# MOTIVIC GALOIS GROUPS

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## 1. Tannakian categories ([26], cf. [7])

$K$ field of characteristic 0, $\mathcal{A}$ rigid tensor $K$-linear *abelian* category, $L$ extension of $K$.

**Definition 1.** An $L$-valued fibre functor is a tensor functor $\omega : \mathcal{A} \to Vec_L$ which is *faithful* and *exact*.

**Definition 2.** $\mathcal{A}$ is

- neutralised Tannakian if one is given a $K$-valued fibre functor
- neutral Tannakian if $\exists$ $K$-valued fibre functor
- Tannakian if $\exists$ $L$-valued fibre functor for some $L$.

**Example 1.** $G$ affine $K$-group scheme, $\mathcal{A} = Rep_K(G)$, $\omega : \mathcal{A} \to Vec_K$ the forgetful functor.

$(\mathcal{A}, \omega)$ neutralised Tannakian category: $G_K := Aut^{\otimes}(\omega)$ is (canonically) the $K$-points of an affine $K$-group scheme $G(\omega)$.

**Theorem 1** (Grothendieck-Saavedra [26]). *a) For $(\mathcal{A}, \omega)$ as in Example 1, $G(\mathcal{A}, \omega) = G$.*
*b) In general $\omega$ enriches into a tensor equivalence of categories*

$$\tilde{\omega} : \mathcal{A} \xrightarrow{\sim} Rep_K(G(\mathcal{A}, \omega)).$$

*c) Dictionary (special case): $\mathcal{A}$ semi-simple $\iff$ $G$ proreductive.*

When $\mathcal{A}$ Tannakian but not neutralised, need replace $G(\mathcal{A}, \omega)$ by a *gerbe* (or a groupoid): Saavedra-Deligne [8].

**Theorem 2** (Deligne [8]). *$\mathcal{A}$ rigid $K$-linear abelian. Equivalent conditions:*

- *$\mathcal{A}$ is Tannakian*
- $\forall M \in \mathcal{A}$, $\exists n > 0$: $\Lambda^n(M) = 0$.
- $\forall M \in \mathcal{A}$, $\dim_{rigid}(M) \in \mathbf{N}$.

## 2. Are motives Tannakian?

Ideally, would like $Mot_{\mathrm{num}}(k, \mathbf{Q})$ Tannakian, fibre functors given by Weil cohomologies $H$. Two problems:

- $$\begin{tikzcd} \mathrm{Mot}_H \arrow[r, hook, "H"] \arrow[d, two heads] & Vec_K^* \\ \mathrm{Mot}_{\mathrm{num}} \arrow[ur, dashed, "???"'] & \end{tikzcd}$$
- $\mathrm{Mot}_{\mathrm{num}}(k, \mathbf{Q})$ is *never Tannakian* because $\dim_{rigid}(X) = \chi(X)$ may be negative (e.g. $X$ curve of genus $g$: $\chi(X) = 2 - 2g$).

Second problem: matter of commutativity constraint – need modify it.

Yields Grothendieck's *standard conjectures* ([13], cf. [20]):

- (HN) $\sim_H = \sim_{\mathrm{num}}$.
- (C) $\forall X$ the Künneth components of $H(\Delta_X)$ are algebraic.

Another conjecture (B) (skipped):

- (HN) $\Rightarrow$ (B) $\Rightarrow$ (C).
- (HN) $\iff$ (B) in characteristic 0.

**Theorem 3** (Lieberman-Kleiman [19]). *Conjecture (B) holds for abelian varieties.*

**Theorem 4** (Katz-Messing [18]). *Conjecture (C) is true if $k$ finite.*

**Corollary 1** (Jannsen [14]). *If $k$ finite, a suitable modification $\widetilde{\mathrm{Mot}}_{\mathrm{num}}(k, \mathbf{Q})$ is (abstractly) Tannakian.*

Apart from this, wide open!

