

The fullness conjectures for products of elliptic curves

By *Bruno Kahn* at Paris

With an appendix by *Cyril Demarche* at Paris

Abstract. We prove all conjectures of Chapter 7 of Yves André’s book (2004) in the case of products of elliptic curves. The proofs given here are simpler and more uniform than the previous proofs in known cases.

Introduction

Two of the most famous conjectures on a smooth projective variety X over a suitable field k are the Hodge and the Tate conjectures: $k = \mathbf{C}$ for the Hodge conjecture and k finitely generated for the Tate conjecture. They are expounded for example by Yves André in his book on motives [1, Chapter 7], where he also describes two other similar conjectures, when k is a number field:

- The “de Rham–Betti” conjecture [1, 7.5.1.1]: it is related to a conjecture of Grothendieck on periods [1, 7.5.2.1] which can be traced back to [12]; see letter of Y. André to C. Bertolin in [3] for historical details.
- The Ogus conjecture [25, Introduction and §4]: it is related to another conjecture of Grothendieck on algebraic solutions to differential equations [25, p. 268].

For the readers’ convenience, I recall these lesser-known conjectures in Section 1.

We shall be concerned here with the case where X a product of elliptic curves. If $k = \mathbf{C}$, the Hodge conjecture is known for X : this is attributed to Tate (unpublished) by Grothendieck in [13, §3 c)]; a full proof was given by Imai in [14].

Let ℓ be a prime number invertible in k , assumed to be finitely generated. The Tate conjecture for ℓ -adic cohomology is known for X in the following cases, given in chronological order:

- k is a number field (Imai [14]);
- k is finite (Spieß [29]).
- k is finitely generated over \mathbf{Q} : as pointed out in [20], this follows from [21, Corollary 1.2].

(In [14, 21], Imai and Lombardo prove the Mumford–Tate conjecture for X , which, together with the Hodge conjecture, implies the Tate conjecture.)

In each case, the result is stronger: the Hodge, or Tate, classes are generated by those of degree 2. The proofs, however, are different: for the Hodge and the Tate conjecture over a number field, Imai uses essentially a Tannakian argument plus results of Shimura–Taniyama, while, over a finite field, Spieß’ proof is obtained from an inequality on elliptic Weil numbers. On the other hand, the proof of the Hodge conjecture given by Imai involves a number of subcases and is especially delicate when dealing with elliptic curves with complex multiplication.

This prompted me to look for a simpler and unified proof, which would also cover the two other conjectures. This was successful.

Theorem 1. *All the above-mentioned conjectures hold for X , in the strong sense that the cohomology classes coming from algebraic cycles are generated by those of degree 2. In particular, the Tate conjecture holds for X over any finitely generated field k .*

For the Ogus conjecture, the special case $k = \mathbf{Q}$, X a power of a non-CM elliptic curve is outlined in [1, 7.4.3.1]; it has inspired part of the proof here. Moreover, an assumption on k in the first version of this article has now been dropped thanks to a theorem of Cyril Demarache (Theorem 4.4).

The case of the de Rham–Betti conjecture in Theorem 1 implies part of a conjecture of Rohrlich: if all elliptic curves involved in X have complex multiplication, then the multiplicative relations between its periods are generated by the “obvious” ones [1, 24.6.3.1]. Another consequence of Theorem 1 is the following corollary.

Corollary 2 (see Proposition 3.2). *Let X be a product of elliptic curves over any field k . Then the algebra of cycles modulo numerical equivalence on X is generated in degree 1.*

Remarks 3. (a) Theorem 1 and its corollary extend to abelian varieties isogenous to products of elliptic curves (see Proposition 3.2).

(b) To avoid a false impression, recall that there are known examples of abelian varieties X such that the Hodge or the Tate conjecture hold for all powers of X but where their “strong form” is false, e.g. [28] or [23, Example 1.8].

(c) It is tempting to try and deduce the full Grothendieck period conjecture in the form of [1, 7.5.2.2] from Theorem 1, say for a product X of CM elliptic curves, reducing to the case of one such curve (Chudnovsky). However, this is doomed to failure unless one knows something on the closure Z of the canonical \mathbf{C} -point in the torsor \mathfrak{P} of periods for X . Namely, a necessary condition is that Z is a sub-torsor of \mathfrak{P} , meaning that its stabiliser in the corresponding Tannakian group has the same dimension as Z ; conversely, this condition is inductively sufficient. Can one prove it?

Some words on the proof. The formalism developed here consists of two steps.

(A) Assuming the ground field k sufficiently large, it works for any “enriched realisation” into a Tannakian category verifying certain axioms: see Theorem 4.1. One then needs to check the axioms case by case: this is done in Section 6. Here the key point is that each conjecture is known in codimension 1 for abelian varieties: in the case of the Hodge conjecture, this is due to Lefschetz and Kodaira, for the Tate conjecture, it is due to Tate, Zarhin and Faltings, and for

the two other conjectures, it was deduced by André in [1] from Wüstholz’s analytic subgroup theorem for the de Rham–Betti conjecture and from results of Bost for the Ogus conjecture.

(B) A descent argument. To be able to use Tannakian arguments, one also needs the source category to be Tannakian. For this, and also to tackle the “strong” form of the conjectures conveniently, we use a category of “Chow–Lefschetz motives” (defined only for abelian varieties) introduced by Milne in [22] and developed in [17].

As in [1, 7.6], an important part of step (A) consists of a group-theoretic argument: it starts with the special case of one elliptic curve (and its powers), and uses a principle due to Goursat, Kolchin and Ribet to pass from there to the general case. The special case works well provided one knows that the Tannakian group attached to any elliptic curve is *connected*. This is trivial for the Hodge conjecture and easy for the Tate conjecture; for the two other conjectures, I borrowed arguments from Yves André (see Sections 6.3 and 6.4).

The general case is where elliptic curves with complex multiplication, whose motivic Galois groups are abelian, have rendered the Goursat–Kolchin–Ribet principle delicate in [14]. However, nobody seems to have used the full strength of Kolchin’s version of this principle [19]: his theorem is powerful enough to create a streamlined proof when the coefficients K of the Weil cohomology involved are “in good position with respect to the multiplications of the CM elliptic curves”: see condition (D) in Theorem 4.1. This is true, in particular, when $K = \mathbf{Q}$, which is the case for the Hodge conjecture (where this approach trivialises Imai’s arguments) and the de Rham–Betti conjecture. For the two other conjectures, one needs more sophisticated arguments to get around this condition.

Step (B) is much simpler than in the first version of this paper, relying on an elementary category-theoretic result (Proposition 5.1).

As far as I have seen, the present method remains unfortunately very special to products of elliptic curves and not prone to generalisation. It raises nevertheless interesting questions about the generality of the *result*. For example, let X be an abelian variety of a type for which one of the conjectures is known “in the strong sense” (for X and all its powers), e.g. one taken from the examples in [23, A.7] lifted to characteristic 0. Can one prove the same for the other conjectures? At least, Corollary 3.3 shows that “weak sense” implies “strong sense” for these other conjectures.

This work was done in 2019, and was given a brief announcement in the algebraic geometry seminar of IMJ-PRG on June 20, 2019. Since then, Kreutz, Shen and Vial have also proven the de Rham–Betti conjecture for products of elliptic curves in [20]. Instead of Chow–Lefschetz motives, they use André’s category of motivic motives and his “Hodge = motivic” theorem for abelian varieties to reduce to the Hodge conjecture.

Notation. We write $\mathbf{Sm}^{\text{proj}}(k) = \mathbf{Sm}^{\text{proj}}$ for the category of smooth projective varieties over a field k and $\mathbf{Ab}(k) = \mathbf{Ab}$ for the category of abelian k -varieties and homomorphisms of abelian varieties. We write Vec_K for the category of finite-dimensional vector spaces over a field K , and $\text{Rep}_K(G)$ for the Tannakian category of finite-dimensional representations of an affine K -group G . If \mathcal{C} is an additive category, we write $\mathcal{C}^{(\mathbf{Z})}$ for the category of \mathbf{Z} -graded objects of \mathcal{C} with finite support. If \mathcal{C} is symmetric monoidal, we provide $\mathcal{C}^{(\mathbf{Z})}$ with the commutativity constraint given by the Koszul rule. A \otimes -functor is a strong unital symmetric monoidal functor between unital symmetric monoidal categories.

A full subcategory \mathcal{D} of an additive category \mathcal{C} is *thick* if it is additive and stable under direct summands.

