ON "HORIZONTAL" INVARIANTS ATTACHED TO QUADRATIC FORMS

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Abstract. We introduce series of invariants related to the dimension for quadratic forms over a field, study relationships between them and prove a few results about them.

This is the TEX-ing of a manuscript from 1993 entitled *Quadratic forms and simple algebras of exponent two.* The original manuscript contained an appendix that has appeared in [K3]: I removed it and replaced references to it by references to [K3]. I also extended Section 2 somewhat, improved Proposition 3.3 a bit and removed a section that did not look too useful. Finally I changed the title to a better-suited one. These are essentially the only changes to the original manuscript.

The main reasons I have to exhume it are that 1) the notion of dimension modulo I^{n+1} has recently been used very conceptually by Vishik (*e.g.* [Vi], to which the reader is referred for lots of highly nontrivial computations) and 2) Question 1.1 has been answered positively by Parimala and Suresh [P-S]. I have included a proof that is different from theirs (see Corollary 2.1).

Vishik has suggested that the invariant λ which is studied here might be replaced by a finer one: the "geometric length", where one takes transfers into account. Namely, if $x \in I^nF/I^{n+1}F$, its geometric length is the smallest integer ℓ such that there exists an étale F-algebra E of degree ℓ and a Pfister form $y = \langle\langle u_1, \dots, u_n\rangle\rangle \in I^nE/I^{n+1}E$ such that $x = Tr_{E/F}(y)$. (By [TR], the two invariants coincide for $n = 2$.) This invariant should definitely be investigated as well.

Everything here is anterior to Voevodsky's proof of the Milnor conjecture, which is not used. I wish to thank Karim Becher for very helpful comments.

Introduction

Let F be a field of characteristic $\neq 2$. The u-invariant $u(F)$ of F is the least integer n such that any quadratic form in more than n variables over F is isotropic, or $+\infty$ if no such integer exists. (A finer version exists for formally real fields, but for simplicity we shall

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not consider it.) In [Me2] (see also [Ti]), Merkurjev disproved a long-standing conjecture of Kaplansky, asserting that the u-invariant should always be a power of 2. In fact, Merkurjev produced for any integer $m \geq 3$ examples of fields of u-invariant $2m$.

A remarkable feature of Merkurjev's examples is that they can be contrived to have 2-cohomological dimension 2. This destroys a naïve belief that $u(F)$ would be $2^{\nu(F)}$, where $\nu(F)$, the ν-invariant of F, is the largest integer n such that $I^\nu F \neq 0$, which was the case in all previously known examples (see *e.g.* [K1, th. 1]). It hints that a good understanding of the u-invariant involves not only the ν-invariant, but also 'horizontal' invariants attached to the quotients $I^nF/I^{n+1}F$.

Introducing such invariants and starting their study of is the object of this paper. Given a quadratic form, one may approximate its anisotropic dimension, the dimension of its kernel forms, by its 'dimension modulo I^{n+1}' for every $n \geq 1$ (the fact that this actually is an approximation is a consequence of the Arason-Pfister theorem). Given an element of $I^nF/I^{n+1}F$, one may study its 'length' or 'linkage index', the smallest number of classes of Pfister forms necessary to express it. The suprema $u_n(F)$ of the former invariants ('u-invariant modulo I^{n+1}') approximate the u-invariant; the suprema $\lambda^n(F)$ of the latter help giving upper bounds for the former. More precisely, one can bound $u_n(F)$ in terms of $\lambda^n(F)$ and $u_{n-1}(F)$ (Proposition 1.2).

The only case in which I can prove a converse to these bounds is $n = 2$, where $u_2(F) = 2\lambda^2(F) + 2$. However, it is not impossible that all the $\lambda^n(F)$, as well as $u(F)$ when F is not formally real, can actually be bounded in terms of $\lambda^2(F)$ (and n for $\lambda^n(F)$). At least this is the case when $\lambda^2(F) = 1$, by a theorem of Elman-Lam [Lam, th. XI.4.10]. I partially generalize this theorem in one direction (Propositions 3.2 and 3.3), but the general case seems quite open.

In all this paper, we use Lam's [Lam] notations for Pfister forms, *i.e.* $\langle\langle a_1, \dots, a_n\rangle\rangle = \langle 1, a_1\rangle \otimes \cdots \otimes \langle 1, a_n\rangle$. We write $\cong$ (resp. $\sim$) for isometry (resp. Witt-equivalence) of quadratic forms.

