A DESCENT THEOREM FOR PURE MOTIVES

BRUNO KAHN

Abstract. We give necessary conditions for a category fibred in pseudo-abelian additive categories over the classifying topos of a profinite group to be a stack; these conditions are sufficient when the coefficients are **Q**-linear. We use this to prove that pure motives à la Grothendieck (with rational coefficients) over a field form a stack for the étale topology; this holds more generally for several motivic categories considered in [5]. Finally, we clarify the construction of Chow-Lefschetz motives given in [6], and simplify the proof of the computation of the motivic Galois group of Lefschetz motives modulo numerical equivalence, given in loc. cit.

Introduction

The first main result of this note is

Theorem 1. *Let k be a field, $\sim$ an adequate equivalence relation on algebraic cycles with rational coefficients and $\mathbf{Mot}_\sim(k)$ the category of pure motives over k modulo $\sim$, in the sense of Grothendieck. Then the assignment*

$$l \mapsto \mathbf{Mot}_\sim(l)$$

defines a stack of rigid $\otimes$-categories over the small étale site of $\operatorname{Spec} k$.

Theorem 1 is so easy to prove that it ought to be part of the folklore. Here is a sketch: for l/k a finite Galois extension with group G, write $\mathbf{Mot}_\sim(l, G)$ for the category of descent data on $\mathbf{Mot}_\sim(l)$ relative to G. We have to prove that the canonical functor $\mathbf{Mot}_\sim(k) \to \mathbf{Mot}_\sim(l, G)$ is an equivalence of categories. Full faithfulness follows from a standard transfer argument, using that the coefficients are $\mathbf{Q}$. For the essential surjectivity, we use the fact that the base change functor $f^* : \mathbf{Mot}_\sim(k) \to \mathbf{Mot}_\sim(l)$ has a right adjoint f_*; if $(C, (b_g)_{g\in G}) \in \mathbf{Mot}_\sim(l, G)$ is a descent datum, with $C \in \mathbf{Mot}_\sim(l)$, the natural action of G on f_*C gives a projector whose image yields the effectivity of the descent datum.

Date: December 4, 2023.
2020 *Mathematics Subject Classification.* 18F20, 18M25, 14C15.
Key words and phrases. Stacks, motives, Tannakian categories.

In fact, such a result and sketch of proof hold in much greater generality, which led me to give them an abstract formulation: this is the subject of Section 1. In Theorem 1.5, we get necessary conditions for a fibered category in pseudo-abelian additive categories over the classifying topos of a profinite group to be a stack, which are sufficient when the categories are **Q**-linear. They use a baby "3 functors formalism" (for Galois étale morphisms!), see Definition 1.4. In Section 2, we show how to weaken the hypotheses of this formalism in the presence of a monoidal structure: this allows us to easily prove the stack property for all motivic theories appearing in [5, Th. 4.3 a)], not just for pure motives (Theorem 3.1). It also applies to the related theories of [2] and [1].

What started me on this work was the desire to clarify and simplify some constructions and reasonings in [6]. In its §4, I construct a category of Chow-Lefschetz motives over a (possibly non separably closed) field in two steps: first a "crude" category and then a better-behaved one. By hindsight, it became likely that the second step was just the process of creating the associated stack, and this is what is checked in Proposition 4.1. The reasoning I wanted to simplify was the rather ugly recourse to continuous descent data in the proof of [6, Th. 5]: this is done here in Theorem 5.3, which also clarifies the proof of [2, Prop. 6.23 (a)] (quoted without comment in [1, 4.6, exemples]), and especially the rôle of semisimplicity.

Note that Theorem 1 does not extend to motives over a base S in the sense, say, of Deninger-Murre [3]; indeed, for X, Y smooth projective over S, the presheaf $U \mapsto CH^*(X_U \times_U Y_U)_{\mathbf{Q}}$ for $U \to S$ étale is already not a sheaf in the Zariski topology! Similarly, Theorem 1 is obviously false if the coefficients are not **Q**-linear. If one wanted to extend it to these two cases, one would probably have to consider stable ∞-categories. This is a sense in which Theorem 1 is "elementary".