1. Review of the de Rham–Betti and the Ogus conjectures

In both conjectures, k is a number field; let Y be a smooth projective k -variety. In the first case, we fix an embedding $k \hookrightarrow \mathbf{C}$ and define a *de Rham–Betti cycle of codimension n* as a pair $(\alpha, \beta) \in H_{\text{dR}}^{2n}(Y/k) \times H_B^{2n}(Y, \mathbf{Q})$ such that $\alpha \otimes 1 \mapsto (2\pi i)^n \beta \otimes 1$ via the period isomorphism $H_{\text{dR}}^{2n}(Y/k) \otimes_k \mathbf{C} \xrightarrow{\sim} H_B^{2n}(Y, \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C}$. The cycle classes of any algebraic cycle of codimension n yield a de Rham–Betti cycle and, conversely, we have the following.

De Rham–Betti conjecture ([1, 7.5.1.1]). *Any de Rham–Betti cycle on Y is algebraic.*

In the second case, we consider de Rham cohomology of Y with extra structure: if v is a finite unramified place of k where Y has good reduction, we have the Berthelot isomorphism (see [1, 3.4.2])

$$H_{\text{dR}}^{2n}(Y_v/k_v) \xrightarrow{\sim} H_{\text{cris}}^{2n}(Y(v)/W(k(v))) \otimes_{W(k(v))} k_v,$$

where k_v (resp. $k(v)$) is the completion (resp. the residue field) of k at v , $Y_v = Y \otimes_k k_v$ and $Y(v)$ is the special fibre of a smooth projective model of Y at v . By transport of structure, the Frobenius automorphism of the right-hand side provides the left-hand side with an automorphism φ_v which is semi-linear with respect to the absolute Frobenius of k_v . An *Ogus cycle* is an element α of $H_{\text{dR}}^{2n}(Y/k)$ such that, for almost all v , one has $\varphi_v(\alpha) = q_v^n \alpha$, where $q_v = |k(v)|$. The cycle class of any algebraic cycle of codimension n is an Ogus cycle and, conversely, we have the following.

Ogus conjecture ([1, 7.4.1.2]). *Any Ogus cycle on Y is algebraic.*

2. Background

Let k be a field, and let **Mot** be the category of pure motives over k modulo algebraic equivalence [1, Chapter 4], with coefficients in a field K of characteristic 0. We shall use the notation **Mot**(k) when it is necessary to specify k , but dispense from writing down coefficients K . We write $\mathbb{L} \in \mathbf{Mot}$ for the Lefschetz motive.

Let \mathcal{B} be a Tannakian category [1, 2.3] over K (this means that $\text{End}_{\mathcal{B}}(\mathbf{1}) = K$). Surprisingly, I could not find a proof of the following result in the literature, while a corresponding result is available when K is of characteristic $p > 0$ (see [8, Theorem 6.1]).

Proposition 2.1. *In \mathcal{B} , the tensor product and duals of two semi-simple objects B, B' are semi-simple.*

Proof. When \mathcal{B} is neutral, this follows from Chevalley's theorem [6, p. 88]. In general, one may assume that \mathcal{B} is generated by B and B' ; by [9, II, Remark 3.10], there exists a fibre functor with values in a finite extension L of K , that we may assume Galois of group G . The Tannakian category $\mathcal{B}_{(L)}$ of [9, pp. 155–156] is neutralised by the canonical extension of this fibre functor along the inclusion $\mathcal{B} \hookrightarrow \mathcal{B}_{(L)}$. Therefore, it suffices to show that if an object $M \in \mathcal{B}$ becomes semi-simple in $\mathcal{B}_{(L)}$, it is semi-simple. Let $i: M' \hookrightarrow M$ be a monomorphism. In $\mathcal{B}_{(L)}$, the inclusion $i_{(L)}$ has a retraction r ; the morphism $\frac{1}{|G|} \sum_{g \in G} grg^{-1}$ is another, G -equivariant, retraction of $i_{(L)}$ which descends to a retraction of i . \square

We can define a (generalised) Weil cohomology \mathbf{H}^* with values in \mathcal{B} just as in [1, 3.3]: see [26, VI.A.1] or [2, 4.2.1]. It induces a \otimes -functor

$$(2.1) \quad \mathbf{H}^*: \mathbf{Mot} \rightarrow \mathcal{B}^{(\mathbf{Z})}$$

(see notation at the end of the introduction). For $A \in \mathcal{B}$ and $p \in \mathbf{Z}$, we write

$$A(p) := A \otimes \mathbf{H}^2(\mathbb{L})^{\otimes -p}.$$

If $\omega: \mathcal{B} \rightarrow \text{Vec}_K$ is a neutral fibre functor, then $H^* = \omega^{(\mathbf{Z})} \circ \mathbf{H}^*$ is a Weil cohomology in the usual sense, and \mathbf{H}^* is an enrichment of H^* in the sense of [1, 7.1.1].

For $X \in \mathbf{Sm}^{\text{proj}}$ and $r \geq 0$, we are interested in the following condition:

($F(X, r)$) the map $A_{\text{alg}}^r(X) \otimes K = \mathbf{Mot}(\mathbf{1}, h(X)(r)) \rightarrow \mathcal{B}(\mathbf{1}, \mathbf{H}^{2r}(X)(r))$ is surjective.

The following is well known, by a duality argument (cf. [1, Chapter 7]).

Lemma 2.2. *Let $\mathbf{Mot}[X]$ be the thick rigid \otimes -subcategory of \mathbf{Mot} generated by the motive $h(X)$ of some $X \in \mathbf{Sm}^{\text{proj}}(k)$. Then condition $(F(X^n, r))$ for all $n, r \geq 0$ is equivalent to the fullness of \mathbf{H}^* restricted to $\mathbf{Mot}[X]$. \square*

Suppose that H -homological equivalence agrees with numerical equivalence on $\mathbf{Mot}[X]$. Then H^* and therefore \mathbf{H}^* factor through the semi-simple \otimes -category $\mathbf{Mot}_{\text{num}}[X]$ (see [1, 4.3]). Moreover, the Künneth projectors of X are algebraic [1, 5.4.2.1]; in (2.1), after changing the commutativity constraints of $\mathbf{Mot}_{\text{num}}[X]$ as usual and those of $\mathcal{B}^{(\mathbf{Z})}$ and $\text{Vec}_L^{(\mathbf{Z})}$ by removing the signs of the Koszul rule, the composition of \mathbf{H}^* with the direct sum functor becomes symmetric monoidal and $H = \bigoplus H^*$ becomes a fibre functor. We may then study condition $(F(X, r))$ by Tannakian methods as in [1, Chapter 7], according to the following proposition.

Proposition 2.3. *Let $\mathbf{H}: \mathcal{A} \rightarrow \mathcal{B}$ be an exact \otimes -functor between Tannakian categories over K : in particular, \mathbf{H} is faithful by [9, II, 1.19]. Let $\omega: \mathcal{B} \rightarrow \text{Vec}_K$ be a neutral fibre functor and $H = \omega \circ \mathbf{H}$. Suppose that \mathcal{A} has a \otimes -generator M and that \mathbf{H} is essentially surjective. Then the Tannakian groups G_H and G_ω of H and ω are both subgroups of $\mathbf{GL}(H(M))$; in particular, $G_\omega \hookrightarrow G_H$. Moreover, the following are equivalent:*

- (i) $G_\omega = G_H$;
- (ii) \mathbf{H} is an equivalence of categories.

Proof. The first claim is clear since a \otimes -automorphism of H or ω is determined by its value on M or $\mathbf{H}(M)$. This said, (ii) \Rightarrow (i) is trivial and (i) \Rightarrow (ii) follows from Tannakian duality. \square

Remark 2.4. If \mathcal{A} is semi-simple, the exactness of \mathbf{H} is automatic.

Suppose that k is of characteristic 0, embeddable in \mathbf{C} , that H^* is a classical Weil cohomology [1, 3.4], and that X is an abelian variety. Then homological equivalence agrees with numerical equivalence on $\mathbf{Mot}[X]$ (see [1, 5.4.1.4]) and we can use Proposition 2.3. This

approach has two drawbacks:

- it fails in positive characteristic;
- in any characteristic, it does not account for the “strong” version of the fullness conjectures as explained in the introduction.

Our solution is to use Chow–Lefschetz motives, as explained next.

