DIVIDED POWERS ON ABELIAN VARIETIES

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Abstract. We prove the existence of divided powers in étale Chow groups of abelian varieties over a separably closed field, and hence of an integral lift of the Fourier transform, away from the characteristic and up to 2-torsion. The method is to lift the Deninger-Murre Chow-Künneth projectors to integral ones, and draw consequences. Several techniques used here are new.

Contents

1. Introduction

Let A be an abelian variety of dimension g over a field k. In [14, Rem. 4.1], Hélène Esnault gave an example, with $g = 2$, where there exists a divisor class $L \in CH^1(A)$ such that L^2 is not divisible by 2 in the Chow group $CH^2(A)$ and even in its étale version $CH^2_{\text{ét}}(A)$, where

$$CH^i_{\text{ét}}(A) := H^{2i}_{\text{ét}}(A, \mathbf{Z}(i))$$

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(see [23] for étale motivic cohomology). This is significant because, for any value of g, the degree of L^g is divisible by $g!$ [31, §16]. We recall her argument in §4 and give a slightly stronger statement, where $CH^2_{\text{ét}}(A)$ is replaced by $H^4_{\text{cont}}(A, \mathbf{Z}_2(2))$ (continuous étale cohomology).

In the same negative direction, Philip Engel, Olivier de Gaay Fortman and Stefan Schreieder have very recently proven in [12, Th. 1.1] that, for $k = \mathbf{C}$, $g \geq 4$ and A very general, principally polarised by the divisor Θ, the cohomology class of any cycle of codimension $c \in [2, g-1]$ is an even multiple of $\operatorname{cl}(\Theta)^c/c!$, hence a fortiori that Θ^c is not divisible by $c!$ in $CH^c(A)$. In [13], the multiple 2 is replaced by 6 for $g = 6$.

Suppose k separably closed. Then $CH^2(A)_{\deg=0}$ is divisible and, moreover, $CH^2(A) \to CH^2_{\text{ét}}(A)$ is an isomorphism thanks to the exact sequence

$$0 \to CH^2(A) \to CH^2_{\text{ét}}(A) \to H^0(A, \mathcal{H}^3_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(2))) \to 0$$

of [23, Prop. 2.9] and the fact that $H^0(A, \mathcal{H}^3_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(2))) = 0$ since $\operatorname{cd}(k(A)) = 2$. Thus Esnault's example cannot exist in this case. Similarly, the following proposition shows that the counterexample of [12] disappears when we replace Chow groups by étale Chow groups.

Proposition 1.1. *Let $x \in CH^i_{\text{ét}}(A)$. Then x^r is divisible by $(r!)_p$ in $CH^{ir}_{\text{ét}}(A)$ for any $r \geq 1$, where p is the exponential characteristic of k and $(n)_p$ denotes the largest divisor of an integer n which is prime to p.*

The proof, given in §3, is almost trivial in view of a result of [23]; I feel sheepish not to have noticed it at the time. (In fact, [21, Th. 1.22 and (1.8)] is already sufficient.)

In Proposition 1.1, we could say that $CH^*_{\text{ét}}(A)$ has "weak" divided powers; note that in view of torsion in this ring, they are by no means uniquely determined. This raises the issue of strong divided powers, i.e. the existence of functions $\gamma_r : CH^i_{\text{ét}}(A) \to CH^{ir}_{\text{ét}}(A)$ such that, identically

(1) $\gamma_0(x) = 1$, $\gamma_1(x) = x$;
(2) $\gamma_n(\lambda x) = \lambda^n \gamma_n(x)$ for any λ;
(3) $\gamma_n(x + y) = \sum_{r+s=n} \gamma_r(x)\gamma_s(y)$:
(4) $\gamma_m(x)\gamma_n(x) = \binom{m+n}{m}\gamma_{m+n}(x)$;
(5) $\gamma_n(\gamma_m(x)) = \frac{(mn)!}{(m!)^n n!}\gamma_{mn}(x)$.

It turns out that the solution to this problem is essentially positive — except that, besides the exponential characteristic, the prime 2 comes to haunt us: one *cannot* refine Proposition 1.1 to obtain such divided powers, see Remark 1.5 below. However, the 2-torsion subgroup ${}_2CH^*_{\text{ét}}(A)[1/p]$ is an ideal in the ring $CH^*_{\text{ét}}(A)[1/p]$; write $\overline{\mathrm{CH}}^*_{\text{ét}}(A)$ for the quotient.[1] Then:

Theorem 1.2. *a) There exists a canonical divided power structure on the ideal $\overline{\mathrm{CH}}^{>0}_{\text{ét}}(A)$ of $\overline{\mathrm{CH}}^*_{\text{ét}}(A)$.*

[1] Since the torsion subgroup of $CH^*_{\text{ét}}(A)[1/p]$ is divisible (Proposition 2.6 c)), $CH^*_{\text{ét}}(A)[1/p]$ and $\overline{\mathrm{CH}}^*_{\text{ét}}(A)$ are isomorphic as groups but probably not as rings.

b) If $f : A \to B$ is a homomorphism of abelian varieties, then $\gamma_n(f^*x) = f^*\gamma_n(x)$ for any $x \in \overline{\mathrm{CH}}^{>0}_{\text{ét}}(B)$.

A consequence of Theorem 1.2 is that the Chern character of any line bundle over A admits a canonical lift to $\overline{\mathrm{CH}}^*_{\text{ét}}(A)$. (Here we use the fact that $CH^*(A)\otimes \mathbf{Q} \xrightarrow{\sim} CH^*_{\text{ét}}(A)\otimes \mathbf{Q}$ [23, Th. 2.6 c)].) As a special case, we get part of the following corollary.

Corollary 1.3. *The Fourier transform of Beauville [4] admits a canonical lift $\mathcal{F}_A$ to $\overline{\mathrm{CH}}^*_{\text{ét}}(A)$ verifying the identities of loc. cit., Prop. 3 (i), (ii) and (iii): inversion formula, exchange of intersection product and Pontryagin product and commutation with isogenies.*

In §12, we show that several identities of Beauville in [4] have integral lifts, and use this to give an integral lift of Scholl's formula in [39] for the Chow-Künneth projectors of Deninger-Murre when A is principally polarised, and of Suh's formula in [41] for Jacobians of curves (see below for more on the Deninger-Murre projectors).

Corollary 1.4. *The augmentation ideal of $\overline{\mathrm{CH}}^{\text{ét}}_*(A)$ provided with the Pontryagin product admits a canonical divided power structure.*

Remark 1.5. Corollary 1.3 is optimal: by [29, Th. 3.11], one cannot lift its Fourier transform to $CH^*_{\text{ét}}(A)[1/p]$ in general, already when A is an elliptic curve (note that Chow groups and étale Chow groups agree in this case).

We now come to Chow-Künneth projectors. Here we don't need to use $\overline{\mathrm{CH}}^*_{\text{ét}}(A)$. For convenience, here is a definition.

Definition 1.6. Let $\mathcal{A} \to S$ be an abelian scheme of relative dimension g, where S is a connected smooth scheme over a field of exponential characteristic p. Orthogonal projectors with sum 1 $(\pi^i)_{0\le i\le 2g}$ in $CH^g(\mathcal{A}\times_S \mathcal{A})$ (resp. in $CH^g_{\text{ét}}(\mathcal{A}\times_S\mathcal{A})[1/p]$) are called *integral* (resp. *étale integral*) Deninger-Murre projectors (DM projectors, for short) if

- For any i and any prime $l \neq p$, $\mathrm{cl}_l(\pi^i)$ is the i-th Künneth projector of $H^*(\mathcal{A}_{\bar\eta}, \mathbf{Q}_l)$, where $\mathcal{A}_{\bar\eta}$ is the geometric generic fibre of $\mathcal{A}$; here cl_l is the l-adic cycle class map (resp. the étale l-adic cycle class map of [23, §3A]).
- For any i and any integer n, $n^*_{\mathcal{A}}\circ\pi^i = n^i\pi^i$, where n_A is multiplication by n on $\mathcal{A}$.

Theorem 1.7. *a) If A is an abelian k-variety with k separably closed, the rational DM projectors of [10] lift to a system of étale integral DM projectors (π^i_A). This set is unique up to conjugation by an element of the form $1+x$ with $2x = 0$.*
b) If B is another abelian k-variety, then $2\pi^i_A \circ f^* = 2f^*\circ\pi^i_B$ in $CH^g_{\text{ét}}(A\times B)[1/p]$ for any homomorphism $f : A \to B$. Here $g = \dim A$.
c) If A is principally polarised in a), we may choose (π^i_A) self-conjugate, *i.e.* ${}^t\pi^i_A = \pi^{2g-i}_A$.

(In b), the factor 2 is rather unsubstantial: among the correspondences $\pi_A^j \circ f^* \circ \pi_B^i$ for $i \neq j$, the only possibly nonzero ones are for $j = i - 1$, and those are 2-torsion.)

The proof of Theorem 1.7 is in two steps. First a weaker statement up to multiplication by 2, Theorem 7.1. Then we get rid of this factor 2 by a deformation argument which rests on the (non trivial!) special case of elliptic curves: Theorem 10.8 and Proposition 10.9. The reader may consult Theorem 11.11 for a more complete and cleaner statement in $\overline{\mathrm{CH}}^*_{\text{ét}}(A)$.

The following corollary was my initial motivation for this work.

Corollary 1.8. *Let* **Ab** *be the category of abelian k-varieties, with morphisms the homomorphisms of abelian varieties. Let $\mathcal{M}$ be the category of effective Chow motives with integral coefficients* [39]. *Then the additive functor of* [39, Cor. 5.2]

$$h^1 \otimes \mathbf{Q} : \mathbf{Ab} \otimes \mathbf{Q} \to \mathcal{M} \otimes \mathbf{Q}$$

induced by the first Chow-Künneth projectors of [10] *lifts to an additive, fully faithful functor*

$$h^1_{\text{ét}} : \mathbf{Ab}[1/p] \to \mathcal{M}_{\text{ét}},$$

where $\mathcal{M}_{\text{ét}}$ is the category of effective étale Chow motives (see §2).

I doubt that this functor lifts to a functor $\mathbf{Ab} \to \mathcal{M}$, although I don't have any counterexample.

Remark 1.9. The proof of the full faithfulness of $h^1_{\text{ét}}$ rests on an adjunction statement closely related to the functors LAlb and RPic of [3]: Theorem A.10. This recovers the full faithfulness of $h^1 \otimes \mathbf{Q}$ [39, Cor. 5.10][2] with a completely different proof, which does not involve a hard Lefschetz isomorphism theorem.

While in [10] the existence of the Chow-Künneth projectors is deduced from that of the Fourier transform, here we proceed in the opposite way, proving first Theorem 1.7, deducing Theorem 1.2 and finally Corollary 1.3. Contrary to my initial expectation, Proposition 1.1 is not used.

Abelian schemes. In §9, we extend the above story to abelian schemes over a smooth base S, up to losing some torsion. It yields the following

Theorem 1.10. *Let S be a smooth scheme over a field k, and let $\mathcal{A}$ be an abelian S-scheme of relative dimension g. Then all the above results hold in $CH^*_{\text{ét}}(\mathcal{A})[1/Mp]$ and $CH^g_{\text{ét}}(\mathcal{A} \times_S \mathcal{A})[1/Mp]$, where M is the product of the primes l such that*

$$l - 1 \mid \inf(\mathrm{cd}(S) + 1, 2g - 1)^{\&}. \tag{1.1}$$

Here we use the notation (for want of an established one)

$$r^{\&} = \mathrm{lcm}\{a \mid 1 \leq a \leq r\}.$$

[2] In the proof of loc. cit., the 4th line of the computation should be dropped.

The proof, given in § 11.5, relies on a finer result, Theorem 9.2. Here $\mathrm{cd}(S)$ is the étale cohomological dimension of S; recall that $\mathrm{cd}(S) \le \mathrm{cd}(k) + 2\dim S$, and even $\mathrm{cd}(S) \le \mathrm{cd}(k) + \dim S$ if S is affine.

A similar bound was obtained in [18, Th. 4.5 and 5.5] for ordinary Chow groups, where M is replaced by $(2g + \dim S)!$. The proofs are completely different: those of [18] rest essentially on Pappas' integral refinement of the Riemann-Roch theorem [33], while the present proofs rest on the excellent control one has on the torsion subgroup of $CH^*_{\text{ét}}(\mathcal{A})[1/p]$, see §§2 and 5.

Algebraic and étale algebraic cohomology classes. We come back to the case of a separably closed base field k. Call an l-adic cohomology class of a smooth k-variety X *algebraic* (resp. *étale algebraic*) if it is in the image of the l-adic cycle class map (resp. of the étale l-adic cycle class map). If $k \subseteq \mathbf{C}$, both cycle class maps admit refinements to maps to Betti cohomology (in the étale case, thanks to [1, Déf. 2.4]), and we can talk of algebraic and étale algebraic classes in Betti cohomology. The subgroups of algebraic and étale algebraic classes coincide rationally and the latter is pure in $H^*(X)$ (see Proposition 2.1 a) below); therefore, when X is projective, it is contained in the subgroup of Hodge or Tate classes. This shows readily that the (rational) Hodge or Tate conjectures are equivalent to their *integral* versions involving étale algebraic classes (see [37] for developments of this remark). When the latter conjectures are not known, étale algebraic classes thus allow us to reformulate the issue of an "integral Hodge or Tate conjecture" in the following unconditional form:

Question 1.11. When are étale algebraic classes algebraic?

In this spirit, the result of [12] mentioned at the beginning of this introduction shows that, in general, algebraic classes are not stable under divided powers. By contrast, the previous results show:

Corollary 1.12. *For abelian varieties, étale algebraic classes are stable under divided powers and the Fourier transform. The Künneth projectors are étale algebraic. The groups of étale algebraic classes of A and its dual $\hat{A}$ are isomorphic.* □

To conclude, these results confirm that étale Chow groups are a receptacle in which (basically) "everything goes well", in contrast to ordinary Chow groups: étalification in some sense removes the asperities of torsion in the latter.[3] These asperities, on the other hand, are sources of counterexamples to rationality, as for example in [12, Cor. 1.4]. My viewpoint is that both theories are relevant and one should play with them according to needs. See §§B.6 and B.7 for possible interactions of the present ideas with ordinary Chow groups.

Acknowledgement. I thank Olivier Benoist for pointing out the preprint [13], which completely changes my perspective on the subject.

[3] For example, the "étale Griffiths group" is divisible by Proposition 2.1 a), contrary to the classical one in characteristic 0 [6, 38].

2. Étale motivic cohomology of abelian varieties

Let k be a field of exponential characteristic p, and take a prime number l, possibly equal to p. Recall

Proposition 2.1 ([23, Cor. 3.5] and [24, Prop. 1]). *Let X be a smooth k-variety. Let $(i,j) \in \mathbf{Z}$.*
a) The étale cycle class map of [23, §3A]

$$\mathrm{cl}_l : H^j_{\text{ét}}(X, \mathbf{Z}(i)) \otimes \mathbf{Z}_l \to H^j_{\text{cont}}(X, \mathbf{Z}_l(i))$$

has divisible kernel and torsion-free cokernel. Here the target is Jannsen's continuous étale cohomology [19] *if $l \neq p$ and logarithmic Hodge-Witt cohomology if $l = p$.*
b) Suppose k algebraically closed and X projective. For $j \neq 2i$, $H^j_{\text{ét}}(X, \mathbf{Z}(i)) \otimes \mathbf{Z}_{(l)}$ is a direct sum of a group of finite exponent and a divisible group; in particular, $H^j_{\text{ét}}(X, \mathbf{Z}(i)) \otimes \mathbf{Q}_l/\mathbf{Z}_l = 0$ and the image of cl_l is finite if $l \neq p$.

Definition 2.2. If M is an abelian group, we say that an element $\alpha \in M \otimes \mathbf{Q}$ is *integral* (with respect to M) if it comes from some element of M.

Remark 2.3. In Definition 2.2,
a) α is integral with respect to M if and only if it is integral with respect to M/M_{tors}, and its lift is unique in the latter case.
b) To be integral is the same as to vanish in $M \otimes \mathbf{Q}_l/\mathbf{Z}_l$ for all l. This shows that α is integral with respect to M if and only if $\alpha \otimes \mathbf{Q}_l$ is integral with respect to $M \otimes \mathbf{Z}_l$ for all l.

Corollary 2.4 (Integrality lemma). *Let $\alpha \in H^j_{\text{ét}}(X, \mathbf{Q}(i))$, and let $N > 0$. Suppose that $\mathrm{cl}_l(\alpha) \in H^j_{\text{cont}}(X, \mathbf{Q}_l(i))$ is integral with respect to $H^j_{\text{cont}}(X, \mathbf{Z}_l(i))$ for all $l \nmid N$. Then α is integral with respect to $H^j_{\text{ét}}(X, \mathbf{Z}(i))[1/N]$.*

Proof. In view of remark 2.3 b), this follows from Proposition 2.1 a) via an easy diagram chase. □

Proposition 2.5 ([27, Th. 15.1]). *Let A be an abelian k-variety of dimension g, with k separably closed. If $l \neq p$, $H^*_{\text{cont}}(A, \mathbf{Z}_l)$ is an exterior algebra on $H^1_{\text{cont}}(A, \mathbf{Z}_l)$; in particular $H^*_{\text{cont}}(A, \mathbf{Z}_l)$ is torsion-free.*

The same holds for Betti cohomology when $k = \mathbf{C}$ [31, §1, (4)].