1. Stacks over a profinite group

1.1. **The set up.** Let $\mathcal{E}$ be a category; recall from [4, §§7, 8] that there is a dictionary between fibered categories over $\mathcal{E}$ and pseudo-functors $\mathcal{E} \to \mathbf{Cat}$ whose comparison 2-cocycle consists of natural isomorphisms; we shall adopt here the latter viewpoint, which is also the one of [5].

Take $\mathcal{E} = B\Pi$, where Π is a profinlte group and $B\Pi$ is its classifying topos, i.e. the category of finite continuous Π-sets. We are interested in (contravariant) pseudo-functors $\mathcal{A}$ from $B\Pi$ to the 2-category of additive categories. We want to give (necessary and) sufficient conditions for such an $\mathcal{A}$ to be a stack for the natural topology on $B\Pi$ (given by finite covers). Recall what this means [8, Def. 026F]; given $S \in B\Pi$:

(1) For any $A, B \in \mathcal{A}(S)$, the presheaf $(U \xrightarrow{f} S) \mapsto \mathcal{A}(U)(f^*A, f^*B)$ on $B\Pi/S$ is a sheaf;
(2) Any descent datum relative to a cover $f : T \to S$ is effective.

Condition (1) implies in particular that $\mathcal{A}$ commutes with coproducts. Assuming this holds, let us translate the above conditions in Galois terms: given a Galois covering $f : T \to S$ of connected Π-sets, with $G = Gal(f)$:

(1G) the natural map $\mathcal{A}(S)(A, B) \xrightarrow{a} \mathcal{A}(T)(f^*A, f^*B)^G$ is an isomorphism for any $A, B \in \mathcal{A}(S)$;
(2G) any descent datum relative to f is effective.

In (1G), let us explain the action of G on the right hand side: for each $g \in G$, the equality $fg = f$ and the pseudo-functor structure of $\mathcal{A}$ yield a natural isomorphism $i_g : f^* \overset{\sim}{\Rightarrow} g^*f^*$; these are compatible with the natural isomorphisms $c_{g,h} : h^*g^* \overset{\sim}{\Rightarrow} (gh)^*$. To $\varphi : f^*A \to f^*B$, one associates $\varphi^g = i_g(B)^{-1}(g^*\varphi)i_g(A)$ (right action!). If φ is of the form $f^*\psi$, then $\varphi^g = \varphi$ by naturality of i_g.

The meaning of (2G) is the following: let $C \in \mathcal{A}(T)$, provided with isomorphisms $b_g : C \xrightarrow{\sim} g^*C$ verifying the usual 1-coboundary condition with respect to the 2-cocycle $c_{g,h}$ (descent datum). Then there exists $B \in \mathcal{A}(S)$ and an isomorphism $f^*B \xrightarrow{\sim} C$ which induces an isomorphism of descent data, for the canonical descent datum on f^*B implicitly used in the previous paragraph.

To formalise this, we introduce the category $\mathcal{A}(T, G)$ of descent data: an object is a descent datum as above, and morphisms are the obvious ones. There is a functor $\tilde{f} : \mathcal{A}(S) \to \mathcal{A}(T, G)$ sending A to $(f^*A, (i_g(A)))$; Condition (1G) amounts to say that $\tilde{f}$ is fully faithful, and (2G) amounts to say that it is essentially surjective.

If $(C, (b_g)), (D, (b'_g)) \in \mathcal{A}(T, G)$ and $\varphi :\in \mathcal{A}(T)(C, D)$, one defines φ^g for $g \in G$ as in the case of effective descent data, generalising the previous construction.

1.2. **Introducing an adjoint.** Suppose that f^* has a right adjoint f_*, with counit ε. For a descent datum $(C, (b_g))$ and $g \in G$, we get an endomorphism $[g]$ of f_*C, corresponding to ε_C^g by adjunction; in formula:

$$[g] = f_*(\varepsilon_C^g) \circ \eta_{f_*C}$$

where η is the unit of the adjunction.