3. Chow–Lefschetz motives

When k is algebraically closed, Milne defined in [22] a rigid \otimes -category **LMot** of “Lefschetz motives”¹⁾ modelled on abelian varieties, for homological equivalence modulo a Weil cohomology H as above; homological and numerical equivalences agree in this category, which therefore does not depend on the choice of H and is abelian semi-simple. In [17], its definition was extended to other adequate equivalence relations and over not necessarily algebraically closed fields; as in [22], correspondences are given by sums of intersections of divisors. In this generality, numerical equivalence even agrees with *algebraic* equivalence [22, Theorem 3]. In particular, if we work modulo algebraic equivalence, then **LMot** maps to **Mot**, becomes Tannakian after changing the commutativity constraint as before, and H induces a fibre functor on **LMot**. As above, we assume that the coefficients of **Mot** and **LMot** are K . We write $\iota: \mathbf{LMot} \rightarrow \mathbf{Mot}$ for the inclusion functor. The additive functor $h^1: \mathbf{Ab} \otimes K \rightarrow \mathbf{Mot}$ of [1, 4.3.3] or [16, §6.11] factors via ι through an additive functor

$$Lh^1(X): \mathbf{Ab} \otimes K \rightarrow \mathbf{LMot}$$

(see [17, Corollary 4.1 and Definition 4.4]).

Lemma 3.1. *Let \mathbf{LMot}^1 be the thick subcategory of **LMot** generated by the $Lh^1(X)$ where X runs through **Ab**, and define **Mot**¹ similarly. Then the functor $\iota^1: \mathbf{LMot}^1 \rightarrow \mathbf{Mot}^1$ induced by ι is an equivalence of categories.*

Proof. First assume that $K = \mathbf{Q}$. By [16, Theorem 6.37], h^1 is fully faithful, and essentially surjective by Poincaré’s complete reducibility. Since ι^1 is faithful, Lh^1 is then fully faithful. Therefore, any idempotent endomorphism of some $Lh^1(X)$ comes from $\mathbf{Ab} \otimes \mathbf{Q}$; hence Lh^1 is an equivalence of categories and so is also ι^1 . This remains true after tensoring morphisms with K and then taking pseudo-abelian envelopes, which concludes the proof since **LMot** and **Mot** are pseudo-abelian. \square

If $X \in \mathbf{Ab}$, the category **LMot**[X] analogous to **Mot**[X] (Lemma 2.2) is \otimes -generated by the Chow–Lefschetz motive $Lh^1(X)$ (see [17, Corollary 4.1]), just as **Mot**[X] is \otimes -generated by $h^1(X)$ (see [1, 4.3.3]). We have a string of \otimes -functors

$$(3.1) \quad \mathbf{LMot}[X] \xrightarrow{\iota} \mathbf{Mot}[X] \xrightarrow{H} \mathcal{B}[X] \xrightarrow{\omega} \mathbf{Vec}_K,$$

¹⁾ This terminology creates an ambiguity regarding the name “Lefschetz motive” given to \mathbb{L} ; we hope this will not cause any confusion.

where $\mathcal{B}[X]$ is the full Tannakian subcategory of \mathcal{B} (see [1, 2.3.5]) generated by $\mathbf{H}^1(X)$. We therefore have inclusions of Tannakian groups

$$(3.2) \quad G_\omega(X) \hookrightarrow G_{H\iota}(X) \hookrightarrow \mathbf{GL}(H^1(X)).$$

Proposition 3.2. *The following statements hold.*

(a) *The categories $\mathbf{LMot}[X]$ and $\mathbf{Mot}[X]$ only depend on the isogeny class of X . The functor $\mathbf{LMot}^1[X] \rightarrow \mathbf{Mot}^1[X]$ is an equivalence of categories, where*

$$\mathbf{Mot}^1[X] = \mathbf{Mot}^1 \cap \mathbf{Mot}[X] \quad \text{and} \quad \mathbf{LMot}^1[X] = \mathbf{LMot}^1 \cap \mathbf{LMot}[X].$$

(b) *The following are equivalent: referring to (3.1),*

- (i) $\mathbf{H}\iota$ is full;
- (ii) *the restriction of \mathbf{H} to $\mathbf{Mot}^1[X]$ is full and, for any $n \geq 0$, the graded algebra $\bigoplus_{r \geq 0} \mathcal{B}(\mathbf{1}, \mathbf{H}^{2r}(X^n))(r)$ is generated in degree 1;*
- (iii) *$(F(X^n, r))$ holds for any $n, r \geq 0$ and the graded algebra $\bigoplus_{r \geq 0} A_H^r(X^n)$ is generated in degree 1, where A_H^* denotes algebraic cycles with K coefficients modulo H -homological equivalence.*

They imply

- (iii') *$(F(X^n, r))$ holds for any $n, r \geq 0$ and the graded algebra $\bigoplus_{r \geq 0} A_{\text{num}}^r(X^n)$ is generated in degree 1, where A_{num}^* denotes algebraic cycles with \mathbf{Q} coefficients modulo numerical equivalence.*

- (c) *The category $\mathcal{B}[X]$ is semi-simple if and only if $\mathbf{H}^1(X)$ is semi-simple. This happens if and only if $G_\omega(X)$ is reductive (not necessarily connected).*
- (d) *If $\mathbf{H}^1(X)$ is semi-simple, conditions (i)–(iii) are equivalent to (iii') and imply that $\mathbf{H}\iota$ is an equivalence of categories.*
- (e) *If $\mathbf{H}^1(X)$ is semi-simple, conditions (i)–(iii) are also equivalent to*

- (iv) *in (3.2), $G_\omega(X) = G_{H\iota}(X)$.*

Proof. (a) follows from Lemma 3.1 and its proof.

(b) (i) \Rightarrow (ii): the first part of (ii) follows from (a), and the second part follows from Lemma 2.2 and the fact that the graded algebra $\bigoplus_{r \geq 0} \mathbf{LMot}(\mathbf{1}, Lh^{2r}(X^n))(r)$ is generated in degree 1 by definition of the morphisms in \mathbf{LMot} . (ii) \Rightarrow (iii): the first claim is obvious and the second one then follows. (iii) \Rightarrow (i): again by Lemma 3.1, the first part of (iii) implies that the restriction of $\mathbf{H}\iota$ to $\mathbf{LMot}^1[X]$ is full, while its second part extends this fullness to all of $\mathbf{LMot}[X]$. (iii) \Rightarrow (iii'): this is clear for algebraic cycles with K coefficients modulo numerical equivalence, and we deduce the case with \mathbf{Q} coefficients from [1, 3.2.7.1].

(c) The first claim follows from Proposition 2.1, and the second one from [9, II, Remark 2.28].

(d) (iii') \Rightarrow (iii) follows from (c) and [1, 7.1.1.1]. For the rest, in view of (i), it suffices to see that $\mathbf{H}\iota$ is essentially surjective. But an object H of $\mathcal{B}[X]$ is a direct sum of direct summands of tensor constructions $\mathbf{H}^1(X)^{\otimes p} \otimes \mathbf{H}^1(X)^{* \otimes q} = \mathbf{H}(Lh^1(X)^{\otimes p} \otimes Lh^1(X)^{* \otimes q})$, and we may assume that H is a single such direct summand. By full faithfulness, the idempo-

tent with image H comes from an idempotent endomorphism of $Lh^1(X)^{\otimes p} \otimes Lh^1(X)^{* \otimes q}$. (This reasoning was also used in the proof of Lemma 3.1; see [2, Lemma 2.3.8] for a general context.)

(e) In view of (d), (i) \Leftrightarrow (iv) was seen in Proposition 2.3. \square

Corollary 3.3. *Let \mathbf{H}, \mathbf{H}' be two enriched realisations. Suppose that $\mathbf{H}^1(X)$ and $\mathbf{H}'^1(X)$ are semi-simple and that $(F(X^n, r))$ holds for any $n, r \geq 0$ for both \mathbf{H} and \mathbf{H}' . Then the conditions of Proposition 3.2 (b) hold for \mathbf{H} if and only if they hold for \mathbf{H}' .*

Proof. Indeed, if (iii) holds for \mathbf{H} , it implies (iii') for \mathbf{H} by Proposition 3.2 (b); hence it also holds for \mathbf{H}' by hypothesis, which in turn implies (iii) for \mathbf{H}' by hypothesis and Proposition 3.2 (d). \square

Remark 3.4. One should beware that the last claim of (iii) in Proposition 3.2 (b) becomes *false* if one replaces homological equivalence with algebraic equivalence, already for $k = \mathbf{C}$ and X the cube of the Fermat elliptic curve; cf. [17, remarks after Theorem 3].

