ANNALS OF K-THEORY

vol. 10 no. 4 2025

An l-adic norm residue epimorphism theorem

Bruno Kahn

msp

A JOURNAL OF THE K-THEORY FOUNDATION

ANNALS OF K-THEORY
Vol. 10 (2025), No. 4, pp. 707–720
DOI: 10.2140/akt.2025.10.707

An l-adic norm residue epimorphism theorem

Bruno Kahn

We show that the continuous étale cohomology groups $H^n_{\mathrm{cont}}(X, \mathbb{Z}_l(n))$ of smooth varieties X over a finite field k are spanned as $\mathbb{Z}_l$-modules by the n-th Milnor K-sheaf locally for the Zariski topology for all $n \geq 0$. Here l is a prime invertible in k. This is the first general unconditional result towards the conjectures of Kahn (1998) which put together the Tate and the Beilinson conjectures relative to algebraic cycles on smooth projective k-varieties.

1. Introduction

Two fundamental conjectures on smooth projective varieties X over a finite field k are

- the Tate conjecture: for any $n \geq 0$, the order of the pole of the zeta function $\zeta(X, s)$ at $s = n$ equals the rank of the group of algebraic cycles of codimension n over X, modulo numerical equivalence;
- the Beilinson conjecture: for any $n \geq 0$, an algebraic cycle of codimension n on X with $\mathbb{Q}$-coefficients which is numerically equivalent to 0 is rationally equivalent to 0.

In the unpublished preprint [Kahn 1998] — inspired by work of Geisser [1998] — I put these two conjectures together and reformulated them into a sheaf-theoretic statement involving all smooth (not necessarily projective) k-varieties.

Actually, there are two reformulations in [Kahn 1998]: one with rational coefficients (Conjecture 8.12) and one with integral coefficients (Conjecture 9.6). The first one is elementary, involving cohomology of Milnor K-sheaves; the second one involves motivic cohomology and also appears in the published paper [Kahn 2002] (Conjecture 3.2 and Theorem 3.4).

Here we shall be interested in the first reformulation. Let me recall it. Let $\mathcal{S}$ denote the big étale site of $\operatorname{Spec} k$ restricted to smooth k-varieties; as in [Kahn 1998, Definition 2.1], write $\mathbb{Z}_l(n)^c$ (resp. $\mathbb{Q}_l(n)^c$) for the object $R \varprojlim(\mu_{l^\nu}^{\otimes n})$ (resp. $\mathbb{Z}_l(n)^c \otimes \mathbb{Q}$) of $D^+(\mathrm{Ab}(\mathcal{S})) =: D^+(\mathcal{S})$. Thus,

$$H^i_{\text{ét}}(X, \mathbb{Z}_l(n)^c) = H^i_{\mathrm{cont}}(X, \mathbb{Z}_l(n)),$$

MSC2020: 11G25, 14C35, 19E15.
Keywords: Milnor K-theory, Tate–Beilinson conjecture, motivic cohomology.

© 2025 The Author, under license to MSP (Mathematical Sciences Publishers).

where the right-hand side is the continuous étale cohomology of Jannsen [1988] (by [Kahn 1998, Lemma 1.1(a)]), and similarly with $\mathbb{Q}_l$ coefficients.

As a first step, we have

$$H^i_{\mathrm{cont}}(X, \mathbb{Q}_l(n)^c) = 0 \quad \text{for } i < n \tag{1-1}$$

for any $X \in \mathcal{S}$ by [Kahn 1998, Corollary 6.10(b)]. It follows that the presheaf $X \mapsto H^n_{\mathrm{cont}}(X, \mathbb{Q}_l(n))$ is an étale sheaf. Then, by [Kahn 1998, Proposition 8.10 and its proof], a version of a theorem of Tate [1976, Theorem 3.1] yields a homomorphism

$$K^M_n(X) \otimes \mathbb{Z}_l \to H^n_{\mathrm{cont}}(X, \mathbb{Z}_l(n)) \tag{1-2}$$

for any smooth X, where $K^M_n(X) := K^M_n(\Gamma(X, \mathcal{O}_X))$, hence a homomorphism of associated Zariski sheaves

$$\mathcal{K}^M_n \otimes \mathbb{Z}_l \to \mathcal{H}^n(\mathbb{Z}_l(n)^c) \tag{1-3}$$

and a fortiori a homomorphism of associated étale sheaves

$$\alpha^* \mathcal{K}^M_n \otimes \mathbb{Z}_l \to \alpha^* \mathcal{H}^n(\mathbb{Z}_l(n)^c),$$

where α is the projection of $\mathcal{S}$ on the big smooth Zariski site. By (1-1), we then get a morphism in $D^+(\mathcal{S})$:

$$\alpha^* \mathcal{K}^M_n[-n] \otimes \mathbb{Q}_l \to \mathbb{Q}_l(n)^c.$$

For $n = 0$, this morphism is not an isomorphism because the right-hand side has two nonzero cohomology sheaves, coming from $H^0(k, \mathbb{Q}_l)$ and $H^1(k, \mathbb{Q}_l)$; compare [Kahn 1998, Theorem 4.6(b)]. To get the correct comparison morphism, we tensor with it to get

$$\alpha^* \mathcal{K}^M_n[-n] \otimes^L \mathbb{Q}_l(0)^c \to \mathbb{Q}_l(n)^c. \tag{1-4}$$

Conjecture 8.12 of [Kahn 1998] states that (1-4) is an isomorphism. Note that $\mathbb{Q}_l(0)^c \simeq \mathbb{Q}_l \oplus \mathbb{Q}_l[-1]$ by [Kahn 1998, Corollary 4.5 and Theorem 4.6]. In concrete terms, (1-4) therefore induces homomorphisms

$$H^{i-n-1}(X, \mathcal{K}^M_n) \otimes \mathbb{Q}_l \oplus H^{i-n}(X, \mathcal{K}^M_n) \otimes \mathbb{Q}_l \to H^i_{\mathrm{cont}}(X, \mathbb{Q}_l(n))$$

for any smooth X, and Conjecture 8.12 predicts that they are isomorphisms. (Here we use the fact that Zariski and étale sections of $\mathcal{K}^M_n \otimes \mathbb{Q}$ agree; compare [Kahn 1998, Corollary 8.6].) This can be viewed as an extension of the cohomological version of Tate's conjecture saying that, in some sense, all continuous étale cohomology groups are generated by "algebraic cycles" (cohomology of Milnor K-sheaves: note that $H^n(X, \mathcal{K}^M_n) \simeq CH^n(X)$) plus one transcendental element: the generator of $H^1_{\mathrm{cont}}(k, \mathbb{Q}_l) = \mathrm{Hom}_{\mathrm{cont}}(G_k, \mathbb{Q}_l)$ which sends Frobenius to 1.

As a special case, this conjecture proposes a description of the first nonzero continuous étale cohomology group $H^n_{\text{cont}}(X, \mathbb{Q}_l(n))$, which ought to be isomorphic to $H^0(X, \mathcal{K}_n^M) \otimes \mathbb{Q}_l$ via (1-3). I realised recently that a refinement of the proof of (1-1) might give enough information on this group to make some progress on this latter conjecture. This was successful, and we even get an integral statement which is the main result of this article.