Proposition 2.6. *In Proposition 2.1, suppose that $X = A$ is an abelian variety. Suppose also k separably closed and $l \neq p$. Then*
c) There is a canonical isomorphism

$$H^{j-1}_{\text{cont}}(A, \mathbf{Z}_l) \otimes \mathbf{Q}_l/\mathbf{Z}_l(i) \xrightarrow{\sim} H^j_{\text{ét}}(A, \mathbf{Z}(i))\{l\}.$$

If $j \neq 2i$, $H^j_{\text{ét}}(A, \mathbf{Z}(i))[1/p]$ is divisible.
d) If $(i,j), (i',j')$ are two pairs, the cup-product

$$H^j_{\text{ét}}(A, \mathbf{Z}(i)) \times H^{j'}_{\text{ét}}(A, \mathbf{Z}(i')) \to H^{j+j'}_{\text{ét}}(A, \mathbf{Z}(i+i'))$$

restricts to 0 *on torsion subgroups.*
e) For $n \in \mathbf{Z}$*, the contravariant action of* $n_A \in \operatorname{End}(A)$ *on* $H^j_{\text{ét}}(A, \mathbf{Z}(i))_{\text{tors}}$ *is multiplication by* n^{j-1} *and its covariant action is multiplication by* n^{2g-j+1}*, at least away from* p*.*
f) If $j \neq 2i$*,* $H^j_{\text{ét}}(A, \mathbf{Z}(i))[1/p]$ *is divisible.*

Proof. In c), the isomorphism is the composition of two isomorphisms

$$H^{j-1}_{\text{cont}}(A, \mathbf{Z}_l)(i) \otimes \mathbf{Q}_l/\mathbf{Z}_l \xrightarrow{\sim} H^{j-1}_{\text{ét}}(A, \mathbf{Q}_l/\mathbf{Z}_l(i)) \xrightarrow{\sim} H^j_{\text{ét}}(A, \mathbf{Z}(i))\{l\}$$

where the second one follows from Proposition 2.1 b) and the first holds by Proposition 2.5. By Proposition 2.1 b), if $j \neq 2i$ then the image of the étale cycle class map is finite, hence 0 by Proposition 2.5, hence $H^j_{\text{ét}}(A, \mathbf{Z}(i))[1/p]$ is divisible by Proposition 2.1 a). d) follows from c), since the tensor product of two torsion divisible groups is 0; similarly, e) follows from c). Finally, f) follows from Propositions 2.1 and 2.5. □

Correspondences and motives. We shall use three categories of (pure, effective) motives: $\mathcal{M}$ (Chow motives), $\mathcal{M}_{\text{ét}}$ (étale Chow motives) and $\hat{\mathcal{M}}$ (cohomological motives). There are ⊗-functors

$$\mathcal{M} \xrightarrow{\alpha^*} \mathcal{M}_{\text{ét}} \xrightarrow{\hat{R}} \hat{\mathcal{M}}.$$

If X is a smooth projective k-variety, we write $h(X)$, $h_{\text{ét}}(X)$, $h_l(X)$ for its motive in $\mathcal{M}, \mathcal{M}_{\text{ét}}, \hat{\mathcal{M}}$ respectively. If Y is another, connected, one, morphisms from its motive to that of X are elements of the groups of correspondences

$$CH^{\dim Y}(Y \times X),\ CH^{\dim Y}_{\text{ét}}(Y \times X)[1/p],\ \hat{H}^{2\dim Y}(Y \times X, \dim Y)$$

where

$$\hat{H}^j(Z, i) := \prod_{l \neq p} H^j_{\text{cont}}(Z, \mathbf{Z}_l(i)).$$

This extends to nonconnected Y by additivity as usual. The functor $\hat{R}$ is induced by the étale cycle class maps cl_l, and $\alpha^* \otimes \mathbf{Q}$ is an equivalence of categories [23, Th. 2.6 c)]. Correspondences γ, δ from Z to Y and from Y to X compose according to the usual rule

$$\delta \circ \gamma = (p_{13})_*(p_{12}^*\delta \cdot p_{23}^*\gamma)$$

where p_{ij} are the projections of $X \times Y \times Z$ on the two-fold factors. To justify that composition is well-defined and associative in $\mathcal{M}_{\text{ét}}$, the simplest is to use the formalism of the six operations for étale motives, as in [2] or [8].[4]

If $k \subseteq \mathbf{C}$, the adic realisation $\hat{R}$ factors through the Betti realisation

$$\mathcal{M}_{\text{ét}} \xrightarrow{R_B} \mathcal{M}_B$$

[4] If one wishes not to invert p in characteristic p, the six operations are not available anymore because $\mathbf{A}^1$-invariance is lost. See [22, App. A] for a programme to define such a refined category; to the best of my knowledge, the required properties have still not been written up.

where $\mathcal{M}_B$ is the category of cohomological motives with respect to Betti cohomology H_B, and $H^j_B(Z,\mathbf{Z}(i))\otimes\prod_l \mathbf{Z}_l \xrightarrow{\sim} \hat{H}^j(Z,i)$ by Artin's comparison theorem. The reader only interested in this case can replace $\hat{H}$ by H_B in the sequel.

We shall also use the notation

$$\mathcal{M}^{\rm ab} \xrightarrow{\alpha^*} \mathcal{M}^{\rm ab}_{\rm \acute{e}t} \xrightarrow{\hat{R}} \hat{\mathcal{M}}^{\rm ab}$$

for the thick (= full, stable under direct summands) subcategories generated by motives of abelian varieties; similarly for $\mathcal{M}_B^{\rm ab}$. We then have

Corollary 2.7. *Composition in $\mathcal{M}^{\rm ab}_{\rm \acute{e}t}$ restricts to 0 on torsion correspondences.*

Proof. This follows from Proposition 2.6 d). □

3. Proof of Proposition 1.1

By Proposition 2.5, the even part of the cohomology algebra $H^*_{\rm cont}(A,\mathbf{Z}_l)$ has a divided power structure [34, 36]. Thus $\mathrm{cl}_l(x^r)=\mathrm{cl}_l(x)^r$ is divisible by $r!$ and we conclude with Corollary 2.4, applied to $\alpha=\frac{x^r}{r!}$. □

Remark 3.1. If $x\in H^j_{\rm \acute{e}t}(A,\mathbf{Z}(i))$ with $j\neq 2i$, then x^r is divisible by *any* positive integer in $H^{jr}_{\rm \acute{e}t}(A,\mathbf{Z}(ir))[1/p]$ by Proposition 2.6 f). If j is odd and $p\neq 2$, we even have $x^2=0$ because $2x^2=0$ *a priori* [26, Th. 15.9].

4. Esnault's example, refined

Here we take $g=2$. Further, we assume that A is the Jacobian of a smooth projective curve C having a rational point c. This provides an embedding

$$i: C\hookrightarrow A$$

sending c to 0 and realising $i(C)$ as a theta divisor.

Proposition 4.1. *Let $K_C\in \mathrm{Pic}(C)$ be the class of the canonical divisor on C, and let x be the class of $i(C)$ in $\mathrm{Pic}(A)$. Then $\deg(x^2)=2$ and the Albanese class of $x^2-2[0]\in CH^2(A)_0$ in $A(k)$ is $K_C-2[c]$. In particular, x^2 is divisible by 2 in $CH^2(A)$ or $CH^2_{\rm \acute{e}t}(A)$ if and only if K_C is divisible by 2 in $\mathrm{Pic}(A)$. If $p\neq 2$, the same holds for $\mathrm{cl}_2(x)$.*

Proof. By [17, Ch. II, Prop. 8.20], we have

$$K_C=i^*(K_A+[i(C)])=i^*i_*[C]$$

using that $K_A=0$ [31, §4, (ii)]. Therefore

$$x^2=i_*i^*i_*[C]\ =i_*K_C.$$

This already gives $\deg(x^2)=2g-2=2$, and moreover

$$x^2-2[0]=i_*(K_C-2[c])$$

hence the conclusion (here we use that the composition $A(k) = \operatorname{Pic}^0(A) \xrightarrow{i_*} CH_0(A)_0 \xrightarrow{a} A(k)$ is the identity, where a is the Albanese map.) To deal with $\mathrm{cl}_2(x)$, we reason as in the proof of Proposition 1.1. □

One example in [14, §4] (attributed to Serre) where the condition of Proposition 4.1 fails is $k = \mathbf{C}(t)$, C given by the affine equation $y^2 = x^6 - x - t$.

5. The torsion bimodule

Let X be a smooth projective k-variety. From now on, we abbreviate $H^j_{\mathrm{cont}}(X, \mathbf{Z}_l(i))$ to $H^j_l(X, i)$ and $H^j_l(X, 0)$ to $H^j_l(X)$ if $l \neq p$. Consider the following two actions:

- the étale intersection product
$$H^{j-1}_{\text{ét}}(X, \mathbf{Q}_l/\mathbf{Z}_l(i)) \times H^{j'}_{\text{ét}}(X, \mathbf{Z}(i')) \to H^{j+j'-1}_{\text{ét}}(X, \mathbf{Q}_l/\mathbf{Z}_l(i+i')),$$
denoted by $\cdot$;
- the l-adic product
$$H^{j-1}_{\text{ét}}(X, \mathbf{Q}_l/\mathbf{Z}_l(i)) \times H^{j'}_l(X, i') \to H^{j+j'-1}_{\text{ét}}(X, \mathbf{Q}_l/\mathbf{Z}_l(i+i')),$$
denoted by $\cup$.

Lemma 5.1. *For $(x, y) \in H^{j-1}_{\text{ét}}(X, \mathbf{Q}_l/\mathbf{Z}_l(i)) \times H^{j'}_{\text{ét}}(X, \mathbf{Z}(i'))$, we have*
$$x \cdot y = \mathrm{cl}_l(x) \cup y.$$

Proof. This follows from the fact that, in [23], the morphism (3-2) (defining cl_l) is compatible with products. □

Given an additive category $\mathcal{A}$, recall that an $\mathcal{A}$-*bimodule* is an additive bifunctor on $\mathcal{A}^{\mathrm{o}} \times \mathcal{A}$. Let
$$(\mathbf{Q}/\mathbf{Z})' = \bigoplus_{l \neq p} \mathbf{Q}_l/\mathbf{Z}_l.$$

For $r \in \mathbf{Z}$, the two actions described above provide the assignment $(Y, X) \mapsto H^{2\dim Y - r}_{\text{ét}}(Y \times X, (\mathbf{Q}/\mathbf{Z})'(\dim Y))$ with structures of bimodule over $\mathcal{M}_{\text{ét}}$ and $\hat{\mathcal{M}}$ respectively, and Lemma 5.1 implies

Lemma 5.2. *These two bimodule structures are compatible via $\hat{R}$.* □

We denote this common bimodule by $t(r)$. For abelian varieties A, B with $\dim A = g$, the proof of Proposition 2.6 c) interprets $t(1)$ both as $H^{2g}_{\text{ét}}(A \times B, \mathbf{Z}(g))[1/p]_{\mathrm{tors}}$ and as $\hat{H}^{2g-1}(A \times B, g) \otimes \mathbf{Q}/\mathbf{Z}$. In the rest of this section, we compute the $\hat{\mathcal{M}}^{\mathrm{ab}}$-bimodule structure of $t(r)$. For this, we note that as a consequence of Proposition 2.5, the Künneth formula for $H^*_l(A \times B)$ holds integrally for all $l \neq p$. Moreover, Poincaré duality also holds integrally for A thanks to [7, §11, Prop. 7] and the fact that the trace map $H_l^{2g}(A,g) \to \mathbf{Z}_l$ is an isomorphism. We therefore have a Künneth decomposition in $\hat{\mathcal{M}}$:

$$\hat{h}(A) = \bigoplus_{i=0}^{2g} \hat{h}^i(A) \tag{5.1}$$

corresponding to the Künneth projectors $\hat{\pi}_A^i$ in

$$\hat{H}^{2g}(A \times A, g) \simeq \prod_{i=0}^{2g} \operatorname{End}_{\hat{\mathbf{Z}}}(\hat{H}^i(A)).$$

We first recall the structure of the tautological $\hat{\mathcal{M}}^{\mathrm{ab}}$-bimodule $t(0) = \operatorname{Hom}_{\hat{\mathcal{M}}^{\mathrm{ab}}}$: in view of (5.1), it may be written $\bigoplus_{i,j} \hat{\mathcal{M}}(\hat{h}^i(A), \hat{h}^j(B))$, where

$$\hat{\mathcal{M}}(\hat{h}^i(A), \hat{h}^j(B)) = \begin{cases} 0 & \text{if } i \neq j \\ \operatorname{Hom}_{\hat{\mathbf{Z}}}(\hat{H}^i(A), \hat{H}^i(B)) & \text{if } i = j. \end{cases}$$

In particular, $\hat{\pi}_B^i \circ f = f \circ \hat{\pi}_A^i$ for any $f \in \hat{\mathcal{M}}(h_l(A), h_l(B))$, and the $\hat{\pi}_A^i$ are central in $\hat{H}^{2g}(A \times A, g)$.

Lemma 5.3. *Let $T(r)$ be the $\hat{\mathcal{M}}^{\mathrm{ab}}$-bimodule induced by $(A,B) \mapsto \hat{H}^{2g-r}(A \times B, g)$. Then*

$$T(r)(\hat{h}^i(A), \hat{h}^j(B)) = \begin{cases} 0 & \text{if } j \neq i-r \\ \operatorname{Hom}_{\hat{\mathbf{Z}}}(\hat{H}^i(A), \hat{H}^{i-r}(B)) & \text{if } j = i-r. \end{cases}$$

In particular, $\hat{\pi}^{i-r}(B) \bullet x = x \bullet \hat{\pi}^i(A)$ for any $x \in T(r)(\hat{h}(A), \hat{h}(B))$, where $\bullet$ denotes the bimodule action.
The same holds, mutatis mutandis, for $t(r) = T(r) \otimes \mathbf{Q}/\mathbf{Z}$.

Proof. Using again the Künneth formula and Poincaré duality, we get an isomorphism

$$\hat{H}^{2g-r}(A \times B, g) \simeq \bigoplus_{i=0}^{2g} \operatorname{Hom}_{\hat{\mathbf{Z}}}(\hat{H}^i(A), \hat{H}^{i-r}(B)) \tag{5.2}$$

and the lemma follows by bookkeeping. □

For later use, we consider a slightly more general situation, replacing B by an arbitrary smooth projective variety X. By Proposition 2.5, the Künneth formula for $\hat{H}^*(A \times X)$ still holds integrally. This, plus Poincaré duality for A, still yields the isomorphism (5.2) for any $r \in \mathbf{Z}$. Torsion in the cohomology of X makes it difficult to talk of its integral Künneth projectors, but those of A still act on the left hand side, yielding the decomposition on the right hand side. For $r = 1$ and $l \neq p$, we have a short exact sequence

$$0 \to H_l^{2g-1}(A \times X, g) \otimes \mathbf{Q}_l/\mathbf{Z}_l \to H_{\text{ét}}^{2g-1}(A \times X, \mathbf{Q}_l/\mathbf{Z}_l(g)) \\ \to H_l^{2g}(A \times X, g)\{l\} \to 0$$

from which the action of π_l^1 (the l-component of $\hat\pi^1$) cuts off the exact sequence

$$0 \to \mathrm{Hom}_{\mathbf{Z}_l}(H_l^1(A), H_l^0(X)) \otimes \mathbf{Q}_l/\mathbf{Z}_l \to H^{2g-1}_{\text{ét}}(A \times X, \mathbf{Q}_l/\mathbf{Z}_l(g)) \circ \pi_l^1 \\ \to \mathrm{Hom}_{\mathbf{Z}_l}(H_l^1(A), H_l^1(X))\{l\} \to 0.$$

But $H_l^1(X))$ is always torsion-free and $H_l^0(X) = \mathbf{Z}_l$, so this exact sequence boils down to isomorphisms

$$A(k)\{l\} \simeq H_l^1(A)^* \otimes \mathbf{Q}_l/\mathbf{Z}_l \xrightarrow{\sim} H^{2g-1}_{\text{ét}}(A \times X, \mathbf{Q}_l/\mathbf{Z}_l(g)) \circ \pi_l^1. \tag{5.3}$$

6. 1-cocycles

6.1. **Hochschild cohomology.** Let $\Lambda = \mathbf{Z}[\mathbf{Z} - \{0\}]$ be the group algebra of the multiplicative monoid $\mathbf{Z} - \{0\}$; if M is a Λ-bimodule, we have as usual its Hochschild cohomology

$$HH^r(\Lambda, M) = \mathrm{Ext}^r_{\Lambda\otimes\Lambda}(\Lambda, M)$$

where Λ is considered as a Λ-bimodule via left and right action. In particular,

$$HH^1(\Lambda, M) = \frac{\{f : \mathbf{Z} - \{0\} \to M \mid f(mn) = m \diamond f(n) + f(m) \diamond n\}}{\{m \mapsto m \diamond a - a \diamond m\}}.$$

Since $\mathbf{Z} - \{0\}$ is commutative, the 1-cocycle relation implies in particular the identity in (m, n)

$$m \diamond f(n) - f(n) \diamond m = n \diamond f(m) - f(m) \diamond n. \tag{6.1}$$

6.2. **Some numerology.**

Definition 6.1. For any $i > j \geq 0$, we set

$$w_{i,j} = \gcd_{m\neq 0}(m^i - m^j)$$

and let $w_{i,j} := 1$ if $i \geq 0$ and $j < 0$.

