# CORRECTION TO "VOEVODSKY'S MOTIVES AND WEIL RECIPROCITY," DUKE MATH. J. 162 (2013), 2751–2796

BRUNO KAHN and TAKAO YAMAZAKI

**Abstract**
*We correct a mistake in Section 2.4 of the said paper.*

In [1, Section 2.4], we wrote: "The category $\mathbf{Z}\,\mathbf{Span}$ is isomorphic to the full subcategory of **Cor** consisting of smooth $k$-schemes of dimension 0." Tom Bachmann kindly pointed out to us that this statement is incorrect. Here we clarify the relationship between the two categories and show that it does not affect any argument about *cohomological* Mackey functors (the only Mackey functors appearing in [1]).

We retain the notation of [1].

**1**
Let $\mathbf{Cor_0}$ be the full subcategory of **Cor** given by 0-dimensional smooth schemes (= étale $k$-schemes). If $f : X \to Y$ is a surjective morphism of degree $d$ of étale $k$-schemes, then we have the formula in $\mathbf{Cor_0}$,

$$^t f \circ f = d. \tag{1}$$

**2**
There is a canonical functor

$$\varepsilon : \mathbf{Z}\,\mathbf{Span} \to \mathbf{Cor_0} \tag{2}$$

which is the identity on objects and sends a span (2.1) from [1],

$$X \xleftarrow{g} Z \xrightarrow{f} Y, \tag{3}$$

to $f \circ {}^t g$.

DUKE MATHEMATICAL JOURNAL
Vol. 164, No. 10, © 2015 DOI 10.1215/00127094-3146068
Received 23 October 2014. Revision received 14 March 2015.
2010 *Mathematics Subject Classification*. Primary 19E15; Secondary 19A22, 18D10.

LEMMA 1

*In (3), assume that $Z$ is irreducible, and let $\bar{Z}$ be its image in $X \times Y$, viewed as an element of* $\mathbf{Cor}_0(X,Y)$. *Then* $\varepsilon(f,g) = [Z : \bar{Z}]\bar{Z}$.

*Proof*

This follows from the formula for the composition of finite correspondences. □

PROPOSITION 2

*Let $M \in$ **Mack** be a Mackey functor, that is, an additive contravariant functor from* **Z Span** *to* **Ab**. *Then $M$ is cohomological if and only if it factors through $\varepsilon$. This yields an equivalence,*

$$\mathbf{Mack}_c \simeq \mathrm{Mod}-\mathbf{Cor_0}\,.$$

*Proof*

If $M$ factors through $\varepsilon$, it is cohomological thanks to (1). Conversely, if $M$ is cohomological, consider a span (3) with $Z$ irreducible, and let $\bar{Z}$ be as in Lemma 1. So we have a commutative diagram:

$$\begin{tikzcd}
& Z \arrow[dl, "g"'] \arrow[dr, "f"] \arrow[dd, "\pi"] & \\
X & & Y \\
& \bar{Z} \arrow[ul, "\bar{g}"] \arrow[ur, "\bar{f}"'] &
\end{tikzcd}$$

Then $M^*(f) = M^*(\pi)M^*(\bar{f})$, $M_*(g) = M_*(\bar{g})M_*(\pi)$, and thus

$$\begin{aligned}
M(f,g) &= M_*(g)M^*(f) \\
&= M_*(\bar{g})M_*(\pi)M^*(\pi)M^*(\bar{f}) \\
&= \deg(\pi)M_*(\bar{g})M^*(\bar{f}) \\
&= \deg(\pi)M(\bar{f},\bar{g}) \\
&= M\big(\varepsilon(f,g)\big)
\end{aligned}$$

by Lemma 1.

(Alternatively, Proposition 2 follows from combining [4, Theorem 4.3] and a version of [3, Proposition 3.4.1].) □

**3**

Proposition 2 justifies and corrects [1, Section 2.4]: the inclusion functor $\mathbf{Cor}_0 \hookrightarrow \mathbf{Cor}$ induces an exact functor

$$\rho : \mathbf{PST} \to \mathbf{Mack}_c\,.$$

(In [1, Section 2.4], it is not necessary to restrict $\rho$ to $\mathbf{HI}$ to get into $\mathbf{Mack}_c$.)

