Applications of Weight-Two Motivic Cohomology

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Received: July 26, 1996

Communicated by Peter Schneider

Abstract. Using Lichtenbaum's complex $\Gamma(2)$, we reprove and extend a little bit some known results relating the kernel of $H^3(F, \mathbf{Q}/\mathbf{Z}(2)) \to H^3(F(X), \mathbf{Q}/\mathbf{Z}(2))$ to the torsion of CH^2X for rational varieties X.

1991 Mathematics Subject Classification: Primary 14C35, secondary 19E20.

Contents

Introduction

Let F be a field and X be a smooth, geometrically integral variety over F. In [6, prop. 3.6], Colliot-Thélène and Raskind produced an exact sequence:

$$(1)\quad H^1_{\mathrm{Zar}}(X, \mathcal{K}_2) \to H^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2)^{G_F} \to H^1(F, K_2(\overline{F}(X))/H^0_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2)) \to \mathrm{Ker}(CH^2X \to CH^2\overline{X}) \to H^1(F, H^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2)) \to H^2(F, K_2(\overline{F}(X))/H^0_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2)).$$

Here, $\overline{X}$ denotes the variety X viewed over the separable closure $\overline{F}$ of F, $\mathcal{K}_2$ is the Zariski sheaf associated to the presheaf $U \mapsto K_2(U)$ and G_F is the absolute Galois group of F. On the other hand, in [17, th. 3.1], we produced an isomorphism

$$(2)\quad H^1(F, K_2(\overline{F}(X))/K_2(\overline{F})) \simeq \mathrm{Ker}(H^3(F, \mathbf{Q}/\mathbf{Z}(2)) \to H^3(F(X), \mathbf{Q}/\mathbf{Z}(2))).$$

In (2), the coefficients $\mathbf{Q}/\mathbf{Z}(2)$ are

$$\varinjlim \mu_n^{\otimes 2} \text{ if } \operatorname{char} F = 0 \text{ and } \varinjlim_{(n,\operatorname{char} F)=1} \mu_n^{\otimes 2} \oplus \varinjlim_r W_r\Omega^2_{\log}[-2] \text{ if } \operatorname{char} F > 0,$$

where $W_r\Omega^2_{\log}$ is the weight-two logarithmic part of the de Rham-Witt complex over the big étale site of $\operatorname{Spec} F$ [13] (see comments at the end of the introduction).

When X is a *complete rational variety, i.e.* the extension $\overline{F}(X)/\overline{F}$ is purely transcendental, the group $H^0_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2)$ coincides with $K_2(\overline{F})$. One may therefore replace the group $H^1(\overline{F}, K_2(F(X))/H^0_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2))$ in (1) by $\operatorname{Ker}(H^3(F, \mathbf{Q}/\mathbf{Z}(2)) \to H^3(F(X), \mathbf{Q}/\mathbf{Z}(2)))$ in this case. The resulting exact sequence has been used in [29] and [30].

Moreover, the left map in (1) is injective when X is a complete rational variety ([6, prop. 4.3] in characteristic 0, [24, prop. 1.5] in general). Putting all this together, one therefore gets an exact sequence:

$$\begin{aligned} 0 \to H^1_{\mathrm{Zar}}(X, \mathcal{K}_2) \to H^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2)^{G_F} & \\ \to \operatorname{Ker}(H^3(F, \mathbf{Q}/\mathbf{Z}(2)) \to H^3(F(X), \mathbf{Q}/\mathbf{Z}(2))) & \\ \to \operatorname{Ker}(CH^2X \to CH^2\overline{X}) \to H^1(F, H^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2)) & \end{aligned}$$

for any complete rational variety X.

In this paper, we use the Lichtenbaum complex $\Gamma(2)$ of [22], [23] to recover this exact sequence directly, and extend it to the right. Our main result is:

Theorem 1. *Let X be a smooth variety over F.*
a) Assume that $K_2(\overline{F}) \xrightarrow{\sim} H^0_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2)$. Let us denote by

$$\eta : H^3(F, \mathbf{Q}/\mathbf{Z}(2)) \to H^0_{\mathrm{Zar}}(X, \mathcal{H}^3(\mathbf{Q}/\mathbf{Z}(2)))$$
$$\xi : CH^2X \to (CH^2\overline{X})^{G_F}$$
$$\mathrm{cl}^2_X : CH^2X \otimes \mathbf{Q}/\mathbf{Z} \to H^4(X, \mathbf{Q}/\mathbf{Z}(2))$$

the natural maps and the divisible cycle class map. Then there is an exact sequence

$$(3) \quad 0 \to H^1_{\mathrm{Zar}}(X, \mathcal{K}_2) \to H^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2)^{G_F} \to \operatorname{Ker}\eta \to \operatorname{Ker}\xi \to H^1(F, H^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2)).$$

b) Assume moreover that $H^0_{\mathrm{Zar}}(\overline{X}, \mathcal{H}^3(\mathbf{Q}/\mathbf{Z}(2)))$ is p-primary torsion, where p is the characteristic exponent of F and $\mathcal{H}^3(\mathbf{Q}/\mathbf{Z}(2))$ is the Zariski sheaf associated to the presheaf $U \mapsto H^3_{\text{ét}}(U, \mathbf{Q}/\mathbf{Z}(2))$ (if $\operatorname{char} F = 0$, this means $H^0_{\mathrm{Zar}}(\overline{X}, \mathcal{H}^3(\mathbf{Q}/\mathbf{Z}(2))) = 0$). Then the exact sequence (3) extends to a complex

$$(4) \quad \operatorname{Ker}\xi \to H^1(F, H^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2)) \to H^4(F, \mathbf{Q}/\mathbf{Z}(2)) \to \operatorname{Coker} \mathrm{cl}^2_X.$$

Let A (resp. B) denote the homology of (4) at $H^1(F, H^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2))$ (resp. at $H^4(F, \mathbf{Q}/\mathbf{Z}(2))$). Then there is another complex

$$(5) \quad 0 \to \operatorname{Coker}\eta \otimes \mathbf{Z}[1/p] \to \operatorname{Coker}\xi \otimes \mathbf{Z}[1/p] \to H^2(F, H^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2)) \otimes \mathbf{Z}[1/p]$$

whose homology at $\operatorname{Coker}\eta \otimes \mathbf{Z}[1/p]$ (resp. at $\operatorname{Coker}\xi \otimes \mathbf{Z}[1/p]$) is $A \otimes \mathbf{Z}[1/p]$ (resp. $B \otimes \mathbf{Z}[1/p]$).
If $H^0_{\mathrm{Zar}}(\overline{X}, \mathcal{H}^3(\mathbf{Q}/\mathbf{Z}(2))) = 0$, we can remove $\otimes \mathbf{Z}[1/p]$ everywhere.

Remark. The assumptions are satisfied if X is a complete rational variety, but also if it is a torsor under a semi-simple, simply connected algebraic group [7]. If $\operatorname{char} k = p > 0$, in the second case the group $H^0_{\mathrm{Zar}}(\overline{X}, \mathcal{H}^3(\mathbf{Q}/\mathbf{Z}(2))$ is in general nonzero, as higher logarithmic Hodge-Witt cohomology is not homotopy invariant; hence the complicated statement of theorem 1. However, we do have $H^0_{\mathrm{Zar}}(\overline{X}, \mathcal{H}^3(\mathbf{Q}/\mathbf{Z}(2)) = 0$ in the first case (compare corollaries 5.3 and 6.2 c)).