**Definition 3.** When $\widetilde{\mathrm{Mot}}_{\mathrm{num}}(k, \mathbf{Q})$ exists, the gerbe that classifies it is called the [pure] motivic Galois group $GMot_k$. $H$ Weil cohomology with coefficients $K$: fibre of $GMot_k$ at $H$ is proreductive $K$-group $GMot_{H,k}$.

More generally, $\mathcal{A}$ thick rigid subcategory of $\mathrm{Mot}_{\mathrm{num}}$, get an "induced" Galois group $GMot(\mathcal{A})$ of $\mathcal{A}$, quotient of the motivic Galois group. E.g. $\mathcal{A}$ thick rigid subcategory generated by $h(X)$: get the motivic Galois group of $X$ $GMot_{H,k}(X)$(of finite type).

**Examples 2.**

(1) $\mathcal{A}$ = Artin motives (generated by $h(\operatorname{Spec} E)$, $[E : k] < \infty$): $GMot(\mathcal{A}) = G_k$.
(2) $\mathcal{A}$ = pure Tate motives (generated by $L$ or $h(\mathbf{P}^1)$): $GMot(\mathcal{A}) = \mathbb{G}_m$.
(3) $\mathcal{A}$ = pure Artin-Tate motives (put these two together): $GMot(\mathcal{A}) = G_k \times \mathbb{G}_m$.
(4) $E$ elliptic curve over $\mathbf{Q}$, $H = H_{Betti}$.

- $E$ not CM $\Rightarrow GMot_{H,\mathbf{Q}}(E) = GL_2$.
- $E$ CM $\Rightarrow GMot_{H,\mathbf{Q}}(E)$ = torus in $GL_2$ or its normaliser.

**Example 3.** Suppose Conjecture (HN) true.

- Characteristic 0: Betti cohomology yields (several) $\mathbf{Q}$-valued fibre functors, as long as $card(k) \leq card(\mathbf{C})$: $\mathrm{Mot}_{\mathrm{num}}(k, \mathbf{Q})$ is neutral. Comparison isomorphisms $\Rightarrow$ isomorphisms between various motivic Galois groups.
- Characteristic $p$: $k \supseteq \mathbf{F}_{p^2}$ finite $\Rightarrow$ $\mathrm{Mot}_{\mathrm{num}}(k, \mathbf{Q})$ is *not neutral*: if $K \subseteq \mathbf{R}$ or $K \subseteq \mathbf{Q}_p$, no $K$-valued fibre functor (Serre: endomorphisms of a supersingular elliptic curve = quaternion $\mathbf{Q}$-algebra nonsplit by $\mathbf{R}, \mathbf{Q}_p$).

## 3. Connection with Hodge and Tate conjectures

### 3.1. Tate conjecture.

$k$ finitely generated, $G_k := Gal(\bar{k}/k)$, $H = H_l$ ($l \neq \operatorname{char} k$): the $\otimes$-functor

$$H_l : \mathrm{Mot}_H \to Vec^*_{\mathbf{Q}_l}$$

enriches into a $\otimes$-functor

$$\hat{H}_l : \mathrm{Mot}_H \longrightarrow Rep^{cont}_{\mathbf{Q}_l}(G_k)^*.$$

Tate conjecture $\iff$ $\tilde{H}_l$ *fully faithful* (it is faithful by definition).

**Proposition 1.** *Tate conjecture $\Rightarrow$ Conjecture (B).*

Hence under Tate conjecture, Conjecture (C) holds and can modify commutativity constraint:

$$\tilde{H}_l : \widetilde{\mathrm{Mot}}_H \longrightarrow Rep^{cont}_{\mathbf{Q}_l}(G_k).$$

$(Rep^{cont}_{\mathbf{Q}_l}(G_k)$, forgetful functor) neutralised Tannakian $\mathbf{Q}_l$-category with fundamental group $\Gamma_k$: for $V \in Rep^{cont}_{\mathbf{Q}_l}(G_k)$, $\Gamma_k(V)$ = Zariski closure of $G_k$ in $GL(V)$.