A stable version appears in [40, 2.8]: Soulé's w_r is the supremum of our $w_{i,j}$ for $i - j = r$.

Here are some elementary properties of $w_{i,j}$:

Lemma 6.2. *a) Let l be a prime number and let $s > 0$. If l is odd, then $l^s \mid w_{i,j} \Rightarrow l^s - l^{s-1} \mid i - j$. For $l = 2$, $4 \mid w_{i,j} \Rightarrow i - j$ is even and $2^s \mid w_{i,j}$ $\Rightarrow 2^{s-2} \mid i - j$ for $s > 2$. In particular, $w_{i,j} = 2$ if $i - j$ is odd.*
b) If $j = 0$, then $w_{i,j} = 1$.
c) The converse to a) holds provided $j \geq s$.

Proof. a) Suppose l odd. Then $(\mathbf{Z}/l^s)^*$ is cyclic of order $l^s - l^{s-1}$. Taking an integer m whose class generates this group gives a) in this case. For $l = 2$, $(\mathbf{Z}/2^s)^* = \{\pm 1\} \times (1 + 4\mathbf{Z}/2^s) \simeq \mathbf{Z}/2 \times \mathbf{Z}/2^{s-2}$ when $s > 1$; taking either $m = -1$ when $s = 2$ or $m = 5$ when $s > 2$ gives a) in that case. If $j = 0$, then $l \nmid l^i - 1$; this proves b). Conversely, to get c) we observe that, under the hypothesis, if $l \nmid m$ then $l^s \mid m^{i-j} - 1$ and if $l \mid m$ then $l^s \mid m^j$. □

6.3. **Cocycles and coboundaries.** For $i>j$, define

$$R_{i,j}=\Lambda\otimes\Lambda/\langle([m]-m^j)\otimes 1, 1\otimes([n]-n^i)\rangle.$$

A Λ-bimodule M is *of level* (i,j) if the action of $\Lambda\otimes\Lambda$ factors through $R_{i;j}$. Here the 1-cocycle relation is

$$f(mn)=m^jf(n)+n^if(m) \tag{6.2}$$

and (6.1) becomes

$$(m^i-m^j)f(n)=(n^i-n^j)f(m). \tag{6.3}$$

For convenience, set

$$\theta(n)=\frac{n^i-n^j}{w_{i,j}}.$$

Then θ is a 1-cocycle with values in $\mathbf{Z}$ provided with its obvious level (i,j)-structure, and so is $n\mapsto\theta(n)b$ for any $b\in M$; we call such 1-cocycles *refined coboundaries.*

Proposition 6.3. *a) Let M be a Λ-bimodule of level (i,j). Then $HH^0(\Lambda,M)$ $={}_{w_{i,j}}M$ and $w_{i,j}HH^1(\Lambda,M)=0$. In particular, $HH^0(\Lambda,M)=HH^1(\Lambda,M)$ $=0$ if $j=0$.*
b) Suppose that $j=i-1>0$ and $w_{i,j}M=2M=0$. Then $f(n)=0$ if n is a square or if $4\mid n$. Moreover, $f(2n)=f(2)$ if n is odd.

Proof. a) The first claim is obvious. Let now f be a 1-cocycle. Take $n_1,\dots,n_r$ such that $\gcd_s(n_s^i-n_s^j)=w_{i,j}$; choose $a_1,\dots a_r$ such that $\sum_s a_s(n_s^i-n_s^j)=w_{i,j}$, and let $b=\sum_s a_sf(n_s)$. Then, for all n,

$$w_{i,j}f(n)=\sum_s a_s(n_s^i-n_s^j)f(n)=\sum_s a_s(n^i-n^j)f(n_s)=(n^i-n^j)b$$

thanks to (6.3), hence the second claim. The last one follows from Lemma 6.2 b).

b) In this case, (6.2) may equally be written

$$f(mn)=mf(n)+nf(m).$$

For $n=m$, we get the first claim; then $f(4m)=m^jf(4)+4^jf(m)=0$. The last identity follows similarly. $\square$

7. A weak form of Theorem 1.7

7.1. A weak version of Theorem 1.7 a) (I).

Theorem 7.1. *a) There exists a system (π_A^i) of orthogonal projectors of sum 1 in $\operatorname{End}_{\mathcal{M}_{\text{ét}}}(h(A))$ such that π_A^0 is given by the origin of A and, for any $n\in\mathbf{Z}$:*

- *$n_A^*\circ\pi_A^1=n\pi_A^1$;*
- *$2n_A^*\circ\pi_A^i=2n^i\pi_A^i$ for $i>1$.*

Moreover, $n_A^*\circ\pi_A^i=n^i\pi_A^i$ for all i if n is a square or if $4\mid n$.
b) Any other such system is conjugate to (π_A^i) by an element of the form $1+z$ with $4z=0$ and $z\circ\pi_A^1=0$. Conversely, any conjugate of (π_A^i) by such an element verifies a).
c) If $x\in \operatorname{End}_{\mathcal{M}_{\text{ét}}}(h(A))_{\text{tors}}$ is such that $(1+x)\circ\pi_A^i=\pi_A^i\circ(1+x)$ for all i, then $x=0$. (Hence the element z of b) is unique.)

We proceed in three steps.

Step 1. Using the adic Künneth projectors $\hat\pi_A$ of §5, Corollary 2.4 shows that the Chow-Künneth projectors of [10] lift from $CH^g(A\times A)\otimes\mathbf{Q}\xrightarrow{\sim} CH^g_{\text{ét}}(A\times A)\otimes\mathbf{Q}$ to a (unique) set of orthogonal projectors with sum 1 in $CH^g_{\text{ét}}(A\times A)[1/p]/\text{tors}$.

Step 2. Using Corollary 2.7 and [20, Lemma 5.4], we may lift the projectors of Step 1 to a set of orthogonal projectors with sum 1 $(\pi_0^i)_{i=0}^{2g}$ in $CH^g_{\text{ét}}(A\times A)[1/p]$.

Step 3. Now we modify the projectors of Step 2 to achieve the conditions of Theorem 7.1. We choose π_A^0 as said: it certainly verifies $m_A^*\pi_A^0=\pi_A^0$. For $i>0$, consider the Λ-bimodule structure on $CH^g_{\text{ét}}(A\times A)[1/p]$ given by $m\diamond x\diamond n=n^im_A^*\circ x$. By [10, Th. 3.1], the 1-coboundary

$$y_i(n)=n_A^*\circ\pi_0^i-n^i\pi_0^i$$

takes values in $CH^g_{\text{ét}}(A\times A)[1/p]_{\text{tors}}$, hence defines a 1-cocycle with values in this bimodule. But

$$y_i(n)=y_i(n)\circ\pi_0^i. \tag{7.1}$$

Therefore, by Lemmas 5.2 and 5.3,

$$y_i(n)\in t_i(1):=t(1)(h^i(A),h^{i-1}(B))$$

and the bimodule $t_i(1)$ is of level $(i,i-1)$ by Proposition 2.6 e). Applying Proposition 6.3 a), we may write

$$y_i(n)=\begin{cases}(n-1)x_1 & \text{if } i=1\\ \frac{n^i-n^{i-1}}{2}x_i'+g_i(n) & \text{if } i>1\end{cases}$$

for some $x_1,x_i'\in t_i$, with $2g_i(n)=0$. We set $g_1(n)=0$.

For $i>1$, using the divisibility of t_i let $x_i\in t_i$ be such that $2x_i=x_i'$, and let $x=\sum_{i>0}x_i$. Define

$$\begin{aligned}\pi^i=(1+x)\circ\pi_0^i\circ(1+x)^{-1}&=\pi_0^i+x\circ\pi_0^i-\pi_0^i\circ x\\ &=\pi_0^i+x\circ\pi_0^i-x\circ\pi_0^{i+1}=\pi_0^i+x_i-x_{i+1}\end{aligned}\tag{7.2}$$

where we used again Corollary 2.7 for the second equality and Lemmas 5.2 and 5.3 for the third. Then, for $n\in\mathbf{Z}$,

$$n_A^*\circ\pi^i-n^i\pi^i=y_i(n)+(n^{i-1}-n^i)x_i=g_i(n)$$

which proves a). The last claim follows from Proposition 6.3 b) applied to $g_i(n)$.

b) Still by [20, Lemma 5.4], any other lift of the projectors of Step 2 is of the form $\tilde\pi^i = (1+z)\pi^i(1+z)^{-1}$ with $z \in CH^g_{\text{ét}}(A\times A)[1/p]_{\text{tors}}$. The same computation as (7.2) yields

$$\tilde\pi^i = \pi^i + z_i - z_{i+1}$$

with $z_i = z\circ \pi^i = \pi^{i-1}\circ z$.

Suppose that the $\tilde\pi^i$ also satisfy a). For $n\neq 0$, let $g_i(n) = (n_A^* - n^i)\circ \pi^i$ as in the proof of a), and let similarly $\tilde g_i(n) = (n_A^* - n^i)\circ\tilde\pi^i$. Then

$$(n_A^* - n^i)\circ(z_i - z_{i+1}) = (n_A^* - n^i)\circ(\tilde\pi^i - \pi^i) = \tilde g_i(n) - g_i(n) \text{ if } i>1;$$
$$(n_A^* - n)\circ(z_1 - z_2) = 0.$$

But $(n_A^* - n^i)\circ(z_i - z_{i+1}) = (n^{i-1} - n^i)z_i$ as in the proof of a). Since $2(\tilde g_i(n) - g_i(n)) = 0$ we get $4z_i = 0$ for $i>1$, and $z_1 = 0$. Conversely, under these conditions, the same computation shows that the $\tilde\pi^i$ also satisfy a).

In c), the condition is equivalent to $x\circ\pi_A^i = \pi_A^i\circ x$; but $x\circ\pi_A^i = \pi_A^{i-1}\circ x$ by Lemma 5.3, hence

$$\pi_A^i\circ x = \pi_A^i\circ\pi_A^i\circ x = \pi_A^i\circ x\circ\pi_A^i = \pi_A^i\circ\pi_A^{i-1}\circ x = 0$$

for all i, and therefore $x=0$. This concludes the proof of Theorem 1.7 a).

7.2. **Another lifting method.** Later we shall need the following proposition:

Proposition 7.2. *Let $(\bar\pi_A^i)_{i=0}^{2g}$ be the set of projectors of Step 1 in the proof of Theorem 7.1, and let $(\pi^i)_{i=0}^{2g}$ be a set of lifts of the $\bar\pi_A^i$, with $\sum \pi^i = 1$. Then the $(\pi^i)^2$ form a set of orthogonal projectors with sum 1.*

Proof. It is similar to that of Theorem 7.1 c), but a little more elaborate. Let $a_{ij} = \pi^i\pi^j - \delta_{ij}\pi^i \in CH^{2g}_{\text{ét}}(A\times A)[1/p]_{\text{tors}}$. For any r and $l\neq p$, we have

$$\pi^r\circ a_{ij} = \hat\pi^r\bullet a_{ij},\quad a_{ij}\circ\pi^r = a_{ij}\bullet\hat\pi^r$$

by Lemma 5.2, and

$$a_{ij}\bullet\hat\pi^r = \hat\pi^{r-1}\bullet a_{ij}$$

by Lemma 5.3.

Consider first the case $i=j$. Then a_{ii} commutes with π^i, hence $\hat\pi^i\bullet a_{ii} = \hat\pi^{i-1}\bullet a_{ii}$, which implies

$$\hat\pi^i\bullet a_{ii} = \hat\pi^{i-1}\bullet a_{ii} = 0 \tag{7.3}$$

hence $\pi^i\circ a_{ii} = \pi^{i-1}\circ a_{ii} = 0$ and in particular

$$(\pi^i)^3 = (\pi^i)^2 \text{ and } (\pi^i)^4 = (\pi^i)^3 = (\pi^i)^2. \tag{7.4}$$

Suppose now that $i\neq j$. Then

$$(\pi^i)^2\circ\pi^j = (\pi^i + a_{ii})\circ\pi^j = a_{ij} + a_{ii}\bullet\hat\pi^j = a_{ij} + \hat\pi^{j-1}\bullet a_{ii}$$
$$= \pi^i\circ(\pi^i\circ\pi^j) = \hat\pi^i\bullet a_{ij}$$

i.e.

$$(1-\hat\pi^i)\bullet a_{ij}+\hat\pi^{j-1}\bullet a_{ii}=0 \tag{7.5}$$

and playing with the equality $\pi^i\circ(\pi^j)^2=(\pi^i\circ\pi^j)\circ\pi^j$, we obtain similarly

$$(1-\hat\pi^{j-1})\bullet a_{ij}+\hat\pi^i\bullet a_{jj}=0. \tag{7.6}$$

If moreover $i\neq j-1$, this implies $\hat\pi^r\bullet a_{ij}=0$ for $r\neq i,j-1$, hence

$$a_{ij}=\hat\pi^i\bullet a_{ij}+\hat\pi^{j-1}\bullet a_{ij}=-\hat\pi^i\bullet a_{jj}-\hat\pi^{j-1}\bullet a_{ii}. \tag{7.7}$$

But we have

$$\pi^j=(\sum_i\pi^i)\pi^j=\pi^j+a_{jj}+\sum_{i\neq j}a_{ij}$$

i.e.

$$a_{jj}+a_{j-1,j}+\sum_{i\neq j,j-1}a_{ij}=0\,\forall\,j. \tag{7.8}$$

Putting (7.7) into (7.8), we get

$$\begin{aligned}a_{jj}+a_{j-1,j}&=\sum_{i\neq j,j-1}(\hat\pi^i\bullet a_{jj}+\hat\pi^{j-1}\bullet a_{ii})=(\sum_{i\neq j,j-1}\hat\pi^i)\bullet a_{jj}+\hat\pi^{j-1}\bullet\sum_{i\neq j,j-1}a_{ii}\\&=(\sum_i\hat\pi^i)\bullet a_{jj}+\hat\pi^{j-1}\bullet\sum_{i\neq j,j-1}a_{ii}=a_{jj}+\hat\pi^{j-1}\bullet\sum_{i\neq j,j-1}a_{ii}\end{aligned}$$

where we used (7.3) for the last but first equality; thus

$$a_{j-1,j}=\hat\pi^{j-1}\bullet\sum_{i\neq j,j-1}a_{ii}=-\hat\pi^{j-1}\bullet a_{jj}-\hat\pi^{j-1}\bullet a_{j-1,j-1}$$

where we used (7.8) again; so (7.7) is also true for $i=j-1$.

This in turn implies, for $i\neq j$

$$(\pi^i)^2\circ\pi^j=-\hat\pi^i\bullet a_{jj}=-\pi^i\circ((\pi^j)^2-\pi^j)=\pi^i\circ\pi^j-\pi^i\circ(\pi^j)^2.$$

Composing with π^i on the left and using (7.4), the latter gives

$$(\pi^i)^2\circ(\pi^j)^2=0. \tag{7.9}$$

We have proven that the $(\pi^i)^2$ form a set of orthogonal projectors; it remains to see that their sum is 1. But

$$1=(\sum_i\pi^i)^2=\sum_i(\pi^i)^2+\sum_{i\neq j}a_{ij}$$

which shows that $x=\sum_{i\neq j}a_{ij}$ is idempotent. But $x\circ x=0$ by Corollary 2.7, so $x=0$ and the proof is complete. □

Corollary 7.3. *Let $(\pi^i)_{i=0}^{2g}$ be such that $\sum\pi^i=1$ and $n_A^*\circ\pi^i=n^i\pi^i$ for all i and all $n\in\mathbf{Z}$. Then the $(\pi^i)^2$ form a set of étale integral DM projectors.*

Proof. By [10, Th. 3.1], the condition $n_A^*\circ\bar\pi^i=n^i\pi^i$ implies the condition of Proposition 7.2. □

7.3. **A weak version of Theorem 1.7 b).**

Lemma 7.4. *Let (π_A^i) be a system of projectors as in Theorem 7.1. Then we have*

$$2\pi_A^i \circ n_A^* = 2n_A^* \circ \pi_A^i \tag{7.10}$$

for any $n \in \mathbf{Z}$.