Corollary. *Let X be as in theorem 1 b).*
1) Suppose $\operatorname{cd} F \leq 3$. Then there is an exact sequence

$$0 \to H^1_{\mathrm{Zar}}(X, \mathcal{K}_2) \to H^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2)^{G_F} \to \operatorname{Ker} \eta \to \operatorname{Ker} \xi \to H^1(F, H^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2)) \\ \to \operatorname{Coker} \eta \to \operatorname{Coker} \xi \to H^2(F, H^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2))$$

after tensorisation by $\mathbf{Z}[1/p]$. The part of this sequence up to $H^1(F, H^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2))$ exists and is exact without tensoring by $\mathbf{Z}[1/p]$.
2) Suppose $\operatorname{cd} F \leq 2$. Then there is an isomorphism

$$H^1_{\mathrm{Zar}}(X, \mathcal{K}_2) \xrightarrow{\sim} H^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2)^{G_F}$$

and an exact sequence

$$0 \to \operatorname{Ker} \xi \to H^1(F, H^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2)) \\ \to H^0_{\mathrm{Zar}}(X, \mathcal{H}^3(\mathbf{Q}/\mathbf{Z}(2))) \to \operatorname{Coker} \xi \to H^2(F, H^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2))$$

after tensorisation by $\mathbf{Z}[1/p]$. The injection $\operatorname{Ker} \xi \hookrightarrow H^1(F, H^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2))$ holds without tensoring by $\mathbf{Z}[1/p]$.
If $H^0_{\mathrm{Zar}}(\overline{X}, \mathcal{H}^3(\mathbf{Q}/\mathbf{Z}(2))) = 0$, the results hold without tensoring by $\mathbf{Z}[1/p]$.

To try and get a relationship between theorem 1 and the last term in (1), we observe that a closer examination of the spectral sequence used in [17, proof of th. 3.1] yields an exact sequence:

$$(6) \quad H^3(F, \mathbf{Q}/\mathbf{Z}(2)) \to \operatorname{Ker}(H^3(F(X), \mathbf{Q}/\mathbf{Z}(2)) \to H^3(\overline{F}(X), \mathbf{Q}/\mathbf{Z}(2))) \\ \to H^2(F, K_2(F(X))/K_2(F)) \to H^4(F, \mathbf{Q}/\mathbf{Z}(2)) \to H^4(F(X), \mathbf{Q}/\mathbf{Z}(2)).$$

How to derive theorem 1 from sequence (6) does not seem obvious, however.

This paper is organized as follows. In section 1, we compute the étale hypercohomology of X with coefficients in $\Gamma(2)$: this is done in theorem 1.1, which is of independent interest. In sections 2 and 3, we introduce two relative complexes $\Gamma(F(X)/X, 2)$ (over $X_{\text{ét}}$) and $\Gamma(X/F, 2)$ (over $(\operatorname{Spec} F)_{\text{ét}}$). Considering the Hochschild-Serre spectral sequence for the hypercohomology of $\Gamma(F(X)/X, 2)$, we get back the Colliot-Thélène-Raskind exact sequence (1) in a straightforward manner (see proposition 2.2). To prove theorem 1, we similarly examine the Hochschild-Serre spectral sequence for the hypercohomology of X with coefficients $\Gamma(X/F, 2)$ (see section 3). In sections 4, 5 and 6, we respectively prove a purity theorem, compute the motivic cohomology of a projective bundle and prove a Bloch-Ogus type theorem. Finally, in section 7, we look at projective homogeneous varieties.

The proof of the isomorphism (2) in [17] consisted of considering the Hochschild-Serre spectral sequence for the hypercohomology of F with coefficients in a relative

Lichtenbaum complex $\Gamma(F(X)/F,2)$, relative to the extension $\overline{F}/F$. What we do here can be considered as a refinement of this method, by factoring the morphism $\operatorname{Spec} F(X) \to \operatorname{Spec} F$ into

$$\operatorname{Spec} F(X) \longrightarrow X \longrightarrow \operatorname{Spec} F.$$

REMARKS ON CHARACTERISTIC p. We have to be a little careful if $\operatorname{char} F > 0$ when defining the coefficients $\mathbf{Q}/\mathbf{Z}(2)$. In characteristic 0, they are defined as $\varinjlim \mu_n^{\otimes 2}$. If $\operatorname{char} F = p > 0$, we set $\mathbf{Z}/p^r(2) = W_r\Omega^2_{\log}[-2]$, where $W_r\Omega^2_{\log}$ is the sheaf of logarithmic de Rham-Witt differentials over the big étale site of $\operatorname{Spec} F$, defined as the subsheaf of the de Rham-Witt sheaf $W_r\Omega^2$ generated locally for the étale topology by sections of the form $d\log \underline{x}_1 \wedge d\log \underline{x}_2$ [13, I.5.7]. So $\mathbf{Z}/p^r(2)$ is a complex of étale sheaves concentrated in degree 2. The Verlagerung maps $V : W_n\Omega^2 \to W_{n+1}\Omega^2$ preserve logarithmic differentials, hence can be used to define $\mathbf{Q}_p/\mathbf{Z}_p(2)$ as $\varinjlim_r \mathbf{Z}/p^r(2)$. Corollaires I.3.5 and I.5.7.5 of [13] yield exact sequences of étale sheaves

$$0 \to \mathbf{Z}/p^r(2) \xrightarrow{V^s} \mathbf{Z}/p^{r+s}(2) \to \mathbf{Z}/p^s(2) \to 0 \tag{7}$$

hence exact sequences

$$0 \to \mathbf{Z}/p^r(2) \to \mathbf{Q}_p/\mathbf{Z}_p(2) \xrightarrow{p^r} \mathbf{Q}_p/\mathbf{Z}_p(2) \to 0. \tag{8}$$

We now define $\mathbf{Q}/\mathbf{Z}(2)$ as $\varinjlim_{(n,\operatorname{char} F)=1} \mu_n^{\otimes 2} \oplus \mathbf{Q}_p/\mathbf{Z}_2(2)$. We sometimes abbreviate $\mathbf{Q}/\mathbf{Z}(2)$ by '2'.

NOTATION. We denote by $\Gamma_{\mathrm{Zar}}(2)$ (*resp.* $\Gamma_{\text{ét}}(2)$) the complex of sheaves over the big Zariski (*resp.* étale) site of $\operatorname{Spec} F$ associated to the presheaf $U \mapsto \Gamma(U,2)$ of [22]. When necessary, we denote by $\Gamma_{\mathrm{Zar}}(X,2)$ (*resp.* $\Gamma_{\text{ét}}(X,2)$) the restriction of $\Gamma_{\mathrm{Zar}}(2)$ (*resp.* $\Gamma_{\text{ét}}(2)$) to the small Zariski (*resp.* étale) site of a scheme X. We drop indices when the context makes it clear what site we are in.

1. MOTIVIC COHOMOLOGY OF SMOOTH VARIETIES

Let X be a smooth, connected variety over a field F. We compute the étale hypercohomology groups $\mathbb{H}^*_{\text{ét}}(X,\Gamma(2)) = \mathbb{H}^*_{\text{ét}}(X,\Gamma_{\text{ét}}(2))$:

1.1. THEOREM. *$\mathbb{H}^i_{\text{ét}}(X,\Gamma(2))$ is*

(i) 0 *for* $i \leq 0$.
(ii) $K_3(F(X))_{\mathrm{ind}}$ *for* $i = 1$.
(iii) $H^0_{\mathrm{Zar}}(X,\mathcal{K}_2)$ *for* $i = 2$.
(iv) $H^1_{\mathrm{Zar}}(X,\mathcal{K}_2)$ *for* $i = 3$
(v) $\operatorname{Coker} \mathrm{cl}^2_X$ *for* $i = 5$
(vi) $H^{i-1}_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(2))$ *for* $i \geq 6$

where cl^2_X *is defined in theorem 1. Moreover, for* $i = 4$ *there is a short exact sequence:*

$$0 \to CH^2X \to \mathbb{H}^4_{\text{ét}}(X,\Gamma(2)) \to H^0_{\mathrm{Zar}}(X,\mathcal{H}^3(\mathbf{Q}/\mathbf{Z}(2))) \to 0. \tag{9}$$

As an immediate application, we get:

1.2. COROLLARY. *In characteristic* 0*, weight-two étale motivic cohomology is homotopy invariant. In characteristic* >0*, this is still true up to (cohomological) degree* 3.