Theorem 1.1. *The morphism (1-3) is an epimorphism of Zariski sheaves and even of presheaves if $n \le 2$.*

This is the first general unconditional result in the direction of [Kahn 1998, Conjecture 8.12]. It can be viewed as an l-adic norm residue epimorphism theorem. As a complement, let us notice that the Zariski and étale sections of both sides coincide by [Mazza et al. 2006, Theorems 14.24 and 22.2] and that, after tensoring with $\mathbb{Q}$, those of the right-hand side on some smooth X are $H^n_{\text{cont}}(X, \mathbb{Q}_l(n))$ by (1-1). So, for $n \le 2$, Theorem 1.1 yields a surjection

$$H^0(X, \mathcal{K}_n) \otimes \mathbb{Q}_l \twoheadrightarrow H^n_{\text{cont}}(X, \mathbb{Q}_l(n))$$

for all smooth X.

To avoid any misunderstanding, let me point out that Theorem 1.1 is *deduced* from its rational version, Theorem 6.2, by using the norm residue isomorphism theorem of [Voevodsky 2011], while the proof of Theorem 6.2 itself has nothing to do with the latter theorem.

One may ask about isomorphy in Theorem 6.2. But this seems much harder: after tensorisation with $\mathbb{Q}$, the global sections of the right-hand side of (1-3) are 0 on X if X is projective (provided $n > 0$), so this would imply the vanishing of $H^0(X, \mathcal{K}_n^M) \otimes \mathbb{Q}$ for such X. Conversely, this vanishing for all smooth projective varieties would imply that (1-3)$\otimes\mathbb{Q}$ is an isomorphism; see the beginning of Section 7. It can actually be proven for certain smooth projective X (Theorem 7.1), but there aren't enough of them to deduce the isomorphy of (1-3)$\otimes\mathbb{Q}$ in general. See nevertheless Corollary 7.2 and Example 7.3.

The proof of Theorem 6.2 is not difficult, but involves a number of ideas. Here is a description. By de Jong's theorem on alterations, we reduce to the case where X has a smooth compactification whose closed complement is the support of a divisor with strict normal crossings. A suitable spectral sequence, plus cohomological purity, then allows us to get a concrete description of $H^n_{\text{cont}}(X, \mathbb{Q}_l(n))$, as in Corollary 2.2(b). This description already shows that these cohomology classes are, in some sense, of an algebraic nature, and the next step is to make the link with (1-3). Here we pass to Voevodsky's theory of homotopy-invariant Nisnevich sheaves with transfers [Mazza et al. 2006] and its extension to *homotopic modules* by Déglise [2011]. It turns out that the collection of the $H^n_{\text{cont}}(X, \mathbb{Q}_l(n))$, for $n \in \mathbb{Z}$, defines a special kind of homotopic module that we call *reduced* (Definition 3.9; see Proposition 5.1).

The Milnor K-sheaves, for their part, form a homotopic module which maps to the latter via (1-3), but this homotopic module is not known to be reduced (this is precisely the vanishing issue mentioned in the previous paragraph). However, any ((−1)-connected) homotopic module admits a universal map to a reduced one, and fortunately this map is epi (Theorem 3.11); the proof involves a generalisation of the theory of triangulated birational motives of [Kahn and Sujatha 2017] to Verdier quotients of Voevodsky's category $\mathbf{DM}^{\text{eff}}(k)$ by higher powers of the Tate object, in the spirit of [Huber and Kahn 2006]. The reduced homotopic module associated to $\mathcal{K}_*^M \otimes \mathbb{Q}_l$ therefore maps to the homotopic module of continuous étale sheaves, and a comparison using an analogue of Corollary 2.2(b) shows that this is an isomorphism.

2. The l-adic computation

Let $\bar{X}$ be a smooth projective geometrically irreducible variety over a field k, $Z = \bigcup_{i\in I} Z_i \subset \bar{X}$ a normal crossing divisor and $X = \bar{X} - Z$. For $J \subseteq I$, write $Z_J = \bigcap_{i\in J} Z_i$; in particular, (by convention) $Z_\varnothing = \bar{X}$.

Next we let $(i, n) \in \mathbb{Z}\times\mathbb{Z}$. If $H^i(V, n)$ denotes the continuous étale cohomology $H^i_{\text{cont}}(V, \mathbb{Q}_l(n))$ of [Jannsen 1988], and similarly for cohomology groups with supports, the exact sequences for cohomology with supports and the reasoning of [Esnault et al. 1998, §3.3] yield a spectral sequence

$$E_1^{p,q} = \bigoplus_{|J|=d-p} H^q_{Z_J}(\bar{X}, n) \Rightarrow H^{p+q-d}(X, n), \tag{2-1}$$

where $d = \dim X$ and where the d^1 differentials are given by Gysin maps. By purity [Jannsen 1988], we have

$$H^q_{Z_J}(\bar{X}, n) \simeq H^{q-2(d-p)}(Z_J, n+p-d). \tag{2-2}$$

This yields the following.

Proposition 2.1. *Suppose that k is finite. Then $E_1^{p,q} = 0$ unless $q \in \{2n, 2n+1\}$, $d-n \le p \le d$ and $n \le d$.*

Proof. The first condition follows from the Weil conjecture and the Hochschild–Serre spectral sequence; compare [Colliot-Thélène et al. 1983, §2.1]. In the second condition, the upper bound is clear, while the lower bound follows from the inequality $q-2(d-p) \ge 0$ and the first condition. For the third condition, the étale cohomological dimension of Z_J is $2(d-|J|)+1 = 2p+1$; hence $E_1^{p,q} = 0$ unless $q-2(d-p) \le 2p+1$, i.e., $q \le 2d+1$, which in turn implies $n \le d$ by the first condition. □

Corollary 2.2. *We have*

(a) *long exact sequences*

$$\cdots \to E_2^{r-2n,2n} \to H^{r-d}(X, n) \to E_2^{r-2n-1,2n+1} \to E_2^{r-2n+1,2n} \to \cdots$$

(b) $H^i(X,n)=0$, *unless* $n\le d$ *and* $i\ge n$, *and an exact sequence*

$$0\to H^n(X,n)\to\bigoplus_{|J|=n}H^0(Z_J,0)\xrightarrow{i_n}\bigoplus_{|J|=n-1}H^2(Z_J,1),$$

where i_n *is given by the Gysin maps in continuous étale cohomology.*

Proof. (a) is obvious from the first condition on q in Proposition 2.1, and (b) then follows from the other conditions. Indeed, all terms in (a) are 0 if $r-2n<d-n$, i.e., if $r-d<n$. If now $r-d=n$, then the middle term is isomorphic to $\mathrm{Ker}(E_1^{d-n,2n}\xrightarrow{d_1}E_1^{d-n+1,2n})$, hence the conclusion. □