In all the cases given in the introduction, the first property of (ii) in Proposition 3.2 is verified (see Lemma 6.1). We draw a consequence, weaker than (iv) in this proposition. Let \bar{K} be an algebraic closure of K ; if G is a closed subgroup of \mathbf{GL}_n , write $\bar{K}G$ for the sub- \bar{K} -algebra of $M_n(\bar{K})$ generated by $G(\bar{K})$. Note that an irreducible representation of G is also $G(\bar{K})$ -irreducible because $G(\bar{K})$ is Zariski-dense in G ; hence a semi-simple representation of G is also $G(\bar{K})$ -semi-simple.

Proposition 3.5. *Suppose that $\mathbf{H}^1(X)$ is semi-simple and that the restriction of \mathbf{H} to $\mathbf{Mot}^1[X]$ is full. Then $\bar{K}G_\omega(X) = \bar{K}G_{H_\ell}(X)$.*

Proof. Both algebras are semi-simple since they admit the faithful semi-simple module $\bar{K} \otimes_K H^1(X)$, and they have the same centraliser by the fullness hypothesis and Lemma 3.1. The conclusion follows from the double centraliser theorem [5, §14, no 5, Theorem 5 a)]. \square

In the next section, we shall use the following definition.

Definition 3.6. Let k_s be a separable closure of k , and $X_s = X \otimes_k k_s$. The field k is *large* relatively to X if $\text{Gal}(k_s/k)$ acts trivially on $\text{End}^0(X_s) := \text{End}(X_s) \otimes \mathbf{Q}$, or equivalently if the injection $\text{End}^0(X) \hookrightarrow \text{End}^0(X_s)$ is bijective.

From now on, we simplify the notation $G_{H_\ell}(X)$ to $G_H(X)$. This group comes with a “Tate” character t_X , given by the \otimes -subcategory of $\mathbf{LMot}[X]$ generated by the Lefschetz motive \mathbb{L} .²⁾ If k is large relatively to X , as a special case of [17, Theorem 5], we have an isomorphism

$$(3.3) \quad G_H(X) \xrightarrow{\sim} \prod_{\mathbb{G}_m} G_H(X_\alpha),$$

²⁾ Note that $\mathbb{L} \in \mathbf{LMot}[X]$, as a direct summand of $Lh^2(X)$ given by the class of an ample line bundle.

where the X_α 's are the isotypic components of X up to isogeny, and $\prod_{\mathbb{G}_m}$ denotes the fibre product over \mathbb{G}_m with respect to the t_{X_α} 's. (See [17, (0.1)] for the precise value of the $G_H(X_\alpha)$.) Note also that k is large with respect to X if and only if it is large with respect to X_α for each α .

Proposition 3.7. *In $\mathcal{B}[X]$, $\mathbf{H}^1(X)$ is semi-simple if and only if all the $\mathbf{H}^1(X_\alpha)$ are semi-simple. If this holds and if k is large with respect to X , the following are equivalent:*

- (1) *condition (iv) of Proposition 3.2 (e) holds for X ;*
- (2) *condition (iv) of Proposition 3.2 (e) holds for each X_α and the obvious homomorphism*

$$(3.4) \quad G_\omega(X) \rightarrow \prod_{\mathbb{G}_m} G_\omega(X_\alpha),$$

analogous to (3.3) is bijective.

Proof. The first claim is obvious. (1) \Rightarrow (2): the fullness of \mathbf{H} on $\mathbf{LMot}[X]$ implies its fullness on $\mathbf{LMot}[X_\alpha]$ for each α ; hence the first condition of (2) follows from Proposition 3.2 (e); its second condition is now obvious. (2) \Rightarrow (1) is also clear, thanks to (3.3). \square

Remark 3.8. (3.4) is a monomorphism, as follows from comparing it with (3.3). Moreover, under the hypotheses of Proposition 3.7, the homomorphism $\pi_\alpha: G_\omega(X) \rightarrow G_\omega(X_\alpha)$ is faithfully flat by [9, II, Remark 2.29], since it corresponds to a full embedding of semi-simple Tannakian categories.

4. The case of a large ground field

We keep the notation of Sections 2 and 3: in particular, \mathcal{B} is a Tannakian category over a field K of characteristic 0, ω is a neutral fibre functor on \mathcal{B} and $\mathbf{H} = \bigoplus \mathbf{H}^n$, where \mathbf{H}^* is a \otimes -functor as in (2.1), that we implicitly restrict to \mathbf{LMot} (see (3.1)).

Theorem 4.1. *Let $X = \prod_{i \in I} E_i$ be a product of elliptic curves over k . Assume that k is large relatively to X and that*

- (A) $G_\omega(E_i)$ is connected for all i ;
- (B) $\mathbf{H}^1(X)$ is semi-simple;
- (C) *the restriction of \mathbf{H}^1 to $\mathbf{Mot}[X]$ is full (equivalently, the restriction of \mathbf{H} to $\mathbf{Mot}^1[X]$ is full);*
- (D) *for all (i, j) such that E_i, E_j are non-isogenous and have complex multiplication (are ordinary if $\text{char } k > 0$), write F_{ij} for the real quadratic subfield of*

$$\text{End}(E_i) \otimes \text{End}(E_j) \otimes \mathbf{Q}.$$

Then $F_{ij} \otimes_{\mathbf{Q}} K$ is a field.

Then, in (3.1), $\mathbf{H}i: \mathbf{LMot}[X] \rightarrow \mathcal{B}[X]$ is an equivalence of categories.

Here are some comments on this statement. The assumption that k be large is made to be able to use (3.3). Note that the conclusion of Theorem 4.1 implies (A), (B) and (C), because

the groups $G_H(E_i)$ are connected by [17, Example 1] and $\mathbf{LMot}[X]$ is semi-simple. So the only ‘‘mysterious’’ condition is (D); its relevance will appear when we apply Kolchin’s theorem below (case (II) in (2)).

Proof. Our strategy is to prove condition (iv) of Proposition 3.2 by the method of Proposition 3.7, i.e. to prove item (2) of this proposition. Thus there are two steps.

(1) X is a single elliptic curve. Note that the Tate character t_X factors through

$$\det: \mathbf{GL}(H^1(X)) \rightarrow \mathbb{G}_m.$$

The group $G_H(X)$ is connected reductive by [17, Example 1], and so is $G_\omega(E)$ by (A) and (B). The connected reductive subgroups G of \mathbf{GL}_2 which map surjectively to \mathbb{G}_m by the determinant are \mathbf{GL}_2 , \mathbb{G}_m (its centre) and maximal tori. These subgroups are distinguished by the \bar{K} -algebras $\bar{K}G$, and we conclude by Proposition 3.5.

(2) The general case. Write M^1 for $\text{Ker}(M \rightarrow \mathbb{G}_m)$, for all groups M appearing in the picture. For simplicity, write also $G = G_\omega(X)$ and $G_i = G_\omega(E_i)$. Let

$$J' = \{i \in I \mid G_i^1 = \{1\}\};$$

projecting onto $\prod_{i \in J - J'} G_i$, we may assume that $J' = \emptyset$.

It now suffices to prove that the inclusion $G^1 \subseteq \prod_i G_i^1$ (see Remark 3.8) is an equality. Assume the contrary and write $\pi_i^1: G^1 \rightarrow G_i^1$ for the projection induced by π_i : since π_i is faithfully flat (Remark 3.8 again), so is π_i^1 . By (A), all G_i ’s are connected, hence isomorphic to \mathbf{GL}_2 (no complex multiplication), $R_{\text{End}(E_i)} \otimes K/K \mathbb{G}_m$ (complex multiplication/ordinary in positive characteristic) or \mathbb{G}_m (supersingular). This ensures that all G_i^1 ’s are quasi-simple (K -simple in the terminology of [19]), so we are in a position to apply Kolchin’s result [19, Theorem]. By this theorem and the remarks following it, there are two possibilities.

(I) There exist two distinct indices $i \neq j$ such that G_i^1, G_j^1 are non-abelian (hence isomorphic to \mathbf{SL}_2), and an isomorphism $\varphi: G_i^1 \xrightarrow{\sim} G_j^1$ such that the diagram

$$\begin{array}{ccc} & G^1 & \\ \pi_i^1 \swarrow & & \searrow \pi_j^1 \\ G_i^1 & \xrightarrow{\varphi} & G_j^1 \end{array}$$

commutes.

(II) There exist l distinct indices $j(1), \dots, j(l)$ with $l \geq 2$ and $G_{j(1)}^1, \dots, G_{j(l)}^1$, each commutative, and l faithfully flat K -homomorphisms $f_\lambda: G_{j(\lambda)}^1 \rightarrow G_{j(l)}^1$ ($1 \leq \lambda \leq l$) such that $\prod_{1 \leq \lambda \leq l} f_\lambda \circ \pi_{j(\lambda)}^1 = 1$.