To prove theorem 1.1, we shall use the Leray spectral sequence

$$E_2^{p,q} = H^p_{\mathrm{Zar}}(X, R^q\alpha_*\Gamma(2)) \Longrightarrow \mathbb{H}^{p+q}_{\text{ét}}(X, \Gamma(2)) \tag{10}$$

associated to the change-of-sites map $\alpha : X_{\text{ét}} \to X_{\mathrm{Zar}}$. For the convenience of the reader, we prove a well-known general lemma:

1.3. Lemma. *Let $\eta \xrightarrow{j} X$ be the generic point of the irreducible normal scheme X, and let A be an étale sheaf over η. Then the cohomology groups $H^q_{\text{ét}}(X, j_*A)$ are torsion for all $q > 0$.*

Proof. Let $\eta = \operatorname{Spec} K$. Consider the Leray spectral sequence for j

$$E_2^{p,q} = H^p_{\text{ét}}(X, R^q j_*A) \Longrightarrow H^{p+q}_{\text{ét}}(K, A).$$

Since the abutment is Galois cohomology, it is torsion for $p + q > 0$ and we have to prove that $R^q j_*A$ is torsion for all $q > 0$. But since X is normal, it is geometrically unibranch and the stalks of $R^q j_*A$ are Galois cohomology of the strict Henselizations of K relatively to the points of X, hence the claim. □

1.4. Lemma. *The Zariski sheaves $R^q\alpha_*\Gamma(2)$ are as follows:*

(i) 0 *for* $q \leq 0$.
(ii) *The constant sheaf* $K_3(F(X))_{\mathrm{ind}}$ *for* $q = 1$.
(iii) $\mathcal{K}_2$ *for* $q = 2$.
(iv) 0 *for* $q = 3$.
(v) $\mathcal{H}^{q-1}(\mathbf{Q}/\mathbf{Z}(2))$ *for* $q \geq 4$.

Proof. (i) is obvious, (iii) is proved in [23, th. 2.10]) and (ii) (*resp.* (iv)) is proved in [23, prop. 2.11] (*resp.* in [23, prop. 2.12]) but only up to 2-torsion. This partially comes from the insistence to deal with $gr^2_\gamma K_3$ rather than with $K_{3,\mathrm{ind}}$. We give proofs of (ii), (iv) and (v).

Denote by $\mathbf{K}_{3,\mathrm{ind}}$ (*resp.* $\mathbf{H}^1(\Gamma(2))$) the étale sheaf associated to the presheaf $R \mapsto K_3(R)_{\mathrm{ind}}$ (*resp.* $R \mapsto H^1(\Gamma(R, 2))$) for étale $\operatorname{Spec} R \to X$. Let $x \in X$. We claim that there is a chain of isomorphisms

$$\begin{aligned} H^1_{\text{ét}}(\mathcal{O}_{X,x}, \Gamma(2)) \xrightarrow{\sim} H^0_{\text{ét}}(\mathcal{O}_{X,x}, \mathbf{H}^1(\Gamma(2))) \xleftarrow{\sim} H^0_{\text{ét}}(\mathcal{O}_{X,x}, \mathbf{K}_{3,\mathrm{ind}}) \\ \xrightarrow{\sim} H^0_{\text{ét}}(K, \mathbf{K}_{3,\mathrm{ind}}) \xleftarrow{\sim} K_3(K)_{\mathrm{ind}}. \end{aligned} \tag{11}$$

The first isomorphism (from the left) simply comes from the fact that $\mathbf{H}^i(\Gamma(2)) = 0$ for $i \leq 0$. The last one is proven in [26, prop. 11.4] (see also [21, th. 4.13]). By [16, theorem], if A is a local ring of a smooth variety, then $K_3(A)_{\mathrm{ind}} \to K_3(K)_{\mathrm{ind}}$ is bijective, where K is the field of fractions of A. Letting $j : \operatorname{Spec} K \hookrightarrow X$ be the inclusion of the generic point, this shows that the map $\mathbf{K}_{3,\mathrm{ind}} \to j_*j^*\mathbf{K}_{3,\mathrm{ind}}$ is an isomorphism, hence the third isomorphism in (11). Finally, by [22, prop. 1.8], for any local ring A whose residue field contains more than 2 elements, there is a surjection

$$K_3(A)_{\mathrm{ind}} \twoheadrightarrow H^1(\Gamma(A, 2))$$

which is bijective if A is a field. Therefore, the commutative diagram

$$\begin{tikzcd} K_3(\mathcal{O}^{sh}_{X,x})_{\mathrm{ind}} \arrow[r, two heads] \arrow[d, "\wr"'] & H^1(\Gamma(\mathcal{O}^{sh}_{X,x}, 2)) \arrow[d] \\ K_3(K^{sh}_x)_{\mathrm{ind}} \arrow[r, "\sim"] & H^1(\Gamma(K^{sh}_x, 2)) \end{tikzcd}$$

where $\mathcal{O}_{X,x}^{sh}$ is the strict Henselisationes of $\mathcal{O}_{X,x}$ and K_x^{sh} is its field of fractions, shows that $K_3(\mathcal{O}_{X,x}^{sh})_{\mathrm{ind}} \to H^1(\Gamma(\mathcal{O}_{X,x}^{sh}, 2))$ is an isomorphism (we used [16] again for the left vertical isomorphism). This proves the second isomorphism in (11), which proves lemma 1.4 (ii).

We note that (iv) follows from (iii), the Merkurjev-Suslin theorem for the local rings of X [22, th. 9.1], the fact that $R^3\alpha_*\Gamma(2)$ is torsion [22, th. 9.2] and the triangles

$$\begin{tikzcd}[column sep=small] \Gamma(2) \arrow[rr, "n"] & & \Gamma(2) \arrow[dl] \\ & \mu_n^{\otimes 2} \arrow[ul] & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] \Gamma(2) \arrow[rr, "p^r"] & & \Gamma(2) \arrow[dl] \\ & \mathbf{Z}/p^r(2) \arrow[ul] & \end{tikzcd} \tag{12}$$

in the derived category (the second triangle in the case $\operatorname{char} F = p > 0$). The first triangle is proven exact in [22] and [23] only for n odd, relying on the computation of torsion and cotorsion in $\mathbf{K}_{3,\mathrm{ind}}$ [22, lemma 8.2]. However, the proof goes through just as well for n even by using the isomorphism from [16] already mentioned. The second triangle is proven exact in [23, lemma 2.7] only for $r = 1$ and $p > 2$ (this fact was overlooked in [17]). However, the proof of [23, lemma 2.7] carries over in the same way, using (ii) and the Bloch-Gabber-Kato isomorphism $K_2(E)/p^r \xrightarrow{\sim} W_r\Omega^2_{E,\log}$ for any field E of characteristic p [2, cor. 2.8].