Proof. We proceed as in the proof of [10, Th. 3.1]: by Theorem 7.1, we have

$$n_A^* = \sum_j n_A^* \circ \pi_A^j = \sum_j (n^j \pi_A^j + g_j(n))$$

with $2g_j(n) = 0$. Moreover, $g_j(n) \circ \pi_A^j = g_j(n)$. Then

$$\pi_A^i \circ n_A^* = \sum_j (n^j \pi_A^i \circ \pi_A^j + \pi_A^i \circ g_j(n)) = n^i \pi_A^i + g_{i+1}(n) = n_A^* \circ \pi_A^i - g_i(n) + g_{i+1}(n)$$

where we used Lemma 5.3. Hence (7.10). □

Theorem 7.5. *Let A, B be two abelian varieties, and let (π_A^i), (π_B^i) be systems of projectors as in Theorem 7.1. Let $f : A \to B$ be a homomorphism. Then*

$$4f^* \circ \pi_B^i = 4\pi_A^i \circ f^*$$

for all i.

(We cannot get less than a factor 4 in view of the indeterminacy in Theorem 7.1 b): take $A = B$, $f = 1_A$.)

Proof. We imitate the proof of [10, Prop. 3.3]. Consider $c_{i,j} = \pi_A^j \circ f^* \circ \pi_B^i$. For $i \neq j$, $c_{i,j}$ is torsion by that proposition, hence 0 unless $j = i - 1$ by Lemma 5.3. In this case, we have

$$\begin{aligned} n^i c_{i,i-1} = \pi_A^{i-1} \circ f^* \circ n^i \pi_B^i &= \pi_A^{i-1} \circ f^* \circ (n_B^* \circ \pi_B^i + h_i(n)) \\ &= \pi_A^{i-1} \circ n_A^* \circ f^* \circ \pi_B^i + \pi_A^{i-1} \circ f^* \circ h_i(n) \\ &= (n^{i-1}\pi_A^{i-1} + g_i(n)) \circ f^* \circ \pi_B^i + \pi_A^{i-1} \circ f^* \circ h_i(n) \\ &= n^{i-1} c_{i,i-1} + g_i(n) \circ f^* \circ \pi_B^i + \pi_A^{i-1} \circ f^* \circ h_i(n) \end{aligned}$$

where h_i is a similar function for B as g_i is for A in the proof of Lemma 7.4, hence, taking the gcd over n (or just $n = -1$),

$$4c_{i,i-1} = 0.$$

From the identity

$$f = \sum_{i,j} c_{i,j} = \sum_i (c_{i,i} + c_{i,i-1})$$

it follows that

$$\pi_A^i \circ f^* - f^* \circ \pi_B^i = c_{i+1,i} - c_{i,i-1} \tag{7.11}$$

is 4-torsion. □

To this, we add

Proposition 7.6. *For $A, B \in \mathbf{Ab}$, $\mathcal{M}_{\text{ét}}(h^i(A), h^j(B))$ is torsion-free if $j \neq i-1$ (in particular for $i = j$), and*

$$\mathcal{M}_{\text{ét}}(h^i(A), h^{i-1}(B))_{\text{tors}} \simeq \operatorname{Hom}_{\hat{\mathbf{Z}}}(\hat{H}^i(A), \hat{H}^{i-1}(B)) \otimes \mathbf{Q}/\mathbf{Z}.$$

Proof. Same as for Theorem 7.1 c): we have

$$\mathcal{M}_{\text{ét}}(h^i(A), h^j(B)) = \{f \in CH^{g_A}_{\text{ét}}(A \times B)[1/p] \mid f = \pi^j_B \circ f = f \circ \pi^i_A\}$$

and $\mathcal{M}_{\text{ét}}(h^i(A), h^j(B))$ is torsion for $i \neq j$ by [10]. But if $f \in \mathcal{M}_{\text{ét}}(h^i(A), h^j(B))$ is torsion, then $f \circ \pi^i_A = \pi^{i-1}_B \circ f$ by Lemma 5.3, hence $f = \pi^j_B \circ \pi^{i-1}_B \circ f = 0$ if $j \neq i-1$. If $j = i-1$, the conclusion follows as usual from Lemma 5.3 and Proposition 2.6 a). □

7.4. **A weak version of Theorem 1.7 a) (II); self-conjugate projectors.** We first give a proof of [10, §3, Rem. 3)]:

Lemma 7.7. *In $CH^g(A \times A) \otimes \mathbf{Q}$,*
a) The operator n_A^* is invertible for any $n \neq 0$.
b) We have

$$\pi^i_A \circ (n_A)_* = n^{2g-i}\pi^i_A \tag{7.12}$$

for all (n, i).
c) We have

$$^t\pi^{2g-i}_A = \pi^i_A \tag{7.13}$$

for all i.

Proof. a) holds thanks to [10, Cor. 3.2]. Since $(n_A)_* n_A^* = n^{2g}$, (7.12) becomes true after composing with n_A^* on the right, hence b) follows from a). This implies

$$n_A^* \, {}^t\pi^i_A = n^{2g-i} \, {}^t\pi^i_A \; \forall (n, i),$$

hence c) by the uniqueness part of [20, Lemma 5.4]. □

Corollary 7.8. *In $CH^g_{\text{ét}}(A \times A)[1/p]$,*
a) The identity (7.12) remains true after multiplication by 2.
b) In Theorem 7.1 a), one can choose the π^i_A self-conjugate.
c) Let B be another abelian variety, and let (π^i_A) (π^i_B) be two systems of self-conjugate projectors verifying Theorem 7.1 a). Let $f : A \to B$ be a homomorphism. Then

$$4\pi^i_B \circ f_* = 4 f_* \circ \pi^i_A$$

for all i.

Proof. a) For $n \in \mathbf{Z} - \{0\}$,

$$g^i(n) = \pi^i_A \circ ((n_A)_* - n^{2g-i})$$

is torsion by Lemma 7.7 b), and we have

$$g^i(mn) = n^{2g-i} g^i(m) + g^i(n) \circ (m_A)_*$$

because $(m_A)_*$ and $(n_A)_*$ commute. But, by the same reasoning as in the proof of Lemma 7.7 b), $(m_A)_*$ acts as multiplication by m^{2g-i+1} on $\hat{H}^{i-1}(A)\otimes \mathbf{Q}$, hence on $\hat{H}^{i-1}(A)$ and finally on $\hat{H}^{i-1}(A)\otimes \mathbf{Q}/\mathbf{Z}$. It follows that $g^i(n)\circ(m_A)_* = m^{2g-i+1}g^i(n)$ and the above identity becomes

$$g^i(mn) = n^{2g-i}g^i(m) + m^{2g-i+1}g^i(n).$$

Applying Proposition 6.3, there exists b such that $b=\pi^i\circ b$ and

$$2g^i(n) = (n^{2g-i}-n^{2g-i+1})b \tag{7.14}$$

for any n, so that

$$(2\pi_A^i - b)\circ(n_A)_* = n^{2g-i}(2\pi_A^i-b).$$

Composing with n_A^* on the right, we get

$$n^{2g}(2\pi_A^i-b) = n^{2g-i}(2\pi_A^i\circ n_A^* - b\circ n_A^*) = n^{2g}2\pi_A^i - n^{2g-i}b\circ\pi_A^*$$

by Lemma 7.4, i.e.

$$n^{2g}b = n^{2g-i}b\circ n_A^*.$$

But since $b\in \pi_A^i\circ CH^g_{\text{ét}}(A\times A)[1/p]_{\text{tors}} = CH^g_{\text{ét}}(A\times A)[1/p]_{\text{tors}}\bullet\hat{\pi}^{i+1}$ (Lemmas 5.2 and 5.3), we have $b\circ n_A^* = n^{i+1}b$, hence $(n^{2g}-n^{2g+1})b=0$ for all n, which finally yields $2b=0$. But then $2g^i(n)=0$ by (7.14).

b) By Theorem 7.1 b) and Lemma 7.7 c), there exists $x\in CH^g_{\text{ét}}(A\times A)[1/p]_{\text{tors}}$ such that

$${}^t\pi_A^{2g-i} = (1+x)\pi_A^i(1+x)^{-1} = \pi_A^i + x\pi_A^i - \pi_A^i x = \pi_A^i + x_{i-1} - x_i \tag{7.15}$$

for all i, where $x_i = \pi_A^i\circ x$ (here we used Lemma 5.3).

We first bound the order of x. Composing (7.15) with n_A^* on the left and using a) and the identities of Theorem 7.1 a), we get

$$2n^i\,{}^t\pi_A^{2g-i} = 2(n^i\pi_A^i + n^{i-1}x_{i-1} - n^i x_i)$$

which, compared with (7.15) multiplied by $2n^i$, yields

$$2(n^i-n^{i-1})x_{i-1} = 0.$$

Taking $n=-1$, we find $4x_{i-1}=0$ for all i, hence $4x=0$.

Now taking the transpose of (7.15) after switching i and $2g-i$, we get

$$\pi_A^i = (1+{}^tx)^{-1}\,{}^t\pi_A^{2g-i}(1+{}^tx) = (1+{}^tx)^{-1}(1+x)\pi_A^i(1+x)^{-1}(1+{}^tx)$$

for all i.

By Theorem 7.1 c), this implies $(1+{}^tx)^{-1}(1+x)=1$, i.e. ${}^tx = x$, hence

$$x_i = \pi_A^i\circ x = \pi_A^i\circ{}^tx = {}^t(x\circ{}^t\pi_A^i) = {}^t(x\bullet{}^t\hat{\pi}_A^i) = {}^t(x\bullet\hat{\pi}_A^{2g-i}) = {}^t(\hat{\pi}_A^{2g-i-1}\bullet x)$$
$$= {}^tx_{2g-i-1}.$$

Let $y=\sum_{i=0}^{g-1}x_i$; noting that $x_{2g}=0$, we have $x={}^ty+y$, or equivalently $1+x={}^t(1+y)(1+y)$. Therefore, if $\tilde{\pi}_A^i = (1+y)\pi_A^i(1+y)^{-1}$, we have ${}^t\tilde{\pi}_A^i = \tilde{\pi}_A^{2g-i}$ for all i, and $4y=0$. Then $y\circ\pi_A^1 = y\bullet\hat{\pi}_A^1 = \hat{\pi}_A^0\bullet y = x_0$ is not necessarily 0, but to achieve this we just replace y by $y-x_0+{}^tx_0$, or equivalently redefine y as $\sum_{i=1}^{g-1} x_i + x_{2g-1}$. Then $(\tilde{\pi}_A^i)$ still verifies Theorem 7.1 a), by part b) of this theorem.

c) follows from taking the transpose of the formula in Theorem 7.5. □

8. Proof of Corollary 1.8

We use the projector π_A^1 of Theorem 7.1. It defines a motive $h^1_{\text{ét}}(A) \in \mathcal{M}_{\text{ét}}$, direct summand of the motive $h(A)$ associated to A. We write $i_A : h^1_{\text{ét}}(A) \to h_{\text{ét}}(A)$ for the inclusion and $p_A : h_{\text{ét}}(A) \to h^1_{\text{ét}}(A)$ for the projection, so that $p_A i_A = 1$ and $i_A p_A = \pi_A^1$.

Let $f : A \to B$ be a homomorphism. We define

$$h^1_{\text{ét}}(f) = p_A f^* i_B$$

where $f^* = h(f) : h_{\text{ét}}(B) \to h_{\text{ét}}(A)$.

To show that $h^1_{\text{ét}}$ is a (contravariant) functor, let $g : B \to C$ be another morphism. We must show that $h^1_{\text{ét}}(g \circ f) = h^1_{\text{ét}}(f) \circ h^1_{\text{ét}}(g)$. It suffices to prove this equality after composing with ι_A on the right and with p_C on the left, hence to prove tie identity

$$\pi_A^1 \circ f^* \circ g^* \circ \pi_C^1 = \pi_A^1 \circ f^* \circ \pi_B^1 \circ g^* \circ \pi_C^1.$$

The difference between the two sides is

$$\begin{aligned}\pi_A^1 \circ (f^* - f^* \circ \pi_B^1) \circ g^* \circ \pi_C^1 &= \pi_A^1 \circ (\pi_A^1 \circ f^* - f^* \circ \pi_B^1) \circ g^* \circ \pi_C^1 \\ &= \pi_A^1 \circ (c_{2,1} - c_{1,0}) \circ g^* \circ \pi_C^1 = (c_{2,1} - c_{1,0}) \circ \pi_B^2 \circ g^* \circ \pi_C^1 \\ &= (c_{2,1} - c_{1,0}) \circ (g^* \circ \pi_C^2 + d_{3,2} - d_{2,1}) \circ \pi_C^1 \\ &= (c_{2,1} - c_{1,0}) \circ (d_{3,2} - d_{2,1}) \circ \pi_C^1 = 0\end{aligned}$$

where $c_{2,1} - c_{1,0}$ is as in (7.11), $d_{3,2} - d_{2,1}$ is similar with respect to g and the final vanishing follows from Corollary 2.7. This proves the existence of the functor, which is obviously additive. Its full faithfulness will be proven in Corollary A.11.

For later use, we record here:

Proposition 8.1. *Then we have an isomorphism*

$$A(k)[1/p]_{\text{tors}} \xrightarrow{\sim} \mathcal{M}_{\text{ét}}(h^1_{\text{ét}}(A), h_{\text{ét}}(X))_{\text{tors}}$$

for any smooth projective k-variety X.

Proof. This follows from Proposition 2.1 b) and the isomorphism (5.3). □

9. Extension to abelian schemes

Let S be a smooth k-scheme, where k is now an arbitrary field.[5] We want to extend what comes before to abelian S-schemes $\mathcal{A}$ in the style of

[5] k could even be a Dedekind domain, see [25, Rem. 1.1], but one would have to invert all primes p that are not invertible in k. This somewhat reduces the interest of this generalisation when k is a localised ring of integers in a number field.

[10]. This is not straightforward, as the torsion of $CH^g_{\text{ét}}(\mathcal{A}\times_S\mathcal{A})$ is more complicated. So we proceed step by step.

Lemma 9.1. *The ideal* $CH^g_{\text{ét}}(\mathcal{A}\times_S\mathcal{A})[1/p]_{\text{tors}}$ *of* $CH^g_{\text{ét}}(\mathcal{A}\times_S\mathcal{A})[1/p]$ *is nilpotent.*

Proof. Consider the Leray spectral sequence

$$H^a_{\text{ét}}(S,R^b(\pi\times_S\pi)_*\mathbf{Z}(g))\Rightarrow H^{a+b}_{\text{ét}}(\mathcal{A}\times_S\mathcal{A},\mathbf{Z}(g))$$

where $\pi:\mathcal{A}\to S$ is the projection. It induces a filtration $F^aCH^g_{\text{ét}}(\mathcal{A}\times_S\mathcal{A})[1/p]$ on $CH^g_{\text{ét}}(\mathcal{A}\times_S\mathcal{A})[1/p]$ of length $\le\inf(\operatorname{cd}(S),2g)$, and it is standard that $F^a\circ F^{a'}\subseteq F^{a+a'}$. So it suffices to show that $CH^g_{\text{ét}}(\mathcal{A}\times_S\mathcal{A})[1/p]_{\text{tors}}\circ CH^g_{\text{ét}}(\mathcal{A}\times_S\mathcal{A})[1/p]_{\text{tors}}\subseteq F^1$. But this follows from Corollary 2.7, applied over a separable closure of $k(S)$. □

Theorem 7.1 a) now extends to $\mathcal{A}$ as follows:

Theorem 9.2. *For all i, let*

$$N_i=\prod_{a=0}^{\inf(\operatorname{cd}(S),2g-1)} w_{i,i-a-1}$$

where the $w_{i,j}$ are as in Definition 6.1 and $\operatorname{cd}(S)$ is the (possibly infinite) étale cohomological dimension of S. Then one can find a system (π^i_A) of orthogonal projectors of sum 1 in $CH^g_{\text{ét}}(\mathcal{A}\times_S\mathcal{A})[1/p]$ such that π^0_A is given by the 0-section of $\mathcal{A}$ and

$$N_in^*_A\circ\pi^i_A=N_in^i\pi^i_A \text{ for any } n\in\mathbf{Z}.$$

Proof. We mimick the proof of Theorem 7.1, starting from the DM projectors of [10]. Thanks to Lemma 9.1, we can go as far as Step 2 in §7.1. For Step 3, we consider this time the Leray spectral sequence

$$H^a_{\text{ét}}(S,R^b(\pi\times_S\pi)_*\mathbf{Q}_l/\mathbf{Z}_l(g))\Rightarrow H^{a+b}_{\text{ét}}(\mathcal{A}\times_S\mathcal{A},\mathbf{Q}_l/\mathbf{Z}_l(g)).$$

By smooth and proper base change, the sheaves $R^b(\pi\times_S\pi)_*\mathbf{Q}_l/\mathbf{Z}_l(g)$ are locally constant; therefore similar computations as in §5 hold, namely

$$(R^b(\pi\times_S\pi)_*\mathbf{Z}_l(g))\otimes\mathbf{Q}_l/\mathbf{Z}_l\xrightarrow{\sim}R^b(\pi\times_S\pi)_*\mathbf{Q}_l/\mathbf{Z}_l(g);$$

$$R^b(\pi\times_S\pi)_*\mathbf{Z}_l(g)\simeq\bigoplus_i\underline{\operatorname{Hom}}(R^i\pi_*\mathbf{Z}_l,R^{i+b-2g}\mathbf{Z}_l).$$

If $(\hat\pi^i(\mathcal{A}))_{i=0}^{2g}$ denote the Künneth projectors of $H^0(S,R^{2g}(\pi\times_S\pi)_*\hat{\mathbf{Z}}(g))$, then, by Lemma 5.3, $\hat\pi^i(\mathcal{A})$ acts on the right on $R^b(\pi\times_S\pi)_*\hat{\mathbf{Z}}(g)$ as $\hat\pi^{i+b-2g}(\mathcal{A})$ acts on the left, and the same holds for their action on $H^a_{\text{ét}}(S,R^b(\pi\times_S\pi)_*(\mathbf{Q}/\mathbf{Z})'(g))$ for all a.