Remark 2.3. If $|I|\le d$ and $n>|I|$, we get a sharper vanishing bound: $H^i(X,n)=0$ for $i<2n-|I|$, and an exact sequence

$$0\to H^{2n-|I|}(X,n)\to H^0(Z_I,0)\to\bigoplus_{|J|=|I|-1}H^2(Z_J,1).$$

3. Reduced homotopic modules

We go back temporarily to a general perfect field k and write **HI** for the category of homotopy-invariant Nisnevich sheaves with transfers over k [Mazza et al. 2006, Lecture 13]. Let $\mathbf{HI}^{\mathrm{o}}$ be the full subcategory of **HI** consisting of birational sheaves [Kahn and Sujatha 2017, Definition 2.3.1].[1] By [Kahn and Sujatha 2017, §7.1 and Theorem 7.3.1], the inclusion functor $\mathbf{HI}^{\mathrm{o}}\hookrightarrow\mathbf{HI}$ has a right-adjoint

$$\mathcal{F}\mapsto\mathcal{F}_{\mathrm{nr}}=R^0_{\mathrm{nr}}\mathcal{F}.$$

Definition 3.1. A sheaf $\mathcal{F}\in\mathbf{HI}$ is *reduced* if $\mathcal{F}_{\mathrm{nr}}=0$.

Lemma 3.2. *Let* $\mathcal{F}\in\mathbf{HI}$. *Then the presheaf with transfers*

$$\mathcal{F}_{\mathrm{rd}}=\mathrm{Coker}(\mathcal{F}_{\mathrm{nr}}\to\mathcal{F})$$

is a reduced (Nisnevich) sheaf, and the functor $\mathcal{F}\mapsto\mathcal{F}_{\mathrm{rd}}$ *is left adjoint to the inclusion of reduced sheaves into* **HI**.

Proof. By [Kahn and Sujatha 2017, Lemma 2.3.2], we have $H^1(X,\mathcal{F}_{\mathrm{nr}})=0$ for any smooth X, hence a short exact sequence

$$0\to\mathcal{F}_{\mathrm{nr}}(X)\to\mathcal{F}(X)\to a\mathcal{F}_{\mathrm{rd}}(X)\to 0,$$

where $a\mathcal{F}_{\mathrm{rd}}$ is the Nisnevich sheaf associated to $\mathcal{F}_{\mathrm{rd}}$; therefore $\mathcal{F}_{\mathrm{rd}}\to a\mathcal{F}_{\mathrm{rd}}$ is an isomorphism of presheaves. Applying now the functor R_{nr} of [Kahn and Sujatha 2017, §3.1] to the exact sequence $0\to\mathcal{F}_{\mathrm{nr}}\to\mathcal{F}\to\mathcal{F}_{\mathrm{rd}}\to 0$, we get an exact triangle in $\mathbf{DM}^{\mathrm{o}}$:

$$R_{\mathrm{nr}}(\mathcal{F}_{\mathrm{nr}}[0])\to R_{\mathrm{nr}}(\mathcal{F}[0])\to R_{\mathrm{nr}}(\mathcal{F}_{\mathrm{rd}}[0])\xrightarrow{+1}.$$

[1] Recall that a presheaf is *birational* if it inverts open immersions; it is then automatically a Nisnevich sheaf.

But $R_{\mathrm{nr}}(\mathcal{F}_{\mathrm{nr}}[0]) = \mathcal{F}_{\mathrm{nr}}[0]$ because $R^0_{\mathrm{nr}}\mathcal{F}[0] \in \mathbf{DM}^{\mathrm{o}}$ by [Kahn and Sujatha 2017, Theorem 4.4.1] (the part on t-structures). Taking the long cohomology exact sequence for the homotopy t-structure of $\mathbf{DM}^{\mathrm{o}}$, it follows that $R^0_{\mathrm{nr}}(\mathcal{F}_{\mathrm{rd}}) = 0$, i.e., that $\mathcal{F}_{\mathrm{rd}}$ is reduced. That it defines the said left adjoint is now obvious. □

Here is a generalisation. Recall that $\mathcal{F} \in \mathbf{HI}^{\mathrm{o}} \Leftrightarrow \mathcal{F}_{-1} = 0$, where $(-)_{-1}$ is Voevodsky's contraction [Kahn and Sujatha 2017, Proposition 2.5.2].

Definition 3.3. (a) A sheaf $\mathcal{F} \in \mathbf{HI}$ is *of coniveau* $< n$ if $\mathcal{F}_{-n} = 0$.[2] Write $\mathbf{HI}_{<n}$ for the full subcategory of $\mathbf{HI}$ consisting of sheaves of coniveau $< n$ (so that $\mathbf{HI}_{<1} = \mathbf{HI}^{\mathrm{o}}$).

(b) A sheaf $\mathcal{F} \in \mathbf{HI}$ is *n-reduced* if its only subsheaf of coniveau $< n$ is 0.

Definition 3.4. We write $\mathbf{DM}^{\mathrm{eff}}_{<n} = \mathbf{DM}^{\mathrm{eff}} / \mathbf{DM}^{\mathrm{eff}}(n)$ (so that $\mathbf{DM}^{\mathrm{eff}}_{<1} = \mathbf{DM}^{\mathrm{o}}$).

The same yoga as in [Kahn and Sujatha 2017] (Brown representability) gives:

Proposition 3.5. *The localisation functor $\nu_{<n} : \mathbf{DM}^{\mathrm{eff}} \to \mathbf{DM}^{\mathrm{eff}}_{<n}$ admits a (fully faithful) right-adjoint ι_n, which itself admits a right-adjoint $R_{<n} : C \mapsto C_{<n}$. Moreover, there are functorial exact triangles*

$$\nu^{\geq n} M \to M \xrightarrow{\varepsilon_M} \iota_n \nu_{<n} M \xrightarrow{+1},$$

where ε_M is the unit of the adjunction $(\nu_{<n}, \iota_n)$ and, as in [Huber and Kahn 2006, (1.1)], $\nu^{\geq n} M = \underline{\mathrm{Hom}}(\mathbb{Z}(n), M)(n)$. □

The key point is the following.