Lemma 1.1. *a) We have*

$$\varepsilon_C^g = \varepsilon_C \circ f^*[g].$$

b) Let $A \in \mathcal{A}(S)$, $(C,(b_g))$ be a descent datum, and let $g \in G$. If $\varphi : f^*A \to C$ and $\psi : A \to f_*C$ correspond to each other by adjunction, then φ^g and $[g]\circ\psi$ correspond to each other by adjunction. In particular (taking $A = f_*C$, $\psi = 1_A$), we have $[gh] = [g][h]$ (sic) and $[g]$ is an automorphism.
c) Suppose that C is an effective descent datum f^*B. For any $g \in G$, we have $\varepsilon^g_{f^*B} \circ f^*\eta_B = 1_{f^*B}$.

Proof. a) This is just the other adjunction identity relating $[g]$ and ε_C^g.

b) We have

$$\varphi = \varepsilon_C \circ f^*\psi, \quad \psi = f_*\varphi \circ \eta_A.$$

The first identity yields

$$\varphi^g = \varepsilon_C^g \circ (f^*\psi)^g = \varepsilon_C^g \circ f^*\psi.$$

By the second identity, the morphism corresponding to φ^g is then

$$f_*(\varphi^g) \circ \eta_A = f_*(\varepsilon_C^g) \circ f_*f^*\psi \circ \eta_A = f_*(\varepsilon_C^g) \circ \eta_{f_*C} \circ \psi = [g] \circ \psi$$

where we used the naturality of η. Hence also the last claim.

c) Indeed,

$$\varepsilon^g_{f^*B} \circ f^*\eta_B = \varepsilon^g_{f^*B} \circ (f^*\eta_B)^g = (\varepsilon_{f^*B} \circ (f^*\eta_B))^g = 1^g_{f^*B} = 1_{f^*B}.$$

□

1.3. **Cartesianity.** Let $C \in \mathcal{A}(T)$ and $g \in G$. We define a morphism $f^*f_*C \to g^*C$ as the composition

$$f^*f_*C \xrightarrow{i_g(f_*C)} g^*f^*f_*C \xrightarrow{g^*\varepsilon_C} g^*C.$$

Collecting over g, we get a morphism

$$f^*f_*C \to \bigoplus_{g\in G} g^*C. \tag{1.1}$$

Definition 1.2. The functor f^* is *Cartesian* if (f_* exists and) (1.1) is a natural isomorphism.

(In view of the isomorphism of Π-sets $\coprod_{g\in G} T \xrightarrow{\sim} T \times_S T$ given by $y_g \mapsto (y, gy)$, Definition 1.2 amounts to saying that the "base change morphism" $f^*f_* \Rightarrow (f \times_S 1)_*(1 \times_S f)^*$ in the diagram

$$\begin{array}{ccc}
\mathcal{A}(T) & \xrightarrow{(1\times_S f)^*} & \mathcal{A}(T\times_S T) \\
{\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle (f\times_S 1)_*} \\
\mathcal{A}(S) & \xrightarrow{f^*} & \mathcal{A}(T)
\end{array}$$

is an isomorphism. One should not confuse this notion with that of a Cartesian morphism in a fibred category.)

Assume f^* Cartesian, and let $(C,(b_g))$ be a descent datum. Composing with the b_g^{-1} in (1.1), we get an isomorphism

$$f^* f_* C \xrightarrow{u_C} \bigoplus_{g\in G} C \tag{1.2}$$

whose g-component is given, by definition, by ε_C^g.

Lemma 1.3. *Let $h \in G$. Then the action of $f^*[h]$ on the left hand side of (1.2) amounts to the action of h by right translation on the indexing set G of its right hand side.*

Proof. Let $g \in G$. Using Lemma 1.1 a) and b), we find

$$\varepsilon_C^g \circ f^*[h] = \varepsilon_C \circ f^*[g] \circ f^*[h] = \varepsilon_C \circ f^*[gh] = \varepsilon_C^{gh}.$$

□

1.4. **Traces.** To formulate the result, we need a further definition:

Definition 1.4. Suppose f^* is Cartesian. A *trace structure* on (f^*, f_*) is a natural transformation $\mathrm{tr} : f_* f^* \Rightarrow \mathrm{Id}_{\mathcal{A}(S)}$ such that, for any $B \in \mathcal{A}(S)$:

(1) the composition

$$B \xrightarrow{\eta_B} f_* f^* B \xrightarrow{\mathrm{tr}_B} B \tag{1.3}$$

is multiplication by $|G|$;

(2) the isomorphism (1.2) (for $C = f^*B$) converts $f^* \mathrm{tr}_B$ into the sum map.