Finally, let us prove (v). By the triangle (12), we have a long exact sequence of Zariski sheaves

$$\cdots \to R^{i-1}\alpha_*\Gamma(2) \otimes \mathbf{Q} \to R^{i-1}\alpha_*\mathbf{Q}/\mathbf{Z}(2) \to R^i\alpha_*\Gamma(2) \to R^i\alpha_*\Gamma(2) \otimes \mathbf{Q} \to \dots$$

so that it is enough to see that $R^i\alpha_*\Gamma(2)$ is torsion for $i \geq 3$. For $i = 3$, this is (iv). For $i > 3$, we have a long exact sequence of sheaves

$$\cdots \to R^{i-1}\alpha_*\mathbf{K}_{3,\mathrm{ind}} \to R^i\alpha_*\Gamma(2) \to R^{i-2}\alpha_*\mathbf{K}_2 \to \dots$$

so it is enough to see that $R^i\alpha_*\mathbf{K}_{3,\mathrm{ind}}$ and $R^i\alpha_*\mathbf{K}_2$ are torsion for $i > 0$. In view of the isomorphism (see above)

$$\mathbf{K}_{3,\mathrm{ind}} \xrightarrow{\sim} j_*j^*\mathbf{K}_{3,\mathrm{ind}}$$

the first one follows from lemma 1.3. We are left with proving that $R^i\alpha_*\mathbf{K}_2$ is torsion for $i > 0$. As in [23, proof of lemma 2.2], we have a "Gersten resolution"

$$0 \to \mathbf{K}_2 \to j_*K_{2,K} \to \coprod_{x\in X^{(1)}} i_x^*\mathbb{G}_m \to \coprod_{x\in X^{(2)}} i_x^*\mathbf{Z} \to 0.$$

This complex of étale sheaves is not exact, but up to torsion it is. Therefore, up to torsion, there is a spectral sequence of Zariski sheaves

$$E_1^{p,q} = R^q\alpha_*C^p \Longrightarrow R^{p+q}\alpha_*\mathbf{K}_2$$

where C^p is the p-th term of the above "resolution" of $\mathbf{K}_2$. Since C^0 is of the form $j_*\mathcal{F}$, the same argument as above shows that $E_1^{0,q}$ is torsion for $q > 0$. The stalks of $E_1^{1,q}$ and $E_1^{2,q}$ are sums of Galois cohomology groups, so are torsion for $q > 0$. This shows that $E_2^{p,q}$ is torsion for $p+q>0$, except perhaps when $q=0$. But, for $x\in X$, the stalks of $E_2^{1,0}$ and $E_2^{2,0}$ at x are the cohomology groups of the complex

$$H^0(K, K_2(\overline{K})) \to \coprod_{y\in Y^{(1)}} F(y)^* \to \coprod_{y\in Y^{(2)}} \mathbf{Z} \to 0 \tag{13}$$

where $Y=\operatorname{Spec}\mathcal{O}_{X,x}$. Comparing with the exact sequence (Gersten's conjecture)

$$K_2(K) \to \coprod_{y\in Y^{(1)}} F(y)^* \to \coprod_{y\in Y^{(2)}} \mathbf{Z} \to 0$$

and using the fact that the map $K_2(K)\to H^0(K, K_2(\overline{K}))$ has torsion kernel and cokernel, we get that (13) has torsion cohomology groups, which concludes the proof of lemma 1.4 (v). □

Proof of theorem 1.1. As indicated above, we use the spectral sequence (10). (i) is obvious in view of lemma 1.4 (i) and so is (ii) in view of the isomorphism

$$\mathbb{H}^1_{\text{ét}}(X,\Gamma(2)) \xrightarrow{\sim} H^0_{\text{Zar}}(X, R^1\alpha_*\Gamma(2))$$

and lemma 1.4 (ii). To get further, we observe that $E_2^{p,1}=0$ for $p>0$ since $R^1\alpha_*\Gamma(2)$ is constant, and $E_2^{p,3}=0$ for all p in view of lemma 1.4 (iv). This and lemma 1.4 (iii) immediately imply (iii) and (iv). Still by lemma 1.4 (iii) and Gersten's conjecture, $E_2^{p,2}=0$ for $p>2$ and $E_2^{2,2}\simeq CH^2X$; this and lemma 1.4 (v) (for $q=4$) gives the exact sequence (9). We now note that the above information and lemma 1.4 (v) imply that $\mathbb{H}^i_{\text{ét}}(X,\Gamma(2))$ is torsion for $i\geq 5$. (v) and (vi) now follow from (9) and the long exact sequence

$$\cdots\to\mathbb{H}^{i-1}_{\text{ét}}(X,\Gamma(2))\otimes\mathbf{Q}\to H^{i-1}_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(2))\to\mathbb{H}^i_{\text{ét}}(X,\Gamma(2))\to\mathbb{H}^i_{\text{ét}}(X,\Gamma(2))\otimes\mathbf{Q}\to\cdots$$

□

1.5. Remark. The same computation gives the cohomology sheaves of $\Gamma_{\text{Zar}}(X,2)$:

$$\begin{aligned}\mathcal{H}^1(\Gamma_{\text{Zar}}(X,2)) &= K_3(K)_{\text{ind}}\\ \mathcal{H}^2(\Gamma_{\text{Zar}}(X,2)) &= \mathcal{K}_2\\ \mathcal{H}^i(\Gamma_{\text{Zar}}(X,2)) &= 0 \,\text{for}\, i\neq 1,2.\end{aligned}$$

From this, we deduce a triangle, precising [23, prop. 3.1]:

$$\begin{tikzcd}[column sep=small] \Gamma_{\text{Zar}}(2) \arrow[rr] & & R\alpha_*\Gamma_{\text{ét}}(2) \arrow[dl] \\ & \tau_{\geq 3}(R\alpha_*\mathbf{Q}/\mathbf{Z}(2))[-1] \arrow[ul] & \end{tikzcd}$$

In particular,

$$\Gamma_{\text{Zar}}(2)\otimes\mathbf{Q} \xrightarrow{\sim} R\alpha_*\Gamma_{\text{ét}}(2)\otimes\mathbf{Q}. \tag{14}$$

We also get the following analogue of theorem 1.1:

1.6. Theorem. $\mathbb{H}^i_{\text{Zar}}(X,\Gamma_{\text{Zar}}(2)) = \begin{cases} K_3(K)_{\text{ind}} & \textit{if } i=1\\ H^{i-2}_{\text{Zar}}(X,\mathcal{K}_2) & \textit{if } 2\leq i\leq 4\\ 0 & \textit{otherwise.}\end{cases}$ □

2. Relative motivic cohomology, I

Let $j : \operatorname{Spec} F(X) \hookrightarrow X$ be the inclusion of the generic point and $\Gamma(F(X)/X, 2)$ be the homotopy fibre of the morphism

$$\Gamma_{\text{ét}}(X, 2) \to Rj_* \Gamma_{\text{ét}}(F(X), 2).$$

Denote the hypercohomology group $\mathbb{H}^i_{\text{ét}}(X, \Gamma(F(X)/X, 2))$ by $\mathbb{H}^i(F(X)/X, \Gamma(2))$, so that we have a long exact sequence

$$\to \mathbb{H}^i(F(X)/X, \Gamma(2)) \to \mathbb{H}^i_{\text{ét}}(X, \Gamma(2)) \to \mathbb{H}^i_{\text{ét}}(F(X), \Gamma(2)) \to \mathbb{H}^{i+1}(F(X)/X, \Gamma(2)) \to$$

This gives:

2.1. Lemma. *The groups $\mathbb{H}^i(F(X)/X, \Gamma(2))$ are 0 for $i \le 2$; there are exact sequences:*

$$0 \to K_2(F(X))/H^0_{\text{Zar}}(X, \mathcal{K}_2) \to \mathbb{H}^3(F(X)/X, \Gamma(2)) \to H^1_{\text{Zar}}(X, \mathcal{K}_2) \to 0$$

$$\mathbb{H}^4(F(X)/X, \Gamma(2)) \xrightarrow{\sim} CH^2 X$$

$$(15) \quad 0 \to H^0_{\text{Zar}}(X, \mathcal{H}^3(2)) \to H^3_{\text{ét}}(F(X), 2) \to \mathbb{H}^5(F(X)/X, \Gamma(2)) \to \operatorname{Coker} \mathrm{cl}^2_X \to H^4_{\text{ét}}(F(X), 2).$$