We show the statement modulo $F^aCH^g_{\text{ét}}(\mathcal{A}\times_S\mathcal{A})[1/p]_{\text{tors}}:=F^a\cap CH^g_{\text{ét}}(\mathcal{A}\times_S\mathcal{A})[1/p]_{\text{tors}}$, by induction on a. Let $(\pi^i_{\mathcal{A}}(a-1))$ be a set of projectors modulo F^{a-1}, and $(\tilde\pi^i_{\mathcal{A}}(a-1))$ a lift modulo F^a. As in Step 3 of the proof of

Theorem 7.1 a), the obstruction for $\pi^i_{\mathcal{A}}(a)$ is a 1-cocycle f with values in gr^a, verifying

$$f(mn) = m^{i-a-1}f(n) + n^i f(m).$$

(If $i-a-1<0$, there is no obstruction.)

By Proposition 6.3, $w_{i,i-a-1}f$ is a coboundary, hence the theorem. □

10. Getting rid of 2-torsion: proof of Theorem 1.7

10.1. Good abelian schemes.

Definition 10.1. An abelian scheme $\mathcal{A}\to S$ is *good* (resp. *étale-good*) if it admits a set of integral (resp. étale integral) DM projectors (Definition 1.6).

Lemma 10.2. *Suppose that $S=\operatorname{Spec} k$ with k separably closed. Then*
a) The set of étale integral DM projectors $(\pi^i_{\mathcal{A}})$ forms a torsor for conjugation by elements of the form $1+x$ with $2x=0$ and $x\circ\pi^1=0$.
b) They commute with $n^*_{\mathcal{A}}$ for any $n\in\mathbf{Z}$, and verify Theorem 7.5 with multiplying by 2 instead of 4.
c) We have the identity

$$\pi^i_{\mathcal{A}}\circ(n_{\mathcal{A}})_* = n^{2g-i}\pi^i_{\mathcal{A}}$$

for any $i\in[0,2g]$ and any $n\in\mathbf{Z}$ except perhaps if $n\equiv 2 \pmod 4$, in which case it holds at least after multiplying by 2.

Proof. a) follows from Theorem 7.1 c), and implies b) as one checks by going through the proofs of Lemma 7.4 and Theorem 7.5 (the functions g_i and h_i used there are 0 in the present case).

For c), recall from Corollary 7.8 a) that $2g^i(n)=0$ for all n, where

$$g^i(n) = \pi^i_{\mathcal{A}}\circ((n_{\mathcal{A}})_* - n^{2g-i}).$$

By b), we have $g^i(n)\circ n^*_{\mathcal{A}}=0$. But, as in the proof of Corollary 7.8 a) for b, $g^i(n)\circ n^*_{\mathcal{A}} = n^{i+1}g^i(n)$. This shows that $g^i(n)=0$ for n odd; also, $g^i(n)=0$ for n divisible by 4 by Proposition 6.3 b). □

Remark 10.3. Still by Proposition 6.3 b), $g^i(2m)=g^i(2)$ if m is odd. So the defect in Lemma 10.2 c) is entirely controlled by $2_{\mathcal{A}}$. Moreover, $g^i(2)$ does not change when one modifies the $\pi^i_{\mathcal{A}}$ by conjugation as in Lemma 10.2 a), as an immediate computation shows; therefore these are invariants of $\mathcal{A}$.

Lemma 10.4. *Suppose that $S=\operatorname{Spec} k$ with k separably closed. If $\mathcal{A}$ is étale good, it enjoys a set of self-conjugate étale integral DM projectors if and only if the invariants $g^i(2)$ of Remark 10.3 vanish. In general, we can at least achieve $2\pi^{2g-i}_A = 2{}^t\pi^i_A$.*

Proof. It improves that of Corollary 7.8:

Let $(\pi^i_{\mathcal{A}})_{i=0}^{2g}$ be a set of étale integral DM projectors. By Lemma 10.2 c) and the hypothesis, $({}^t\pi^{2g-i}_{\mathcal{A}})_{i=0}^{2g}$ is another set of étale integral DM projectors.

Reasoning as in the proof of Corollary 7.8 b), we find y with $2y = 0$ such that the projectors

$$\tilde{\pi}^i_{\mathcal{A}} = (1+y)\pi^i_{\mathcal{A}}(1+y)^{-1}$$

are self-conjugate. But the $\tilde{\pi}^i_{\mathcal{A}}$ are also DM projectors, thanks to Lemma 10.2 a). Conversely, if $\mathcal{A}$ has a set of self-conjugate étale integral DM projectors, the invariants of Remark 10.3 must obviously vanish. □

Definition 10.5. Under the conditions of Lemma 10.4, we say that $\mathcal{A} \to \operatorname{Spec} k$ is *very good*.

Lemma 10.6. *If $\mathcal{A}, \mathcal{B}$ are good, then $\mathcal{A} \times_S \mathcal{B}$ is good; similarly for "étale-good" and "very good". The converse is true for étale-good if $S = \operatorname{Spec} k$ with k separably closed.*

Proof. The 'if' part of is obvious by taking the usual choice for the CK projectors

$$\pi^i_{\mathcal{A}\times_S\mathcal{B}} = \sum_{j+k=i} \pi^j_{\mathcal{A}} \times_S \pi^k_{\mathcal{B}}.$$

Conversely, let $(\pi^*_{\mathcal{A}\times\mathcal{B}})$ be a set of étale integral DM projectors (here we assume $S = \operatorname{Spec} k$, k separably closed). Let $\iota : \mathcal{A} \hookrightarrow \mathcal{A} \times \mathcal{B}$ be the inclusion $a \mapsto (a, 0)$, and let $\pi : \mathcal{A} \times \mathcal{B} \to \mathcal{A}$ be the projection. Let

$$\pi^i = \iota^* \circ \pi^i_{\mathcal{A}\times\mathcal{B}} \circ \pi^*.$$

Then π^i lifts the i-th l-adic Künneth projector of $\mathcal{A}$ for all $l \neq p$. By Proposition 7.2, the $\pi^i_A = (\pi^i)^2$ form a set of orthogonal projectors of sum 1. Moreover, for $n \in \mathbf{Z}$ one has

$$n^*_A \circ \pi^i = \iota^* \circ n^*_B \circ \pi^i_{\mathcal{A}\times\mathcal{B}} \circ \pi^* = \iota^* \circ n^i \circ \pi^i_{\mathcal{A}\times\mathcal{B}} \circ \pi^* = n^i \pi^i$$

hence $n^*_A \circ \pi^i_A = n^i \pi^i_A$ as well. □

10.2. **Elliptic curves.** Let C be a curve over the algebraically closed field k, $c \in C(k)$ and let X be a smooth projective k-variety. The standard choice of Chow-Künneth projectors

$$\pi^0 = \{c\} \times C, \pi^2 = C \times \{c\}, \pi^1 = \Delta_C - \pi^0 - \pi^2$$

acts on the group of correspondences $CH^1(C \times X)$ by composition on the right. On the other hand, we have the decomposition

$$CH^1(C \times X) = p_1^* CH^1(C) \oplus \operatorname{Corr}((C, c), (X, x)) \oplus p_2^* CH^1(X)$$

attached to a point $x \in X(k)$ as in [31, §6, Remark], where p_i is the i-th projection and $\operatorname{Corr}((C, c), (X, x))$ is the group of divisorial correspondences. Write (p^0, p^1, p^2) for the orthogonal projectors on $CH^1(C \times X)$ with images these summands. More specifically, let $p : C \to \operatorname{Spec} k$ be the structural morphism and $i : \operatorname{Spec} k \to C$ be the section at c; let $q : X \to \operatorname{Spec} k$ be the structural morphism and $j : \operatorname{Spec} k \to X$ be the section at x. We have $p_2 = p \times 1_X$, $p_1 = 1_C \times q$ and

$$p^0 = (1_C \times (jq))^*, \quad p^2 = ((ip) \times 1_X)^*.$$

We have a further decomposition

$$p^0 = p^0_1 + p^0_2$$

where $\operatorname{Im} p^0_1 = p_1^* \operatorname{Pic}^0(C)$ and $\operatorname{Im} p^0_2 = \mathbf{Z} p_1^*[c]$.

Lemma 10.7. *a) For $\xi \in CH^1(C \times X)$, we have $\xi \circ \pi^2 = p^2(\xi)$ and $\xi \circ \pi^0 = p^0_2(\xi)$, hence $\xi \circ \pi^1 = p^1(\xi) + p^0_1(\xi)$.*
b) If $f : X \to C$ is a morphism, then

$$p^0_1({}^t\Gamma_f) = p_1^*([c] - [f(x)]).$$

Proof. We have

$$\xi \circ \pi^2 = ((ip) \times 1_X)^* \xi, \quad \xi \circ \pi^0 = ((ip) \times 1_X)_* \xi.$$

The first equality gives the first claim of a), which also implies $p^2(\xi) \circ \pi^0 = 0$. Then

$$\begin{aligned} p^0(\xi) \circ \pi^0 &= ((ip) \times 1_X)_*(1_C \times (jq))^* \xi = (i \times 1)_* q^* p_* (1 \times j)^* \xi \\ &= (i \times 1)_* q^* j^* (p \times 1)_* \xi = (i \times 1)_* (p \times 1)_* \xi = \xi \circ \pi^0 \end{aligned}$$

hence $p^1(\xi) \circ \pi^0 = 0$. Finally, note that

$$(p \times 1_X)_* \xi = d_1(\xi)[X]$$

(which defines $d_1(\xi)$), hence $\xi \circ \pi^0 = d_1(\xi)[c \times X]$. If $\xi = p_1^* \eta$, then $d_1(\xi) = \deg(\eta)$; in particular, $p_1^* \eta \circ \pi^0 = 0$ if $\eta \in \operatorname{Pic}^0(C)$. This proves the second claim of a), hence its third claim. We then compute

$$p^0({}^t\Gamma_f) = (1 \times q)^* (1 \times j)^{* t}\Gamma_f = (1 \times q)^* [f(x))] = p_1^* [f(x))]$$

hence b). ☐

Theorem 10.8. *Elliptic curves $\mathcal{E} \to S$ are good. More precisely, the standard choice of projectors: $\pi^0_{\mathcal{E}} = \{0\} \times_S \mathcal{E}$, $\pi^2_{\mathcal{E}} = \mathcal{E} \times_S \{0\}$ and $\pi^1_{\mathcal{E}} = \Delta_{\mathcal{E}} - \pi^0_{\mathcal{E}} - \pi^2_{\mathcal{E}}$ is a set of integral DM projectors.*

Proof. We proceed in three steps.

Step 1. $S = \operatorname{Spec} k$ with k algebraically closed.

Obviously, $n^*_{\mathcal{E}} \circ \pi^0_{\mathcal{E}} = (1 \times n_{\mathcal{E}})^* \{0\} \times \mathcal{E} = \{0\} \times \mathcal{E}$.

Next, $n^*_{\mathcal{E}} \circ \pi^2_{\mathcal{E}} = (1 \times n_{\mathcal{E}})^* \mathcal{E} \times \{0\} = \mathcal{E} \times {}_n\mathcal{E}$. We reduce to two cases:

- n **is invertible in** k**:** by the theorem of the square, the divisor class $[{}_n\mathcal{E}] - n^2[0]$ equals $n^2([\sum_{x \in {}_n\mathcal{E}} x] - [0])$. But $\sum_{x \in {}_n\mathcal{E}(k)} x = 0$ since ${}_n\mathcal{E}(k)$ is not cyclic, and $n^*_{\mathcal{E}} \circ \pi^2_{\mathcal{E}} = n^2 \pi^2_{\mathcal{E}}$.
- $n = p = \operatorname{char} k$**:** Here the group scheme ${}_n\mathcal{E}$ is nonreduced. According as $\mathcal{E}$ is supersingular or not, it is supported by 0 with multiplicity p^2, or is of the form $F \times G$ with $F \simeq \mathbf{Z}/p$ and $G \simeq \mu_p$ [31, §15, p. 247]. In the first case, we obviously have $[{}_n\mathcal{E}] = p^2[0]$, while in the second, we have

$$[{}_n\mathcal{E}] - n^2[0] = p[F] - p^2[0] = p[\sum_{x \in F(k)} x] - p^2[0] = 0$$

because $\sum_{x\in\mathbf{Z}/p} x$ equals 0 when p is odd and 1 when $p=2$, but $2([1]-[0])=0$ in this case (phew!)

There remains the case of $\pi^1_{\mathcal{E}}$. For this, we apply Lemma 10.7 with $(C,c)=(X,x)=(\mathcal{E},0)$ and $f=n_{\mathcal{E}}$. Since $n_{\mathcal{E}}(0)=0$, we find that $\operatorname{Im}\pi^1_{\mathcal{E}}=\operatorname{Corr}((\mathcal{E},0),(\mathcal{E},0))$. But the map $\operatorname{Corr}((\mathcal{E},0),(\mathcal{E},0))\to\operatorname{End}(\mathcal{E})$ given by the action of divisorial correspondences is an isomorphism of rings; therefore $n^*_{\mathcal{E}}\circ\pi^1_{\mathcal{E}}=n\pi^1_{\mathcal{E}}$.

Step 2. $S=\operatorname{Spec}k$ with k arbitrary. We simply use the fact that $\operatorname{Pic}(\mathcal{E}\times_k\mathcal{E})\to\operatorname{Pic}((\mathcal{E}\times_k\mathcal{E})\times_k\bar{k})$ is injective for an algebraic closure $\bar{k}/k$.

Step 3. The general case. Let η be the generic point of S. We have an exact sequence

$$0\to\operatorname{Pic}(S)\xrightarrow{p^*_{(2)}}\operatorname{Pic}(\mathcal{E}\times_S\mathcal{E})\to\operatorname{Pic}(\mathcal{E}_\eta\times_\eta\mathcal{E}_\eta)\to 0$$

where $p_{(2)}:\mathcal{E}\times_S\mathcal{E}\to S$ is the structural map.

Let n be an integer $\neq 0$. For $i=0$, the identity $n^*_{\mathcal{E}}\circ\pi^0_{\mathcal{E}}=\pi^0_{\mathcal{E}}$ is obvious. Suppose that $i=1$. By Step 2 applied over η, $n^*_{\mathcal{E}}\circ\pi^1_{\mathcal{E}}-n\pi^1_{\mathcal{E}}\in\operatorname{Pic}(S)$.

Let $x\in\operatorname{Pic}(S)$ and $\gamma\in\operatorname{Pic}(\mathcal{E}\times_S\mathcal{E})$. An easy computation shows that

$$p^*_{(2)}x\circ\gamma=d_1(\gamma)p^*_{(2)}x$$

where $d_1(\gamma)$ is given by the equality $(p_1)_*\gamma=d_1(\gamma)[\mathcal{E}]$ in $CH^0(\mathcal{E})$ for $p_1:\mathcal{E}\times_S\mathcal{E}\to S$ the first projection. In particular, $p^*_{(2)}x\circ\pi^1_{\mathcal{E}}=0$ and $n^*_{\mathcal{E}}\circ\pi^1_{\mathcal{E}}-n\pi^1_{\mathcal{E}}=(n^*_{\mathcal{E}}\circ\pi^1_{\mathcal{E}}-n\pi^1_{\mathcal{E}})\circ\pi^1_{\mathcal{E}}=0$.