More precisely, (I) and (II) correspond to (ii) and (iii), respectively, of [19, Theorem]³⁾, except that (ii) is stated modulo finite subgroups of G_i^1 and G_j^1 , but the remark in [19, Theorem, p. 1154] shows that one can get rid of these finite subgroups. In (II), the only thing we are going to use is the existence of a nontrivial homomorphism $f_\lambda: G_{j(\lambda)}^1 \rightarrow G_{j(l)}^1$ for some $\lambda < l$.

³⁾ Part (i) is excluded by step (1).

In case (I), the image of G^1 in $G_i^1 \times G_j^1$ must be the diagonal, hence the image of G in $G_i \times_{\mathbb{G}_m} G_j$ must be the fibred diagonal Δ . But, inside $M_4(\bar{K})$, the two subalgebras

$$\begin{aligned}\bar{K}\Delta &= \left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \mid u \in M_2(\bar{K}) \right\}, \\ \bar{K}(\mathbf{GL}_2 \times_{\mathbb{G}_m} \mathbf{GL}_2) &= \left\{ \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \mid u, v \in M_2(\bar{K}) \right\}\end{aligned}$$

are distinct; hence this case contradicts Proposition 3.5 in view of (3.3).

In case (II), recall that 1-dimensional K -tori are classified by quadratic characters of $\mathrm{Gal}(\bar{K}/K)$ via Cartier duality. Let χ_λ be the quadratic character corresponding to $G_{j(\lambda)}^1$: this is also the quadratic character corresponding to the étale K -algebra $\mathrm{End}(E_{j(\lambda)}) \otimes K$. Suppose $\lambda < l$. Since f_λ is not constant, we have $\chi_\lambda = \chi_l$ or equivalently $\chi_\lambda \chi_l = 1$, which contradicts (D) since $\chi_\lambda \chi_l$ defines the étale K -algebra $F_{j(\lambda)j(l)} \otimes_{\mathbb{Q}} K$. \square

Remark 4.2. The proof of Theorem 4.1 shows that condition (D) is only needed when X has at least two CM/ordinary factors.

For the proof of the Tate conjecture (Section 6.2), we shall also need the following.

Proposition 4.3. *Assume that conditions (A), (B) and (C) of Theorem 4.1 are satisfied. Write $X = X_1 \times X_2$, where X_1 is the product of the CM factors of X (ordinary factors in positive characteristic) and X_2 is the product of the other factors. Then the inclusion $G_\omega(X) \subseteq G_\omega(X_1) \times_{\mathbb{G}_m} G_\omega(X_2)$ is an equality.*

Proof. Since the homomorphisms $G_\omega(X) \rightarrow G_\omega(X_i)$ are faithfully flat by Remark 3.4, so are the homomorphisms $\pi_i: G_\omega^1(X) \rightarrow G_\omega^1(X_i)$. Let $N_i = \mathrm{Ker} \pi_i$: its image in $G_\omega^1(X_j)$ ($j \neq i$) is a normal subgroup. This time we apply Goursat's lemma [11]: the image of $G_\omega^1(X)$ in $G_\omega^1(X_1) \times G_\omega^1(X_2)$ is the graph of an isomorphism $G_\omega^1(X_1)/N_2 \xrightarrow{\sim} G_\omega^1(X_2)/N_1$. But, by step (1) in the proof of Theorem 4.1 and Remark 3.8, $G_\omega^1(X_1)/N_2$ is a group of multiplicative type, and by Remark 4.2, $G_\omega^1(X_2)/N_1$ is a product of copies of \mathbf{SL}_2 . Therefore, these two groups are trivial; hence $G_\omega^1(X)$ contains $G_\omega^1(X_1) \times \{1\}$ and $\{1\} \times G_\omega^1(X_2)$ and the conclusion follows. \square

For the proof of the Ogus conjecture, we need the following result of Cyril Demarche.

Theorem 4.4. *Let F be a number field and let \mathcal{G} be a connected algebraic gerbe over F (see [9, II, Appendix]⁴⁾. Then, for any finite (field) extension R/F , there exists a finite extension K/F , linearly disjoint from R/F , such that $\mathcal{G}(K) \neq \emptyset$.*

Proof. See Proposition A.2 in the appendix. \square

⁴⁾ Here “connected algebraic” means that, for any extension E of F such that $\mathcal{G}(E) \neq \emptyset$ and for any $X \in \mathcal{G}(E)$, the affine E -group G_X such that $G_X(K) = \mathrm{Aut}(X_K)$ for any extension K/E is of finite type and connected. This is equivalent to the condition “with connected linear band” in Proposition A.2.

5. Descent

Suppose that we have an enrichment as in Definition 2.1, but do not assume k large. We would like to reduce Theorem 1 to Theorem 4.1; this is the aim of this section.

The idea. Let Γ be a finite group and let $f^*: \mathcal{A} \rightarrow \mathcal{A}'$ be a Γ -equivariant functor, where \mathcal{A} and \mathcal{A}' are categories provided with a (pseudo-)action of Γ , this action being trivial on \mathcal{A} . Then f^* induces a functor from \mathcal{A} to the category $\mathcal{A}'[\Gamma]$ of descent data with respect to the action of Γ ; we say that f^* has descent if this functor is an equivalence of categories [18, end of §1.1]. Let l/k be a finite Galois extension and $\Gamma = \text{Gal}(l/k)$. As observed in the proof of [18, Proposition 7.1], it follows from the definition of **LMot** (see [17, Definition 4.4]) and [17, proof of Theorem 5 in §5.5] that $f^*: \mathbf{LMot}(k) \rightarrow \mathbf{LMot}(l)$ has descent, where $f: \text{Spec } l \rightarrow \text{Spec } k$ is the structural morphism. This fact restricts to the categories $\mathbf{LMot}(k)[X]$ and $\mathbf{LMot}(l)[X]$ for any $X \in \mathbf{Ab}(k)$.

Suppose now that we have a naturally commutative diagram of rigid abelian \otimes -categories and \otimes -functors

$$\begin{array}{ccc} \mathbf{LMot}(l) & \xrightarrow{\mathbf{H}(l)} & \mathcal{B}(l) \\ \uparrow f^* & & \uparrow f_{\mathcal{B}}^* \\ \mathbf{LMot}(k) & \xrightarrow{\mathbf{H}} & \mathcal{B} \end{array}$$

in which $\mathbf{H}(l)$ restricts to an equivalence of categories on the rigid subcategories generated by $Lh^1(X)$ and its images, as in the previous sections. If $f_{\mathcal{B}}^*$ also has descent and $\mathbf{H}(l)$ is Γ -equivariant, we get the same conclusion for \mathbf{H} .

The problem in this strategy is that, in practice, the category $\mathcal{B}(l)$ which is used to formulate a given fullness conjecture fails to carry a Γ -action, so that the above argument does not make sense. As an example, for the Tate conjecture, we use $\mathcal{B} = \text{Rep}_{\mathbb{Q}_\ell}(G_k)$ (continuous representations), where $G_k = \text{Gal}(k_s/k)$ is the absolute Galois group of k relative to some separable closure k_s . But the action of G_k by conjugation on $\text{Rep}_{\mathbb{Q}_\ell}(G_l)$, where l is supposed to be a subextension of k_s/k , does not factor through its quotient Γ . The case of de Rham–Betti realisations is similar.

Fortunately, there is a catch-all solution to this issue, which is given by the following proposition.

Proposition 5.1. *Consider a naturally commutative diagram of categories and functors*

$$(5.1) \quad \begin{array}{ccccc} \mathcal{A}' & \xrightarrow{\mathbf{H}'} & \mathcal{B}' & \xrightarrow{\rho'} & \mathcal{C}' \\ \uparrow f^* & & \uparrow f_{\mathcal{B}}^* & & \uparrow f_{\mathcal{C}}^* \\ \mathcal{A} & \xrightarrow{\mathbf{H}} & \mathcal{B} & \xrightarrow{\rho} & \mathcal{C} \end{array}$$

in which a finite group Γ acts on \mathcal{A}' and \mathcal{C}' . We assume that

- $\rho' \mathbf{H}'$ is Γ -equivariant, as well as f^* and $f_{\mathcal{C}}^*$ for the trivial action of Γ on \mathcal{A} and \mathcal{C} ;
- ρ' and $f_{\mathcal{B}}^*$ are faithful;
- the functor $\mathcal{A} \rightarrow \mathcal{A}'[\Gamma]$ induced by f^* is fully faithful (in particular, f^* is faithful).

If \mathbf{H}' is fully faithful, so is \mathbf{H} .