1. A few quadratic invariants

As in Arason [A], for a quadratic form q we denote by $\operatorname{diman}(q)$ the rank of the unique anisotropic quadratic form whose class in $W(F)$ equals the class of q (the kernel form of q). As will be seen in Proposition 1.1, Definition 1.1 generalises this definition.

Definition 1.1. a) Let q be a quadratic form over F and J an ideal of $W(F)$. The *J-dimension* of q is $\dim_J(q) = \inf\{\dim(q') \mid q' \equiv q \pmod J\}$.

b) Let $n \geq 0$ be an integer. The *n-dimension* of q is $\dim_n(q) = \dim_{I^{n+1}F}(q)$.[1]

Remark 1.1. $\dim_n(q)$ only depends on the class of q modulo $I^{n+1}F$. For all n, one has $\dim_n(q) \leq \mathrm{diman}(q)$.

Definition 1.2. For J as in Definition 1.1 a), a quadratic form q is *anisotropic modulo* J if $\dim_J(q) = \dim(q)$.

It is clear that for two quadratic forms q, q', one has $\dim_n(q \perp q') \leq \dim_n(q) + \dim_n(q')$. The following lemma strengthens this result when $q' \in I^nF$. Despite its simplicity, it is basic in much of this section.

Lemma 1.1. *Let* $(q, q') \in W(F) \times I^nF$ *with* $q, q' \neq 0$. *Then* $\dim_n(q \perp q') \leq \dim_n(q) + \dim_n(q') - 2$.

Proof. Without loss of generality, we may assume that q and q' are anisotropic modulo $I^{n+1}F$. Since $q' \in I^nF$, $q' \equiv \langle a \rangle q' \pmod{I^{n+1}F}$ for any $a \in F^*$. For suitable a, the form $q \perp \langle a \rangle q'$ is isotropic. Hence

$$\begin{aligned}\dim_n(q \perp q') &= \dim_n(q \perp \langle a \rangle q') \leq \mathrm{diman}(q \perp \langle a \rangle q') \\ &\leq \dim(q) + \dim(q') - 2 = \dim_n(q) + \dim_n(q') - 2.\end{aligned}$$

□

Lemma 1.2. *Let* $q \in W(F)$ *be anisotropic modulo* $I^{n+1}F$. *Then its only subforms belonging to* I^nF *are* 0 *and possibly* q.

Proof. This follows from Lemma 1.1. □

Definition 1.3. Let $n \geq 0$ and $x \in I^nF/I^{n+1}F$. The *length* of x is

$$\lambda(x) = \inf\{r \mid x \text{ is a sum of } r \text{ classes of } n\text{-fold Pfister forms}\}.$$

If $q \in I^nF$, we write $\lambda(q)$ for $\lambda(x)$, where x is the image of q in $I^nF/I^{n+1}F$.

Proposition 1.1. *a)* $\dim_0(q) = \begin{cases} 0 & \textit{if } \dim(q) \textit{ is even} \\ 1 & \textit{if } \dim(q) \textit{ is odd.} \end{cases}$

b) $\dim_1(q) = \begin{cases} 0 & \textit{if } \dim(q) \textit{ is even and } d_\pm(q) = 1 \\ 1 & \textit{if } \dim(q) \textit{ is odd} \\ 2 & \textit{if } \dim(q) \textit{ is even and } d_\pm(q) \neq 1. \end{cases}$

c) If $q \in I^nF - I^{n+1}F$, *then* $2^n \leq \dim_n(q) \leq (2^n - 2)\lambda(q) + 2$.

d) If $q \notin I^nF$, *then* $\dim_n(q) \leq (2^n - 2)\lambda(q') + \dim_{n-1}(q)$ *for some*

[1]One should be careful that Vishik's notation in [Vi, Def. 6.8] is different: our $\dim_n(q)$ is his $\dim_{n+1}(q)$.

$q' \in I^nF$.
e) If $q \in I^2F - I^3F$, then $\dim_2(q) = 2\lambda(q) + 2$.
f) If $2^n \geq \operatorname{diman}(q)$, then $\dim_n(q) = \operatorname{diman}(q)$.

Proof. a) is obvious. Let $\dim(q)$ be odd. Then, for the right choice of $\varepsilon = \pm 1$, $q \perp \langle \varepsilon d_\pm(q)\rangle \in I^2F$, so $\dim_1(q) = 1$. Let $\dim(q)$ be even. Then $q \in I^2F$ if and only if $d_\pm(q) = 1$; if $q \in I^2F$, then $q \perp \langle -1, d_\pm(q)\rangle \in I^2F$, so $\dim_1(q) = 2$. This proves b).