**Proposition 2** (folklore, cf. [27], [17])**.** *Assume Tate conjecture. Equivalent conditions:*

- *Conjecture (HN);*
- $Im\tilde{H}_l \subseteq Rep^{cont}_{\mathbf{Q}_l}(G_k)_{ss}$ *(full subcategory of semi-simple representations).*

Under these conditions, $\mathrm{Mot}_{\mathrm{num}}$ Tannakian, reduce to $\Gamma_k^{ss}$ (for $Rep^{cont}_{\mathbf{Q}_l}(G_k)_{ss}$) proreductive and canonical epimorphism

$$\Gamma_k^{ss} \longrightarrow\!\!\!\!\!\rightarrow GMot_{H_l,k}.$$

In particular, $\forall X$, $GMot_{H_l,k}(X)$ = Zariski closure of $G_k$ in $GL(H_l(X))$.

**Delicate question**: essential image of $\tilde{H}_l$? Conjectural answers for $k$ finite (see below) and $k$ number field (Fontaine-Mazur [11]).

## 3.2. Hodge conjecture.

$\sigma : k \hookrightarrow \mathbf{C}$, $H = H_\sigma$: this time enriches into $\otimes$-functor

$$\hat{H}_\sigma : \mathrm{Mot}_{H_\sigma} \to PHS^*_{\mathbf{Q}}$$

(graded pure Hodge structures over $\mathbf{Q}$). Hodge conjecture $\iff$ $\hat{H}_\sigma$ *fully faithful.*

**Proposition 3.** *Hodge conjecture $\Rightarrow$ Conjecture (B) $\iff$ Conjecture (HN).*

Hence, under Hodge conjecture, get modified fully faithful tensor functor

$$\tilde{H}_\sigma : \widetilde{\mathrm{Mot}}_{\mathrm{num}} \to PHS_{\mathbf{Q}}.$$

Latter category semi-simple neutralised Tannakian (via forgetful functor). If extend scalars to $\mathbf{R}$, fundamental group = Hodge torus $S = R_{\mathbf{C}/\mathbf{R}}\mathbb{G}_m$. Over $\mathbf{Q}$ it is the Mumford-Tate group $MT$: for $V \in PHS_{\mathbf{Q}}$, $MT(V) = \mathbf{Q}-$ Zariski closure of $S$ in $GL(V)$.

Hodge conjecture $\iff \forall X$, $GMot_{k,H_\sigma}(X) = MT(X) \subseteq GL(H_\sigma(X))$.

Sometimes gives proof of Hodge conjecture (for powers of $X$, $X$ abelian variety)!

# 4. Unconditional motivic Galois groups

Want an unconditional theory of motives (not assuming the unproven standard conjectures)

## 4.1. First approach (Deligne, André).

Both are in characteristic 0.

- **Deligne** [10]**:** replace motives by systems of compatible realisations: motives for absolute Hodge cycles (systems of cohomology classes corresponding to each other by comparison isomorphisms). Gives semi-simple Tannakian category.
  Hodge conjecture $\Rightarrow$ absolute Hodge cycles are algebraic so same category.
- **André** [3]: only adjoin to algebraic cycles the inverses of the Lefschetz operators: motives for cycles. Gives semi-simple Tannakian category.
  Conjecture (B) $\Rightarrow$ motivated cycles are algebraic so same category.
  (Hodge conjecture $\Rightarrow$ Conjecture (B) so cheaper approach!)

$A$ abelian variety over number field:

**Theorem 5** (Deligne [9]). *Every Hodge cycle on $A$ is absolutely Hodge.*

**Corollary 2.** *Tate conjecture $\Rightarrow$ Hodge conjecture on $A$.*

Better:

**Theorem 6** (André [3]). *Every Hodge cycle on $A$ is motivated.*

**Corollary 3.** *Conjecture (B) for abelian fibrations on curves $\Rightarrow$ Hodge conjecture on $A$.*

Tannakian arguments:

**Theorem 7** (Milne [23]). *Hodge conjecture for complex CM abelian varieties $\Rightarrow$ Tate conjecture for all abelian varieties over a finite field.*

**Theorem 8** (André [4]). *$A$ abelian variety over a finite field: every Tate cycle is motivated.*

## 4.2. Second approach (André-K): tensor sections.

$\mathcal{A}$ pseudo-abelian $\mathbf{Q}$-linear category, $\mathcal{R}$ Kelly radical of $\mathcal{A}$ (like Jacobson radical of rings): smallest ideal such that $\mathcal{A}/\mathcal{R}$ semi-simple.