Proof. This is a standard categorical diagram chase, that we make explicit for the benefit of the reader. For simplicity, we reason as if (5.1) were strictly commutative.

Let $A, A' \in \mathcal{A}$ and let $w, w' \in \mathcal{A}(A, A')$ be such that $\mathbf{H}(w) = \mathbf{H}(w')$. Then we have $\mathbf{H}'f^*(w) = \mathbf{H}'f^*(w')$; hence $w = w'$ since f^* and \mathbf{H}' are faithful. This proves that \mathbf{H} is faithful.

Let now $u \in \mathcal{B}(\mathbf{H}(A), \mathbf{H}(A'))$. By hypothesis, there exists $v \in \mathcal{A}'(f^*A, f^*A')$ such that $\mathbf{H}'(v) = f_{\mathcal{B}}^*(u)$. I claim that $gv = v$ for any $g \in \Gamma$; by the faithfulness of ρ' and \mathbf{H}' , it suffices to show this after applying $\rho'\mathbf{H}'$. But $\rho'\mathbf{H}'(v) = \rho'f_{\mathcal{B}}^*(u) = f_{\mathcal{C}}^*\rho(u)$; hence

$$\rho'\mathbf{H}'(gv) = g\rho'\mathbf{H}'(v) = gf_{\mathcal{C}}^*\rho(u) = f_{\mathcal{C}}^*\rho(u) = \rho'\mathbf{H}'(v),$$

where we used the Γ -equivariance of $\rho'\mathbf{H}'$. Using now the hypothesis on f^* , we find that $v = f^*w$ for some $w \in \mathcal{A}(A, B)$. Hence

$$f_{\mathcal{B}}^*\mathbf{H}(w) = \mathbf{H}'f^*(w) = \mathbf{H}'(v) = f_{\mathcal{B}}^*(u)$$

and $\mathbf{H}(w) = u$ by the faithfulness of $f_{\mathcal{B}}^*$. This proves that \mathbf{H} is full. \square

Remark 5.2. Proposition 5.1 does not give the essential surjectivity of \mathbf{H} even if we assume it for \mathbf{H}' . Theorem 1 does not state such essential surjectivity. Yet, one can deduce it from Proposition 3.2 (d) by the semi-simplicity of $\mathbf{H}^1(X)$ (Lemma 6.1) if one wishes.

6. Proof of Theorem 1

First we have the following lemma.

Lemma 6.1 ([1, Theorem 7.1.7.5]). *Conditions (B) and (C) of Theorem 4.1 hold for each conjecture of [1, Chapter 7].* \square

(This was the starting point of this paper.)

We now go case by case by using Proposition 5.1 to reduce to the case where k is large enough, so that we may apply Theorem 4.1.

6.1. The Hodge conjecture [1, 7.2]. Here $k = \mathbf{C}$, H is Betti cohomology, \mathcal{B} is the category **Hdg** of pure polarisable \mathbf{Q} -Hodge structures. Since \mathbf{C} is algebraically closed, there is no descent issue. Condition (A) holds because Mumford–Tate groups are connected (cf. [1, 7.1.2.1 1]), and condition (D) is trivial because $K = \mathbf{Q}$.

6.2. The Tate conjecture [1, 7.3]. Here k is finitely generated over its prime field, H is ℓ -adic cohomology for some prime number ℓ invertible in k , \mathcal{B} is the category of continuous representations of $G_k = \text{Gal}(k_s/k)$ on finite-dimensional \mathbf{Q}_ℓ -vector spaces, and $K = \mathbf{Q}_\ell$.

We may replace the latter by the equivalent Tannakian category $R_\ell(k)$ of continuous representations of the absolute Galois groupoid Π_k , as in [15, 2.2.3]. The corresponding functor \mathbf{H} sends $h(X)$, for $X \in \mathbf{Sm}^{\text{proj}}(k)$, to the functor from Π_k to $\text{Vec}_{\mathbf{Q}_\ell}$ which sends a separable algebraic closure k_s of k to $H_{\text{ét}}^*(X_{k_s}, \mathbf{Q}_\ell)$. If l/k is a finite Galois extension with group Γ , then Γ

acts on $R_\ell(l)$ by the functoriality of the étale fundamental groupoid. We may then apply Proposition 5.1 with $\mathcal{A} = \mathbf{LMot}(k)$, $\mathcal{A}' = \mathbf{LMot}(l)$, $\mathcal{B} = \mathcal{C} = R_\ell(k)$ and $\mathcal{B}' = \mathcal{C}' = R_\ell(l)$. The faithfulness of $f_{\mathcal{B}}^*$ is trivial and the functor \mathbf{H}' corresponding to \mathbf{H} “over l ” is Γ -equivariant. To prove Theorem 1 in the case of the Tate conjecture, we may therefore assume k large relatively to X , and further enlarge it if needed.

For $M \in \mathcal{B}$, $G_\omega(M)$ is the Zariski closure of the action of G_k on $\omega(M)$ (see [1, 7.1.3]); it follows that the composition $G_k \rightarrow G_\omega(M)(k) \rightarrow \pi_0(G_\omega(M))$ is surjective, which implies condition (A) up to enlarging k to a finite Galois extension k_ℓ , possibly depending on ℓ .

By quadratic reciprocity, condition (D) holds for a set of prime numbers in a suitable union of arithmetic progressions depending on X , which completes the proof of the Tate conjecture (in its strong form) for these ℓ . We are now going to prove it for *any* ℓ by avoiding condition (D), in several steps.

6.2.1. k is finite. By the above, there exists at least one prime ℓ_0 such that the conclusion of Theorem 4.1 holds for ℓ_0 -adic cohomology. In particular, the ℓ_0 -adic Tate conjecture holds for X , and $H_{\ell_0}^1(X)$ is semi-simple. By [30, Theorem 2.9(c)], the same then holds for any ℓ . Therefore, the conclusion of Theorem 4.1 holds for any ℓ thanks to Corollary 3.3.

6.2.2. k is a number field. Write $X = X_1 \times X_2$, where X_1 is the product of the CM factors of X and X_2 is the product of the other factors. By Theorem 4.1, we know that $G_\omega(X_2) = G_H(X_2)$ (because condition (D) is empty for X_2), and by Proposition 4.3, we know that $G_\omega(X) \xrightarrow{\sim} G_\omega(X_1) \times_{\mathbb{G}_m} G_\omega(X_2)$. Therefore, by (3.3), it suffices to prove that $G_\omega(X_1) = G_H(X_1)$. Choose an embedding σ of k_s into \mathbf{C} , and let $H_B : \mathbf{LMot} \rightarrow \mathbf{Vec}_{\mathbf{Q}}$ be the corresponding Betti realisation, which factors as

$$\mathbf{LMot} \xrightarrow{H_B} \mathbf{Hdg} \xrightarrow{\omega'} \mathbf{Vec}_{\mathbf{Q}},$$

where **Hdg** is as in Section 6.1. Recall M. Artin’s comparison isomorphism $H \simeq H_B \otimes_{\mathbf{Q}} \mathbf{Q}_\ell$. We have $G_{\omega'}(X_1) = G_{H_B}(X_1)$ by Section 6.1 and $G_{\omega'}(X_1) \otimes_{\mathbf{Q}} \mathbf{Q}_\ell = G_\omega(X_1)$ by [14, p. 371] or [31, §4], which concludes the proof.

6.2.3. The general case. Let k_0 be the field of constants of k . Write $X = X_1 \times X_2$ as above, where “CM” means “ordinary” in positive characteristic. As in Section 6.2.2, we reduce to the case $X = X_1$. Note that X is then defined over a finite extension of k_0 ; hence, taking k sufficiently large, we may assume that X is actually defined over k_0 . Then the isomorphism $G_\omega(X) \xrightarrow{\sim} G_H(X)$ over k_0 remains valid over k , because the left and the right-hand sides do not change under such extension of scalars: the surjectivity of $G_k \rightarrow G_{k_0}$ implies $\mathbf{LMot}(k_0)[X] \xrightarrow{\sim} \mathbf{LMot}(k)[X]$ on the right-hand side, while on the left-hand side, it implies that G_k and G_{k_0} have the same Zariski closure in $\mathbf{GL}(H^1(X))$.