In c), the lower bound for $\dim_n(q)$ is a consequence of the Arason-Pfister theorem [AP]. The upper bounds in c) and d) follow from Lemma 1.1 by induction on $\lambda(q)$ (for d), take $q' = q \perp -q_1$ with $q_1 \equiv q \pmod{I^nF}$ and $\dim q_1 = \dim_{n-1} q$). Let us prove equality in the case $n = 2$ (*cf.* [Me1, lemma]). We may assume $\dim(q) = \dim_2(q) = 2m$. We argue by induction on m. The case $m = 1$ is impossible. Assume $m \geq 2$. We may write $q = \langle a, b, c\rangle \perp q'$. Then q is Witt-equivalent to $\langle a, b, c, abc\rangle \perp (\langle -abc\rangle \perp q')$. The first summand is $\langle a\rangle\langle\langle ab, ac\rangle\rangle \equiv \langle\langle ab, ac\rangle\rangle \pmod{I^3F}$, while the second one $q'' = \langle -abc\rangle \perp q'$ has dimension $2m-2$, so that $\dim_2(q'') \leq 2m - 2$. By induction, $2\lambda(q'') + 2 \leq \dim_2(q'')$, so $2\lambda(q) + 2 \leq 2\lambda(q'') + 4 \leq \dim_2(q'') + 2 \leq \dim_2(q)$.

Note that this argument fails for $n \geq 3$.

Let us prove f). We may assume that q is anisotropic. Assume that $\dim_n(q) < \dim(q)$. Then there exists q' with $q' \equiv q \pmod{I^{n+1}F}$ and $\dim(q') < \dim(q)$. Therefore, $q \perp -q' \in I^{n+1}F$. But $\dim(q \perp -q') < 2\dim(q) \leq 2^{n+1}$: therefore, by the Arason-Pfister theorem [AP], $q \perp -q'$ is hyperbolic. This means that q is Witt-equivalent to q': this is impossible, since q is anisotropic and $\dim(q') < \dim(q)$. □

Definition 1.4. Let $n \geq 1$ and $k \leq n$. The *k-restricted u-invariant modulo I^{n+1} of F* is $u_n^k(F) = \sup\{\dim_n(q) \mid q \in I^k(F)\}$. If $k = 0$, we write $u_n(F)$ for $u_n^k(F)$ and call it the *u-invariant of F modulo I^{n+1}*.

Remark 1.2. When k is fixed, the $u_n^k(F)$'s form a non-decreasing sequence for increasing n. In particular the $u_n(F)$ form a non-decreasing sequence. Similarly, when n is fixed, the $u_n^k(F)$ form a non-increasing sequence for increasing k.

Definition 1.5. Let $n \geq 1$. The *n-th λ-invariant* of F is $\lambda^n(F) = \sup\{\lambda(x) \mid x \in I^nF/I^{n+1}F\}$.

Proposition 1.2. *a) For $k > 0$ and $n \geq k$, $I^kF \neq 0 \iff u_n^k(F) \neq 0 \iff u_n^k(F) \geq 2^k$. If $I^nF \neq 0$, $u_n^k(F) \geq 2^n$. In particular, if $I^nF \neq 0$, $u_n(F) \geq 2^n$.*
b) For $k \geq 0$, $\sup_{n\geq k} u_n^k(F) = u^k(F) := \sup\{\operatorname{diman}(q) \mid q \in I^k(F)\}$; in particular, $\sup_{n\geq 0} u_n(F) = u(F)$.

c) $u_n^n(F) \leq (2^n - 2)\lambda^n(F) + 2$.
d) For any $k < n$, $u_n^k(F) \leq (2^n - 2)\lambda^n(F) + u_{n-1}^k(F)$ *(if* $k = 0$ *we assume* $n \geq 2$*).*
e) $u_0(F) = 1$.
f) If $IF \neq 0$, $u_1(F) = u_1^1(F) = 2$.
g) If $I^2F \neq 0$*, then* $u_2(F) = u_2^2(F) = 2\lambda^2(F) + 2$.

(I am indebted to O. Gabber for pointing out d).)

Proof. It is clear that $I^kF = 0 \Rightarrow u_n^k(F) = 0$ for all $n \geq k$. By Remark 1.2, $u_n^k(F) \geq u_k^k(F)$ when $n \geq k$. Assume $I^kF = I^{k+1}F$. Then any k-fold Pfister form belongs to $I^{k+1}F$. By [AP], such a form must be hyperbolic, hence $I^kF = 0$. This shows that if $I^kF \neq 0$, there exists a form $q \in I^kF - I^{k+1}F$. By Proposition 1.1 c), we have $\dim_k(q) \geq 2^k$, hence $u_k^k(F) \geq 2^k$ and $u_n^k(F) \geq 2^k$. The last two claims of a) follow by Remark 1.2 ($u_n(F) \geq u_n^k(F) \geq u_n^n(F)$ when $k \leq n$). This proves a).