Proof. The first claim is clear for $i \le 0$; for $i = 1$ and 2 it follows from theorem 1.1 and the injectivity of $H^0_{\text{Zar}}(X, \mathcal{H}^2) \to K_2(F(X))$. For $i = 3$, it follows from theorem 1.1 again, plus the vanishing of $\mathbb{H}^3(F(X), \Gamma(2))$. For $i = 4, 5$, we have a cross of exact sequences:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \mathbb{H}^4(F(X)/X, \Gamma(2)) \arrow[d] & & \\
0 \arrow[r] & CH^2 X \arrow[r] & \mathbb{H}^4_{\text{ét}}(X, \Gamma(2)) \arrow[r] \arrow[d] & H^0_{\text{Zar}}(X, \mathcal{H}^3(2)) \arrow[r] & 0 \\
 & & H^3_{\text{ét}}(F(X), 2) \arrow[d] & & \\
 & & \mathbb{H}^5(F(X)/X, \Gamma(2)) \arrow[d] & & \\
 & & \operatorname{Coker} \mathrm{cl}^2_X \arrow[d] & & \\
 & & H^4_{\text{ét}}(F(X), 2) & &
\end{tikzcd}$$

The map $\mathbb{H}^4_{\text{ét}}(X, \Gamma(2)) \to H^3_{\text{ét}}(F(X), 2)$ factors through $H^0_{\text{Zar}}(X, \mathcal{H}^3(2)) \to H^3_{\text{ét}}(F(X), 2)$, which is injective. A diagram chase concludes the proof. □

For simplicity, let us denote by $\overline{K}_2(F(X))$ the group $K_2(F(X))/H^0_{\mathrm{Zar}}(X,\mathcal{K}_2)$. Using the "Hochschild-Serre" (hypercohomology) spectral sequence

$$H^p_{\text{ét}}(F,\mathbb{H}^q(\overline{F}(X)/\overline{X},\Gamma(2)))\Rightarrow \mathbb{H}^{p+q}(F(X)/X,\Gamma(2))$$

and the vanishing of $\mathbb{H}^i(F(X)/X,\Gamma(2))$ for $i\leq 2$, we get an isomorphism

$$\mathbb{H}^3(F(X)/X,\Gamma(2))\xrightarrow{\sim} H^0(F,\mathbb{H}^3(\overline{F}(X)/\overline{X},\Gamma(2)))$$

and an 5-terms exact sequence

$$\begin{aligned}0\to H^1(F,\mathbb{H}^3(\overline{F}(X)/\overline{X},\Gamma(2)))\to \mathbb{H}^4(F(X)/X,\Gamma(2))\\ \to H^0(F,\mathbb{H}^4(\overline{F}(X)/\overline{X},\Gamma(2)))\to H^2(F,\mathbb{H}^3(\overline{F}(X)/\overline{X},\Gamma(2)))\to \mathbb{H}^5(F(X)/X,\Gamma(2))\end{aligned}$$

hence, using lemma 2.1:

2.2. Proposition. *There are exact sequences:*

$$\begin{aligned}0\to \overline{K}_2(F(X))\to \overline{K}_2(\overline{F}(X))^{G_F}\to H^1_{\mathrm{Zar}}(X,\mathcal{K}_2)\to H^1_{\mathrm{Zar}}(\overline{X},\mathcal{K}_2)^{G_F}\\ \to H^1(F,\overline{K}_2(F(X)))\to H^1(F,\mathbb{H}^3(\overline{F}(X)/\overline{X},\Gamma(2))\\ \to H^1(F,H^1_{\mathrm{Zar}}(\overline{X},\mathcal{K}_2))\to H^2(F,\overline{K}_2(\overline{F}(X)))\end{aligned}$$

$$\begin{aligned}0\to H^1(F,\mathbb{H}^3(\overline{F}(X)/\overline{X},\Gamma(2))\to CH^2X\to (CH^2\overline{X})^{G_F}\\ \to H^2(F,\mathbb{H}^3(\overline{F}(X)/\overline{X},\Gamma(2))\to \mathbb{H}^5(F(X)/X,\Gamma(2)).\end{aligned}$$

□

The exact sequence (1) follows immediately. Moreover, we also get [6, lemma 4.1].

3. Relative motivic cohomology, II

We recall some notation:

- As above, $H^i(X,j)$ (*resp.* $\mathcal{H}^i(j)$) is shorthand for $H^i_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(j))$ (*resp.* for $\mathcal{H}^i(\mathbf{Q}/\mathbf{Z}(j))$).
- η is the map $H^3(F,2)\to H^0(X,\mathcal{H}^3(2))$.
- ξ is the map $CH^2X\to (CH^2\overline{X})^{G_F}$.

We also denote by $\overline{H}^0(X,\mathcal{K}_2)$ the group $H^0(X,\mathcal{K}_2)/K_2(F)$.

Let $\pi: X\to \operatorname{Spec} F$ be the structural morphism and $\Gamma(X/F,2)$ be the homotopy fibre (in the derived category) of the morphism

$$\Gamma_{\text{ét}}(F,2)\to R\pi_*\Gamma_{\text{ét}}(X,2).$$

Denote the hypercohomology group $\mathbb{H}^i_{\text{ét}}(F,\Gamma(X/F,2))$ by $\mathbb{H}^i(X/F,\Gamma(2))$, so that we have a long exact sequence

(16)
$$\cdots\to \mathbb{H}^i(X/F,\Gamma(2))\to \mathbb{H}^i_{\text{ét}}(F,\Gamma(2))\to \mathbb{H}^i_{\text{ét}}(X,\Gamma(2))\to \mathbb{H}^{i+1}(X/F,\Gamma(2))\to\cdots$$

This gives:

3.1. Lemma. *The groups* $\mathbb{H}^i(X/F,\Gamma(2))$ *are:*

(i) 0 *for* $i\leq 1$.
(ii) $K_3(F(X))_{\mathrm{ind}}/K_3(F)_{\mathrm{ind}}$ *for* $i=2$.
(iii) $H^0_{\mathrm{Zar}}(X,\mathcal{K}_2)/K_2(F)$ *for* $i=3$.

Moreover, there is a complex

$$\begin{aligned}(17)\quad 0 \to H^1_{\mathrm{Zar}}(X,\mathcal{K}_2) \to \mathbb{H}^4(X/F,\Gamma(2)) \to \operatorname{Ker}\eta \\ \to CH^2X \to \mathbb{H}^5(X/F,\Gamma(2)) \to H^4(F,\mathbf{Q}/\mathbf{Z}(2)) \to \operatorname{Coker}\mathrm{cl}^2_X\,.\end{aligned}$$

This complex is exact, except perhaps at $\mathbb{H}^5(X/F,\Gamma(2))$*, where its homology is* $\operatorname{Coker}\eta$*. In particular, we have an isomorphism*

$$H^1_{\mathrm{Zar}}(\overline{X},\mathcal{K}_2) \xrightarrow{\sim} \mathbb{H}^4(\overline{X}/\overline{F},\Gamma(2))$$

and a short exact sequence

$$0 \to CH^2\overline{X} \to \mathbb{H}^5(\overline{X}/\overline{F},\Gamma(2)) \to H^0_{\mathrm{Zar}}(\overline{X},\mathcal{H}^3(2)) \to 0. \tag{18}$$

Proof. (i), (ii) and (iii) immediately follow from theorem 1.1 and the exact sequence (16). The complex (17) and the value of its homology follow from the cross of exact sequences ((9) and (16))