1.5. **Main result.**

Theorem 1.5. *Suppose that $\mathcal{A}(S)$ is pseudo-abelian for all S.*
a) If $\mathcal{A}$ is a stack, then

(i) *$\mathcal{A}$ commutes with coproducts;*
(ii) *For any Galois covering $f : T \to S$ in $B\Pi$, with S, T connected, f^* is Cartesian and has a trace structure.*

b) The converse is true if $|Gal(f)|$ is invertible in the coefficients of $\mathcal{A}$ for any f as in (ii).

Proof. a) The forgetful functor $\mathcal{A}(T,G) \to \mathcal{A}(T)$ sending $(C,(b_g))$ to C has the right adjoint $D \mapsto \bigoplus_{g\in G} g^*D$ provided with the descent datum (b_h) given by the isomorphisms $c_{g,h}(D) : h^*g^*D \xrightarrow{\sim} (gh)^*D$ of Subsection 1.1; the unit of this adjunction is given by the inverses of the b_g's. Cartesianity is tautologically true, and the trace morphism is given by $\bigoplus_{g\in G} g^*C \xrightarrow{(b_g)} C$ for $(C,(b_g))\in \mathcal{A}(T,G)$. Condition (1) of Definition 1.4 is immediate, and Condition (2) is left to the reader.

b) Let $G = Gal(f)$ as before. We check Conditions (1G) and (2G) of Subsection 1.1:

(1G) By adjunction, the map $\mathcal{A}(S)(A,B)\to \mathcal{A}(T)(f^*A,f^*B)$ may be rewritten as the map

$$\mathcal{A}(S)(A,B)\xrightarrow{a}\mathcal{A}(S)(A,f_*f^*B)$$

induced by the unit morphism η_B. Using (1.3), we get a map b in the opposite direction such that ba is multiplication by $|G|$; hence a is injective since on $\mathcal{A}(S)(A,B)$ is a $\mathbf{Q}$-vector space.

I now claim that $ab=\sum_{g\in G} g$ for the action of G on $\mathcal{A}(T)(f^*A,f^*B)$ explained before. By Lemma 1.1 b), it suffices to prove that the composition

$$f_*f^*B\xrightarrow{\operatorname{tr}_B} B\xrightarrow{\eta_B} f_*f^*B$$

is $\sum_{g\in G}[g]$. By the faithfulness of f^* which has just been established, it suffices to do this after applying f^*. By Condition (2) of the trace structure, this translates as a composition

$$\bigoplus_{g\in G} f^*B\xrightarrow{\Sigma} f^*B\xrightarrow{\Delta}\bigoplus_{g\in G} f^*B$$

in which Σ is the sum map and Δ is the diagonal map by Lemma 1.1 c); the claim now follows from Lemma 1.3.

Coming back to the proof of (1G), we find that the composition

$$\mathcal{A}(T)(f^*A,f^*B)^G\hookrightarrow \mathcal{A}(T)(f^*A,f^*B)\xrightarrow{ab}\mathcal{A}(T)(f^*A,f^*B)^G$$

is also multiplication by $|G|$, hence the desired bijectivity of a.

(2G) Let $(C,(b_g))$ be a descent datum. Consider the idempotent $e=\frac{1}{|G|}\sum_{g\in G}[g]$ in $\operatorname{End} f_*C$, and let $A=\operatorname{Im} e$. The adjoint of the inclusion $\iota: A\hookrightarrow f_*C$ yields a morphism $\tilde\iota: f^*A\to C$. Let us check that this is a morphism of descent data, and an isomorphism.

The first point amounts to say that $\tilde\iota^g=\tilde\iota$ for all g which, by Lemma 1.1 b), amounts to $[g]\circ\iota=\iota$ for all g: this is true by definition of ι.