To prove b), first assume that $u^k(F)$ is finite. Let n be such that $2^n \geq u^k(F)$. By Proposition 1.1 f), $\dim_n(q) = \operatorname{diman}(q)$ for any $q \in I^kF$. In particular, $u_n^k(F) = u^k(F)$ for all such n. Assume now that the sequence $(u_n^k(F))_{n \geq k}$ is bounded, say by N. Let $q \in I^kF$ and choose n such that $2^n \geq \dim(q)$. Applying Proposition 1.1 f) again, we have $\dim_n(q) = \operatorname{diman}(q)$. This shows that $\operatorname{diman}(q) \leq N$, hence that $u^k(F) \leq N$.

Part c) is an immediate consequence of Proposition 1.1 c). To prove d), we may assume that $IF \neq 0$, otherwise it is trivial. We may further assume that $I^kF \neq 0$. Let $q \in I^kF$. We distinguish two cases:

(i) $q \notin I^nF$. By Proposition 1.1 d), $\dim_n(q) \leq (2^n - 2)\lambda^n(F) + \dim_{n-1}(q) \leq (2^n - 2)\lambda^n(F) + u_{n-1}^k(F)$.

(ii) $q \in I^nF$. By Proposition 1.1 c), $\dim_n(q) \leq (2^n - 2)\lambda^n(F) + 2$. This is $\leq (2^n - 2)\lambda^n(F) + u_{n-1}^k(F)$ provided $u_{n-1}^k(F) \geq 2$. If $k \geq 1, u_{n-1}^k(F) \geq 2^k \geq 2$ by a). If $k = 0$, $u_{n-1}(F) \geq u_1(F) \geq 2$ since $IF \neq 0$ (see f)).

e) and f) follow from Proposition 1.1 a) and b). It remains to prove g). First we prove that $u_2(F)$ cannot be odd, i.e. $u_2(F) = u_2^1(F)$. This is a consequence of Proposition 1.1 e), Proposition 1.2 e) and the following lemma.

Lemma 1.3. *a)* $u_2^2(F) = \infty \iff u_2(F) = \infty \iff \lambda^2(F) = \infty$.
b) Assume that $u_2(F) < \infty$ *and* $IF \neq 0$*. Then* $u_2(F) = u_2^1(F)$.

Proof. a) is a consequence of Proposition 1.1, b) d) and e). For b), let q be such that $\dim(q) = \dim_2(q) = u_2(F)$. We show that $\dim(q)$ cannot be odd, unless $IF = 0$. If it is, then as in the proof of Proposition 1.1 b) we choose $\varepsilon = \pm 1$ such that $q' = q \perp \langle \varepsilon d_\pm(q) \rangle \in I^2F$. By assumption,

$\dim_2(q') \leq \dim(q)$, so that $q' \equiv q''$ (mod I^3F), where $q'' \in I^2F$ is such that $\dim(q'') \leq \dim(q)$. Then $q \equiv q'' \perp \langle -\varepsilon d_\pm(q)\rangle$ (mod I^3F). If $q'' = 0$, then $u_2(F) = 1$, which implies $IF = 0$ (Proposition 1.2, b) and remark 1.2). Otherwise, by Lemma 1.1 we have $\dim_2(q) \leq \dim(q'') - 1 < \dim(q)$, a contradiction. □

We now prove that $u_2^1(F) = u_2^2(F)$. By lemma 1.3 we may assume that $u_2(F) < \infty$. Let $q \in IF$ be such that $\dim_2(q) = u_2^1(F)$. We may assume that q is anisotropic modulo I^3F. Then $q' = q \perp \langle -1, d_\pm(q)\rangle \in I^2F$, with $\dim(q') = \dim(q) + 2$. Since $\dim(q) = u_2^1(F) = u_2(F)$, we have $q \perp \langle -1\rangle \equiv q''$ (mod I^3F), where q'' is such that $\dim(q'') \leq \dim(q)$. But $\dim(q'') \equiv \dim(q \perp \langle -1\rangle)$ (mod 2), so that $\dim(q'')\langle\dim(q)$. Therefore, $q' \equiv q'''$ (mod I^3F), with $\dim(q''') < \dim(q) = \dim_2(q)$, and $q \equiv q''' \perp \langle 1, -d_\pm(q)\rangle$ (mod I^3F). If $q''' = 0$, $u_2^1(F) = 2$, but then $I^2F = 0$. Otherwise, since $q''' \in I^2F$, $\dim_2(q) = \dim_2(q''' \perp \langle 1, -d_\pm(q)\rangle) \leq \dim_2(q''')$ by Lemma 1.1. Hence $\dim_2(q''') = \dim_2(q)$ and $u_2^1(F) = u_2^2(F)$. □

Remark 1.3. The proof of g) is borrowed from [Lam, ch XI, proof of lemma 4.9].