Proposition 3.6. *The homotopy t-structure on $\mathbf{DM}^{\mathrm{eff}}$ induces a t-structure on $\mathbf{DM}^{\mathrm{eff}}_{<n}$ via ι_n, with heart $\mathbf{HI}_{<n}$.*

Proof. For $C \in \mathbf{DM}^{\mathrm{eff}}$, write $C_{-n} = \underline{\mathrm{Hom}}(\mathbb{Z}(n)[n], C)$. Then $C \in \iota_n \mathbf{DM}^{\mathrm{eff}}_{<n}$ if and only if $C_{-n} = 0$. But this functor is t-exact as the n-fold composition of the t-exact functor $(-)_{-1}$ [Déglise 2011, Theorem 5.2]. □

Proposition 3.7. (a) *The inclusion $\mathbf{HI}_{<n} \hookrightarrow \mathbf{HI}$ has a right-adjoint $\mathcal{F} \mapsto \mathcal{F}_{<n}$. Moreover, we have $(\mathcal{F}_{-1})_{<n-1} = (\mathcal{F}_{<n})_{-1}$ as subsheaves of $\mathcal{F}_{-1}$.*

(b) *The inclusion of n-reduced sheaves in $\mathbf{HI}$ has a left-adjoint $\mathcal{F} \mapsto \mathcal{F}_{n\text{-rd}}$, and the unit morphism $\mathcal{F} \to \mathcal{F}_{n\text{-rd}}$ is an epimorphism of sheaves.*

Proof. Since $(-)_{-n}$ is exact and commutes with infinite direct sums, $\mathbf{HI}_{<n}$ is stable under arbitrary colimits; defining $\mathcal{F}_{<n} = \varinjlim \mathcal{G}$, where $\mathcal{G}$ runs through the subsheaves of $\mathcal{F}$ which belong to $\mathbf{HI}_{<n}$, proves the first part of (a). For the second part, the exactness of $(-)_{-1}$ gives an inclusion $(\mathcal{F}_{<n})_{-1} \subseteq (\mathcal{F}_{-1})_{<n-1}$; conversely, the inclusion $(\mathcal{F}_{-1})_{<n-1} \subseteq \mathcal{F}_{-1}$ yields by adjunction a morphism $(\mathcal{F}_{-1})_{<n-1} \otimes \mathbb{G}_m \to \mathcal{F}$,

[2] This terminology will be justified by Lemma 4.4(a).

which factors through $\mathcal{F}_{<n}$ by the cancellation theorem [Voevodsky 2010]; hence $(\mathcal{F}_{-1})_{<n-1} \subseteq (\mathcal{F}_{<n})_{-1}$ by adjunction again.

For (b), define $\mathcal{F}_{n\text{-rd}} = \operatorname{Coker}(\mathcal{F}_{<n} \to \mathcal{F})$. Using Proposition 3.6, the same reasoning as in the proof of Lemma 3.2 shows that $\mathcal{F}_{n\text{-rd}}$ is n-reduced, hence defines the desired left adjoint. □

Remark 3.8. Contrary to Lemma 3.2, the map $\mathcal{F} \to \mathcal{F}_{n\text{-rd}}$ may not be an epimorphism of presheaves if $n > 1$.

Recall from [Déglise 2011] that a *homotopic module* is an Ω-$\mathbb{G}_m$-spectrum in **HI**, i.e., a sequence $(\mathcal{F}_n)_{n\in\mathbb{Z}}$ of objects of **HI** provided with isomorphisms $\mathcal{F}_n \xrightarrow{\sim} (\mathcal{F}_{n+1})_{-1}$. We shall say that a homotopic module $(\mathcal{F}_n)$ is (-1)-*connected* if $\mathcal{F}_n = 0$ for $n < 0$. Write $\mathbf{HI}_*$ for the category of homotopic modules and $\mathbf{HI}_*^c$ for its full subcategory of (-1)-connected homotopic modules.

Definition 3.9. A (-1)-connected homotopic module $(\mathcal{F}_n)_{n\geq 0}$ is *reduced* if $\mathcal{F}_n$ is reduced for all $n > 0$. Write $\mathbf{HI}_*^{\mathrm{rd}}$ for the full subcategory of $\mathbf{HI}_*^c$ formed of reduced homotopic modules.

Lemma 3.10. *If $(\mathcal{G}_n) \in \mathbf{HI}_*^c$ is reduced, then $\mathcal{G}_n$ is n-reduced for all $n \geq 0$.*

Proof. We proceed by induction on n. The case $n = 0$ is trivial. Suppose that the statement is true for $n-1 \geq 0$, and let $\mathcal{H} \subseteq \mathcal{G}_n$ with $\mathcal{H} \in \mathbf{HI}_{<n}$. Then $\mathcal{H}_{-1} \subseteq (\mathcal{G}_n)_{-1}$ is 0 since $\mathcal{H}_{-1} \in \mathbf{HI}_{<n-1}$. As $\mathcal{G}_n$ is reduced, we have $\mathcal{H} = 0$. □

Theorem 3.11. *The inclusion $\mathbf{HI}_*^{\mathrm{rd}} \hookrightarrow \mathbf{HI}_*^c$ has a left-adjoint $(\mathcal{F}_*) \mapsto (\mathcal{F}_*)^{\mathrm{rd}}$. The unit of this adjunction is an epimorphism of graded sheaves.*

Proof. For $(\mathcal{F}_*) \in \mathbf{HI}_*^c$ and $n \geq 0$, define

$$\mathcal{F}_n^{\mathrm{rd}} = (\mathcal{F}_n)_{n\text{-rd}}.$$

By Proposition 3.7(a), the isomorphisms $\mathcal{F}_{n-1} \xrightarrow{\sim} (\mathcal{F}_n)_{-1}$ induce isomorphisms $(\mathcal{F}_{n-1})_{<n-1} \xrightarrow{\sim} ((\mathcal{F}_n)_{<n})_{-1}$, hence they induce isomorphisms $\mathcal{F}_{n-1}^{\mathrm{rd}} \xrightarrow{\sim} (\mathcal{F}_n^{\mathrm{rd}})_{-1}$ by Proposition 3.7(b). Thus $(\mathcal{F}_*)^{\mathrm{rd}} := (\mathcal{F}_*^{\mathrm{rd}}) \in \mathbf{HI}_*^c$, and this homotopic module is reduced. Its universal property now follows from Lemma 3.10. □

4. Cohomology

Lemma 4.1. *Let $\mathcal{F} \in \mathbf{HI}$. If F is a smooth closed subset of pure codimension c in a smooth k-scheme X, there are isomorphisms $H_F^i(X, \mathcal{F}) \simeq H^{i-c}(F, \mathcal{F}_{-c})$, hence a long exact sequence for $U = X - F$:*

$$\cdots \to H^{i-c}(F, \mathcal{F}_{-c}) \to H^i(X, \mathcal{F}) \to H^i(U, \mathcal{F}) \xrightarrow{\partial} H^{i+1-c}(F, \mathcal{F}_{-c}) \to \cdots.$$

In particular, we have $\mathcal{F}(X) \xrightarrow{\sim} \mathcal{F}(U)$ *if* $c > 1$ *and an exact sequence*

$$0 \to \mathcal{F}(X) \to \mathcal{F}(U) \to \mathcal{F}_{-1}(F)$$

if $c = 1$.