**4**

To obtain [1, (2.9)], it remains to show that $\varepsilon^* : \mathbf{Mack}_c \to \mathbf{Mack}$ is symmetric monoidal with respect to the tensor structures induced by those of $\mathbf{Z}\,\mathbf{Span}$ and $\mathbf{Cor}_0$. (Recall these tensor structures: on objects they are given by the product of étale $k$-schemes; the tensor product of two spans $(f, g)$ and $(f', g')$ is $(f \times f', g \times g')$, and the tensor product of finite correspondences is the usual one.) This is obvious if $k$ is algebraically closed, because $\varepsilon$ is then a $\otimes$-isomorphism of $\otimes$-categories. The general case follows from the next proposition.

PROPOSITION 3
*Let $\varepsilon : \mathcal{A} \to \mathcal{B}$ be a full $\otimes$-functor between rigid symmetric monoidal categories, which is the identity on objects. Then the natural morphism*

$$\varepsilon^* M \otimes_{\mathcal{A}} \varepsilon^* N \to \varepsilon^*(M \otimes_{\mathcal{B}} N) \tag{4}$$

*is an isomorphism for any $M, N \in \mathrm{Mod}-\mathcal{B}$.*

Before starting the proof, let us clarify the somewhat improper use of "dummy" in [1, last part of the proof of Proposition A.14].

LEMMA 4
*Let $\mathcal{A}$ be an additive category, and let $M \in \mathrm{Mod}-\mathcal{A}$. Then there is a canonical isomorphism*

$$\theta : \int^{B\in\mathcal{A}} M(B) \otimes \mathcal{A}(A, B) \xrightarrow{\sim} M(A)$$

*for any $A \in \mathcal{A}$.*

*Proof*
The "evaluation" morphisms $M(B) \otimes \mathcal{A}(A, B) \to M(A)$ mapping $m \otimes f$ to $f^*m$ are linked by commutative diagrams like diagram (3) of [2, p. 219]: this provides the map $\theta$. Let us show that the map $\lambda : M(A) \to \int^{B\in\mathcal{A}} M(B) \otimes \mathcal{A}(A, B)$ given by $\lambda(m) =$ (the class of) $m \otimes 1_A \in M(A) \otimes \mathcal{A}(A, A)$ is inverse to $\theta$. It is obvious

that $\theta\lambda$ is the identity. To check $\lambda\theta = \mathrm{id}$, take $B \in \mathcal{A}$, $m \in M(B)$, and $g \in \mathcal{A}(A, B)$. Then we have $\lambda\theta(m \otimes g) = g^*(m) \otimes 1_A = m \otimes g$, where the last equality holds because $m \otimes 1_A \in M(B) \otimes \mathcal{A}(A, A)$ is mapped to $m \otimes g \in M(B) \otimes \mathcal{A}(A, B)$ (resp., to $g^*(m) \otimes 1_A \in M(A) \otimes \mathcal{A}(A, A)$) by $1 \otimes g_*$ (resp., by $g^* \otimes 1$). □

We also have the following lemma.

LEMMA 5
*Let $\varepsilon : \mathcal{A} \to \mathcal{B}$ be a functor, and let $T : \mathcal{B}^{\mathrm{op}} \times \mathcal{B} \to$* **Set** *be a bifunctor. Then there is a canonical morphism $\int^{A\in\mathcal{A}} T(\varepsilon A, \varepsilon A) \to \int^{B\in\mathcal{B}} T(B, B)$. If $\varepsilon$ is surjective on objects, then this is a surjection; if $\varepsilon$ is moreover full and bijective on objects, then this is a bijection.*

*Proof*
We may interpret coends as colimits by the dual of [2, Proposition 1, p. 224]. The first statement is then obvious (cf. formula (1) in [2, p. 217]), and the second one follows by inspection. (The surjectivity of $\varepsilon$ on objects gives surjectivity on generators, its bijectivity gives bijectivity on generators, and its fullness gives surjectivity on relations.) Alternatively, this can also be shown by using the final functor theorem of [2, Theorem 1, p. 217]; the details are left to the interested readers. □