Remark 1.4. The statement $u_n(F) = u_n^1(F)$ is equivalent to "$u_n(F)$ is even".

Remark 1.5. These proofs prompt the definition of quadratic forms universal modulo I^{n+1}, round modulo I^{n+1}. This is left to the reader.

Corollary 1.1 (cf [Lam, ch. XI, lemma 4.9])**.** *If $u_2(F) > 1$, it is even.*

Example 1.1. For all $n \geq 0$, $u_n(\mathbf{R}) = 2^n$.

Question 1.1. For $n > 2$, can one bound $\lambda^n(F)$ in terms of n and $u_n(F)$? More precisely, let $q \in I^nF$. Can one bound $\lambda(q)$ purely in terms of $\dim_n q$?

For $n = 3$, one would like to prove this by using the following generic argument. Let k be a base field, m a fixed integer, $F_0 = k(T_1, \dots, T_{2m})$, Q the quadratic form $\langle T_1, \dots, T_{2m}\rangle$ over F_0, $F_1 = F_0(\sqrt{(-1)^m T_1 \dots T_{2m}})$ and F_2 the function field of the Severi-Brauer variety of the Clifford algebra of Q_{F_1}. Then, by Merkurjev's theorem [Me] $Q_{F_2} \in I^3F_2$ and is a 'generic element of rank $2m$ in I^3'. Show that, for any field F containing k and any $q \in I^3F$ of rank $2m$, one has $\lambda(q) \leq \lambda(Q_{F_2})$.

This has been achieved by Parimala and Suresh [P-S] with a general position argument: we shall give a different argument avoiding general position in the next section (see Cor. 2.1).

Question 1.2. By Elman-Lam [EL2], if F is not formally real and $\lambda^2(F) = 1$ then $u(F) = 1, 2, 4$ or 8. Is it true that, if $\lambda^2(F) = 2$, one has $u(F) < \infty$?

In the next section, we give some evidence that the answer to this question might be positive.

2. Discrete valuations; iterated power series

Let A be a complete discrete (rank 1) valuation ring, E its quotient field anf F its residue field. We assume that $\operatorname{char} F \neq 2$.

Proposition 2.1. *a) For all $n \geq 1$, $\lambda^n(E) \leq \lambda^n(F) + \lambda^{n-1}(F)$.*
b) For all $n \geq 1$ and $0 \leq k \leq n$, $u_n^k(E) \leq u_n^{k-1}(F) + u_{n-1}^{k-1}(F)$.

Note. In b), one should interpret $u_n^{-1}(F)$ as $u_n(F)$.
Proof. Let π be a prime element of E. Every quadratic form q over E can be written $q \cong q_1 \perp \pi q_2$, where q_1 and q_2 are classes of unimodular forms over A. Alternatively, q can be written up to Witt equivalence $q_1' \perp \langle\langle \pi \rangle\rangle q_2$, still with q_1' integral. If $q \in I^n E$, then $q_1' \in I^n A$ and $q_2 \in I^{n-1} A$ [S]. Since $W(A) \xrightarrow{\sim} W(F)$ is a filtered isomorphism, this proves a).
To see b), let $q \in I^n E$ and q_1', q_2 as above. Let $\bar{q}_2$ be the residue image of q_2 over F. Take $\bar{q}_2' \equiv \bar{q}_2 \pmod{I^n F}$ with $\bar{q}_2' \in I^{k-1} F$ and $\dim(\bar{q}_2') \leq u_{n-1}^{k-1}(F)$. Let q_2' be a lift of $\bar{q}_2'$ to A. Then $q_2' \equiv q_2 \pmod{I^n A}$ and $q \equiv q_1' \perp \langle\langle \pi \rangle\rangle q_2' = q_1' \perp -q_2' \perp \langle \pi \rangle q_2' \pmod{I^{n+1} F}$. Now let $q_1'' = q_1' \perp -q_2' \in I^{k-1} A$; choose $q_1''' \in I^{k-1} A$ such that $\bar{q}_1''' \equiv \bar{q}_1'' \pmod{I^{n+1} F}$ and $\dim(\bar{q}_1''') \leq u_n^{k-1}(F)$. Then $q_1''' \equiv q_1'' \pmod{I^{n+1} A}$, $q \equiv q_1''' \perp \langle \pi \rangle q_2' \pmod{I^{n+1} F}$ and $\dim(q_1''' \perp \langle \pi \rangle q_2') \leq u_n^{k-1}(F) + u_{n-1}^{k-1}(F)$. □