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & H^1_{\mathrm{Zar}}(X,\mathcal{K}_2) \arrow[d] & & \\
 & & \mathbb{H}^4(X/F,\Gamma(2)) \arrow[d] & & \\
 & & H^3_{\text{ét}}(F,2) \arrow[d] \arrow[dr, "\eta"] & & \\
0 \arrow[r] & CH^2X \arrow[r] & \mathbb{H}^4_{\text{ét}}(X,\Gamma(2)) \arrow[r] \arrow[d] & H^0_{\mathrm{Zar}}(X,\mathcal{H}^3(2)) \arrow[r] & 0 \\
 & & \mathbb{H}^5(X/F,\Gamma(2)) \arrow[d] & & \\
 & & H^4_{\text{ét}}(F,2) \arrow[d] & & \\
 & & \operatorname{Coker}\mathrm{cl}^2_X & &
\end{tikzcd}$$

and the "lemma of the 700th" [27]. □

We now consider the hypercohomology spectral sequence

$$H^p(F,\mathbb{H}^q(\overline{X}/\overline{F},\Gamma(2))) \Longrightarrow \mathbb{H}^{p+q}(X/F,\Gamma(2)). \tag{19}$$

Note that $E_2^{p,2}=0$ for $p>0$, since the group $K_3(F(X))_{\mathrm{ind}}/K_3(F)_{\mathrm{ind}}$ is uniquely divisible by [26, prop. 11.6]. Hence we get an isomorphism

$$\overline{H}^0_{\mathrm{Zar}}(X,\mathcal{K}_2) \xrightarrow{\sim} \overline{H}^0_{\mathrm{Zar}}(\overline{X},\mathcal{K}_2)^{G_F}$$

and an exact sequence

$$0 \to H^1(F, \overline{H}^0_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2)) \to \mathbb{H}^4(X/F, \Gamma(2)) \to H^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2)^{G_F} \to H^2(F, \overline{H}^0_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2))$$

(noting that $H^4(\overline{X}/\overline{F}, \Gamma(2)) = H^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2)$ by lemma 3.1). The isomorphism is Suslin's [32, cor. 5.9], but we get it here by a formal argument, in the vein of [17, th. 3.1 (a)]. The cross of complexes (the above exact sequence and (17)):

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
 & & H^1_{\mathrm{Zar}}(X, \mathcal{K}_2) & & & & \\
 & & \downarrow & & & & \\
0 \to H^1(F, \overline{H}^0_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2)) & \to & \mathbb{H}^4(X/F, \Gamma(2)) & \to & H^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2)^{G_F} & \to & H^2(F, \overline{H}^0_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2)) \\
 & & \downarrow & & & & \\
 & & \operatorname{Ker} \eta & & & & \\
 & & \downarrow & & & & \\
 & & CH^2 X & & & & \\
 & & \downarrow & & & & \\
 & & \mathbb{H}^5(X/F, \Gamma(2)) & & & & \\
 & & \downarrow & & & & \\
 & & H^4_{\text{ét}}(F, 2) & & & & \\
 & & \downarrow & & & & \\
 & & \operatorname{Coker} \mathrm{cl}^2_X & & & &
\end{array}$$

contains, via the lemma of the 700th, all the information one can easily get in this generality.

Assume now that $\overline{H}^0_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2) = 0$. Then the exact row in the above diagram reduces to an isomorphism $\mathbb{H}^4(X/F, \Gamma(2)) \xrightarrow{\sim} H^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2)^{G_F}$, hence we get a complex:

$$0 \to H^1_{\mathrm{Zar}}(X, \mathcal{K}_2) \to H^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2)^{G_F} \to \operatorname{Ker} \eta \to CH^2 X \to \mathbb{H}^5(X/F, \Gamma(2)) \to H^4_{\text{ét}}(F, 2) \to \operatorname{Coker} \mathrm{cl}^2_X \tag{20}$$

with homology $\operatorname{Coker} \eta$ at $\mathbb{H}^5(X/F, \Gamma(2))$ and 0 elsewhere.

Moreover the spectral sequence (19) and lemma 3.1 give an exact sequence

$$0 \to H^1(F, H^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2)) \to \mathbb{H}^5(X/F, \Gamma(2)) \to (\mathbb{H}^5(\overline{X}/\overline{F}, \Gamma(2)))^{G_F} \to H^2(F, H^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2)).$$

Putting (20) and () together, we get a cross of complexes (the horizontal one exact, the vertical one exact except perhaps at the crossing point):

$$\begin{tikzcd}[column sep=small]
 & & 0 \arrow[d] & & \\
 & & H^1_{\mathrm{Zar}}(X,\mathcal{K}_2) \arrow[d] & & \\
 & & H^1_{\mathrm{Zar}}(\overline{X},\mathcal{K}_2)^{G_F} \arrow[d] & & \\
 & & \operatorname{Ker}\eta \arrow[d] & & \\
 & & CH^2X \arrow[d] \arrow[dr, "\xi'"] & & \\
0 \arrow[r] & H^1(F,H^1_{\mathrm{Zar}}(\overline{X},\mathcal{K}_2)) \arrow[r] & \mathbb{H}^5(X/F,\Gamma(2)) \arrow[r] \arrow[d] & (\mathbb{H}^5(\overline{X}/\overline{F},\Gamma(2)))^{G_F} \arrow[r] & H^2(F,H^1_{\mathrm{Zar}}(\overline{X},\mathcal{K}_2)) \\
 & & H^4_{\text{ét}}(F,2) \arrow[d] & & \\
 & & \operatorname{Coker}\mathrm{cl}^2_X\,. & &
\end{tikzcd}$$

Note that $\operatorname{Ker}\xi = \operatorname{Ker}\xi'$ by (18). We get theorem 1 a) from this cross and the latter remark, by a diagram chase analogous to the lemma of the 700th. The same diagram chase gives us the complex (4), and shows that its cohomology coincides with that of a complex

$$0 \to \operatorname{Coker}\eta \to \operatorname{Coker}\xi' \to H^2(F, H^1_{\mathrm{Zar}}(\overline{X},\mathcal{K}_2)).$$

Notice the short exact sequence from (18)

$$0 \to \operatorname{Coker}\xi \to \operatorname{Coker}\xi' \to H^0_{\mathrm{Zar}}(\overline{X},\mathcal{H}^3(2))^{G_F}.$$

Using this exact sequence, we easily conclude the proof of theorem 1. □

4. Purity

In this section, we establish a purity theorem for Zariski and étale weight-two motivic cohomology, generalizing results of [23]. Recall that $\Gamma(1)$ is defined as $\mathbb{G}_m[-1]$ and $\Gamma(0)$ as $\mathbf{Z}[0]$ (in both the Zariski and étale topologies). We also need such complexes for $i < 0$:

4.1. Definition. *For $i < 0$, we define:*

$$\Gamma_{\mathrm{Zar}}(i) = 0;$$
$$\Gamma_{\text{ét}}(i) = \mathbf{Q}/\mathbf{Z}(i)[-1] \qquad \textit{(no p-primary part in characteristic p).}$$

The following theorem extends and precises [23, th. 4.5]; the method of proof is different.

4.2. Theorem. *Let X be a smooth variety over a field and let $Z \xrightarrow{i} X$ be a closed immersion, with Z smooth of codimension c.*

a) There is an isomorphism (in the derived category of complexes of sheaves over Z_{Zar})

$$\Gamma_{\mathrm{Zar}}(Z, 2-c)[-2c] \xrightarrow{\sim} Ri^!_{\mathrm{Zar}}\Gamma_{\mathrm{Zar}}(X, 2).$$

b) There is a map (in the derived category of complexes of sheaves over $Z_{\text{ét}}$)

$$\Gamma_{\text{ét}}(Z, 2-c)[-2c] \to Ri^!_{\text{ét}}\Gamma_{\text{ét}}(X, 2)$$

whose homotopy cofibre is concentrated in degree $c+4$ and has p-primary torsion cohomology, where p is the characteristic exponent of F. In particular, if $\operatorname{char} F = 0$, *this map is an isomorphism.*

4.3. Lemma. *Let $Z \overset{i}{\hookrightarrow} X$ be a smooth subvariety of X of codimension c. Then:*
a) For any constant sheaf A over X_{Zar}, $R^p i^!_{\mathrm{Zar}} A = 0$ for all p.
b) For any n, $R^p i^!_{\mathrm{Zar}}\mathcal{K}_n = \begin{cases} 0 & \text{for } p \neq c \\ \mathcal{K}_{n-c} & \text{for } p = c, \end{cases}$ where $\mathcal{K}_{n-c} := 0$ if $n < c$.