Proof. Let $M_F(X) = \operatorname{cone}(M(U) \to M(X))$. We have a Gysin isomorphism $M_F(X) \simeq M(F)(c)[2c] \simeq M(F) \otimes \mathbb{G}_m^{\otimes c}[c]$ [Mazza et al. 2006, Theorem 15.15]; hence

$$\begin{aligned} H^i_F(X, \mathcal{F}) &\simeq \mathbf{DM}^{\mathrm{eff}}(M_F(X), \mathcal{F}[i]) \simeq \mathbf{DM}^{\mathrm{eff}}(M(F) \otimes \mathbb{G}_m^{\otimes c}[c], \mathcal{F}[i]) \\ &\simeq \mathbf{DM}^{\mathrm{eff}}(M(F), \mathcal{F}_{-c}[i-c]) \simeq H^{i-c}(F, \mathcal{F}_{-c}). \end{aligned}$$

□

Proposition 4.2. *Let* $p : Y \to X$ *be an alteration of smooth* k*-schemes. Then, for any* $\mathcal{F} \in \mathbf{HI}$, *there exists a map* $p_* : \mathcal{F}(Y) \to \mathcal{F}(X)$, *natural in* $\mathcal{F}$, *such that the composition* $\mathcal{F}(X) \xrightarrow{p^*} \mathcal{F}(Y) \xrightarrow{p_*} \mathcal{F}(X)$ *is multiplication by the generic degree* δ.

Proof. If p is finite, this follows from the transfer structure on $\mathcal{F}$. In general, let $Y \xrightarrow{q} Z \xrightarrow{r} X$ be the Stein factorisation of p. Considering the normalisation of Z, we see that Z is normal. Therefore, by the valuative criterion of properness there exists a closed subset $F \subset Z$ of codimension ≥ 2 such that q is an isomorphism above $Z - F$. Then $F' = r(F)$ is of codimension ≥ 2 in X, and $F'' = r^{-1}(F')$ is still of codimension ≥ 2 in Z. Let $G = q^{-1}(F'')$; then $q_{|Y-G} : Y - G \to Z - F''$ is an isomorphism; hence $p' := p_{|Y-G} : Y - G \to X - F'$ is finite. We define p_* as the composition

$$\mathcal{F}(Y) \to \mathcal{F}(Y - G) \xrightarrow{p'_*} \mathcal{F}(X - F') \xrightarrow{\sim} \mathcal{F}(X),$$

where the last map is the inverse of the isomorphism of Lemma 4.1. Using the commutative diagram

$$\begin{array}{ccc} \mathcal{F}(Y) & \longrightarrow & \mathcal{F}(Y - G) \\ {\scriptstyle p^*}\uparrow & & \uparrow{\scriptstyle p'^*} \\ \mathcal{F}(X) & \xrightarrow{\sim} & \mathcal{F}(X - F') \end{array} \tag{4-1}$$

we see that $p_* p^*$ is multiplication by δ. □

Remark 4.3. If p is birational, p'^* is an isomorphism in (4-1). Since its top map is injective (Lemma 4.1), p^* is an isomorphism.

The following lemma will not be used in the sequel but seems worth noting. It generalises [Kahn and Sujatha 2017, Lemma 2.3.2], which is its special case $n = 1$.

Lemma 4.4. *Let* $\mathcal{F} \in \mathbf{HI}_{<n}$. *Then*:

(a) *If* $Z \subset X$ *is a closed pair of smooth varieties, with* Z *of codimension* $\geq n$, *then* $H^*(X, \mathcal{F}) \xrightarrow{\sim} H^*(X - Z, \mathcal{F})$.

(b) $H^i(X, \mathcal{F}) = 0$ *for* $i \geq n$ *and any smooth* X.

Proof. (a) follows from Lemma 4.1 and the definition of $\mathbf{HI}_{<n}$. For (b), by induction on n, the first and last group in the exact sequence of this lemma are 0 for $i-c \geq n-c$, hence for $i \geq n$. By a standard argument of successive singular loci, this implies that $H^i(X, \mathcal{F}) \xrightarrow{\sim} H^i(U, \mathcal{F})$ when $i \geq n$ for any open immersion $U \hookrightarrow X$; but the functor $X \mapsto H^i(X, \mathcal{F})$ is effaceable for $i > 0$ in the sense that every cohomology class vanishes locally for the Zariski topology, hence the conclusion. □

Remark 4.5. The above results can be deduced more elementarily from the Gersten resolution of Voevodsky [2000a, Theorem 4.37].

Theorem 4.6. *Let $n > 0$. For any smooth projective variety X, the counit map of the adjunction of Proposition 3.6*

$$\iota_n \nu_{<n} M(X) \to M(X)$$

becomes an isomorphism after applying the truncation functor $\tau_{\leq -n}$ (cohomological notation).

Proof. It suffices to show that $\tau_{>-n}\nu^{\geq n} M(X) = 0$. Writing

$$\begin{aligned}\nu^{\geq n} M(X) &= \underline{\mathrm{Hom}}(\mathbb{Z}(n)[n], M(X))(n)[n] \\ &= \underline{\mathrm{Hom}}(\mathbb{Z}(n)[n], M(X)) \otimes \mathbb{G}_m^{\otimes n}\end{aligned}$$

and noting that tensor product is right t-exact in $\mathbf{DM}^{\mathrm{eff}}$, it suffices to show that $\tau_{>-n}\,\underline{\mathrm{Hom}}(\mathbb{Z}(n)[n], M(X)) = 0$. This result was proven in [Kahn and Sujatha 2018, Proposition 2.3]. □

Corollary 4.7. *Let X be a smooth projective variety. Then, for any $\mathcal{F} \in \mathbf{HI}$, the counit map*

$$H^i(X, \iota_n R_{<n}\mathcal{F}) \to H^i(X, \mathcal{F})$$

is an isomorphism for $i < n$. If $\mathcal{F} \in \mathbf{HI}_{n\text{-rd}}$, both sides are 0 for $i = 0$.

Proof. The first point follows directly from Theorem 4.6. The second follows from the first, since $R^0_{<n}\mathcal{F} = \mathcal{F}_{<n}$ for any $\mathcal{F}$. □

Let $\mathcal{F}$ be any abelian sheaf on the big Zariski site on smooth k-varieties. For $(\overline{X}, Z, X)$ as in the beginning of Section 2, we have a spectral sequence similar to (2-1):

$$E_1^{p,q} = \bigoplus_{|J|=d-p} H^q_{Z_J}(\overline{X}, \mathcal{F}) \Rightarrow H^{p+q-d}(X, \mathcal{F}). \tag{4-2}$$

If $\mathcal{F} = \mathcal{F}_n$ is part of a homotopic module, Lemma 4.1 yields this time

$$H^q_{Z_J}(\overline{X}, \mathcal{F}_n) \simeq H^{p+q-d}(Z_J, \mathcal{F}_{n+p-d}). \tag{4-3}$$

Proposition 4.8. *Suppose* $(\mathcal{F}_n) \in \mathbf{HI}_*^{\mathrm{rd}}$ (see Definition 3.9). *Then, for* $p+q=d$, *we have* $E_1^{p,q}=0$ *except for* $p=d-n$, *hence an exact sequence*

$$0 \to \mathcal{F}_n(X) \to \bigoplus_{|J|=n} \mathcal{F}_0(Z_J) \xrightarrow{i_n} \bigoplus_{|J|=n-1} H^1(Z_J, \mathcal{F}_1),$$

where i_n *is induced by the boundary maps* ∂ *of Lemma 4.1.*

Proof. The first claim follows from Lemma 3.10 and Corollary 4.7, and the second follows from the first. □

5. Back to Section 2

We now make the link with the situation in that section, so assume again k finite. For any smooth k-scheme X, write

$$\mathcal{H}_n(X) = H^n_{\mathrm{cont}}(X, \mathbb{Z}_l(n)).$$

Proposition 5.1. *The presheaf* $\mathcal{H}_n$ *has a transfer structure and is* $\mathbb{A}^1$*-invariant; after tensoring with* $\mathbb{Q}$*, it becomes an étale sheaf, and the collection* $(\mathcal{H}_n \otimes \mathbb{Q})_{n\in\mathbb{Z}}$ *is an object of* $\mathbf{HI}_*^{\mathrm{rd}}$.