An example. Start from a field K and set $K_d = K((t_1)) \dots ((t_d))$, a field of iterated power series. Then K_d is complete for a discrete valuation, with residue field K_{d-1}. Proposition 2.1 allows one to get upper bounds for the invariants of K_d in terms of d and those of K, by induction on d.

However, these inductive bounds are by no means sharp in general. Computing, or at least estimating $\lambda^n(K_d)$ and $u_n(K_d)$ turns out to be "global" in d. To illustrate this, we now consider the case where K is algebraically closed.

Definition 2.1. Let k be a field, V a d-dimensional k-vector space and n an integer $\leq d$. Let $\Lambda^n(V)$ be the n-th exterior power of V and let

$x \in \Lambda^n(V)$ be an n-vector. The *length* of x is the smallest integer $\ell(x)$ such that x is the sum of $\ell(x)$ pure n-vectors. We denote by $N(k,d,n)$ the supremum of $\ell(x)$ when x runs through V (this is independent of V).

The following proposition is clear.

Proposition 2.2. *a) Let $V = K_d^*/K_d^{*2}$, viewed as an $\mathbf{F}_2$-vector space with basis $t_1,\dots,t_d$. Then there are canonical isomorphisms*

$$I^nK_d/I^{n+1}K_d \simeq \Lambda^n(V)$$

mapping Pfister forms to pure n-vectors.
b) For $x \in I^nK_d/I^{n+1}K_d$, with image x' in $\Lambda^n(V)$, $\lambda(x) = \ell(x')$, where $\lambda(x)$ is as in Definition 1.3 and $\ell(x')$ is as in Definition 2.1.
c) $\lambda^n(K_d) = N(\mathbf{F}_2, d, n)$. □

The following information on $N(k,d,n)$ is collected from [K2].

Proposition 2.3. *a) $N(k,d,n) = N(k,d,d-n)$.*
b) $N(k,d,0) = N(k,d,1) = 1$.
c) $N(k,d,2) = [d/2]$.
d) $N(k,6,3) = 3$ for any field k.
e) If k is algebraically closed, $N(k,7,3) \leq 4$ (probably $= 4$); $N(\mathbf{R},7,3) = 5$; for any field k, $N(k,7,3) \leq 6$ (probably ≤ 5).
f) If k is algebraically closed of characteristic 0, $N(k,8,3) = 5$; $N(\mathbf{R},8,3) \leq 8$; for any k, $N(k,8,3) \leq 10$.
g) If k is algebraically closed of characteristic 0, $N(k,9,3) \leq 9$.
h) There exists a polynomial f_n of degree $\leq n-2$ such that, for any field k, $N(k,d,n) \leq \dfrac{d^{n-1}}{2(n-1)!} + f_n(d)$. In particular, for any k, $\limsup_{d\to\infty} \dfrac{N(k,d,n)}{d^{n-1}} \leq \dfrac{1}{2(n-1)!}$.
i) If k is infinite, $N(k,d,n) \geq \dfrac{\binom{d}{n}}{n(d-n)+1}$. If k is finite with q elements, $N(k,d,n) \geq \dfrac{\binom{d}{n}}{n(d-n)+1+\varepsilon(q)}$, where

$$\varepsilon(q) = \log_q\Big(\prod_{i=2}^{\infty}(1-q^{-i}) - 1\Big).$$

(So $\varepsilon(2) \approx 0.75$.) In particular, for any field k, $\liminf_{d\to\infty} \dfrac{N(k,d,n)}{d^{n-1}} \geq \dfrac{1}{n.n!}$.

This proposition enables us to list values of $\lambda^n(K_d)$ for $d \leq 6$:

$d\backslash n$	0	1	2	3	4	5	6
1	1	1					
2	1	1	1				
3	1	1	1	1			
4	1	1	2	1	1		
5	1	1	2	2	1	1	
6	1	1	3	3	3	1	1

From this table, we see that the inequality

$$u(K_d) \leq \sum_{n=2}^{d} (2^n - 2)\lambda^n(K_d) + 2$$

from Proposition 1.2 d) quickly becomes completely inaccurate.