If $\mathcal{A}$ tensor category, $\mathcal{R}$ may or may not be stable under $\otimes$. True e.g. if $\mathcal{A}$ Tannakian.

**Theorem 9** (André-K [6]). *Suppose that $\mathcal{R}$ is $\otimes$-ideal, $\mathcal{A}(\mathbf{1},\mathbf{1}) = \mathbf{Q}$ and $\mathcal{R}(M,M)$ nilpotent ideal of $\mathcal{A}(M,M)$ for all $M$. Then the projection functor*

$$\mathcal{A} \to \mathcal{A}/\mathcal{R}$$

*has tensor sections, and any two are tensor-conjugate.*

**Application:**
$H$ classical Weil cohomology,

$$\begin{aligned}\mathcal{A} &= \mathrm{Mot}_H^{\pm}(k,\mathbf{Q})\\ &:= \{M \in \mathrm{Mot}_H(k,\mathbf{Q}) \mid \text{ sum of even Künneth projectors of } M \text{ algebraic}\}.\end{aligned}$$

Then $\mathcal{A}$ satisfies assumptions of Theorem 9: in characteristic 0 by comparison isomorphisms, in characteristic $p$ by Weil conjectures.

**Theorem 10** (André-K [5]). *a)* $\mathrm{Mot}_{\mathrm{num}}^{\pm} := Im(\mathrm{Mot}_H^{\pm} \to \mathrm{Mot}_{\mathrm{num}})$ *independent of* $H$.
*b) Can modify commutativity constraints in* $\mathrm{Mot}_H^{\pm}$ *and* $\mathrm{Mot}_{\mathrm{num}}^{\pm}$*, yielding* $\widetilde{\mathrm{Mot}}_H^{\pm}$ *and* $\widetilde{\mathrm{Mot}}_{\mathrm{num}}^{\pm}$.
*c) Projection functor* $\widetilde{\mathrm{Mot}}_H^{\pm} \to \widetilde{\mathrm{Mot}}_{\mathrm{num}}^{\pm}$ *has tensor sections* $\sigma$*; any two are tensor-conjugate.*

$$\begin{tikzcd} \mathrm{Mot}_H \arrow[r, hook, "H"] \arrow[d] & Vec_K^* \\ \mathrm{Mot}_{\mathrm{num}} \arrow[ur, dashed, "???"'] & \end{tikzcd} \qquad \begin{tikzcd} \widetilde{\mathrm{Mot}}_H^{\pm} \arrow[r, hook, "H"] \arrow[d, shift right] & Vec_K \\ \widetilde{\mathrm{Mot}}_{\mathrm{num}}^{\pm} \arrow[u, shift right, "\sigma"'] \arrow[ur, "H\circ\sigma"'] & \end{tikzcd}$$

Variant with

$$\mathrm{Mot}_H^*(k,\mathbf{Q}) := \{M \in \mathrm{Mot}_H(k,\mathbf{Q}) \mid \text{ all Künneth projectors of } M \text{ algebraic}\}.$$

## 5. Description of motivic Galois groups

Assume all conjectures (standard, Hodge, Tate).

### 5.1. In general:

Short exact sequence
$$1 \to GMot_{\bar{k}} \to GMot_k \to G_k \to 1$$
Last morphism: $G_k$ corresponds to motives of 0-dimensional varieties (Artin motives). The group $GMot_{\bar{k}}$ is connected, hence $= GMot_k^0$.

