# An approach to the Tate conjecture for surfaces over a finite field

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# The Tate conjecture

$k$ finitely generated field, $X$ smooth projective $k$-variety, $l$ prime number $\neq$ char $k$.

**Conjecture (Tate, 1964)**

*$n \geq 0$, $CH^n(X)$ = Chow group of cycles of codimension $n$ modulo rational equivalence: the cycle class map*

$$CH^n(X) \otimes \mathbb{Q}_l \to H^{2n}(X_s, \mathbb{Q}_l(n))^G$$

*is surjective.*

Here $G = Gal(k_s/k)$ for a separable closure $k_s$ of $k$ and $X_s = X \otimes_k k_s$.

# Known cases

1. $n = 1$: abelian varieties (Tate, Zarhin/Mori, Faltings).
2. $n = 1$, stable under product and domination, birationally invariant.
3. $n = 1$, $k$ of char. 0 or finite or : K3 surfaces (Ramakrishnan, Nygaard-Ogus, Artin-Swinnerton Dyer, Charles...)
4. $n > 1$: several examples using Tannakian ideas.

$k$ finite: the Tate conjecture (for a given $X$) is independent of $l$.

# Reduction to surfaces

**Theorem (Morrow, Ambrosi, K.)**

*For $n = 1$, the Tate conjecture follows from the special case of surfaces over $\mathbb{Q}$ and $\mathbb{F}_p$.*

# Main result

For any variety $S$, write $H^i(S,j) := H^i_{\text{ét}}(S, \mathbb{Q}_l/\mathbb{Z}_l(j))$.

## Theorem

*$X$ smooth projective surface over $k = \mathbb{F}_q$; assume that $G$ acts trivially on $\mathrm{NS}(X_s)$. Then, equivalent conditions:*

1. *The Tate conjecture holds for $X$.*
2. *For any affine open $U \subset X$ such that $\mathrm{Pic}(U) = 0$, one has $H^3(U,1) = 0$.*
3. *For any affine open $U \subset X$ such that $\mathrm{Pic}(U) = 0$ and any smooth irreducible divisor $Z \subset U$, the map $H^3(U,1) \to H^3(U - Z, 1)$ is injective.*

(Hypothesis sufficient for the Tate conjecture.)
May assume $X$ geometrically connected.

# The Brauer group of $X$

For any variety $S$, $\mathrm{Br}_l(S) := H^2_{\text{ét}}(S, \mathbb{G}_m)\{l\}$, the $l$-primary part of the cohomological Brauer group.

**Proposition (works in any dimension and over any f.g. field $k$)**

*The Tate conjecture for $X$ in codimension 1 $\iff$ $\mathrm{Br}_l(X_s)^G$ is finite.*

**Proof.**

Kummer exact sequence yields short exact sequence

$$0 \to \mathrm{NS}(X_s) \otimes \mathbb{Q}_l \to H^2(X_s, \mathbb{Q}_l(1)) \to V_l(\mathrm{Br}_l(X)) \to 0$$

Take Galois cohomology and observe that

$$H^1(G, \mathrm{NS}(X_s) \otimes \mathbb{Q}_l) = H^1(G, \mathrm{NS}(X_s)) \otimes \mathbb{Q}_l = 0,$$

so Tate $\iff$ $V_l(\mathrm{Br}_l(X))^G = 0$. This is equivalent to finiteness of $\mathrm{Br}_l(X_s)^G$ because $\mathrm{Br}_l(X_s)$ is of cofinite type. □

# The Brauer group of $U$

Back to $k$ finite and $X$ surface.
$U \subset X$ open subset, $Z$ closed complement (reduced): short exact sequence

$$0 \to \mathrm{Br}_l(X_s) \to \mathrm{Br}_l(U_s) \to \bigoplus_{x \in Z \cap X^{(1)}} H^1((Z'_x)_s, \mathbb{Q}_l/\mathbb{Z}_l(0))$$

where, for all $x \in Z \cap X^{(1)}$, $Z'_x$ = intersection of smooth locus of $Z$ with its irreducible component corresponding to $x$.