Proof. That finite correspondences act on étale cohomology with coefficients in twisted roots of unity follows from [Mazza et al. 2006, Theorem 10.3]. Since this action commutes with change of coefficients, it induces one on $\mathcal{H}_n$.[3] Its $\mathbb{A}^1$-invariance is classical, and moreover $(\mathcal{H}_n)_{-1} \simeq \mathcal{H}_{n-1}$ by the projective line formula in étale cohomology. With the notation of the introduction, $H^n(\mathbb{Q}_l(n)^c)$ is the étale sheaf associated to $\mathcal{H}_n \otimes \mathbb{Q}$, which is therefore already an étale sheaf by (1-1). Moreover, the Weil conjectures imply that $\mathbb{Q}_l(n)^c = 0$ for $n<0$ [Kahn 1998, Corollary 6.10(b)]; hence $(\mathcal{H}_n \otimes \mathbb{Q}) \in \mathbf{HI}_*^c$. Finally, $\mathcal{H}_n \otimes \mathbb{Q}$ is reduced for $n>0$ once again by the Weil conjectures plus the theorem of de Jong [1996, Theorem 4.1] since $\mathcal{F}_{<1}(X) = \mathcal{F}(\bar{X})$ for any $\mathcal{F} \in \mathbf{HI}$ if X has a smooth compactification $\bar{X}$ [Kahn and Sujatha 2017, Corollary 7.3.2]. □

Since $(\mathcal{K}_n^M) \in \mathbf{HI}_*^c$, (1-3) for all $n \geq 0$ factors through a morphism in $\mathbf{HI}_*^{\mathrm{rd}}$

$$(\mathcal{K}_n^M) \otimes \mathbb{Q}_l \twoheadrightarrow (\mathcal{K}_n^M)^{\mathrm{rd}} \otimes \mathbb{Q}_l \to (\mathcal{H}_n \otimes \mathbb{Q}) \tag{5-1}$$

by Theorem 3.11 and Proposition 5.1. Here we forget that $\mathcal{H}_n \otimes \mathbb{Q}$ is an étale sheaf and only remember its Nisnevich sheaf structure.

Theorem 5.2. *Let* $(\bar{X}, Z, X)$ *be as in the beginning of Section 2. Then the second map of (5-1) is an isomorphism when evaluated at* X.

[3] Since k is finite, one can use the isomorphisms $H^n_{\mathrm{cont}}(X, \mathbb{Z}_l(n)) \xrightarrow{\sim} \varprojlim H^n_{\mathrm{cont}}(X, \mu_{l^\nu}^{\otimes n})$ to simplify one's life.

Proof. By functoriality, Propositions 2.1 and 4.8 yield via (5-1) a commutative diagram of short exact sequences

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathcal{F}_n(X) & \longrightarrow & \bigoplus_{|J|=n} \mathcal{F}_0(Z_J) & \longrightarrow & \bigoplus_{|J|=n-1} H^1(Z_J, \mathcal{F}_1) \\
& \downarrow a & & \downarrow b & & \downarrow c \\
0 \longrightarrow & \mathcal{H}_n(X)\otimes\mathbb{Q} & \longrightarrow & \bigoplus_{|J|=n} \mathcal{H}_0(Z_J)\otimes\mathbb{Q} & \longrightarrow & \bigoplus_{|J|=n-1} H^1(Z_J, \mathcal{H}_1\otimes\mathbb{Q}) \\
& \| & & \| & & \downarrow d \\
0 \longrightarrow & \mathcal{H}_n(X)\otimes\mathbb{Q} & \longrightarrow & \bigoplus_{|J|=n} H^0(Z_J, 0) & \xrightarrow{i_n} & \bigoplus_{|J|=n-1} H^2(Z_J, 1)
\end{array}$$

where $\mathcal{F}_n = (\mathcal{K}_n^M)^{\mathrm{rd}} \otimes \mathbb{Q}_l$. Here the map between the last two rows follows from comparing the spectral sequences (2-1) and (4-2) via (1-3). But $\mathcal{K}_0^M = \mathbb{Z}$; hence b is an isomorphism, and $\mathcal{K}_1^M = \mathbb{G}_m$; in particular, $\mathcal{K}_1^M \otimes \mathbb{Q}_l$ is reduced because E^* is finite for any finite extension E/k. It follows from the construction in [Grothendieck and Deligne 1977, §2.1] that $d \circ c$ is the cycle class map for divisors; therefore it is injective since we are over a finite field. A diagram chase now shows that a is bijective. □

6. Proof of Theorem 1.1

The field k is still finite.

Theorem 6.1. *The second map of (5-1) is an isomorphism for any $n \in \mathbb{Z}$. If $n \le 2$, the composition is an epimorphism of presheaves.*

Proof. Let X be smooth irreducible. By [de Jong 1996, Theorem 4.1], applied with $Z = \varnothing$, there is an alteration $p : X_1 \to X$ and a dense open immersion $X_1 \subseteq \bar{X}_1$ such that $\bar{X}_1$ is smooth projective and $\bar{X}_1 - X_1$ is the support of a divisor with strict normal crossings. By Theorem 5.2, the statement is true at X_1; hence it is true at X thanks to Proposition 4.2. For $n \le 2$, the claim follows from Lemma 3.2. □

Theorem 6.2. *The morphism (1-3) is an epimorphism after tensorisation with $\mathbb{Q}$.*

Proof. By Theorem 6.1 and the epimorphy in Theorem 3.11, (1-3) is an epimorphism of Nisnevich sheaves, hence also of Zariski sheaves by [Mazza et al. 2006, Theorem 22.2]. □

Proposition 6.3. *The cokernel of $H^n(X, \mathbb{Z}(n)) \otimes \mathbb{Z}_l \to H^n_{\mathrm{cont}}(X, \mathbb{Z}_l(n))$ is torsion-free, and its kernel is divisible. Here the left-hand side denotes motivic cohomology.*

Proof. By [Kahn 2012, Corollary 3.5], the map $H^i_{\text{ét}}(X, \mathbb{Z}(n)) \otimes \mathbb{Z}_l \to H^i_{\mathrm{cont}}(X, \mathbb{Z}_l(n))$ has the said properties for any i. On the other hand, the norm residue isomorphism theorem implies that the map $H^n(X, \mathbb{Z}(n)) \otimes \mathbb{Z}_l \to H^n_{\text{ét}}(X, \mathbb{Z}(n)) \otimes \mathbb{Z}_l$ is an isomorphism (Beilinson–Lichtenbaum conjecture, [Voevodsky 2011]). □