If $k \subseteq k'$, $GMot_{k'}^0 \twoheadrightarrow GMot_k^0$ (but not iso unless $k'/k$ algebraic: otherwise, "more" elliptic curves over $k'$ than over $k$).
Conjecture (C) $\Rightarrow$ weight grading on $\mathrm{Mot}_{\mathrm{num}} \iff$ central homomorphism
$$w : \mathbb{G}_m \to GMot_k.$$
On the other hand, Lefschetz motive gives homomorphism
$$t : GMot_k \to \mathbb{G}_m$$
and $t \circ w = 2$ ($-2$ with Grothendieck's conventions).

## 5.2. Over a finite field:

**Theorem 11** (cf. [22]). *a)* $\mathrm{Mot}_{\mathrm{num}}$ *generated by Artin motives and motives of abelian varieties.*
*b) Essential image of* $\tilde{H}_l$*: $l$-adic representations of* $G_k$ *whose eigenvalues are Weil numbers.*

Uses Honda's theorem [16]: every Weil orbit corresponds to an abelian variety.

**Corollary 4.** $GMot_k^0 =$ *group of multiplicative type determined by action of* $G_{\mathbf{Q}}$ *on Weil numbers.*

Even though $\widetilde{\mathrm{Mot}}_{\mathrm{num}}$ not neutral, $GMot_k^0$ abelian so situation not so bad!

### 5.3. Over a number field:

$S := (GMot_k^0)^{ab}$: the Serre protorus: describe its character group $X(S)$:

$$\mathbf{Q}^{cm} = \bigcup \{E \mid E \text{ CM number field}\}$$

Complex conjugation $c$ central in $Gal(\mathbf{Q}^{cm}/\mathbf{Q})$ (largest Galois subfield of $\bar{\mathbf{Q}}$ with this property).

**Definition 4.** $f : Gal(\mathbf{Q}^{cm}/\mathbf{Q}) \to \mathbf{Z}$ CM type if $f(s) + f(cs)$ independent of $s$. $G_{\mathbf{Q}}$ acts on CM types by $\tau f(s) = f(\tau s)$.

**Theorem 12** ([24]). *$X(S) = \mathbf{Z}[CM\ types]$.*

Can also describe the centre $C$ of $GMot_k^0$ (pro-isogenous to $S$), etc.: cf. [25].

## 6. Mixed (Tate) motives

Expect Tannakian category of mixed motives

$$\mathrm{Mot}_{\mathrm{num}}(k, \mathbf{Q}) \subset \mathrm{MMot}(k, \mathbf{Q})$$

with socle $\mathrm{Mot}_{\mathrm{num}}(k, \mathbf{Q})$, classifying non smooth projective varieties. Corresponding motivic Galois group extension of $GMot_k$ by a pro-unipotent group (or gerbe).

Constructions of MMot:

- Conjecturally, heart of "motivic $t$-structure" on $DM$ (Deligne, Beilinson: cf. Hanamura [15]).
- In characteristic 0: explicit category constructed by Nori.
- Over a finite field: Tate conjecture $\Rightarrow$ $\mathrm{Mot}_{\mathrm{num}} = \mathrm{MMot}$ (cf. [22]).
- Can settle for subcategory: mixed Tate motives $\mathrm{TMMot}_k$. Exists unconditionally if $k$ number field (cf. Levine's talk and [21]).

**Goncharov** [12]**:** $\mathrm{TMMot}_{\mathbf{Z}}$ (mixed Tate motives over $\mathbf{Z}$) defined as full subcategory of $\mathrm{TMMot}_{\mathbf{Q}}$ by non-ramification conditions.

$\Gamma$ the motivic Galois group corresponding to $\mathrm{TMMot}_{\mathbf{Z}}$: Proreductive quotient of $\Gamma$ is $\mathbb{G}_m$ (see above).

**Theorem 13** (Goncharov [12])**.** *Action of $\mathbb{G}_m$ on prounipotent kernel $U$ yields a grading on $Lie(U)$: for this grading, $Lie(U)$ is free with one generator in every odd degree $\leq -3$.*

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