6.3. The de Rham–Betti conjecture [1, 7.5]. Here k is a number field embedded in \mathbf{C} , H is Betti cohomology, \mathcal{B} is the category $\mathbf{Vec}_{k, \mathbf{Q}}$ of [1, 7.1.6]. Here again, let l/k be a finite Galois extension with group Γ . We apply Proposition 5.1 with $\mathcal{A} = \mathbf{LMot}(k)$, $\mathcal{A}' = \mathbf{LMot}(l)$, $\mathcal{B} = \mathbf{Vec}_{k, \mathbf{Q}}$, $\mathcal{C}' = \mathbf{Vec}_k$, ρ the forgetful functor, and similarly for \mathcal{B}' , \mathcal{C}' and ρ' replacing k with l . As in Section 6.2, the faithfulness of $f_{\mathcal{B}}^*$ is trivial and so is that of ρ' ; $\rho' \mathbf{H}'$ is the de Rham realisation which is Γ -equivariant. We may therefore assume k to be large.

Since $K = \mathbf{Q}$, condition (D) is trivial. It remains to prove condition (A) up to extending k . Let $\bar{\mathbf{Q}} \subset \mathbf{C}$ be the set of algebraic numbers, so that $k \subset \bar{\mathbf{Q}}$. We start with a lemma.

Lemma 6.2. *Write G_N for the Tannakian group of the full Tannakian subcategory $\langle N \rangle \subset \text{Vec}_{k,\mathbf{Q}}$ generated by an object $N = (W, V, \varpi)$. For an extension l of k contained in $\bar{\mathbf{Q}}$, write $N_l = (l \otimes_k, W, V, \varpi)$ for the image of N under the obvious functor $\text{Vec}_{k,\mathbf{Q}} \rightarrow \text{Vec}_{l,\mathbf{Q}}$. Then there exists a finite extension l/k such that the natural homomorphism $G_{N_{\bar{\mathbf{Q}}}} \rightarrow G_{N_l}$ is an isomorphism.*

Proof. The affine group schemes G_N, G_{N_l} are algebraic by [9, II, Proposition 2.20 (b)], and the homomorphism $G_{N_l} \rightarrow G_N$ is a closed immersion because the criterion of [9, II, Proposition 2.21 (b)] is trivially verified by the description of the objects of $\langle N \rangle$ and $\langle N_l \rangle$ as subquotients of direct sums of tensor constructions [9, I, 3.1a]. Therefore, the inverse system (G_{N_l}) , where l runs through the finite subextensions of $\bar{\mathbf{Q}}/k$, is stationary, and it suffices to show that its inverse limit is $G_{N_{\bar{\mathbf{Q}}}}$. It suffices to verify this on R -points for any \mathbf{Q} -algebra R (or even for $R = \bar{\mathbf{Q}}$). But this is clear, since $\langle N_{\bar{\mathbf{Q}}} \rangle$ is the 2-colimit of the $\langle N_l \rangle$'s. \square

We are now left to show that any finite quotient G of the Tannakian group of $\text{Vec}_{\bar{\mathbf{Q}},\mathbf{Q}}$ is reduced to $\{1\}$.

The following simple argument was kindly communicated by Y. André. Let (W, V, i) be the object of $\text{Vec}_{\bar{\mathbf{Q}},\mathbf{Q}}$ corresponding to a representation of G . Then i is defined over $\bar{\mathbf{Q}}$ (indeed, W, V and their tensor constructions define a torsor under G , and i defines a complex point, hence a $\bar{\mathbf{Q}}$ -point of this finite torsor). The choice of a basis of V then identifies (W, V, i) with a sum of copies of the unit object in $\text{Vec}_{\bar{\mathbf{Q}},\mathbf{Q}}$.

6.4. The Ogus conjecture. Here k is a number field embedded in \mathbf{C} , $H = H_{\text{dR}}$ is de Rham cohomology, \mathcal{B} is the category $\mathbf{Og}(k)$ of [1, 7.1.5] and $K = \mathbf{Q}$ (see [1, p. 72, footnote (4)]). Since $\mathbf{Og}(l)$ carries a natural action of Γ and the Ogus realisation is Γ -equivariant, we proceed directly with this category as in Section 6.2, and may assume k large.

The fibre functor $\omega: \mathbf{Og}(k) \rightarrow \text{Vec}_k$ is not neutral (unless $k = \mathbf{Q}$), but it becomes so after extending it to the Tannakian category over k , $\mathbf{Og}(k)_{(k)}$, of [9, pp. 155–156] (notation ω_k).

We prove condition (A) for the corresponding Tannakian group $G_{\omega_k}(E)$ of an elliptic factor E of X , by proving directly the equality $G_{\omega_k}(E) = G_H(E)$. This follows from the argument in [1, 7.4.3.1]. A detailed sketch: choose a prime \mathfrak{p} of k , unramified over \mathbf{Q} and with good and ordinary reduction for E , so that Tsuji's p -adic period isomorphism

$$H_{\text{dR}}^1(E/k) \otimes_k B_{\text{pst}} \simeq H_{\text{dR}}^1(E_{k_{\mathfrak{p}}}/k_{\mathfrak{p}}) \otimes_{k_{\mathfrak{p}}} B_{\text{pst}} \simeq H_{\text{ét}}^1(\bar{E}_{k_{\mathfrak{p}}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\text{pst}}$$

(see [1, 3.4.2]) is defined. The action of $G_{\omega_k}(E)$ on the left group is compatible with the action of $\text{Gal}(\bar{k}_{\mathfrak{p}}/k_{\mathfrak{p}})$ on the right group. If E is not CM, [27, Corollary 1, p. IV-44] says that the Lie algebra of the latter action is a Borel subalgebra of $\mathfrak{gl}_2(\mathbf{Q}_p)$. Hence

$$G_{\omega_k}(E) = \mathbf{GL}_2 (= G_H(E)),$$

since it is reductive. If E has complex multiplication, [27, Corollary 2] says that this Lie algebra is a split Cartan algebra; hence $G_{\omega_k}(E)$ contains a maximal torus, but then it must be equal to the maximal torus $G_H(E)$ (see [17, Example 1]).

Let now \mathcal{G} be the gerbe associated to $\mathcal{B}[X]$. If R is the compositum of the fields F_{ij} appearing in condition (D), there exists by Theorem 4.4 another fibre functor $\omega': \mathbf{Og}(k) \rightarrow \mathbf{Vec}_K$, where K is a number field linearly disjoint from R . Replacing $\mathbf{Og}(k)$ by $\mathbf{Og}(k)_{(K)}$, we get a neutral fibre functor ω'_K ; since gerbes are locally connected, condition (A) for ω_k implies condition (A) for ω'_K . By construction, condition (D) is satisfied; hence \mathbf{H} becomes fully faithful after extending scalars from \mathbf{Q} to K , and therefore it is fully faithful.

A. Appendix: Sections of algebraic gerbes

Lemma A.1. *Let F be a number field and E, F_1, \dots, F_r finite field extensions of F . For all $1 \leq i \leq r$, let $\alpha_i \in \mathrm{Br}(F_i)$. Then there exists a finite field extension K/F , linearly disjoint from E and from the F_i 's over F , and such that α_i vanishes in $\mathrm{Br}(K \otimes_F F_i)$ for all i .*

Proof. For all i , let S_i denote the finite set of places v of F_i such that $\alpha_{i,v} \neq 0$ in $\mathrm{Br}(F_{i,v})$. For all i and all $v \in S_i$, denote by $n_{i,v}$ the order of $\alpha_{i,v}$ in $\mathrm{Br}(F_{i,v})$. Let S be the (finite) set of places v of F for which there exists an i and a place $w \in S_i$ dividing v . For each $v \in S$, let n_v denote the lcm of the $n_{i,w}[F_{i,w} : F_v]$ for all i and $w \in S_i$ such that w divides v . Let E'/F denote the Galois closure of the compositum of the fields E, F_1, \dots, F_r , and denote its Galois group by G . By the Chebotarev theorem, for all $g \in G$, there exists a place v_g of F outside S and not dividing 2 such that E'/F is unramified at v_g and the Frobenius at v_g lies in the conjugacy class of g . One can assume that $v_g \neq v_h$ if $g \neq h$. Define S' to be the union of S and all the places v_g for $g \in G$. By the Grunwald–Wang theorem, there exists a (cyclic) field extension K/F such that, for all $v \in S$, all places w of K above v , the local degree $[K_w : F_v]$ is divisible by n_v , and for all $v \in S' \setminus S$, the extension K/F splits completely at v . Then, by construction, the extensions K and E' are linearly disjoint over F , and for all i , the image of α_i in $\mathrm{Br}(K \otimes_F F_i)$ vanishes at all places of $K \otimes_F F_i$ by a restriction–corestriction argument; hence it vanishes globally. \square

We refer to [4, 10] for the notions of gerbes, non-abelian H^2 and k -kernels.