For the second point, we define a morphism $j: C\to f^*A$ as follows. Let $\pi: f_*C\to A$ be the projection associated to the idempotent e. Then j is the composition

$$C\xrightarrow{\Delta}\bigoplus_{g\in G} C\xrightarrow{\sim} f^*f_*C\xrightarrow{f^*\pi} f^*A$$

where the first morphism is the diagonal map and the second one is the inverse of the isomorphism (1.2). It remains to show that j is inverse to $\tilde{\iota}$.

By the first point, we have $f^*[g] \circ f^*\iota = f^*\iota$ which means, by Lemma 1.3, that all the components of $f^*\iota$ on (1.2) are equal, i.e. that $\Delta\varepsilon_C f^*\iota = f^*\iota$. Therefore, with an abuse of notation,

$$j\tilde{\iota} = f^*\pi\Delta\varepsilon_C f^*\iota = f^*\pi f^*\iota = 1_{f^*A}.$$

Finally, we have $f^*\iota f^*\pi = f^*e = \frac{1}{|G|}\sum_{g\in G} f^*[g]$, hence

$$\tilde{\iota}j = \varepsilon_C f^*\iota f^*\pi\Delta = \frac{1}{|G|}\sum_{g\in G}\varepsilon_C f^*[g]\Delta = 1_C$$

as desired. □

2. The monoidal case

In this section, we assume that the categories $\mathcal{A}(S)$ are symmetric monoidal (unital) and that the base change functors f^* are strong symmetric monoidal (unital). We then have a "projection morphism"

$$A \otimes f_*C \to f_*(f^*A \otimes C)$$

for $(A, C) \in \mathcal{A}(S) \times \mathcal{A}(T)$, constructed as the adjoint of

$$f^*(A \otimes f_*C) \xrightarrow{\sim} f^*A \otimes f^*f_*C \xrightarrow{1\otimes\varepsilon_C} f^*A \otimes C.$$

where the first isomorphism is the inverse of the monoidal structure of f^*. For $C = \mathbf{1}_T$, we thus get a morphism

$$A \otimes f_*\mathbf{1}_T \xrightarrow{w_A} f_*f^*A. \tag{2.1}$$

Definition 2.1. We say that f^* verifies the *weak projection formula* if w_A is an isomorphism for any $A \in \mathcal{A}(S)$, and is *weakly Cartesian* if (1.2) is an isomorphism for $C = \mathbf{1}_T$.

Lemma 2.2. *Suppose that f^* verifies the weak projection formula and is weakly Cartesian. Then f^* is Cartesian in the sense of Definition 1.2.*

Proof. For $A \in \mathcal{A}(S)$, consider the commutative diagram

$$\begin{array}{ccc} f^*A \otimes f^*f_*\mathbf{1}_T & \xrightarrow{f^*w_A} & f^*f_*f^*A \\ {\scriptstyle 1\otimes u_{\mathbf{1}_T}}\downarrow & & \downarrow{\scriptstyle u_{f^*A}} \\ f^*A \otimes \bigoplus_{g\in G}\mathbf{1}_T & \xrightarrow{\bigoplus_{g\in G} e_{f^*A}} & \bigoplus_{g\in G} f^*A \end{array} \tag{2.2}$$

where e_{f^*A} is the unit constraint. The bottom horizontal map is an isomorphism; so are the top one and the left vertical one by assumption. Therefore u_{f^*A} is also an isomorphism. □

Suppose that f^* is Cartesian and admits a trace structure in the sense of Definition 1.4. Then there is a morphism $\mathrm{tr} : f_*\mathbf{1}_T \to \mathbf{1}_S$ such that

(1u) the composition
$$\mathbf{1}_S \xrightarrow{\eta_{\mathbf{1}_S}} f_*\mathbf{1}_T \xrightarrow{\mathrm{tr}} \mathbf{1}_S$$
is multiplication by $|G|$;

(2u) the isomorphism (1.2) (for $C = \mathbf{1}_T$) converts $f^*\mathrm{tr}$ into the sum map.