Proof. a) is trivial and b) follows in a well-known way from Gersten's conjecture (*e.g.* [9, § 7]). □

Proof of theorem 4.2 a). Apply $Ri^!$ to the triangle

$$\begin{array}{ccc} (\mathcal{K}_3)_{\mathrm{ind}}[-1] & \longrightarrow & \Gamma_{\mathrm{Zar}}(2) \\ & \nwarrow \quad \swarrow & \\ & \mathcal{K}_2[-2] & \end{array}$$

and apply lemma 4.3, noting that the Zariski sheaf $(\mathcal{K}_3)_{\mathrm{ind}}$ is constant.

For the proof of theorem 4.2 b), we need some facts on étale cohomological purity. For all $m \geq 1$, there is a morphism

$$\mathbf{Z}/m(2-c)[-2c] \to Ri^!_{\text{ét}}\mathbf{Z}/m(2). \tag{21}$$

For m prime to the characteristic exponent of F, this morphism is the classical purity isomorphism of SGA4, e.g. [28, th. 6.1]. For $\operatorname{char} F = p > 0$ and m a power of p, it is comes from Gros' thesis [10, II.3.5]: its homotopy cofibre is concentrated in degree $c+3$. In the general case, we define the morphism component-wise, on the prime-to-p and p-primary parts.

The following rather trivial lemma is very useful:

4.4. Lemma. *a) Let $f : S \to T$ be a morphism of sites and $Rf_* : \mathcal{D}^+(S) \to \mathcal{D}^+(T)$ the functor induced from the bounded below derived category of Abelian S-sheaves to that of Abelian T-sheaves. Let C be a bounded below complex of Abelian groups, that we view as a complex of constant sheaves over S. Then there is a natural isomorphism of functors*

$$Rf_* \circ (C \overset{L}{\otimes} ?) \approx C \overset{L}{\otimes} (Rf_* ?)$$

and a natural morphism of functors

$$f^* \circ (C \overset{L}{\otimes} ?) \to C \overset{L}{\otimes} (f^* ?).$$

b) Denote by i_λ ($\lambda = \mathrm{Zar}$ or $\mathrm{\acute{e}t}$) the map corresponding to i from Z_λ to X_λ (small sites). Then, with C as in a), there is a natural isomorphism of functors

$$Ri^!_\lambda \circ (C \overset{L}{\otimes} ?) \approx C \overset{L}{\otimes} (Ri^!_\lambda ?).$$

PROOF. a) For A, B two Abelian groups, let $A * B$ denote $Tor_1^{\mathbf{Z}}(A, B)$. We note that $*$ is left exact and its unique nonzero higher derived functor is $R^1 * = \otimes$. Hence there is a natural isomorphism

$$C \overset{L}{\otimes} D \approx C \overset{R}{*} D[1]$$

for all $C, D \in \mathcal{D}(Ab)$.

Therefore the natural isomorphism of the lemma is equivalent to a natural isomorphism of functors

$$Rf_* \circ (C \overset{R}{*} ?) \approx C \overset{R}{*} (Rf_* ?)$$

which in turn will follow from a natural isomorphism

$$f_*(A * \mathcal{F}) \approx A * f_*\mathcal{F} \tag{22}$$

for any Abelian group A and any sheaf $\mathcal{F}$ over S. Note that, since $*$ is left exact, the presheaf $U \mapsto A * \mathcal{G}(U)$ is a sheaf for any sheaf $\mathcal{G}$ over any site. Therefore, given $U \in S$, both sides of (22) evaluated on U are $A * \mathcal{F}(f^{-1}(U))$. Finally, the second natural transformation, say, follows from the first one by adjunction.

b) Follows from a), considering the triangle of functors (with $j : X - Z \hookrightarrow X$ the complementary open immersion)

$$i_* Ri^! \to Id_{X_\lambda} \to Rj_* j^* \to i_* Ri^![1] \tag{23}$$

and the fact that i_* is fully faithful. Here we dropped the index $_\lambda$ for notational simplicity. □

Note that the triangle (12) and its analogues for $i = 0, 1$ can be reformulated as quasi-isomorphisms

$$\Gamma_{\mathrm{\acute{e}t}}(i) \overset{L}{\otimes} \mathbf{Z}/m \xrightarrow{\sim} \mathbf{Z}/m(i) \quad (0 \leq i \leq 2) \tag{24}$$

over the big étale site of $\operatorname{Spec} F$. Note also the obvious quasi-isomorphisms

$$\alpha^* \Gamma_{\mathrm{Zar}}(i) \xrightarrow{\sim} \Gamma_{\mathrm{\acute{e}t}}(i) \quad (0 \leq i \leq 2). \tag{25}$$

Using (24) and lemma 4.4, they give by adjunction morphisms

$$\Gamma_{\mathrm{Zar}}(i) \overset{L}{\otimes} \mathbf{Z}/m \to R\alpha_* \mathbf{Z}/m(i) \quad (o \leq i \leq 2) \tag{26}$$

over the big Zariski site of $\operatorname{Spec} F$.

Let finally $\alpha_X : X_{\mathrm{\acute{e}t}} \to X_{\mathrm{Zar}}$ and $\alpha_Z : Z_{\mathrm{\acute{e}t}} \to Z_{\mathrm{Zar}}$ be the natural morphisms of (small) sites. Note the natural isomorphism of functors

$$Ri^!_{\mathrm{Zar}} R(\alpha_X)_* \xrightarrow{\sim} R(\alpha_Z)_* Ri^!_{\mathrm{\acute{e}t}} \tag{27}$$

over the small Zariski site of Z. (It can be obtained for example with the help of (23); compare [14, II.6.14].)

There is a diagram

$$(28)\qquad \begin{array}{ccc} \Gamma_{\mathrm{Zar}}(Z,2-c)[-2c]\overset{L}{\otimes}\mathbf{Z}/m & \longrightarrow & Ri^!_{\mathrm{Zar}}\Gamma_{\mathrm{Zar}}(X,2)\overset{L}{\otimes}\mathbf{Z}/m \\ \downarrow & & \downarrow \\ R(\alpha_Z)_*\mathbf{Z}/m(2-c)[-2c] & \longrightarrow & Ri^!_{\mathrm{Zar}}R(\alpha_X)_*\mathbf{Z}/m(2) \end{array}$$

where the vertical maps are given by (26), the top horizontal map by theorem 4.2 a) and the bottom horizontal map is defined by applying $R(\alpha_Z)_*$ to (21) and using (27). The notation in the top right corner is unambiguous, thanks to lemma 4.4.