## Proposition

*The groups $H^1((Z'_x)_s, \mathbb{Q}_l/\mathbb{Z}_l(0))^G$ are finite.*

## Proof.

Follows from Weil's Riemann hypothesis applied to the smooth completions of the $Z'_x$. □

# The Brauer group of $U$ (continued)

**Proposition**

*Tate* $\iff \forall U\ \mathrm{Br}_l(U_s)^G$ *is finite* $\iff \exists U\ \mathrm{Br}_l(U_s)^G$ *is finite.* □

**Proposition**

$\mathrm{Br}_l(U_s)^G$ *finite* $\iff \mathrm{Br}_l(U_s)_G$ *finite.*

True for any $G$-module of cofinite type (because $G$ is procyclic).

# Passing from $\mathrm{Br}_l(U_s)_G$ to $H^3(U,1)$

If $\mathrm{NS}(U_s)$ is torsion, then $H^2(U_s,1) \xrightarrow{\sim} \mathrm{Br}_l(U_s)$, hence short exact sequence (Hochschild-Serre):

$$0 \to \mathrm{Br}_l(U_s)_G \to H^3(U,1) \to H^3(U_s,1)^G \to 0.$$

## Lemma

*If moreover $U$ is affine, then isomorphism of divisible groups*

$$\mathrm{Br}_l(U_s)_G \xrightarrow{\sim} H^3(U,1).$$

## Proof.

Follows from M. Artin's "affine Lefschetz" ($cd_l(U_s) = 2$) applied twice!. □

# The condition $\mathrm{Pic}(U) = 0$

## Lemma

*Suppose that $G$ acts trivially on $\mathrm{NS}(X_s)$. Then $\mathrm{Pic}(U) = 0 \Rightarrow \mathrm{NS}(U_s)$ is torsion.*

## Proof.

If $G$ acts trivially on $\mathrm{NS}(X_s)$, it acts trivially on its quotient $\mathrm{NS}(U_s)$; also $\mathrm{Pic}^0(X_s) \to \mathrm{Pic}^0(U_s)$ is surjective hence $\mathrm{Pic}^0(U_s)$ is torsion. Finally, $\mathrm{Coker}(\mathrm{Pic}(U) \to \mathrm{Pic}(U_s)^G)$ is torsion by a transfer argument. Conclusion is easy. □

## Remark

Can always reduce to this case after finite extension of $k$ since $\mathrm{NS}(X_s)$ is finitely generated. Sufficient for the Tate conjecture.

# Proof of 1 $\iff$ 2

## Hypotheses

$G$ acts trivially on $NS(X_s)$, $U$ affine and $Pic(U) = 0$.

Tate $\iff$ $Br_l(X_s)^G = 0$ $\iff$ $Br_l(U)_G$ finite $\iff$ $Br_l(U)_G = 0$ (because divisible) $\iff$ $H^3(U, 1) = 0$.

## Remarks

a) Quasi-affine is not sufficient: by purity, $H^3(\mathbb{A}^2 - \{0\}, 1) = H^0(k, -1)$, $\neq 0$ in general.

b) $\exists U$ because $Pic(X)$ finitely generated.

# From 2 to 3

Still assume $G$ acts trivially on $\mathrm{NS}(X_s)$ (blanket assumption now).

**A.**

Condition 2 equivalent to: For any affine open $U \subset X$ such that $\mathrm{Pic}(U) = 0$, and any open $V \subseteq U$, the map $H^3(U, 1) \to H^3(V, 1)$ is injective..