We call this a *weak trace structure.*

Conversely:

Proposition 2.3. *Suppose that f^* verifies the weak projection formula and is weakly Cartesian. Then a weak trace structure yields a trace structure on f^* by the formula $\mathrm{tr}_A = (1_A \otimes \mathrm{tr}) \circ w_A^{-1}$.*

Proof. The first identity of Definition 1.2 is clear from (1u), and the second one follows from (2u) by using Diagram (2.2) again. □

3. Motivic theories

The following generalises Theorem 1 of the introduction:

Theorem 3.1. *All motivic theories $\mathcal{A}$ of* [5, Th. 4.3 a)] *are stacks for the étale topology on* $\operatorname{Spec} k$*.*

Proof. By Theorem 1.5, Lemma 2.2 and Proposition 2.3, it suffices to check that, for any finite Galois extension $f : T = \operatorname{Spec} l \to S = \operatorname{Spec} k$, f^* verifies the weak projection formula, is weakly Cartesian and has a weak trace structure. Use M generically to denote the "motive" functor $\mathbf{Sm}(-) \to \mathcal{A}(-)$. Since f_* coincides with the left adjoint $f_\#$ of [5, Th. 4.1] which commutes with naïve restriction of scalars on $\mathbf{Sm}(-)$ via M, we always have
$$f_*M(X) = M(X_{(S)})$$
where $X_{(S)}$ denotes X viewed as an S-scheme, for any T-scheme X.

That (2.1) is a natural isomorphism is checked on pseudo-abelian generators of $\mathcal{A}(S)$. Also, f^* commutes with Tate twists when they are present in the theory $\mathcal{A}$. We thus may take $A = M(X)$ for $X \in \mathbf{Sm}(k)$ or $\mathbf{Sm}^{\mathrm{proj}}(k)$; then (2.1) becomes
$$M(X) \otimes M(T) \to M(X \times_S T)$$

which is an isomorphism because M is monoidal in all cases. That f^* is weakly Cartesian holds for the same reason (plus Galois theory).

Finally, we define the weak trace tr as the counit of the adjunction $(f_\#, f^*)$ on $\mathbf{1}_S = M(\operatorname{Spec} k)$ (recall that $f_\# = f_*$). The latter is the (finite) correspondence given by the graph of the projection $T \to S$, while $\varepsilon_{\mathbf{1}_S}$ is given by the transpose of this graph (this is the only geometric input in this story!) From this, the weak cartesianity and the axioms of the weak trace structure follow readily. □

4. Chow-Lefschetz motives

4.1. **The associated stack.** Let $\mathcal{A}_0$ be a fibred category over a site Σ. Recall that there is an "associated stack" $\mathcal{A}$ together with a fibered functor $\mathcal{A}_0 \to \mathcal{A}$ which is 2-universal for fibered functors from $\mathcal{A}_0$ to stacks. The stack $\mathcal{A}$ is constructed from $\mathcal{A}_0$ in two steps:

Associated prestack: $\mathcal{A}_1$: same objects as $\mathcal{A}_0$; for $S \in \Sigma$ and $X, Y \in \mathcal{A}_0(S)$, $\mathcal{A}_1(S)(X,Y)$ is the sheaf associated to the presheaf $T \to S \mapsto \mathcal{A}_0(T)(X_T, Y_T)$.

Associated stack (cf. [7, Lemma 3.2]): starting from $\mathcal{A}_1$, for $S \in \Sigma$ an object of $\mathcal{A}(S)$ is a descent datum of $\mathcal{A}_1$ for a suitable cover $(U_i)_{i\in I} \to S$; morphisms are given by refining covers. This operation is fully faithful (loc. cit., Remark 3.2.1).

In the case $\Sigma = B\Pi$, these two constructions translate as follows, with the notation of Section 1: in Step 1, one replaces the groups $\mathcal{A}_0(S)(A,B)$ by $\varinjlim_T \mathcal{A}_0(T)(f^*A, f^*B)^{Gal(f)}$, where $f : T \to S$ runs through the (finite) Galois coverings of S; for Step 2, we take the 2-colimit of the categories of descent data on $\mathcal{A}_1$.

4.2. **The case of Chow-Lefschetz motives.** In [6] we introduced categories of "Chow-Lefschetz motives" $\mathbf{LMot}_\sim(k)$ over a field k (modulo an adequate equivalence relation $\sim$) in two steps: a) by defining "crude" categories $\mathbf{LMot}_\sim(k)_0$ [6, §4.1]; b) by refining this construction [6, §4.2].