Proof of Theorem 1.1. By Theorem 6.2, the cokernel of (1-3) is torsion. On the other hand, that of Proposition 6.3 remains torsion-free after Zariski-sheafification (an exact functor). But (1-3) factors as a composition

$$\mathcal{K}_n^M \otimes \mathbb{Z}_l \to \mathcal{H}^n(\mathbb{Z}(n)) \otimes \mathbb{Z}_l \to \mathcal{H}^n_{\mathrm{cont}}(\mathbb{Z}_l(n)), \tag{6-1}$$

where the first morphism is epi by [Mazza et al. 2006, Theorem 5.1] and [Elbaz-Vincent and Müller-Stach 2002, Proposition 4.3]. Thus the cokernel of (1-3) is 0. □

Remark 6.4. In (6-1), the kernel of the second morphism is divisible (same reasoning as before), while the kernel of the first morphism is killed by $(n-1)!$ (see, e.g., parts (6) and (11) of [Kerz 2010, Proposition 10]), where its failure to be 0 comes from too-small residue fields (part (5) of [loc. cit.]). To correct this and obtain divisibility of the kernel of (1-3), one may replace the sheaves of Milnor K-groups by sheaves of improved Milnor K-groups as in [loc. cit.].

7. The global sections of Milnor K-sheaves

Recall that k is finite. To say that the sheaf $\mathcal{K}_n^M \otimes \mathbb{Q}$ is reduced is exactly to say that

$$H^0(X, \mathcal{K}_n^M) \otimes \mathbb{Q} = 0 \tag{7-1}$$

for any connected smooth projective k-variety X. This is true for $n = 1$ because this group is $E^* \otimes \mathbb{Q}$, where E is the field of constants of X and E is finite (this fact was used in the proof of Theorem 5.2). For $n > 1$, it is open but still true for certain smooth projective X: recall that X is *of abelian type* if its Chow motive is a direct summand of that of an abelian variety (possibly after a finite extension of k). Then:

Theorem 7.1. *Let $n \geq 2$. Suppose that X is of abelian type and that the Tate conjecture holds for X in codimension n. Then (7-1) holds.*

Proof. It is analogous to that of [Kahn 2003, Lemma 1.6] or [Kahn 2023, Theorem 5.4], so we only sketch it. We have

$$H^0(X, \mathcal{K}_n^M \otimes \mathbb{Q}) \simeq \mathbf{DM}^{\mathrm{eff}}(M(X), \mathcal{K}_n^M[0] \otimes \mathbb{Q}), \tag{7-2}$$

where $\mathcal{K}_n^M \otimes \mathbb{Q} \simeq \mathcal{H}^n(\mathbb{Q}(n))$ (see Remark 6.4) is viewed as a homotopy-invariant Nisnevich sheaf with transfers. Write $\mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}}$ (resp. $\mathcal{M}^{\mathrm{eff}}_{\mathrm{num}}$) for the category of effective pure motives over k with rational coefficients modulo rational (resp. numerical) equivalence [Scholl 1994]. Let $\bigoplus_{i\in I} S_i$ be a decomposition of $h_{\mathrm{num}}(X) \in \mathcal{M}^{\mathrm{eff}}_{\mathrm{num}}$ into a direct sum of simple motives. By the nilpotence theorem of Kimura [2005, Proposition 7.5 and Example 9.1], lift this decomposition to an isomorphism $h_{\mathrm{rat}}(X) \simeq \bigoplus_{i\in I} \widetilde{S}_i$ in $\mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}}$. If $\Phi : \mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}} \to \mathbf{DM}^{\mathrm{eff}}$ is the composition of the functor $\mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}} \to \mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ of [Voevodsky 2000b, Proposition 2.1.4] with the full embedding $\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}} \hookrightarrow \mathbf{DM}^{\mathrm{eff}}$ of [loc. cit., Theorem 3.2.6], we thus have

$$M(X) \simeq \bigoplus_{i\in I} \Phi(\widetilde{S}_i).$$

On the direct summand $\mathbf{DM}^{\mathrm{eff}}(\Phi(\widetilde{S}_i), \mathcal{K}_n^M[0])$ of the right-hand side of (7-2), the action of the absolute Frobenius on the left and right term of the Hom induces the same action on the Hom, by naturality. Its action on $\mathcal{K}_n^M[0]$ is multiplication by q^n (where $q = |k|$), while its action on $\Phi(\widetilde{S}_i)$ is killed by a suitable power of the minimal polynomial Π_i of the Frobenius endomorphism of S_i. Therefore, if $\Pi_i(q^n) \neq 0$, then this direct summand is torsion; compare [Kahn 2003, Lemma 1.6].

The remaining case is the one which involves the Tate conjecture. Namely, suppose that $\Pi_i = T - q^n$. Then, inside $H^{2n}(X_{\bar{k}}, \mathbb{Q}_l)$, the geometric Frobenius acts on the summand $H^{2n}((\widetilde{S}_i)_{\bar{k}}, \mathbb{Q}_l)$ by multiplication by q^n. By the Tate conjecture this corresponds to an element of $CH^n(X) \otimes \mathbb{Q}_l$, hence to a nonzero morphism $\widetilde{S}_i \to \mathbb{L}^n$; by Schur's lemma it is an isomorphism modulo numerical equivalence, hence also modulo rational equivalence again by Kimura nilpotence. But $\Phi(\mathbb{L}^n) = \mathbb{G}_m^{\otimes n}[n]$; hence $\mathbf{DM}^{\mathrm{eff}}(\Phi(\widetilde{S}_i), \mathcal{K}_n^M[0]) = 0$ again. □

Since the Tate conjecture obviously holds if $n \geq \dim X$, we get the following.

Corollary 7.2. *Let X be a smooth k-surface which is birational to a smooth projective surface of abelian type. Then the map*

$$H^0(X, \mathcal{K}_n^M) \otimes \mathbb{Q}_l \to H^n_{\mathrm{cont}}(X, \mathbb{Q}_l(n))$$

induced by (1-3) is bijective for any $n \geq 0$.

Proof. By Abhyankar resolution (and embedded resolution of curves), we are in the situation at the beginning of Section 2; moreover, since smooth projective curves are of abelian type, to be of abelian type is a birational invariant of smooth projective surfaces, so that $\overline{X}$ and the Z_i are of abelian type in loc. cit. By Theorem 7.1, we may therefore run the proof of Theorem 5.2 by taking $\mathcal{F}_n = \mathcal{K}_n^M \otimes \mathbb{Q}_l$ instead of $(\mathcal{K}_n^M)^{\mathrm{rd}} \otimes \mathbb{Q}_l$. □

Example 7.3. A smooth projective surface such that $b_2 = \rho$ is of abelian type if and only if it verifies Bloch's conjecture (e.g., if it is not of general type). Fermat surfaces are of abelian type, etc.

