider Diagram (2.2) again. For  $(c, \gamma) \in \int A_1$  and  $(d, \delta) \in \int A_1$  (with  $c, d \in \mathcal{A}$  and  $\gamma \in A_1(c)$ ,  $\delta \in A_1(d)$ ), we have

$$\begin{aligned} T^a y(c, \gamma)(d, \delta) &= a^* y(c, a(\gamma))(d, \delta) = y(c, a(\gamma))(d, a(\delta)) \\ &= \{\varphi \in \mathcal{A}(d, c) \mid \varphi^* a(\gamma) = a(\delta)\} \end{aligned}$$

and

$$\begin{aligned} (a_2)_* a_1^* y(c, \gamma)(d, \delta) &= \varinjlim_{(e, \eta) \in (d, \delta) \downarrow a_2} a_1^* y(c, \gamma)(e, \eta) \\ &= \varinjlim_{(e, \eta) \in (d, \delta) \downarrow a_2} y(c, \gamma)(e, a_1(\eta)) \\ &= \varinjlim_{(e, \eta) \in (d, \delta) \downarrow a_2} \{\psi \in \mathcal{A}(e, c) \mid \psi^* \gamma = a_1(\eta)\}. \end{aligned}$$

We have

$$(d, \delta) \downarrow a_2 = \{(e, \eta, \eta_2, \theta) \in \mathcal{A} \times A_1(e) \times_{A_0(e)} A_1(e) \times \mathcal{A}(d, e) \mid \theta^* \eta_2 = \delta\}.$$

This category has the initial set  $\{(d, \eta_1, \delta, 1_d) \mid a(\eta_1) = a(\delta)\}$ , so

$$\begin{aligned} (a_2)_* a_1^* y(c, \gamma)(d, \delta) &= \coprod_{\{(\eta_1 \in A_1(d) \mid a(\eta_1) = a(\delta))\}} \{\varphi \in \mathcal{A}(d, c) \mid \varphi^* \gamma = \eta_1\} \\ &= \{\varphi \in \mathcal{A}(d, c) \mid a(\varphi^* \gamma) = a(\delta)\} \end{aligned}$$

and the map  $(a_2)_* a_1^* y(c, \gamma)(d, \delta) \rightarrow (a_2)_* (a^{12})^* y(c, \gamma)(d, \delta)$  is clearly equal to the identity.

*General case:* let us write more precisely  $\mathbf{M}^{\mathcal{C}}(A) = \mathbb{CAT}(\int A, \mathcal{C})$ . The functors  $L$  and  $\Omega$  induce pairs of adjoint functors (same notation)

$$L : \mathbf{M}^{\mathbf{Set}}(A) \rightleftarrows \mathbf{M}^{\mathcal{C}}(A) : \Omega.$$

These two functors commute with pull-backs; as  $L$  is a left adjoint, it also commutes with direct images. Therefore, in the above situation, the base change morphism  $\chi_M : (a_2)_* a_1^* M \rightarrow T^a M$  is an isomorphism when  $M \in \mathbf{M}^{\mathcal{C}}(A_1)$  is of the form  $LX$  for  $X \in \mathbf{M}^{\mathbf{Set}}(A_1)$ . For any  $M$ , we have its canonical presentation [12, (5) p. 153]

$$(6.1) \quad (L\Omega)^2 M \rightrightarrows L\Omega M \rightarrow M$$

whose image by  $\Omega$  is a split coequaliser (*loc. cit.*). Given the hypothesis that  $\Omega$  creates such coequalisers, (6.1) is a coequaliser. Since pull-backs are cocontinuous, as well as direct images (again, as left adjoints), (6.1) remains a coequaliser after applying the functors  $(a_2)_*(a^{12})^*$  and  $T^a$ . Finally, a coequaliser of isomorphisms is an isomorphism.  $\square$

*Examples 6.2* (for  $\mathcal{C}$ ). Varieties (category of groups, abelian groups, rings...): [12, VI.8, Th. 1].

*Example 6.3* (for  $\mathcal{A}$ ). The category with one object  $\underline{G}$  associated with a group  $G$ : then  $\mathbf{A}$  is the category of  $G$ -sets. Let us take for  $\mathcal{C}$  the category of  $R$ -modules where  $R$  is a commutative ring. If  $A \in \mathbf{A}$  is  $G$ -transitive,  $\int A$  is a connected groupoid, which is equivalent to  $\underline{H}$  for the stabilizer  $H$  of any element of  $A$ ; thus,  $\mathbf{M}(A)$  is equivalent to  $\mathbf{Rep}_R(H)$ . If  $a : A_1 \rightarrow A_0$  is the morphism of  $\mathbf{A}$  defined by an inclusion  $K \subset H \subset G$  ( $A_1 = G/K$ ,  $A_0 = G/H$ ), then  $a^*$  is restriction from  $H$  to  $K$  and  $a_*$  is induction  $V \mapsto RH \otimes_{RK} V$ . From Proposition 6.1, we thus recover conceptually the Mackey formula of [15, 7.3, Prop. 22], proven “by hand” in *loc. cit.*

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