For π^2, we have to see that $[{}_n\mathcal{E}]=n^2[0]\in\operatorname{Pic}(\mathcal{E})$. The argument is the same as over an algebraically closed field, since the theorem of the cube still holds [31, §10]. □

10.3. **A deformation result.**

Proposition 10.9. *Let $\pi:\mathcal{A}\to S$ be an abelian scheme, where S is a smooth variety over a separably closed field k. Let $s,t\in S(k)$. If $\mathcal{A}_s$ is étale-good, so is $\mathcal{A}_t$, and similarly for "very good".*

Proof. If $S=\operatorname{Spec}k$, Step 3 in the proof of Theorem 7.1 a) shows that we have a collection of obstruction classes

$$c_i(\mathcal{A})\in HH^1(\Lambda,M_i(\mathcal{A}))\,(2\le i\le 2g)$$

with

$$M_i(\mathcal{A})=\operatorname{Hom}(H^i_2(\mathcal{A}),H^{i-1}_2(\mathcal{A}))\otimes\mathbf{Q}/\mathbf{Z}$$

such that $\mathcal{A}$ is étale-good if and only if $c_i(\mathcal{A})=0$ for all i. This extends to any abelian scheme $\pi:\mathcal{A}\to S$:

$$c_i(\mathcal{A})\in HH^1(\Lambda,CH^i_{\text{ét}}(\mathcal{A}))[1/p]_{\text{tors}}).$$

Sheafifying for the étale topology, we get local obstruction classes

$$\tilde{c}_i(\mathcal{A})\in H^0_{\text{ét}}(S,HH^1(\Lambda,R^{2i}\pi_*\mathbf{Z}(i))[1/p]_{\text{tors}}).$$

By Proposition 2.6 c) and "Gersten's principle" [3, 2.4], the natural maps

$$R^{j-1}\pi_*\mathbf{Z}_l \otimes \mathbf{Q}_l/\mathbf{Z}_l(i) \to R^j\pi_*\mathbf{Z}(i))\{l\}$$

remain isomorphisms for $l \neq p$. By smooth and proper base change, $c_i(\mathcal{A}_s) = 0 \iff \tilde{c}_i(\mathcal{A}) = 0 \iff c_i(\mathcal{A}_t) = 0$. Same reasoning for the invariants of Remark 10.3, hence for "very good" by Lemma 10.4. □

10.4. **Proof of Theorem 1.7.** All abelian varieties are supposed to be over k (separably closed). By Lemma 10.6 and Theorem 10.8, products of elliptic curves are good, hence étale-good. By the connectedness of the moduli scheme of principally polarised abelian varieties [15, Ch. IV, Cor. 5.10] and Proposition 10.9, all principally polarised abelian varieties are étale-good. But then so is any abelian variety A by Lemma 10.6 again, since $(A\times \hat{A})^4$ is principally polarised (Zarhin's trick). The same argument shows that all principally polarised abelian varieties are very good, since the DM projectors for elliptic curves in Theorem 10.8 are self-conjugate. □

11. Proofs of Theorems 1.2 and 1.10

We give ourselves a system (π_A^j) of projectors verifying the conditions of Theorem 7.1.

11.1. Beauville's decomposition, p-integrally.

Lemma 11.1. *Let $i \geq 0$. Then,*
a) We have $\pi_A^{2i-1}x = x$ for any $x \in CH^i_{\text{ét}}(A)[1/p]_{\text{tors}}$.
b) For $j \neq 2i$, $\pi_A^j CH^i_{\text{ét}}(A)[1/p]$ is divisible.
c) For $j \neq 2i-1$, $\pi_A^j CH^i_{\text{ét}}(A)[1/p]$ is torsion-free.

Proof. a) write $x = \sum_j x^j$, with $x^j = \pi_A^j x$, and let $n \in \mathbf{Z}$. Then each x^j is torsion hence, by Proposition 2.6 c), we have $n_A^* x^j = n^{2i-1}x^j$ for any j. On the the other hand, $2n_A^* x^j = 2n^j x^j$. Therefore, $2(n^j - n^{2i-1})x^j = 0$. For $j \neq 2i-1$, this means that $\pi_A^j CH^i_{\text{ét}}(A)[1/p]_{\text{tors}}$ has finite exponent. But then it is 0, as a quotient of a divisible group.

b) Let $N = \bigcap_{l\neq p} \operatorname{Ker} \operatorname{cl}_l$; we have $N = \bigoplus_{j=0}^{2g} \pi_A^j N$. Since N is divisible by Proposition 2.1 a), all these summands are divisible. But $\pi_A^j CH^i_{\text{ét}}(A)[1/p]) \subset N$ if $j \neq 2i$, hence $\pi_A^j CH^i_{\text{ét}}(A)[1/p]) = \pi_A^j N$ for those j.

c) follows from a). □

In order to respect Beauville's numbering, we set

Definition 11.2. $CH^i_{\text{ét},s}(A) = \pi_A^{2i-s} CH^i_{\text{ét}}(A)[1/p]$.

Note that $CH^i_{\text{ét}}(A)[1/p] = \bigoplus_{s=2(i-g)}^{2i} CH^i_{\text{ét},s}(A)$.

Proposition 11.3. *Let $B_s = \{x \in CH^i_{\text{ét}}(A)[1/p] \mid n_A^* x = n^{2i-s}x \,\forall n \in \mathbf{Z}\}$. Then*
a) For $s \neq 0$ we have

$$B_s = F_s \oplus CH^i_{\text{ét},s}(A)$$

where $F_1 = 0$ *and* $F_s = {}_{w_{2i-s,2i-1}}CH^i_{\text{ét}}(A)[1/p]$ *for* $s \neq 1$.[6]
b) For $s = 0$*, we have inclusions*

$$2CH^i_{\text{ét},0}(A) \oplus {}_2CH^i_{\text{ét}}(A)[1/p] \subseteq B_0 \subseteq CH^i_{\text{ét},0}(A) \oplus {}_2CH^i_{\text{ét}}(A)[1/p].$$

If (π^j_A) *is a system of étale integral DM projectors, we even have*

$$B_0 = CH^i_{\text{ét},0}(A) \oplus {}_2CH^i_{\text{ét}}(A)[1/p]. \tag{11.1}$$

Note that in a) and b), the sums are direct since $CH^i_{\text{ét},s}(A)$ is torsion-free by Lemma 11.1 c).

Proof. If $x \in B_s$, let $x_t = \pi^{2i-t}_A x$ for all t. Let $n \in \mathbf{Z}$. On the one hand, $2n^*_A x_t = 2n^{2i-t}x_t$ by Theorem 7.1 a). On the other hand, $2n^*_A x_t = 2\pi^{2i-t}_A n^*_A x$ by Lemma 7.4. So $2n^*_A x_t = 2n^{2i-s}x_t$ and $2(n^{2i-t} - n^{2i-s})x_t = 0$, thus x_t is of torsion bounded independently of x if $t \neq s$. Thus there is an integer $N > 0$ such that $N(x - x_s) = 0$ for all $x \in B_s$, i.e. $B_s \subseteq CH^i_{\text{ét},s}(A) + {}_NCH^i_{\text{ét}}(A)[1/p]$. We also have $2CH^i_{\text{ét},s}(A) \subseteq B_s$ by Theorem 7.1 a).

Suppose that $s \neq 0$; then Lemma 11.1 b) implies that $CH^i_{\text{ét},s}(A) \subseteq B_s$ and that this inclusion is split. Therefore $B_s = CH^i_{\text{ét},s}(A) \oplus F_s$ where F_s is of finite exponent. But $F_s \subset CH^i_{\text{ét},1}(A)$ by Lemma 11.1 a); this shows that $F_1 = 0$ and that, for $s \neq 1$, F_s is killed by $w_{2i-s,2i-1}$. Conversely, ${}_{w_{2i-s,2i-1}}CH^i_{\text{ét}}(A)[1/p]$ is clearly contained in B_s. This proves a), and b) is obtained by the same reasonings: more precisely, the integer N of the beginning may be taken as $w_{2i,2i-1} = 2$.

In the case of a system of étale integral DM projectors, the inclusion $2CH^i_{\text{ét},0}(A) \subseteq B_0$ improves to $CH^i_{\text{ét},0}(A) \subseteq B_0$, hence the last claim. □

Corollary 11.4. *a) The subgroup* $CH^i_{\text{ét},s}(A)$ *of* $CH^i_{\text{ét}}(A)[1/p]$ *does not depend on the choice of* (π^i_A) *if* $s \neq 0$*. So does the subgroup* $CH^i_{\text{ét},0}(A) \oplus {}_2CH^i_{\text{ét}}(A)[1/p]$ *in the case of étale integral DM projectors.*
b) Let $(\tilde\pi^j_A)$ *be another set of projectors verifying Theorem 7.1; for* $x \in CH^i_{\text{ét}}(A)[1/p]$ *and* s*, let* $x_s = \pi^{2i-s}x$ *and* $\tilde x_s = \tilde\pi^{2i-s}_A x$*. Then* $x_s = \tilde x_s$ *for* $s \neq 0, 1$ *and* $\tilde x_0 - x_0 = x_1 - \tilde x_1$ *belongs to* ${}_4CH^i_{\text{ét}}(A)[1/p]$*, and even to* ${}_2CH^i_{\text{ét}}(A)[1/p]$ *in the case of étale integral DM projectors.*

Proof. a) Indeed, $CH^i_{\text{ét},s}(A)$ is the maximal divisible subgroup of B_s for $s \neq 0$ by Lemma 11.1 b) and Proposition 11.3 a). For $s = 0$, this follows directly from (11.1).

b) For any s, the operator $\tilde\pi^{2i-s} - \pi^{2i-s}$ is torsion by Theorem 7.1 b); therefore it vanishes on the maximal divisible subgroup of $CH^i_{\text{ét}}(A)[1/p]$, hence on $CH^i_{\text{ét},t}(A)$ for $t \neq 0$. On the other hand, for $s \neq 0, 1$, x_s and $\tilde x_s$ belong to a common uniquely divisible subgroup by a), hence so does their difference which is therefore 0. This gives the first claim, which implies

[6] Here, we write by convention $w_{a,b} := w_{b,a}$ if $a < b$.

$x_0 + x_1 = \tilde{x}_0 + \tilde{x}_1$. But $4(\tilde{\pi}^{2i} - \pi^{2i}) = 0$, and even $2(\tilde{\pi}^{2i} - \pi^{2i}) = 0$ if (π^j) and $(\tilde{\pi}^j)$ are étale integral DM projectors. □

Corollary 11.5. *We have $CH^i_{\text{ét},s}(A) = 0$ if $s \notin [i-g, i]$.*

Proof. By [5, Théorème], $B_s \otimes \mathbf{Q}$ vanishes in this range. Therefore so does $CH^i_{\text{ét},s}(A) \otimes \mathbf{Q}$ by Proposition 11.3, and also $CH^i_{\text{ét},s}(A)$ by Lemma 11.1 c). (Note that $1 \in [i-g, i]$ except for $i = 0$ and that $CH^0_{\text{ét},1}(A) = 0$ trivially.) □

Corollary 11.6. *We have $CH^1_{\text{ét},1}(A) = \mathrm{Pic}^0(A)$.*

Proof. We have $CH^1_{\text{ét},1}(A) = B_1$ by Proposition 11.3 a): this is $\mathrm{Pic}^0(A)$ since $\mathrm{NS}(A)$ is torsion-free. □

Corollary 11.7. *Let $f : A \to B$ be an isogeny. Then $f_* CH^i_{\text{ét},s}(A) \subseteq CH^i_{\text{ét},s}(B)$ for $s \neq 0$. If the (π^i_A) and the (π^i_B) are both étale integral DM projectors, $f_* CH^i_{\text{ét},0}(A) \subseteq CH^i_{\text{ét},0}(B) \oplus {}_2CH^i_{\text{ét}}(B)[1/p]$.*

Proof. Since $n_B^* f_* = f_* n_A^*$ for any $n \in \mathbf{Z}$, we have $f_* B_s(A) \subseteq B_s(B)$ for any s. The conclusion follows as in the proof of Corollary 11.4. □

For the next corollary, note that $B^i_s \cdot B^j_t \subset B^{i+j}_{s+t}$ for all i, j, s, t, with obvious notation.

Corollary 11.8. *We have $CH^i_{\text{ét},s}(A) \cdot CH^j_{\text{ét},t}(A) \subseteq CH^{i+j}_{\text{ét},s+t}(A)$ if $(s,t) \neq (0,0)$.*

Proof. Indeed, in this case either $CH^i_{\text{ét},s}(A)$ or $CH^j_{\text{ét},t}(A)$ is divisible by Lemma 11.1 b), hence so are their tensor product and its image in B^{i+j}_{s+t}. This image is therefore contained in the largest divisible subgroup of B^{i+j}_{s+t}, which is contained in $CH^{i+j}_{\text{ét},s+t}(A)$ by Proposition 11.3 (even if $s+t = 0$). □

11.2. **The good divided powers.** Let $x \in CH^i_{\text{ét}}(A)[1/p]$ with $i > 0$, and let $n \geq 2$. We first define $\gamma_n(x)$ when $x \in CH^i_{\text{ét},s}(A)$ for some s. Suppose $s \neq 0$. Then $x^n \in CH^{ni}_{\text{ét},ns}(A)$ by Corollary 11.8. Since $ns \neq 0, 1$, this group is uniquely divisible by Lemma 11.1 and we just set $\gamma_n(x) = \frac{x^n}{n!}$.

Suppose now that $s = 0$. Then $x^n = x^{[n]} + y_n$, with $x^{[n]} \in CH^{ni}_{\text{ét},0}(A)$ and $2y_n = 0$ (unique decomposition). In the torsion-free group $CH^{ni}_{\text{ét},0}(A)$, $x^{[n]}$ is divisible by $n!$ (same reasoning as in the proof of Proposition 1.1, using Corollary 2.4). We define

$$\gamma_n(x) = \frac{x^{[n]}}{n!}.$$

In general, write $x = \sum_s x_s$ with $x_s \in CH^i_{\text{ét},s}(A)$. We set

$$\gamma_T(x,s) = \sum_{n \geq 0} \gamma_n(x_s) T^n \in CH^*_{\text{ét}}(A)[1/p][T]$$

where $\gamma_0(x_s) := 1$ and $\gamma_1(x_s) := x_s$,

$$\gamma_T(x) = \prod_s \gamma_T(x,s)$$

and $\gamma_n(x) =$ the n-th coefficient of $\gamma_T(x)$.

We may extend the γ_n by the same trick to inhomogeneous elements of $CH^{>0}_{\text{ét}}(A)[1/p]$ (this will not be used in the rest of this article).

Remark 11.9. This construction depends on the choice of (π^i_A), and does not satisfy the divided power identities in $CH^{>0}_{\text{ét}}(A)[1/p]$. To get them, we must go modulo 2-torsion as in the next subsection.

11.3. **Étale motives modulo 2-torsion.** We introduce here a category which helps formulate our results:

Definition 11.10. For an abelian variety A and $i \geq 0$, we set

$$\overline{\mathrm{CH}}^i_{\text{ét}}(A) = CH^i_{\text{ét}}(A)[1/p]/_2CH^i_{\text{ét}}(A)[1/p].$$

If B is another abelian variety of dimension g_B, we set

$$\overline{\mathrm{Corr}}_{\text{ét}}(A,B) = \overline{\mathrm{CH}}^{g_B}_{\text{ét}}(B \times A).$$

We write $\bar{\mathcal{M}}^{\mathrm{ab}}_{\text{ét}}$ for the pseudo-abelian hull of the additive category with objects abelian varieties and morphisms the $\overline{\mathrm{Corr}}_{\text{ét}}(A,B)$, and $\bar{h}_{\text{ét}}(A)$ for the image of A in $\bar{\mathcal{M}}^{\mathrm{ab}}_{\text{ét}}$.

Note that the functors $\hat{R} : \mathcal{M}^{\mathrm{ab}}_{\text{ét}} \to \hat{\mathcal{M}}^{\mathrm{ab}}$ and $R_B : \mathcal{M}^{\mathrm{ab}}_{\text{ét}} \to \mathcal{M}^{\mathrm{ab}}_B$ both factor through $\bar{\mathcal{M}}^{\mathrm{ab}}_{\text{ét}}$. One could equally define a larger category $\bar{\mathcal{M}}_{\text{ét}}$ encompassing all smooth projective varieties, but the above would fail in general, so this does not seem too useful.