Proposition 4.1. $\mathbf{LMot}_\sim(-)$ *is the stack associated to* $\mathbf{LMot}_\sim(-)_0$.

Proof. We first prove that $\mathbf{LMot}_\sim(-)$ is a stack. This is essentially done in [6]: the descent property for morphisms is loc. cit., (4.4) and the effectivity of descent data is shown in the proof of Theorem 5 in loc. cit., §5.5 in the same way as here (we were inspired here by this argument). If we want to apply Theorem 1.5 of the present paper, we can note that the existence of a Cartesian *left* adjoint $f_\#$ to f^* is proven in [6, Lemma 4.5] (so we apply Theorem 1.5 to the opposite categories to $\mathbf{LMot}_\sim(-)$). To be complete, it remains in view of Proposition 2.3 to prove the weak projection formula and to give a weak trace structure. The proof of the first is the same as in Theorem 3.1, and the second follows tautologically from the definition of morphisms in [6, (4.3)].

In remains to show that the canonical fibred functor $\mathbf{LMot}_\sim(-)_0 \to \mathbf{LMot}_\sim(-)$ induces an equivalence on the associated stacks; it suffices to do it for the fibred functor $\mathbf{LCorr}(-)_0 \to \mathbf{LCorr}(-)$ on categories of correspondences. After forming the associated prestack as in §4.1, this functor becomes fully faithful, and it remains to show that it becomes essentially surjective after forming the associated stack. But an object of $\mathbf{LCorr}(k)$ is an abelian scheme over an étale k-algebra, which clearly defines a descent datum for abelian varieties over (finite separable extensions of) k. □

5. Tannakian categories

Let $\Pi, \mathcal{A}$ be as in Section 2, with the $\mathcal{A}(T)$ rigid and abelian; we assume that $K = \operatorname{End}_{\mathcal{A}}(\mathbf{1})$ is a field of characteristic 0. We define $\mathcal{A}_\infty$ as 2-$\varinjlim_U \mathcal{A}(\Pi/U)$, where U runs through the normal open subgroups of Π: it has the same properties. Let $\omega_\infty : \mathcal{A}_\infty \to \mathbf{Vec}_K$ be a fibre functor (exact and faithful) to the category of finite-dimensional K-vector spaces: by restriction, it defines a fibre functor ω on $\mathcal{A}(*) =: \mathcal{A}$. Let $G = \mathbf{Aut}^\otimes(\omega)$ be its Tannakian group and $H = \mathbf{Aut}^\otimes(\omega_\infty)$ be the one of ω_∞.

Definition 5.1. An *Artin object* of $\mathcal{A}$ is an object A such that $f^*A \simeq n\mathbf{1}_T$ for some $f : T \to *$ and some $n \geq 0$. Artin objects form a (full) rigid $\otimes$-subcategory of $\mathcal{A}$, denoted by $\mathcal{A}^0$.

Let $A \in \mathcal{A}^0$ be an Artin object; in Definition 5.1, we may choose $T = G = \Pi/U$ with U normal in Π. Then G acts on $K^n \simeq \operatorname{Hom}(\mathbf{1}_T, f^*A) \simeq \operatorname{Hom}(f^*\mathbf{1}_*, f^*A)$ as in §1.1. If $\mathcal{A}^0(G)$ denotes the full subcategory of Artin objects split by f, this defines a $\otimes$-functor

$$\mathcal{A}^0(G) \to \mathbf{Rep}_K(G). \tag{5.1}$$

Lemma 5.2. *This functor is an equivalence of categories.*

Proof. Full faithfulness: let $A, B \in \mathcal{A}^0(G)$, and let $\varphi : \operatorname{Hom}(\mathbf{1}_T, f^*A) \to \operatorname{Hom}(\mathbf{1}_T, f^*B)$ be a G-equivariant homomorphism. Using the isomorphism $f^*A \simeq n\mathbf{1}_T$, we get a homomorphism $\varphi' : \operatorname{Hom}(f^*A, f^*A) \to \operatorname{Hom}(f^*A, f^*B)$. Then $\varphi'(1_{f^*A})$ maps to φ by (5.1) and is G-equivariant, hence comes from a (unique) morphism $A \to B$ by descent.