If true, then $H^3(U, 1) \hookrightarrow H^3(K, 1)$ $(K = k(X) = k(U))$, but

**Theorem (K., 1991 Lake Louise K-theory proceedings)**

$H^3(K, 1) = 0.$

## Proof.

Hochschild-Serre ⇒ exact sequence

$$0 \to H^2(Kk_s, 1)_G \to H^3(K, 1) \to H^3(Kk_s, 1)^G \to 0.$$

Right hand side 0 because $cd(Kk_s) = 2$; For left hand side, *Bloch-Kato theorem*

$$K_2(Kk_s)/l^\nu \twoheadrightarrow H^2(Kk_s, \mu_{l^\nu}^{\otimes 2})\, \forall \nu \geq 1$$

(predates Merkurjev-Suslin!), hence

$$(K_2(Kk_s) \otimes \mathbb{Q}_l/\mathbb{Z}_l(-1))_G \twoheadrightarrow H^2(Kk_s, 1)_G$$

but left hand side is 0 by Tate's lemma. □

## B.

A. equivalent to same statement, but with $Z := (U - V)_{\text{red}}$ irreducible of dimension 1.

## Proof.

$D_1, \dots, D_n$ irreductible components of codimension 1 of $Z$. For $0 \le i \le n$, $U_i$ inductively defined as $U_{i-1} \setminus D_i$, with $U_0 = U$. Chain of open subsets

$$U \supset U_1 \supset \dots U_n \supseteq V$$

each $U_i$ affine since $D_i$ principal, $\operatorname{Pic}(U_i) = 0$, and $U_n - V$ of codimension $\ge 2$ in $U_n$. By B., $H^3(U_i, 1) \hookrightarrow H^3(U_{i+1}, 1)$ for all $i$, and also $H^3(U_n, 1) \hookrightarrow H^3(V, 1)$ by cohomological purity. □

# End of proof that 2 $\iff$ 3

## C.

B. equivalent to same statement, but with $Z$ smooth.

## Proof.

$\bar{Z}$ closure of $Z$ in $X$, $F$ its singular locus. By Poonen (Bertini theorems over finite fields), $\exists\ C_0 \subset X$ smooth projective curve containing $F$; a fortiori, $C = C_0 \cap U$ is smooth. Apply C. to $(U, C)$ and then to $(U - C, Z \setminus C)$ (note that $Z \setminus C$ is smooth): we get that the composition

$$H^3(U, 1) \to H^3(U - C, 1) \to H^3(U - (C \cup Z), 1)$$

is injective. A fortiori, $H^3(U, 1) \to H^3(U - Z, 1)$ is injective. □

# Going further

Gysin exact sequence

$$H^2(V,1) \xrightarrow{\partial} H^1(Z,0) \xrightarrow{i_*} H^3(U,1) \xrightarrow{j^*} H^3(V,1)$$

## Proposition

*In this sequence,*
*a) Image of $\partial$ contains image of $i^* : H^1(U,0) \to H^1(Z,0)$.*
*b) $i_*$ factors through the finite group $H^1(Z_s,0)^G$.*
*c) $i_* = 0$ (hence $j^*$ injective) for $l \geq l_0$, where $l_0$ prime number depending on $Z$.*

# Proof of a)

$f \in \Gamma(U, \mathbb{G}_a)$ equation of $Z$ in $U$. Then $f$ is inversible on $V$.
$(f) \in H^1(V, \mathbb{Z}_l(1))$ its Kummer class: composition

$$H^1(U,0) \xrightarrow{j^*} H^1(V,0) \xrightarrow{\cup (f)} H^2(V,1) \xrightarrow{\partial} H^1(Z,0)$$

equals $i^*$ (follows from definition of the purity isomorphism).

# Proof of b)

$k_Z$ field of constants of $Z$. Commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(Z_s,0)_G & \longrightarrow & H^1(Z,0) & \longrightarrow & H^1(Z_s,0)^G & \longrightarrow & 0 \\ & & i^*\uparrow & & i^*\uparrow & & i^*\uparrow & & \\ 0 & \longrightarrow & H^0(U_s,0)_G & \longrightarrow & H^1(U,0) & \longrightarrow & H^1(U_s,0)^G & \longrightarrow & 0 \end{array}$$

where left vertical arrow = multiplication by $[k_Z : k]$ in $\mathbb{Q}_l/\mathbb{Z}_l$, hence surjective. By a), image of $\partial$ conains $H^0(Z_s,0)_G$.