More generally, let us consider a field F provided with a discrete valuation v of rank d. Let K be the residue field of v, L the value group of v and $\hat{F}$ the completion of F at v. Assume that $\operatorname{char} K \neq 2$. Then we have

$$\hat{F} \simeq K((t_1)) \dots ((t_d))$$

where $t_1, \dots, t_d \in \hat{F}$ are elements such that $v(t_1), \dots, v(t_d)$ form a basis of L.

Let $\bar{K}$ be an algebraic closure of K and $\tilde{F} = \bar{K}((t_1)) \dots ((t_d))$. By weak approximation, the map $F^*/F^{*2} \to \hat{F}^*/\hat{F}^{*2}$ is surjective, hence so is the composition

$$I^n F/I^{n+1} F \to I^n \hat{F}/I^{n+1} \hat{F} \to I^n \tilde{F}/I^{n+1} \tilde{F}$$

for any $n \geq 1$. It follows that $N(\mathbf{F}_2, d, n)$ is a *lower bound* to $\lambda^n(F)$. I don't know if it is even true that

$$\lambda^n(F) \geq \lambda^n(F_0) + N(\mathbf{F}_2, d, n)?$$

We also have:

Proposition 2.4. *Let μ denote the place from F to F_0 associated to v. Let $Q \in I^n F$ have good reduction at μ. Then $\lambda(\mu_* Q) \leq \lambda(Q)$.*

Proof. Since v is a composition of discrete valuations of rank 1, we may reduce to v of rank 1. Pick a prime element π. Then we have a ring homomorphism (*e.g.* see [M-H, Ch. IV, §1])

$$\begin{aligned} \Delta_{v,\pi} : W(F) &\to W(F_0) \\ \langle \pi^r u \rangle &\mapsto \langle \bar{u} \rangle \end{aligned}$$

where u is a unit and $\bar{u}$ is its image in F_0^*. Clearly $\Delta_{v,\pi}$ preserves n-fold Pfister forms, hence $\lambda(\Delta_{v,\pi}(Q)) \leq \lambda(Q)$ for any $Q \in I^n F$, and the claim is a special case. □

Corollary 2.1. *The answer to Question 1.1 is positive for $n = 3$.*

Indeed, the strategy outlined just after the question works as follows: given $q = \langle a_1, \dots, a_{2m} \rangle \in I^3 F$, consider the place μ_0 from F_0 to F sending T_i to a_i. It is well-known that F_2 is a generic splitting field for the Clifford algebra of Q_{F_1}; hence μ_0 extends to a place μ from F_2 to F sending Q_{F_2} to q. Since the valuation associated to μ_0 is discrete, so is the one associated to μ, and we may apply Proposition 2.4.

If any element of $I^n F / I^{n+1} F$ has a generic splitting field also for $n > 3$, then the above argument applies verbatim to answer Question 1.1 positively.

Question 2.1. Does Proposition 2.4 remain true when the valuation v is not discrete?

3. Relationships between the u-invariant, the ν-invariant and the λ-invariants

Lemma 3.1 ([EL1, th. 4.5])**.** *Let φ be an m-fold Pfister form and ψ be an n-fold Pfister form. Assume that φ and ψ are r-linked but not $(r+1)$-linked. Then, for any $a, b \in F^*$, $i(\langle a \rangle \varphi \perp \langle b \rangle \psi) = 2^r$. Here i denotes the Witt index.* □

Proposition 3.1. *Assume that $u(F) \leq 2^n$. Then $\lambda^n(F) \leq 1$.*

Proof. Let φ and ψ be two n-fold Pfister forms. Then ψ is universal, so $\psi \equiv -\psi$, and $i(\varphi \perp \psi) = i(\varphi \perp -\psi) \geq 2^{n-1}$. By Lemma 3.1, φ and ψ are $(n-1)$-linked, so $\varphi \perp \psi$ is isometric to an n-fold Pfister form. □

Using a result of Bloch and independently Kato, we can deduce a nice corollary to this proposition, generalising a well-kown result for global fields:

Corollary 3.1. *Let F be a function field in n variables over an algebraically closed field, or in $n-1$ variables over a finite field. Then every element of $H^n(F, \mathbf{Z}/2)$ is a symbol.*[2]

[2] As Karim Becher pointed out, thanks to Voevodsky's theorem it is now sufficient to assume that F is C_n.