We can now reformulate some of the previous results as follows:

Theorem 11.11. *For any abelian variety A of dimension g,*
a) A system of self-conjugate projectors (π^i_A) as in Corollary 7.8 b) induces a system of self-conjugate projectors in $\overline{\mathrm{Corr}}_{\text{ét}}(A,A)$ having the property of [10] with respect to multiplications; we call it a system of DM projectors modulo 2-torsion*. We have*

$$\pi^i_A \circ n^*_A = n^*_A \pi^i_A, \quad \pi^i_A \circ (n_A)_* = n^{2g-i}\pi^i_A$$

for any i and any $n \in \mathbf{Z}$.
b) For $i \geq 0$, the direct sum decomposition

$$\overline{\mathrm{CH}}^i_{\text{ét}}(A) = \bigoplus_s \overline{\mathrm{CH}}^i_{\text{ét},s}(A)$$

where $\overline{\mathrm{CH}}^i_{\text{ét},s}(A)$ is the image of $CH^i_{\text{ét},s}(A)$ in $\overline{\mathrm{CH}}^i_{\text{ét}}(A)$, does not depend on the choice of (π^i_A); $\overline{\mathrm{CH}}^i_{\text{ét},s}(A)$ is divisible for $s \neq 0$ and torsion-free for $s \neq 1$.
c) If (π^i_A) is étale integral (Definition 10.1), its image in $\overline{\mathrm{Corr}}(A,A)$ is self-conjugate and does not depend on the choice; we call it the canonical system of DM projectors *for A.*

d) If B is another abelian variety provided with a system (π_B^i) as in a) and if $f : A \to B$ is a homomorphism, we have

$$2f^*\circ\pi_B^i = 2\pi_A^i\circ f^* \in \overline{\mathrm{Corr}}(B,A), \quad 2\pi_B^{2g_B-i}\circ f_* = 2f_*\circ\pi_A^{2g_A-i} \in \overline{\mathrm{Corr}}(A,B)$$

for all i, where g_A and g_B are the dimensions of A and B. If the systems (π_A^i) and (π_B^i) are canonical, we can remove the factor 2 in both identities; in particular, $f^*\,\overline{\mathrm{CH}}^i_{\text{ét},s}(B) \subseteq \overline{\mathrm{CH}}^i_{\text{ét},s}(A)$ and $f_*\,\overline{\mathrm{CH}}^{g_A-i}_{\text{ét},s}(A) \subseteq \overline{\mathrm{CH}}^{g_B-i}_{\text{ét},s}(B)$ for any (i,s).

Proof. a) follows from Lemma 7.4 and Corollary 7.8 a). b) follows from Corollary 11.4 a). c) follows from Lemmas 10.2 and 10.4. d) follows from Theorem 7.5, Corollary 7.8 c) and Lemma 10.2 b). □

We now push $\gamma_T(x)$ in $\overline{\mathrm{CH}}^*_{\text{ét}}(A)[T]$. We have:

Proposition 11.12. *The function $\gamma_T : CH^{>0}_{\text{ét}}(A)[1/p] \to \overline{\mathrm{CH}}^*_{\text{ét}}(A)[T]$ factors through $\overline{\mathrm{CH}}^{>0}_{\text{ét}}(A)$.*

Proof. Let $i > 0$ and $x, y \in CH^i_{\text{ét}}(A)[1/p]$, with $2y = 0$: we must show that $2\gamma_T(x+y) = 2\gamma_T(x)$. Recall that $y \in CH^i_{\text{ét},1}(A)$ (Lemma 11.1 a)). This already shows that $\gamma_T(x,s) = \gamma_T(x+y,s)$ for $s \neq 1$; but for $s = 1$ and $n > 1$,

$$(x_1+y)^n = x_1^n + nyx_1^{n-1} = x_1^n$$

where the first equality comes from Proposition 2.6 d) and the second one from the divisibility of $CH^{ni}_{\text{ét},n}(A)$ (Lemma 11.1 b)). Therefore $\gamma_n(x_1+y) = \gamma_n(x_1)$. □

Proof of Theorem 1.2. a) The divided power identities of the introduction are all homogeneous of a certain degree N: $N = 0, 1$ in (1), $N = n$ in (2) and (3), $N = m+n$ in (4) and $N = mn$ is (5).

They are true tautologically after tensoring with $\mathbf{Q}$, as well as in degree $N = 1$. For the identities of degree $N > 1$, the target group is torsion free (in $\overline{\mathrm{CH}}^*_{\text{ét}}(A)$!), as seen in the definition of the γ_n's, so they remain true.

b) Same argument as in a). □

Definition 11.13. Although the γ_i's now verify the divided power identities, they still depend on the choice of (π_A^j) in $\overline{\mathrm{CH}}^{>0}_{\text{ét}}(A)$. If we choose the canonical system of DM projectors as in Theorem 11.11 c), we shall talk of the *canonical divided powers*.

11.4. **Proof of Corollary 1.3.** From now on, all abelian varieties are provided with their canonical divided powers (Definition 11.13).

For $x \in \overline{\mathrm{CH}}^{>0}_{\text{ét}}(A)$, set

$$e^x = \sum_n \gamma_n(x).$$

This is the polynomial $\gamma_T(x)$ evaluated at $T = 1$. By Property (3) of divided powers, we have the identity

$$e^{x+y} = e^x e^y. \tag{11.2}$$

Definition 11.14. Let $\ell \in \operatorname{Pic}(A \times \hat{A})$ be the class of the (normalised) Poincaré bundle. We define $\mathcal{F}_A$ as the (inhomogeneous) correspondence given by e^ℓ.

To prove Corollary 1.3 (i) and (ii), we want to follow the arguments of [4, Proof of Proposition 3']. However, this proof rests on Corollary 2 to Proposition 3 in the said article, which, of course, rests on this proposition. Therefore, we must first prove an integral version of that corollary.

Proposition 11.15. *Let $\pi : A \times \hat{A} \to A$ be the first projection. Then $\pi_*(e^\ell) = [0] \in \overline{\mathrm{CH}}^g_{\text{ét}}(A)$.*

Proof. We have to show that

$$\pi_*\gamma_i(\ell) = \begin{cases} 0 & \text{if } i \neq g \\ [0] & \text{if } i = g, \end{cases}$$

this being known after tensoring with $\mathbf{Q}$ by [4, Cor. 2 to Prop. 3]. Since $\ell \in \overline{\mathrm{CH}}^1_{\text{ét},0}(A \times \hat{A})$, $\gamma_i(\ell) \in \overline{\mathrm{CH}}^i_{\text{ét},0}(A \times \hat{A})$ by construction. By Theorem 11.11 d), $\pi_*\gamma_i(\ell) \in \overline{\mathrm{CH}}^{i-g}_{\text{ét},0}(A)$. Since this group is torsion-free, we win. □

Thanks to Proposition 11.15 and its dual (transpose e^ℓ), Corollary 1.3 (i) and (ii) are proven as said before, by using (11.2) and Theorem 1.2 a). Moreover, (iii) is given by the argument of Mukai [30, proof of (3.4)] and Theorem 1.2 b).

Remark 11.16. If we relax the condition of taking the canonical divided powers, Proposition 11.15, and therefore Corollary 1.3, remain true after multiplying by 2.

11.5. **Back to abelian schemes: proof of Theorem 1.10.** Consider the situation of Section 9. Let l be a prime number. By Lemma 6.2, $l \mid w_{i,i-a-1} \iff i-a-1>0$ and $l-1 \mid a+1$. Therefore l divides the integer N_i of Theorem 9.2 if and only if $i > 1$ and

$$l-1 \mid \inf(\operatorname{cd}(S)+1, i-1)^{\&}.$$

This implies the statement for the DM projectors, and we get the others by the same arguments as above.

We leave it to the interested reader to refine the latter results in the style of Theorem 9.2.

12. Further properties of the Fourier transform

12.1. **DM projectors.** Let (π_A^i) be a system of projectors on A modulo 2-torsion (see Theorem 11.11 a)). We then get a system of projectors in $\overline{\mathrm{CH}}^g_{\text{ét}}(\hat{A} \times \hat{A})$

$$\pi^i_{\hat{A}} = \mathcal{F}_A \circ \pi_A^{2g-i} \circ \mathcal{F}_A^{-1}.$$

Proposition 12.1. *The $\pi^i_{\hat{A}}$ also form a system of DM projectors modulo 2-torsion.*

Proof. This follows from the commutation of $\mathcal{F}_A$ and $\mathcal{F}_{\hat{A}}$ with isogenies and from Corollary 7.8 a). □

Remark 12.2. It seems reasonable to expect that $(\pi^i_{\hat{A}})$ is canonical if (π^i_A) is, but I don't see a way to prove it.

Corollary 12.3 (cf. [5, Prop. 1])**.** *a) The Fourier transforms $\mathcal{F}_A$ and $\mathcal{F}_{\hat{A}}$ exchange $\overline{\mathrm{CH}}^i_{\text{ét},s}(A)$ and $\overline{\mathrm{CH}}^{g-i+s}_{\text{ét},s}(\hat{A})$ for all (i,s).*
b) The Pontryagin divided powers γ^*_n of Corollary 1.4 respect these subgroups.

Proof. a) follows from Proposition 12.1 and Theorem 11.11 b); b) follows from a) and the same property for the γ_n. □

Corollary 12.4 (cf. [4, Cor. 1 to Prop. 3])**.** *Let $c \in \overline{\mathrm{CH}}^i_{\text{ét}}(A)$ be a symmetric (resp. antisymmetric) element. Then the component of $\mathcal{F}_A(c)$ on $\overline{\mathrm{CH}}^{g-i+r}_{\text{ét}}(A)$ is 0 if r is odd (resp. even).*

Proof. Indeed, c belongs to $\bigoplus_{s \text{ even}} \overline{\mathrm{CH}}^i_{\text{ét},s}(A)$ (resp. to $\bigoplus_{s \text{ odd}} \overline{\mathrm{CH}}^i_{\text{ét},s}(A)$). □

12.2. Beauville's identities.

Proposition 12.5. *Suppose that A is provided with a polarisation $\phi : A \to \hat{A}$ of degree ν, induced by the class $d \in CH^1(A)$ of an ample symmetric divisor. Then we have, for any $i \in [0,g]$*

$$\nu\mathcal{F}_A(\gamma_i(d)) = (-1)^{g-i}\phi_*(\gamma_{g-1}(d)), \quad \nu\gamma^*_i(c) = \nu^i\gamma_{g-i}(d)$$

where $c = \gamma_{g-1}(d)$. In particular,

$$\gamma_g(d) = \nu[0]. \tag{12.1}$$

Proof. Same as for Proposition 11.15: the identities hold after tensoring with $\mathbf{Q}$ by [4, Prop. 6 and Cor. 2], but both of their terms are respectively in the torsion-free groups $\overline{\mathrm{CH}}^{g-i}_{\text{ét},0}(\hat{A})$ and $\overline{\mathrm{CH}}^i_{\text{ét},0}(A)$ (for the first identity, by Theorem 11.11 d) and Corollary 12.3 a); for the second one, by Corollary 12.3 b)). □

12.3. Application: Scholl's formula for the DM projectors.

Proposition 12.6. *In Proposition 12.5, suppose that the polarisation ϕ is principal. Consider the correspondences*

$$p_i = (-1)^i \sum_{2a+b=2g-i \text{ and } b+2c=i} \gamma_a(\pi_1^*d)\cdot\gamma_b((1\times\phi)^*\ell)\cdot\gamma_c(\pi_2^*d) \tag{12.2}$$

in $\overline{\mathrm{CH}}^g_{\text{ét}}(A\times A)$, where π_1,π_2 are the two projections $A\times A\to A$ (recall that ℓ is the class of the normalised Poincaré bundle). Let $\pi^i_A = p_i \circ p_i$. Then (π^i_A) is a system of DM projectors modulo 2-torsion.

Proof. By [39, 5.9], the statement is true after tensoring with $\mathbf{Q}$. Moreover, (12.1) implies as in loc. cit. that $\sum p_i = 1$ (this is the key point). We note that Proposition 7.2 works equally well in $\overline{\mathrm{CH}}^g_{\text{ét}}(A \times A)$. Hence the claim. □

Remark 12.7. I cannot extend Proposition 12.6 to a non-principal polarisation as in [39], except of course up to multiplying by its degree. I am also not sure what condition is needed on d to ensure that (π^i_A) is canonical.

12.4. **Jacobians of curves: Suh's formula.** Let C be a (smooth projective) curve of genus g, with Jacobian J. Choose a rational point on c and, for every $i \in [0, g]$, let $W^{[i]} = [C^{(g-i)}] \in CH^i(J)$ be the corresponding cycle class (image of $S^{g-i}(C)$ by the canonical map).

Theorem 12.8 (Étale integral Poincaré formula)**.** *a) We have $W^{[i]} = \gamma_i(d)$ in $\overline{\mathrm{CH}}^i_{\text{ét}}(J)$, where $d = W^{[1]}$.*
b) We have $W^{[r]} \cdot W^{[s]} = \binom{r+s}{r} W^{[r+s]}$ for any (r, s).

Proof. a) We first prove this after tensoring with $\mathbf{Q}$. Then $W^{[i]} = \gamma^*_{g-i}(c)$ where $c = [C]$ [29, (1.2.1], and the result follows from [4, Cor. 2 to Prop. 6] (the rational version of the second identity in Proposition 12.5). But both terms belong to the torsion-free group $\overline{\mathrm{CH}}^i_{\text{ét},0}(J)$: for $\gamma_i(d)$ this is because $d \in \overline{\mathrm{CH}}^1_{\text{ét},0}(J)$, and for $W^{[i]}$ this is because $n^*_A W^{[i]} = n^{2i} W^{[i]}$ for all $n \in \mathbf{Z}$, and $\overline{\mathrm{CH}}^i_{\text{ét},0}(J) = \overline{B}^i_0(J)$ with obvious notation (which follows from (11.1)). Thus the formula holds integrally.

b) This follows from a) and the identity (4) in the introduction for divided powers. □

The following corollary is an étale integral analogue (modulo 2-torsion) of J. Suh's formula for the Künneth projectors of J [41, Th. 4.2.3]:

Corollary 12.9. *Consider the classes*

$$P_i = (-1)^i \sum_{2a+b=2g-i \text{ and } b+2c=i} p_1^* W^{[a]} \cdot p_2^* W^{[c]} \times \sum_{d+e+f=b} (-1)^{d+f} p_1^* W^{[d]} \cdot \mu^* W^{[e]} \cdot p_2^* W^{[f]} \in \mathrm{Corr}(J, J). \tag{12.3}$$

Then the $\pi^i_A := P_i \circ P_i$ form a system of DM projectors modulo 2-torsion in $\overline{\mathrm{Corr}}_{\text{ét}}(J, J)$.

Proof. I claim that $P_i = p_i$, where p_i is as in (12.2). The argument is the same as in [41]: in (12.3), the term indexed by (d, e, f) is an expansion of the term $\gamma_b((1 \times \phi)^* \ell)$ in (12.2) given in Proposition 12.6, thanks to Theorem 12.8 a) and the identity $(1 \times \phi)^* \ell = p_1^* d + p_2^* d - \mu^* d$. □

Remark 12.10. Another formula for P_i in $\overline{\mathrm{Corr}}_{\text{ét}}(J,J)$ is

$$P_i = \sum_{r,s}(-1)^{g-r-s}\binom{r+s}{s+i-g} p_1^*W^{[r]}\cdot\mu^*W^{[g-r-s]}\cdot p_2^*W^{[s]}. \tag{12.4}$$

This is proven by using Theorem 12.8 b) and the Chu-Vandermonde identity; details are left to the reader.

Appendix A. The Picard and Albanese functors

Definition A.1. We let **AbS** denote the full subcategory of commutative k-group schemes whose objects are extensions of a constant, finitely generated group scheme by an abelian variety. If $P\in\mathbf{AbS}$, we write P^0 for its identity component and $\bar P$ for P/P^0. We say that P is *discrete* if $P^0=0$. Let $P,Q\in\mathbf{AbS}$. We say that a homomorphism $P(k)\to Q(k)$ is *representable* if it is induced by a morphism in **AbS**.

The aim of this appendix is to prove Theorems A.2 and A.10 below.

Theorem A.2. *There exists a naturally commutative diagram of additive functors*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\mathrm{Pic}} & \mathbf{AbS}\\ \downarrow & & \downarrow\\ \mathcal{M}_{\text{ét}} & \xrightarrow{\mathrm{Pic}_{\text{ét}}} & \mathbf{AbS}[1/p] \end{array}$$

such that $\underline{\mathrm{Pic}}(h(X))=\underline{\mathrm{Pic}}_{\text{ét}}(h_{\text{ét}}(X))=\mathrm{Pic}_{X/k}$ *for any smooth projective* X.