# Proof of c)

Needs

**Lemma**

*The order of $H^1(Z_s, 0)^G$ is bounded independently of $l$.*

**Proof.**

Again, follows from Riemann hypothesis applied to smooth completion of $Z$ (bound depends on $k_Z$, the genus and the divisor at infinity). □

# First idea fails

Recall: the Tate conjecture is independent of $l$. So we won! No, because $l_0$ a priori not bounded independently of $Z$.

# Second idea

Inspired by Gillet's proof of Gersten's conjecture for dvr's (for $K$-theory with finite coefficients), J. Alg., 1986.

In three parts: first two parts work but not last.

# Motivation: Gabber rigidity

## Theorem (Gabber)

*$(A_h, I)$ Henselian pair, $U_h = \mathrm{Spec}(A_h)$, $Z = \mathrm{Spec}(A_h/I)$, $i : Z \hookrightarrow U_h$ the closed immersion. Then for any torsion abelian étale sheaf $F$ on $U_h$ and for all $q > 0$, $H^q(U_h, F) \xrightarrow{i^*} H^q(Z, F)$ is bijective.*

Coming back to our $(U, Z)$: recall Nisnevich neighbourhood of $Z \hookrightarrow U$: Cartesian square

$$\begin{array}{ccc} V_1 & \xrightarrow{j_1} & U_1 \\ p \downarrow & & \downarrow q \\ V & \xrightarrow{j} & U \end{array} \tag{1}$$

$q$ etale and $q^{-1}(Z) \xrightarrow{\sim} Z$.
$(U_h, Z)$ henselisation of pair $(U, Z)$: filtering colimit of such Nisnevich squares.

# Second idea: first part

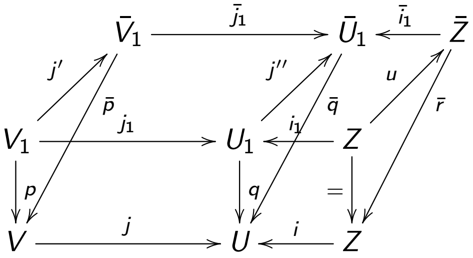
**Corollary**

*$\exists$ $(U_1, q)$ such that $H^1(U_1, 0) \to H^1(Z, 0)$ is surjective. Therefore $\partial_1 : H^2(V_1, 1) \to H^1(Z, 0)$ surjective (see proposition p. 16).*

Unfortunately, not sufficient: how do we go down? No push-forward for $p$.

# Normalising (1)

$\bar{U}_1$ normalisation of $U$ in $q$; more complicated diagram




(2)

$j''$ open immersion, $\bar{q}$ fini (since $q$ étale), $\bar{V}_1 = V \times_U \bar{U}_1$, $\bar{Z} = \bar{U}_1 - \bar{V}_1$ and other arrows follow. In particuliar, $j'$ and $u$ also open immersions, $\bar{Z}$ closed and $\bar{p}$, $\bar{r}$ also finite.

Note:

- Finite $\Rightarrow$ affine, hence $\bar{U}_1$, $\bar{V}_1$ are affine. All vertices of (2) are affine.
- In particular, closed immersion $\bar{i}_1$ purely of codimension 1.
- $\bar{r}$ separated $\Rightarrow$ open immersion $u$ is also closed, hence $\bar{Z} = Z \coprod T$ for some other closed subset $T$.
- $\bar{U}_1$ and $\bar{V}_1$ normal surfaces $\Rightarrow$ $\bar{p}$ and $\bar{q}$ *flat* (Serre's normality criterion $\Rightarrow$ Cohen-Macaulay, etc.).