Proof. A theorem of Kato [Ka, p. 609, prop. 3] (see also Bloch's argument in [B, Lecture 5]) shows that, for a field as in the statement, $H^n(F, \mathbf{Z}/2)$ is generated by symbols. It is then sufficient to prove that every element of $K_n^M(F)/2$ is a symbol. Notice that F is C_n in the sense of Lang [G], hence $u(F) \leq 2^n$. In view of Proposition 3.1, it is then sufficient to have:

Lemma 3.2 ([EL1, th. 6.1])**.** *Let F be a field, $n \geq 1$ and $x, y, z \in K_n^M(F)/2$ be three symbols. Assume that $\nu_n(x) + \nu_n(y) = \nu_n(z)$. Then $x + y = z$.* □

In this lemma, $\nu_n : K_n^M(F)/2 \to I^nF/I^{n+1}F$ is the homomorphism defined in [Mi].

The following is no more than [EL2, lemma 2.3 and cor. 2.5].

Proposition 3.2. *Assume that $\lambda^n(F) \leq 1$. Then $\lambda^{n+1}(F) \leq 1$. If furthermore F is not formally real, then $I^{n+2}F = 0$.* □

The next proposition is the main result of this section.

Proposition 3.3. *Assume that $\lambda^n(F) = m < \infty$ and that F is not formally real. Then $I^{n(m+1)+1}F = 0$. If -1 is a square in F, then $I^{n(m+1)}F = 0$.*

In other words, if F is not formally real then $\nu(F) \leq n(\lambda^n(F) + 1)$ for all n. For $n = 2$, the right hand side is $u_2(F)$ by Proposition 1.2 d). When -1 is a square in F, this bound is improved to $\nu(F) \leq n(\lambda^n(F) + 1) - 1$.

Proof. By Milnor [Mi], there are surjective homomorphisms $\nu_n : K_n^M(F)/2 \to I^nF/I^{n+1}F$, and ν_2 is an isomorphism. Let $K_*'^M(F) = K_*^M(F)/\{-1\}K_*(F)$: as explained in [K3, Appendix], the commutative ring $K_*'^M(F)/2$ enjoys graded divided power operations $x \mapsto x^{[i]}$ which vanish on symbols: by [the argument of the proof of] [K3, Prop. 1 (8)], $K'^M_{n(m+1)}(F)/2 = 0$, hence every element of $I^{n(m+1)}F/I^{n(m+1)+1}F$ is a multiple of (the 1-fold Pfister form) $\langle\langle 1 \rangle\rangle$. So

If -1 is a square in F, then $\langle\langle 1 \rangle\rangle$ is hyperbolic and $I^{n(m+1)}F/I^{n(m+1)+1}F = 0$; using the Arason-Pfister theorem, we deduce that $I^{n(m+1)}F = 0$. Assume now that -1 is not a square in F; let $E = F(\sqrt{-1})$. By [BT, Cor. 5.3], $K^M_{n(m+1)+1}(E)$ is generated by symbols $\{a_1, \ldots, a_{n(m+1)+1}\}$, with $a_1, \ldots, a_{n(m+1)}$ in F^*. Since every element of $K^M_{n(m+1)}(F)/2$ is a multiple of $\{-1\}$, every element in $K^M_{n(m+1)+1}(E)/2$ is a multiple of $\{-1\} = 0$, *i.e.* $K^M_{n(m+1)+1}(E)/2 = 0$, hence $I^{n(m+1)+1}E/I^{n(m+1)+2}E = 0$ and $I^{n(m+1)+1}E = 0$. If now F is not formally real, [A, Satz 3.6 (ii)] implies that $I^{n(m+1)+1}F = 0$. □

Remark 3.1. For $n = 2$ Prop. 3.3 is optimal, at least when -1 is a square in F. For example, let $F = \mathbf{C}((t_1))\dots((t_d))$ be the field of iterated formal power series in d variables over $\mathbf{C}$. Then $\nu(F) = d$ and, by Section 2, $\lambda^2(F) = [d/2]$ which is also the least integer greater than $\frac{d-1}{2}$.

Question 3.1. Is there a universal bound for $\lambda^n(F)$ in terms of n and $\lambda^2(F)$? In view of Propositions 1.2 and 3.3, this would provide a negative answer to question 1.2. For example, it seems plausible that if $\lambda^n(F) = m$, then $\lambda^{nm}(F) \le 1$. This is true in the test example of Remark 3.1. If one could prove it in general, then the estimate $\nu(F) \le n(\lambda^n(F) + 1)$ or $\nu(F) \le n(\lambda^n(F) + 1) - 1$ of Proposition 3.3 would be improved to $\nu(F) \le n\lambda^n(F) + 1$ thanks to Proposition 3.2 (note that this is no improvement if $n = 2$ and -1 is a square in F).

It is clear that divided power operations have not been used up to their full potential.

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