References

[Colliot-Thélène et al. 1983] J.-L. Colliot-Thélène, J.-J. Sansuc, and C. Soulé, "Torsion dans le groupe de Chow de codimension deux", *Duke Math. J.* **50**:3 (1983), 763–801. MR

[Déglise 2011] F. Déglise, "Modules homotopiques", *Doc. Math.* **16** (2011), 411–455. MR

[Elbaz-Vincent and Müller-Stach 2002] P. Elbaz-Vincent and S. Müller-Stach, "Milnor K-theory of rings, higher Chow groups and applications", *Invent. Math.* **148**:1 (2002), 177–206. MR

[Esnault et al. 1998] H. Esnault, B. Kahn, M. Levine, and E. Viehweg, "The Arason invariant and mod 2 algebraic cycles", *J. Amer. Math. Soc.* **11**:1 (1998), 73–118. MR

[Geisser 1998] T. Geisser, "Tate's conjecture, algebraic cycles and rational K-theory in characteristic p", *K-Theory* **13**:2 (1998), 109–122. MR

[Grothendieck and Deligne 1977] A. Grothendieck and P. Deligne, "La classe de cohomologie associée à un cycle", pp. 129–153 in *Cohomologie étale*, Lecture Notes in Math. **569**, Springer, 1977. MR

[Huber and Kahn 2006] A. Huber and B. Kahn, "The slice filtration and mixed Tate motives", *Compos. Math.* **142**:4 (2006), 907–936. MR

[Jannsen 1988] U. Jannsen, "Continuous étale cohomology", *Math. Ann.* **280**:2 (1988), 207–245. MR

[de Jong 1996] A. J. de Jong, "Smoothness, semi-stability and alterations", *Inst. Hautes Études Sci. Publ. Math.* 83 (1996), 51–93. MR

[Kahn 1998] B. Kahn, "A sheaf-theoretic reformulation of the Tate conjecture", 1998. arXiv 9801017

[Kahn 2002] B. Kahn, "The Geisser–Levine method revisited and algebraic cycles over a finite field", *Math. Ann.* **324**:3 (2002), 581–617. MR

[Kahn 2003] B. Kahn, "Équivalences rationnelle et numérique sur certaines variétés de type abélien sur un corps fini", *Ann. Sci. École Norm. Sup.* (4) **36**:6 (2003), 977–1002. MR

[Kahn 2012] B. Kahn, "Classes de cycles motiviques étales", *Algebra Number Theory* **6**:7 (2012), 1369–1407. MR

[Kahn 2023] B. Kahn, "On the injectivity and non-injectivity of the l-adic cycle class maps", 2023. To appear in *Moduli*, *motives and bundles*: *new trends in algebraic geometry*, London Mathematical Society Lecture Note Series, Cambridge University Press. arXiv 2307.13281

[Kahn and Sujatha 2017] B. Kahn and R. Sujatha, "Birational motives, II: Triangulated birational motives", *Int. Math. Res. Not.* **2017**:22 (2017), 6778–6831. MR

[Kahn and Sujatha 2018] B. Kahn and R. Sujatha, "The derived functors of unramified cohomology", *Selecta Math.* (*N.S.*) **24**:2 (2018), 1121–1146. MR

[Kerz 2010] M. Kerz, "Milnor K-theory of local rings with finite residue fields", *J. Algebraic Geom.* **19**:1 (2010), 173–191. MR

[Kimura 2005] S.-I. Kimura, "Chow groups are finite dimensional, in some sense", *Math. Ann.* **331**:1 (2005), 173–201. MR

[Mazza et al. 2006] C. Mazza, V. Voevodsky, and C. Weibel, *Lecture notes on motivic cohomology*, Clay Mathematics Monographs **2**, Amer. Math. Soc., Providence, RI, 2006. MR

[Scholl 1994] A. J. Scholl, "Classical motives", pp. 163–187 in *Motives* (Seattle, 1991), edited by U. Jannsen et al., Proc. Sympos. Pure Math. **55**, Amer. Math. Soc., Providence, RI, 1994. MR

[Tate 1976] J. Tate, "Relations between K_2 and Galois cohomology", *Invent. Math.* **36** (1976), 257–274. MR

[Voevodsky 2000a] V. Voevodsky, "Cohomological theory of presheaves with transfers", pp. 87–137 in *Cycles*, *transfers*, *and motivic homology theories*, edited by J. N. Mather and E. M. Stein, Ann. of Math. Stud. **143**, Princeton Univ. Press, 2000. MR

[Voevodsky 2000b] V. Voevodsky, "Triangulated categories of motives over a field", pp. 188–238 in *Cycles*, *transfers*, *and motivic homology theories*, Ann. of Math. Stud. **143**, Princeton Univ. Press, 2000. MR

[Voevodsky 2010] V. Voevodsky, "Cancellation theorem", *Doc. Math.* (2010), 671–685. MR

[Voevodsky 2011] V. Voevodsky, "On motivic cohomology with $\mathbb{Z}/l$-coefficients", *Ann. of Math.* (2) **174**:1 (2011), 401–438. MR

Received 3 Mar 2025. Revised 6 Sep 2025. Accepted 8 Oct 2025.

BRUNO KAHN: bruno.kahn@imj-prg.fr
CNRS, IMJ-PRG Sorbonne Université and Université Paris Cité, Paris, France

msp

ANNALS OF K-THEORY

msp.org/akt

EDITORIAL BOARD

Joseph Ayoub	(Managing Editor) Universität Zürich Zürich, Switzerland joseph.ayoub@math.uzh.ch
Paul Balmer	University of California, Los Angeles, USA balmer@math.ucla.edu
Guillermo Cortiñas	(Managing Editor) Universidad de Buenos Aires and CONICET, Argentina gcorti@dm.uba.ar
Hélène Esnault	Freie Universität Berlin, Germany liveesnault@math.fu-berlin.de
Eric Friedlander	University of Southern California, USA ericmf@usc.edu
Huaxin Lin	University of Oregon, USA livehlin@uoregon.edu
Birgit Richter	Universität Hamburg, Germany birgit.richter@uni-hamburg.de
Jonathan Rosenberg	University of Maryland, USA jmr@math.umd.edu
Marco Schlichting	University of Warwick, UK m.schlichting@warwick.ac.uk
Charles Weibel	Rutgers University, USA weibel@math.rutgers.edu
Guoliang Yu	Texas A&M University, USA guoliangyu@math.tamu.edu

PRODUCTION

Silvio Levy	(Scientific Editor) production@msp.org

Annals of K-Theory is a journal of the K-Theory Foundation (ktheoryfoundation.org). The K-Theory Foundation acknowledges the precious support of Foundation Compositio Mathematica, whose help has been instrumental in the launch of the Annals of K-Theory.

See inside back cover or msp.org/akt for submission instructions.

The subscription price for 2025 is US $650/year for the electronic version, and $735/year (+$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Annals of K-Theory (ISSN 2379-1681 electronic, 2379-1683 printed) at Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004, is published continuously online.

AKT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

© 2025 Mathematical Sciences Publishers

ANNALS OF K-THEORY

2025 vol. 10 no. 4