# Second idea, second part

Since $\bar{p}, \bar{q}$ finite and flat, trace maps available in étale cohomology; commutative diagram

$$
\begin{array}{ccccc}
H^2(V,1) & \xleftarrow{\bar{p}_*} & H^2(\bar{V}_1,1) & \xrightarrow{j'^*} & H^2(V_1,1) \\
\downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \bar{\partial}_1} & & \downarrow{\scriptstyle \partial_1} \\
H^3_Z(U,1) & \xleftarrow{\bar{q}_*} & H^3_Z(\bar{U}_1,1) \oplus H^3_T(\bar{U}_1,1) & \xrightarrow{j''^*} & H^3_Z(U_1,1) \\
 & \nwarrow{\scriptstyle \sim} & \uparrow{\scriptstyle a} & \nearrow{\scriptstyle \sim} & \\
 & & H^1(Z,0) & &
\end{array}
\qquad (3)
$$

where $\partial_1$ surjective (as seen) and $a$, an isomorphism on first summand defined by excision ($H^3_Z(\bar{U}_1,1) \xrightarrow{\sim} H^3_Z(U_1,1)$), and 0 on second. Left square commutes e.g. by proper (finite) base change.

## Corollary

$\operatorname{Im} \partial \supseteq \operatorname{Im}(\bar{q}_* \circ \bar{\partial}_1)$.

# Second idea, third part

If could show that $\operatorname{Im}\bar{\partial}_1 \supseteq \operatorname{Im} a$, would win. Would like to use surjectivity of $\partial_1$, but not sufficient. Would work if

1. the composition

$$\bar{i}^*_{1,Z} : H^1(\bar{U}_1, 0) \xrightarrow{j''^*} H^1(U_1, 0) \xrightarrow{i_1^*} H^1(Z, 0)$$

   is surjective, and
2. $\exists\ f_1 \in \Gamma(\bar{U}_1, \mathbb{G}_a)$ such that $Z$ principal of equation $f_1$ in $\bar{U}_1$, and $f_1 \equiv 1 \pmod{T}$.

2 looks very expensive, but maybe 1 can be achieved (by enlarging $U_1$). Note that it is true for $l$ large enough, because this holds for $i^*$ (see again prop. p. 16).

# Third idea: from below

Inspired by Gabber's geometric presentation lemma to prove Gersten's conjecture.
Suppose that we can construct an "ante-Nisnevich neighbourhood" of $i$:

$$\begin{array}{ccc} Z & \xrightarrow{i} & U \\ & \searrow^{i_1} & \downarrow{\scriptstyle v} \\ & & U_1 \end{array} \tag{4}$$

$i_1$ closed immersion, $v$ Nisnevich neighbourhood of $Z$, $U_1$ affine open in smooth projective surface for which Tate's conjecture is known. Then $(i_1)_* = 0$, hence $i_* = v^*(i_1)_* = 0$ (functoriality of Gysin maps).
In fact, "Nisnevich neighbourhood" not necessary: by the functoriality of Gysin morphisms, $v$ may be any morphism such that

$$Z = v^{-1}(\overline{v(Z)}), \quad Z \xrightarrow{\sim} \overline{v(Z)} \tag{5}$$

(scheme-theoretically). Moreover, $v(Z)$ is constructible by Chevalley, but $Z$ curve, hence $v(Z)$ open in its closure.

Gabber's lemma: this with $U_1 = \mathbb{A}^2$, but up to an open subset. Version over finite fields by Hogadi-Kulkarni (Crelle 2020):

**Proposition**

*$\exists\, v : U \to \mathbb{A}^2$ and open subset $W \subseteq \mathbb{A}^2$ such that*

1. *$v_{|v^{-1}(W)}$ is étale*
2. *$Z \cap v^{-1}(W) \xrightarrow{v} W$ is a closed immersion.*

But cannot afford to "lose" a closed subset in $U$ (of codimension 2, à la rigueur...) So look at situation for $v$ on the whole of $U$. Second condition of (5) is (essentially) achieved, but not first: can be extra components – and will be in general, because $v$ has generic degree $> 1$ unless birational...

Similar problem as in second idea!

*That's all!*