Proposition A.2. *Let k be a number field and \mathcal{G} an étale (algebraic) gerbe over k , with connected linear band (or k -kernel) L . Let E be a finite extension of k . Then there exists a finite extension K/k linearly disjoint from E/k such that $\mathcal{G}(K) \neq \emptyset$.*

Proof. Consider the class α of \mathcal{G} in the non-abelian cohomology set $H^2(k, L)$ and its image $\alpha_F \in H^2(F, L)$. Let T be the k -torus associated to the k -band L (see [4, 1.7 and 6.1] for instance) and let $\alpha' \in H^2(k, T)$ be the image of α .

By [4, Proposition 6.5], for any totally imaginary finite field extension K/k , $\mathcal{G}(K) \neq \emptyset$ if and only if $\alpha'_K = 0$ in $H^2(K, T)$. So we are reduced to find a totally imaginary finite extension K/k , linearly disjoint from E over k , such that α' vanishes in $H^2(K, T)$.

First, there exists a totally imaginary finite field extensions F/k that is linearly disjoint from E over k (one can work as in Lemma A.1). So we are reduced to find a finite extension K/F , linearly disjoint from $E' := EF$ over F , such that α' vanishes in $H^2(K, T)$.

There exists an exact sequence of k -tori (a flasque resolution of T for instance; see [7, Proposition 1.3]) $0 \rightarrow S \rightarrow P \rightarrow T \rightarrow 0$ such that P is quasi-trivial, i.e. isomorphic to

a finite product of tori of the shape $R_{k_i/k}(\mathbf{G}_m)$ for some finite field extensions k_i/k . The field F is totally imaginary; hence $H^3(F, S) = 0$ by [24, Chapter I, Corollary 4.21]. Therefore, the map $H^2(F, P) \rightarrow H^2(F, T)$ is surjective. Let $\beta' \in H^2(F, P)$ be a lift of the class $\alpha' \in H^2(F, T)$. Writing the quasi-trivial F -torus P_F as $\prod_{i=1}^r R_{F_i/F}(\mathbf{G}_m)$ for some finite field extensions F_i/F , we get that

$$H^2(F, P) \cong \bigoplus_{i=1}^r \text{Br}(F_i).$$

To conclude, we apply the lemma to the elements $(\beta_i)_{1 \leq i \leq r} \in \bigoplus_{i=1}^r \text{Br}(F_i)$ corresponding to the class $\beta' \in H^2(F, P)$ and we get a finite extension K/F , linearly disjoint of E' and the F_i 's over F , such that the image of β' in $H^2(K, P) \cong \bigoplus_{i=1}^r \text{Br}(K \otimes_F F_i)$ vanishes. Then K and E are linearly disjoint over k and the proof is complete. \square

Acknowledgement. Bruno Kahn thanks Yves André for several discussions around this paper, Giuseppe Ancona for pointing out [20], Cyril Demarche for kindly writing up the appendix and the referee for several comments which helped improve the exposition.

References

- [1] *Y. André*, Une introduction aux motifs (motifs purs, motifs mixtes, périodes), Panor. Synthèses **17**, Société Mathématique de France, Paris 2004.
- [2] *L. Barbieri-Viale* and *B. Kahn*, Universal Weil cohomology, preprint 2024, <https://arxiv.org/abs/2401.14127>.
- [3] *C. Bertolin*, Third kind elliptic integrals and 1-motives, J. Pure Appl. Algebra **224** (2020), no. 10, Article ID 106396.
- [4] *M. V. Borovoi*, Abelianization of the second nonabelian Galois cohomology, Duke Math. J. **72** (1993), no. 1, 217–239.
- [5] *N. Bourbaki*, Éléments de mathématique. Algèbre. Chapitre 8. Modules et anneaux semi-simples, Springer, Berlin 2012.
- [6] *C. Chevalley*, Théorie des groupes de Lie. Tome III. Théorèmes généraux sur les algèbres de Lie, Hermann & Cie, Paris 1954.
- [7] *J.-L. Colliot-Thélène* and *J.-J. Sansuc*, Principal homogeneous spaces under flasque tori; applications, J. Algebra **106** (1987), no. 1, 148–205.
- [8] *P. Deligne*, Semi-simplicité de produits tensoriels en caractéristique p , Invent. Math. **197** (2014), no. 3, 587–611.
- [9] *P. Deligne*, *J. S. Milne*, *A. Ogus* and *K.-Y. Shih*, Hodge cycles, motives, and Shimura varieties, Lecture Notes in Math. **900**, Springer, Berlin 1982.
- [10] *C. Demarche* and *G. Lucchini Arteche*, Le principe de Hasse pour les espaces homogènes: réduction au cas des stabilisateurs finis, Compos. Math. **155** (2019), no. 8, 1568–1593.
- [11] *E. Goursat*, Sur les substitutions orthogonales et les divisions régulières de l'espace, Ann. Sci. Éc. Norm. Supér. (3) **6** (1889), 9–102.
- [12] *A. Grothendieck*, On the de Rham cohomology of algebraic varieties, Publ. Math. Inst. Hautes Études Sci. **29** (1966), 95–103.
- [13] *A. Grothendieck*, Hodge's general conjecture is false for trivial reasons, Topology **8** (1969), 299–303.
- [14] *H. Imai*, On the Hodge groups of some abelian varieties, Kodai Math. Sem. Rep. **27** (1976), no. 3, 367–372.
- [15] *B. Kahn*, Motifs et adjoints, Rend. Semin. Mat. Univ. Padova **139** (2018), 77–128.
- [16] *B. Kahn*, Zeta and L -functions of varieties and motives, London Math. Soc. Lecture Note Ser. **462**, Cambridge University, Cambridge 2020.
- [17] *B. Kahn*, Chow–Lefschetz motives, Indag. Math. (2024), DOI 10.1016/j.indag.2024.04.007.
- [18] *B. Kahn*, Galois descent for motivic theories, preprint 2024, <https://arxiv.org/abs/2312.01825>.
- [19] *E. R. Kolchin*, Algebraic groups and algebraic dependence, Amer. J. Math. **90** (1968), 1151–1164.

- [20] *T. Kreutz, M. Shen and C. Vial*, de Rham–Betti classes on products of elliptic curves over a number field are algebraic, preprint 2023, <https://arxiv.org/abs/2206.08618>.
- [21] *D. Lombardo*, On the ℓ -adic Galois representations attached to nonsimple abelian varieties, *Ann. Inst. Fourier (Grenoble)* **66** (2016), no. 3, 1217–1245.
- [22] *J. S. Milne*, Lefschetz motives and the Tate conjecture, *Compos. Math.* **117** (1999), no. 1, 45–76.
- [23] *J. S. Milne*, The Tate conjecture for certain abelian varieties over finite fields, *Acta Arith.* **100** (2001), no. 2, 135–166.
- [24] *J. S. Milne*, Arithmetic duality theorems, 2nd ed., BookSurge, Charleston 2006.
- [25] *A. Ogus*, Hodge cycles and crystalline cohomology in: *Hodge cycles, motives, and Shimura varieties* Lecture Notes in Math. **900**, Springer, Berlin (1982), 357–414.
- [26] *N. Saavedra Rivano*, Catégories Tannakiennes, Lecture Notes in Math. **265**, Springer, Berlin 1972.
- [27] *J.-P. Serre*, Abelian ℓ -adic representations and elliptic curves, W. A. Benjamin, New York 1968.
- [28] *T. Shioda*, Algebraic cycles on abelian varieties of Fermat type, *Math. Ann.* **258** (1981/82), no. 1, 65–80.
- [29] *M. Spieß*, Proof of the Tate conjecture for products of elliptic curves over finite fields, *Math. Ann.* **314** (1999), no. 2, 285–290.
- [30] *J. Tate*, Conjectures on algebraic cycles in ℓ -adic cohomology, in: *Motives (Seattle 1991)*, Proc. Sympos. Pure Math. **55**, American Mathematical Society, Providence (1994), 71–83.
- [31] *C.-F. Yu*, A note on the Mumford–Tate conjecture for CM abelian varieties, *Taiwanese J. Math.* **19** (2015), no. 4, 1073–1084.

Bruno Kahn, Sorbonne Université and Université Paris Cité, CNRS, IMJ-PRG, F-75005 Paris, France
<https://orcid.org/0000-0001-9223-8080>
 e-mail: bruno.kahn@imj-prg.fr

Cyril Demarche, Sorbonne Université and Université Paris Cité, CNRS, IMJ-PRG, F-75005 Paris, France
 e-mail: cyril.demarche@imj-prg.fr

Eingegangen 15. März 2023, in revidierter Fassung 19. September 2024