The functor $X\mapsto\mathrm{Pic}(X)=H^2_{\text{ét}}(X,\mathbf{Z}(1))$ from smooth projective varieties to abelian groups extends to a functor on $\mathcal{M}$, and even on $\mathcal{M}_{\text{ét}}$ when we invert p; we want to show that these functors induce the desired functors. For this, we need a few lemmas:

Lemma A.3. *The subfunctor* $X\mapsto\mathrm{Pic}^0(X)$ *of* Pic *extends to a functor on* $\mathcal{M}$*, and even on* $\mathcal{M}_{\text{ét}}$ *after inverting* p*.*

Proof. If $\alpha: X\to Y$ is a Chow correspondence, then $\alpha(\mathrm{Pic}^0(X))\subseteq\mathrm{Pic}^0(Y)$ because algebraic equivalence is an adequate equivalence relation. If α is now an étale Chow correspondence, there is an integer $n>0$ such that $n\alpha$ is algebraic. Since $\mathrm{Pic}^0(Y)$ is the maximal divisible subgroup of $\mathrm{Pic}(Y)$, we also have $\alpha(\mathrm{Pic}^0(X)[1/p])\subseteq\mathrm{Pic}^0(Y)[1/p]$. □

Lemma A.4. *The category* **AbS** *is pseudo-abelian.*

Proof. Indeed, any idempotent endomorphism e of **AbS** has a kernel, hence an image (= kernel of $1-e$). □

Lemma A.5. *Let* A,B *be two abelian varieties, and let* $\phi: A(k)[1/p]\to B(k)[1/p]$ *be a homomorphism. If there exists an integer* $n>0$ *prime to* p *such that* $n\phi$ *is representable, then so is* ϕ*.*

Proof. Since $B(k)$ is Zariski dense in B, the homomorphism $\mathbf{Ab}(A,B) \to \operatorname{Hom}(A(k), B(k))$ is injective. The conclusion follows from a chase in the commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0\to & \mathbf{Ab}(A,B) & \xrightarrow{n} & \mathbf{Ab}(A,B) & \longrightarrow & \operatorname{Hom}({}_nA,B) \\ & \downarrow & & \downarrow & & \wr\downarrow \\ 0\to & \operatorname{Hom}(A(k),B(k)) & \xrightarrow{n} & \operatorname{Hom}(A(k),B(k)) & \longrightarrow & \operatorname{Hom}({}_nA,B(k)) \end{array}$$

where $\operatorname{Hom}({}_nA,B)$ is computed in the category of commutative k-group schemes, and the last vertical map is an isomorphism. (We used that A and $A(k)$ are n-divisible.) □

Lemma A.6. *For any $P \in \mathbf{AbS}$, the inclusion $P^0 \hookrightarrow P$ has a retraction.*

Proof. Let $\bar{P} = P/P^0$ and P^τ be the inverse image of $\bar{P}_{\rm tors}$ (a finite group) in P. Then $P^0 \hookrightarrow P^\tau$ has a retraction because P^0 is connected, and $P^\tau \hookrightarrow P$ has a retraction because $\operatorname{Ext}(\bar{P}/\bar{P}_{\rm tors}, -) = 0$. The lemma follows. □

Lemma A.7. *Let $P, Q \in \mathbf{AbS}$ and let $\phi : P(k) \to Q(k)$ be a homomorphism. Assume that the restriction of ϕ to $P^0(k)$ is representable. Then ϕ is representable by a unique morphism $f : P \to Q$.*

Proof. First, notice that the representing morphism $f^0 : P^0 \to Q$ is unique, because $P^0(k)$ is Zariski dense in P^0. Consider now the commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0\to & \mathbf{AbS}(\bar{P},Q) & \longrightarrow & \mathbf{AbS}(P,Q) & \xrightarrow{a} & \mathbf{AbS}(P^0,Q) \\ & \wr\downarrow & & \downarrow & & \downarrow \\ 0\to & \operatorname{Hom}(\bar{P},Q(k)) & \longrightarrow & \operatorname{Hom}(P(k),Q(k)) & \longrightarrow & \operatorname{Hom}(P^0(k),Q(k)) \end{array} \tag{A.1}$$

where the left vertical map is an isomorphism. By Lemma A.6, a is surjective and the claim follows from diagram chase. □

Proof of Theorem A.2. We first deal with Pic. By Lemma A.4, it suffices to define it on motives of smooth projective varieties. For this, we use a theorem of Friedlander-Voevodsky (cf. [42, proof of Prop. 2.1.4]): for two (connected) smooth projective varieties X, Y, the group $CH^{d_X}(X \times Y)$ is generated by classes of finite correspondences. Let α be such a finite correspondence, assumed to be integral: the map

$$\alpha : \operatorname{Pic}(Y) \to \operatorname{Pic}(X)$$

factors as a composition

$$\operatorname{Pic}(Y) \xrightarrow{q^*} \operatorname{Pic}(Z) \xrightarrow{p_*} \operatorname{Pic}(X)$$

where p and q are the projections of Z on X and Y (here the existence of p_* follows from the normality of X, [16, Déf. 21.5.5 and (21.5.5.3)][7]). By Yoneda's lemma, to show that q^* and p_* are representable it suffices to show

[7]In loc. cit., §21.5, finite morphisms should presumably be assumed to be surjective.

that they extend to morphisms of Picard functors. This is trivial for q^*, while for p_* it follows from [16, Prop. 21.5.8].

More directly, thanks to the finiteness of p, "Schapiro's lemma" yields an isomorphism

$$H^1_{\text{ét}}(Z, \mathbb{G}_m) \simeq H^1_{\text{ét}}(X, p_*\mathbb{G}_m)$$

as well as a norm map $N : p_*\mathbb{G}_m \to \mathbb{G}_m$ induced by the norm on the function fields; their composition is p_*. This is obviously compatible with base change.

To construct $\underline{\text{Pic}}_{\text{ét}}$, it suffices to prove the representability of morphisms $\alpha : h_{\text{ét}}(X) \to h_{\text{ét}}(Y)$ for X, Y smooth projective. By the existence of $\underline{\text{Pic}}$, $n\alpha$ is representable for some $n > 0$, and therefore so is α by Lemmas A.5 and A.7. □

In order to get a correct adjunction statement, we need to introduce a rather baroque category:

Definition A.8. a) We write $\widetilde{\mathbf{AbS}}$ for the category where

- objects are triples (P, Q, α) where $P, Q \in \mathbf{AbS}$, $\bar{P}$ free, and α is an isomorphism $\widehat{P^0} \xrightarrow{\sim} Q^0$;
- a morphism $(P, Q, \alpha) \to (P', Q', \alpha')$ is a pair of morphisms $f : P' \to P$, $g : Q \to Q'$ such that $\alpha' \circ \widehat{f^0} = g^0 \circ \alpha$.

b) Let Σ be the set of cyclic groups of the form $\mathbf{Z}/l^n$ where l is a prime number different from p, $n \geq 1$ and $\mathbf{Z}/l^n$ is a direct summand of $\text{NS}(X)$ for some smooth projective X. We write $\widetilde{\mathbf{AbS}}'$ for the full subcategory of $\widetilde{\mathbf{AbS}}$ whose objects are the triples (P, Q, α) such that any cyclic direct summand of $\bar{Q}_{\text{tors}}[1/p]$ of prime power order belongs to Σ.

The explanation of this category is given by the following definition:

Definition A.9. We write $\widetilde{\text{Pic}}_{\text{ét}}$ for the functor $\mathcal{M}_{\text{ét}} \to \widetilde{\mathbf{AbS}}'[1/p]$ induced by the rule

$$\widetilde{\text{Pic}}_{\text{ét}}(h_{\text{ét}}(X)) = (\widetilde{\text{Alb}}_X, \underline{\text{Pic}}_{\text{ét}}(h_{\text{ét}}(X)), \alpha)$$

for X smooth projective, where $\widetilde{\text{Alb}}_X$ is the Albanese scheme of X and $\alpha : \widehat{\text{Alb}_X} \xrightarrow{\sim} \text{Pic}^0_X$ is the canonical isomorphism between the Picard variety of X and the dual of its Albanese variety.

(The existence of the component $\widetilde{\text{Alb}}_X$ as a functor on $\mathcal{M}_{\text{ét}}$ is shown as for $\underline{\text{Pic}}_{\text{ét}}$.)

We now have the following complement toi Theorem A.2.

Theorem A.10. *The functor $\widetilde{\text{Pic}}_{\text{ét}}$ has a fully faithful left adjoint h^1_+.*

Corollary A.11. *The functor $h^1_{\text{ét}}$ is fully faithful.* □

Proof of Theorem A.10. The category $\widetilde{\mathbf{AbS}}'[1/p]$ is additive; to prove the existence of h^1_+ it suffices in view of Lemma A.6 to show that it is defined at objects of the form $(A, \hat{A}, 1_A)$, $(P, 0, 1)$ and $(0, Q, 1)$ where $A \in \mathbf{Ab}$, P and Q are discrete and P is free.

We set

(i) $h^1_+(P, 0, 1) = P^* \otimes \mathbf{1}$ where $\mathbf{1}$ is the unit motive and P^* is the $\mathbf{Z}$-dual of P,

(ii) $h^1_+(0, Q, 1) = Q \otimes \mathbb{L}$ where $\mathbb{L}$ is the Lefschetz motive; this makes sense since $\mathcal{M}_{\text{ét}}$ is pseudo-abelian,

(iii) $h^1_+(A, \hat{A}, 1_A) = h^1_{\text{ét}}(A)$,

and check the adjunction property in each case; it is enough to check it "against" motives of the form $h_{\text{ét}}(X)$ for X smooth projective and connected.

We also have an obvious unit isomorphism

$$\eta_{(P,Q,\alpha)} : (P, Q, \alpha) \xrightarrow{\sim} \widetilde{\mathrm{Pic}}_{\text{ét}} h^1_+(P, Q, \alpha)$$

in each special case. We now check that the composition

$$\rho : \mathcal{M}_{\text{ét}}(h^1_+(P, Q, \alpha)), h_{\text{ét}}(X)) \to \widetilde{\mathbf{AbS}}'[1/p](\widetilde{\mathrm{Pic}}_{\text{ét}} h^1_+(P, Q, \alpha)), \widetilde{\mathrm{Pic}}_{\text{ét}} h_{\text{ét}}(X))$$
$$\xrightarrow{\eta^*_{(P,Q,\alpha)}} \widetilde{\mathbf{AbS}}'[1/p]((P, Q, \alpha)), \widetilde{\mathrm{Pic}}_{\text{ét}} h_{\text{ét}}(X))$$

is an isomorphism in each case:

Case (i).

$$\mathcal{M}_{\text{ét}}(P^* \otimes \mathbf{1}, h_{\text{ét}}(X)) = \mathrm{Hom}(P^*, CH^0(X))$$
$$= \mathrm{Hom}(P^*, \mathbf{Z}) = P = \widetilde{\mathbf{AbS}}'[1/p]((P, 0, 1), \widetilde{\mathrm{Pic}}_{\text{ét}}(h_{\text{ét}}(X)).$$

Case (ii).

$$\mathcal{M}_{\text{ét}}(Q \otimes \mathbb{L}, h_{\text{ét}}(X)) = \mathrm{Hom}(Q, CH^1(X))$$
$$\widetilde{\mathbf{AbS}}'[1/p]((0, Q, 1), \widetilde{\mathrm{Pic}}_{\text{ét}}(h_{\text{ét}}(X)) = \mathbf{AbS}(Q, \mathrm{Pic}_X).$$

The isomorphism follows from a diagram chase similar to that in the proof of Lemma A.7.

Case (iii).

$$\mathcal{M}_{\text{ét}}(h^1_{\text{ét}}(A), h_{\text{ét}}(X)) = CH^g_{\text{ét}}(A \times X)[1/p] \circ \pi^1_A$$
$$\widetilde{\mathbf{AbS}}'[1/p]((A, \hat{A}, 1_A), \widetilde{\mathrm{Pic}}_{\text{ét}}(h_{\text{ét}}(X)) = \mathbf{AbS}(\widetilde{\mathrm{Alb}}_X, A)$$

where $g = \dim A$. The map

$$\rho : CH^g_{\text{ét}}(A \times X)[1/p] \circ \pi^1_A \to \mathbf{AbS}(\widetilde{\mathrm{Alb}}_X, A)[1/p]$$

is induced by the functor $\widetilde{\mathrm{Alb}}$. We have an exact sequence

$$0 \to A(k) \to \mathbf{AbS}(\widetilde{\mathrm{Alb}}_X, A) \to \mathbf{Ab}(\mathrm{Alb}_X, A) \to 0. \tag{A.2}$$

For $l \neq p$, ρ is an isomorphism on l-primary torsion by Proposition 8.1 and the isomorphism

$$\mathbf{AbS}(\widetilde{\mathrm{Alb}}_X, A)\{l\} = A(k)\{l\}$$

stemming from (A.2). On the other hand, $\operatorname{Coker} \rho$ is torsion-free, as one sees by using cl_l and Proposition 2.1 a).

Suppose first that $A = J(C)$ for a curve C. Pick a point $x \in X(k)$. Then Lemma 10.7 yields an isomorphism

$$\mathcal{M}_{\text{ét}}(h^1_{\text{ét}}(A), h_{\text{ét}}(X)) \simeq \mathrm{Pic}^0(C) \oplus \mathrm{Corr}((C, c), (X, x))$$

which, in view of (A.2), implies that ρ is an isomorphism. In general, let C be an ample curve traced on A. Then A is almost a direct summand of $J(C)$, hence ρ has torsion kernel and cokernel and is therefore an isomorphism.

The fact that the unit morphism is an isomorphism implies that h^1_+ is fully faithful. □

Appendix B. Open questions

B.1. **p-torsion in characteristic p.** Suppose k algebraically closed. If the cohomology algebra $\bigoplus_{i\geq 0} H^{2i}_{\mathrm{cont}}(A, \mathbf{Z}_p(i))$ also has divided powers, we can replace $(r!)_p$ with $r!$ in Proposition 1.1. If, on the other hand, one looks for a counterexample, the simplest possible instance is for $p = 2$ and a divided square on $H^2_{\mathrm{cont}}(A, \mathbf{Z}_p(1))$. More specifically, one may ask the following question:

Question B.1. Is there an abelian variety A over an algebraically closed field of characteristic 2 and a divisor D on A such that $x^2 \neq 0$ in $H^2(A, \nu(2))$, where x is the cycle class of D in $H^1(A, \nu(1))$?

Here, $\nu(n) = \Omega^n_{\log}$. Note that one must have $\dim A \geq 3$ in view of Roĭtman's theorem, which is independent of the characteristic by [28].

In general, it would be interesting to understand what happens if one does not invert p (see also footnote 4).

B.2. **Self-conjugate DM projectors.**

Question B.2. Can one remove the condition that A is principally polarised in Theorem 1.7 c)? This would be possible if, in Lemma 10.6, the converse statement could be extended from "étale good" to "very good". I was not able to prove this.

B.3. **Pontryagin divided powers.**

Question B.3. In Corollary 1.4, we constructed a system of divided powers on the ring $\overline{\mathrm{CH}}^{\text{ét}}_*(A)$ provided with the Pontryagin product. In [29], the same was done on $CH_*(A)$ by a completely different (geometric) method. Does the natural map $CH_*(A) \to \overline{\mathrm{CH}}^{\text{ét}}_*(A)$ commute with divided powers?

B.4. **Torsion in Néron-Severi groups.**

Question B.4. How large is the set Σ in Definition A.8 b)?

B.5. **Étale Picard motive.** Let $M \in \mathcal{M}_{\text{ét}}$. The counit of the adjunction of Theorem A.10 is a morphism

$$h^1_+ \widetilde{\operatorname{Pic}_{\text{ét}}} M \to M$$

which yields in particular a morphism $h^1_{\text{ét}}(M) \to M$.

Question B.5. After tensoring with $\mathbf{Q}$, this morphism becomes split for $M = h(X)$, X smooth projective, by a Hard Lefschetz argument (see [39, Th. 4.4 (ii)]), and one recovers Murre's Picard motive from [32]. Does it already split in $\mathcal{M}_{\text{ét}}$?

B.6. **Comparison with ordinary Chow groups.** For a smooth projective k-variety X of dimension g, we have the following result ([9, Th. 3.11], [35, Cor. 4.10]):

Proposition B.6. *For any prime l, there is a surjection with divisible kernel*

$$H^{g-3}(X, \mathcal{H}^g_{\text{ét}}(\mathbf{Q}_l/\mathbf{Z}_l(g-1))) \twoheadrightarrow C_{\text{tors}}$$

where $C = \operatorname{Coker}(CH^{g-1}(X) \otimes \mathbf{Z}_l \xrightarrow{\text{cl}_l} H^{2g-2}_{\text{cont}}(X, \mathbf{Z}_l(g-1)))$.

Suppose that X is an abelian variety A. Understanding the group C_{tors} is relevant in view of the results of [11] and [12]. For any $n \in \mathbf{Z}$, n_A^* acts on this group by multiplication by n^{2g-2}. In view of §§6 and 7, this motivates the following

Question B.7. What is the action of n_A^* on $H^{g-3}(A, \mathcal{H}^g_{\text{ét}}(\mathbf{Q}_l/\mathbf{Z}_l(g-1)))$?

Any information on the said action (except if it involves multiplication by n^{2g-2}) would bound the order of C_{tors}.

B.7. **Jacobians of curves.**

Question B.8. Do the P_i of (12.3) define integral DM projectors already in $\operatorname{Corr}(J, J) = CH^g(J \times J)$? Does the identity (12.4) already hold in this group? (In [29, Th. 2.3], the identity of Theorem 12.8 b) is proven in $CH^{r+s}(J)$, modulo 2-torsion, for certain hyperelliptic Jacobians).

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