Essential surjectivity: for $V \in \mathbf{Rep}_K(G)$, the choice of a basis of V yields a descent datum. □

The $\otimes$-functor $F^* : \mathcal{A} \to \mathcal{A}_\infty$ induces a homomorphism $i : H \to G$. Lemma 5.2 yields an equivalence of $\otimes$-categories $\mathcal{A}^0 \xrightarrow{\sim} \mathbf{Rep}_K(\Pi)$, which is induced by ω since $\omega = \omega_\infty \circ F^*$ (indeed, $\mathcal{A}_\infty(\mathbf{1}_\infty, B)$ is functorially isomorphic to $\omega_\infty(B)$ for any split $B \in \mathcal{A}_\infty$). Whence a homomorphism $p : G \to \Pi$.

Theorem 5.3. *In the sequence*

$$H \xrightarrow{i} G \xrightarrow{p} \Pi \tag{5.2}$$

i is a monomorphism and p is faithfully flat; if the $\mathcal{A}(X)$'s are semi-simple, (5.2) *is exact at G.*

Proof. It is the same as for [6, Prop. 5.16], using [2, Prop. 2.21 (a) and (b)] and [6, Prop. 5.12]. For p, noting that $\mathcal{A}^0$ is semi-simple since Π is profinite, we must show that every subobject $B \in \mathcal{A}$ of an object $A \in \mathcal{A}^0$ belongs to $\mathcal{A}^0$; but this is obvious by restricting to $\mathcal{A}_\infty$, since $\mathbf{1}_\infty$ is simple by [2, Prop. 1.17]. For i, we must show that any object C of $\mathcal{A}_\infty$ is a direct summand of an object of the form F^*A for $A \in \mathcal{A}$; since C comes from $\mathcal{A}(G)$ for some $G = \Pi/U$, it suffices to prove this for $C \in \mathcal{A}(G)$ and $f^* : \mathcal{A} \to \mathcal{A}(G)$. But C is a direct summand of $f^* f_* C$ by cartesianity. If the $\mathcal{A}(X)$'s are semi-simple, so is $\mathcal{A}_\infty$; by [6, Prop. 5.12], for the exactness of (5.2) at G it suffices to show that, for any simple $S \in \mathcal{A}$, $\omega_\infty(S)^H \neq 0 \Rightarrow \omega(S)^{\operatorname{Ker} p} \neq 0$. But $\omega_\infty(S)^H \neq 0$ implies that H acts trivially on $\omega_\infty(S)$, i.e. that F^*S is trivial, i.e. that $S \in \mathcal{A}^0$, which concludes the proof. □

References

[1] Y. André *Pour une théorie inconditionnelle des motifs*, Publ. Math. IHÉS **83** (1996), 5–49.

[2] P. Deligne and J. Milne *Tannakian categories*, *in* Hodge Cycles, Motives, and Shimura Varieties, Lect. Notes in Math. **900**, Springer, 1982, 101–228.

[3] C. Deninger and J. P. Murre *Motivic decomposition of abelian schemes and the Fourier transform*, J. Reine Angew. Math. **422** (1991), 201–219.

[4] A. Grothendieck *Catégories fibrées et descente*, Exposé VI of Revêtements étales et groupe fondamental (SGA 1), revised edition, Doc. Math. **3**, SMF, 2003, 119–152.

[5] B. Kahn *Motifs et adjoints*, Rend. Sem. mat. Univ. Padova **139** (2018), 77–128.

[6] B. Kahn *Chow-Lefschetz motives*, preprint, 2022; https://arxiv.org/abs/2302.08327.

[7] G. Laumon, L. Moret-Bailly Champs algébriques, Springer, 2000.

[8] The Stacks project, https://stacks.math.columbia.edu.

CNRS, Sorbonne Université and Université Paris Cité, IMJ-PRG, Case 247, 4 place Jussieu, 75252 Paris Cedex 05, France

Email address: bruno.kahn@imj-prg.fr