**5**
We can now prove Proposition 3. As recalled in [1, Section A.3], $\varepsilon^*$ has a right adjoint, and hence commutes with arbitrary colimits. The two tensor products $\varepsilon^* M \otimes_{\mathcal{A}}$—and $M \otimes_{\mathcal{B}}$—also commute with arbitrary colimits, as seen from [1, Section A.10]. Thus we are reduced to the case where $N$ is representable, say, $N = y_{\mathcal{B}}(C)$ for $C \in \mathcal{B}$ (where $y_{\mathcal{B}} : \mathcal{B} \to \mathrm{Mod}-\mathcal{B}$ is the additive Yoneda embedding). For any $P \in \mathrm{Mod}-\mathcal{B}$ and any $A \in \mathcal{A}$, we have, by definition,

$$\varepsilon^* P(A) = P(\varepsilon A) = P(A)$$

since $\varepsilon$ is the identity on objects. Using [1, (A.4)], this first yields

$$\varepsilon^*\big(M \otimes_{\mathcal{B}} y_{\mathcal{B}}(C)\big)(A) = M(A \otimes_{\mathcal{B}} C^*),$$

where $C^*$ is the dual of $C$. Using now [1, (A.3)], we compute

$$\begin{aligned}&\big(\varepsilon^* M \otimes_{\mathcal{A}} \varepsilon^* y_{\mathcal{B}}(C)\big)(A)\\ &\quad = \int^{B\in\mathcal{A}} M(\varepsilon B) \otimes y_{\mathcal{B}}(C)\big(\varepsilon(A \otimes_{\mathcal{A}} B^*)\big)\end{aligned}$$

$$
\begin{aligned}
&= \int^{B\in\mathcal{A}} M(\varepsilon B)\otimes y_{\mathcal{B}}(C)\big(\varepsilon A\otimes_{\mathcal{B}}(\varepsilon B)^*\big) \quad \text{(monoidality of } \varepsilon\text{)}\\
&= \int^{B\in\mathcal{A}} M(B)\otimes \mathcal{B}(A\otimes_{\mathcal{B}} B^*, C)\\
&= \int^{B\in\mathcal{A}} M(B)\otimes \mathcal{B}(A\otimes_{\mathcal{B}} C^*, B) \quad \big(\text{rigidity, compare [1, bottom p. 2791]}\big)\\
&= \int^{B\in\mathcal{B}} M(B)\otimes \mathcal{B}(A\otimes_{\mathcal{B}} C^*, B) \quad \text{(Lemma 5)}\\
&= M(A\otimes_{\mathcal{B}} C^*) \quad \text{(Lemma 4)}.
\end{aligned}
$$

With these identifications, it is clear that (4) becomes the identity map.

**6**

To summarize this discussion: cohomological Mackey functors are exactly modules over $\mathbf{Cor}_0$; the relations on the tensor product coming from the full transfer structure of Mackey functors are redundant as long as we work with cohomological Mackey functors.

**7**

Here are more minor errata:

- In the second diagram of Section 2.1, the arrows $f^*$ and $f'^*$ should point in the opposite direction.
- In the diagram in Section A.8, the left (resp., right) vertical map should read $\bullet$ (resp., $\bullet_! \circ \boxtimes$).
- Throughout the Appendix, the citation [2, Example 1] should read [2, Exposé 1].

## References


[1] B. KAHN and T. YAMAZAKI, *Voevodsky's motives and Weil reciprocity*, Duke Math. J. **162** (2013), 2751–2796. MR 3127813. DOI 10.1215/00127094-2381379. *(2093, 2095, 2096, 2097)*

[2] S. MAC LANE, *Categories for the Working Mathematician*, 2nd ed., Springer, New York, 1998. MR 1712872. *(2095, 2096)*

[3] V. VOEVODSKY, "Triangulated categories of motives over a field" in *Cycles, Transfers, and Motivic Cohomology Theories*, Ann. of Math. Stud. **143**, Princeton Univ. Press, 2000, 188–238. MR 1764202. *(2094)*

[4] T. YOSHIDA, *On G-functors (II): Hecke operators and G-functors*, J. Math. Soc. Japan **35** (1983), 179–190. MR 0679083. DOI 10.2969/jmsj/03510179. *(2094)*

*Kahn*
CNRS, Institut de Mathématiques de Jussieu–Paris Rive Gauche, UMR 7586, Sorbonne Universités, UPMC Université Paris 06, Université Paris Diderot, Sorbonne Paris Cité, Paris, France; bruno.kahn@imj-prg.fr

*Yamazaki*
Institute of Mathematics, Tohoku University, Aoba, Sendai, Japan; ytakao@math.tohoku.ac.jp