4.5. Lemma. *Diagram* (28) *commutes up to sign.*

Proof. As in the proof of lemma 4.3, this boils down to the fact that the Gersten complex for K-theory is compatible with the Gersten complex for étale cohomology via the Galois symbol (m prime to $\operatorname{char} F$) or the differential symbol (m a power of $\operatorname{char} F$). The first case is well-known; see [11, cor. 1.6 and proof of lemma 4.11] for the second one. □

Proof of theorem 4.2 b). We first construct the map. There is a tautological natural transformation (stemming from (27))

$$(29)\qquad \alpha_Z^* Ri^!_{\mathrm{Zar}} \to Ri^!_{\mathrm{\acute{e}t}}\alpha_X^*$$

hence a morphism (in the derived category of étale sheaves over Z)

$$(30)\qquad \alpha_Z^*\Gamma_{\mathrm{Zar}}(Z,2-c)[-2c] \to Ri^!_{\mathrm{\acute{e}t}}\Gamma_{\mathrm{\acute{e}t}}(X,2)$$

where we used a) and (25). On the other hand, the triangle

$$(31)\qquad \begin{array}{ccc} \Gamma(2) & \longrightarrow & \Gamma(2)\otimes\mathbf{Q} \\ & \nwarrow \quad \swarrow & \\ & \mathbf{Q}/\mathbf{Z}(2) & \end{array}$$

deduced from (12) yields a map

$$(32)\qquad Ri^!_{\mathrm{\acute{e}t}}\mathbf{Q}/\mathbf{Z}(2)[-1] \to Ri^!_{\mathrm{\acute{e}t}}\Gamma_{\mathrm{\acute{e}t}}(X,2).$$

Passing to the colimit in (21), we get a morphism

$$(33)\qquad \mathbf{Q}/\mathbf{Z}(2-c)[-2c] \to Ri^!_{\mathrm{\acute{e}t}}\mathbf{Q}/\mathbf{Z}(2)$$

whose homotopy cofibre is concentrated in degree $c+3$ and has p-primary torsion cohomology. Shifting and composing with (32), we get a morphism

$$(34)\qquad \mathbf{Q}/\mathbf{Z}(2-c)[-1-2c] \to Ri^!_{\mathrm{\acute{e}t}}\Gamma_{\mathrm{\acute{e}t}}(X,2).$$

For $c \le 2$, we use (30) to define the map of b), noting that it becomes then

$$\Gamma_{\mathrm{\acute{e}t}}(Z,2-c)[-2c] \to Ri^!_{\mathrm{\acute{e}t}}\Gamma_{\mathrm{\acute{e}t}}(X,2)$$

via (25). For $c > 2$, we use (34) to define this map.

We now prove the property of the map of b) as claimed in the statement of theorem 4.2. It is enough to do this after tensoring (30) and (34) by $\mathbf{Q}$ and $\mathbf{Z}/m$ for all m (in the derived sense). Since $R(\alpha_Z)_*$ is fully faithful, we may even apply this functor to the situation.

Suppose first that $c \leq 2$. Using (27), (14) and a), we see that the morphism

$$R(\alpha_Z)_*\Gamma_{\text{ét}}(Z, 2-c)[-2c] \otimes \mathbf{Q} \to R(\alpha_Z)_* Ri^!_{\text{ét}}\Gamma_{\text{ét}}(X,2) \otimes \mathbf{Q}$$

is a quasi-isomorphism. On the other hand, there is a $\pm$-commutative diagram

$$\begin{tikzcd}
\Gamma_{\text{ét}}(Z,2-c)\overset{L}{\otimes}\mathbf{Z}/m[-2c] \arrow[r] \arrow[d, "\wr"] & \alpha_Z^* Ri^!_{\text{Zar}}\Gamma_{\text{Zar}}(X,2)\overset{L}{\otimes}\mathbf{Z}/m \arrow[r] \arrow[d] & Ri^!_{\text{ét}}\Gamma_{\text{ét}}(X,2)\overset{L}{\otimes}\mathbf{Z}/m \arrow[dl, "\approx"'] \\
\mathbf{Z}/m(2-c)[-2c] \arrow[r] & Ri^!_{\text{ét}}\mathbf{Z}/m(2). &
\end{tikzcd}$$

In this diagram, the left square is obtained via (25) and (27) by applying adjunction to (28) and using lemma 4.5; the right triangle is obtained via (25) and (29). The left vertical map and the southwest map come from the triangle (12).

The bottom horizontal map is none else than (21): its homotopy cofibre is p-primary torsion and concentrated in degree $c+3$. The left vertical map and the south-west map are quasi-isomorphisms by (24), hence the top composite has the same cofibre as the bottom map. This proves theorem 4.2 b) in the case $c \leq 2$.

Suppose now that $c > 2$. We first have

$$R(\alpha_Z)_* Ri^!_{\text{ét}}\Gamma_{\text{ét}}(X,2)\otimes\mathbf{Q} \approx Ri^!_{\text{Zar}}R(\alpha_X)_*\Gamma_{\text{ét}}(X,2)\otimes\mathbf{Q} \approx Ri^!_{\text{Zar}}\Gamma_{\text{Zar}}(X,2)\otimes\mathbf{Q} = 0$$

by (14) and a). On the other hand, tensoring (34) by $\mathbf{Z}/m$ and using (31) yields

$$\mathbf{Z}/m(2-c)[-2c] \to Ri^!_{\text{ét}}\Gamma_{\text{ét}}(X,2)\overset{L}{\otimes}\mathbf{Z}/m.$$

Using (24), we get a composition

$$\mathbf{Z}/m(2-c)[-2c] \to Ri^!_{\text{ét}}\Gamma_{\text{ét}}(X,2)\overset{L}{\otimes}\mathbf{Z}/m \xrightarrow{\sim} Ri^!_{\text{ét}}\mathbf{Z}/m(2)$$

which is clearly (33). This concludes the proof of theorem 4.2 b). □

5. Cohomology of projective bundles

Let $E \to X$ be a vector bundle of rank n, and $P \xrightarrow{\pi} X$ the associated projective bundle. Our aim in this section is to compute $R\pi_*\Gamma_{\text{Zar}}(P,2)$ and $R\pi_*\Gamma_{\text{ét}}(P,2)$.

In order to state the theorem, we remark that there are pairings ($i \leq 2$):

$$\Gamma_{\text{Zar}}(i-1)\overset{L}{\otimes}\Gamma_{\text{Zar}}(1) \to \Gamma_{\text{Zar}}(i) \tag{35}$$

over the big Zariski site of $\operatorname{Spec} F$, if F has more than two elements, and

$$\Gamma_{\text{ét}}(i-1)\overset{L}{\otimes}\Gamma_{\text{ét}}(1) \to \Gamma_{\text{ét}}(i) \tag{36}$$

over the big étale site of $\operatorname{Spec} F$.

For $i = 2$, (35) and (36) are the pairings of [22, prop. 2.5]; for $i = 1$ they are tautological. For $i < 0$ (and in the étale case), the triangle analogous to (31) for $\Gamma(1)$

shows that, for all i, the morphism $\mathbf{Q}/\mathbf{Z}(i-1)\overset{L}{\otimes}\mathbf{Q}/\mathbf{Z}(1)[-1]\to\mathbf{Q}/\mathbf{Z}(i-1)\overset{L}{\otimes}\Gamma_{\text{ét}}(1)$ is a quasi-isomorphism. Therefore it suffices to define morphisms

$$\mathbf{Q}/\mathbf{Z}(i-1)\overset{L}{\otimes}\mathbf{Q}/\mathbf{Z}(1)\to\mathbf{Q}/\mathbf{Z}(i)[1]$$

for all $i\in\mathbf{Z}$. This is nothing else than Tate twists of the natural isomorphisms (in $\mathcal{D}(Ab)$)

$$\mathbf{Q}_l/\mathbf{Z}_l\overset{L}{\otimes}\mathbf{Q}_l/\mathbf{Z}_l\approx\mathbf{Q}_l/\mathbf{Z}_l[1]$$

for $l\neq\operatorname{char}F$. Finally, for $i=0$, the pairing is defined similarly, using the natural map

$$\mathbf{Q}/\mathbf{Z}[-1]\to\mathbf{Z}[0]=\Gamma_{\text{ét}}(0).$$

Let L be a line bundle over an F-scheme S. Let $\lambda=\mathrm{Zar}$ or ét. Via (36), its class $[L]\in H^1_\lambda(S,\mathbb{G}_m)=\mathbb{H}^2_\lambda(S,\Gamma(1))$ defines morphisms of complexes

$$\Gamma_\lambda(i-j)_{|S}[-2j]\xrightarrow{[L]^j}\Gamma_\lambda(i)_{|S}$$

where $_{|S}$ means "restriction to the big λ site of S". In particular, for $S=P$ and $L=\mathcal{O}(1)$, we get maps

$$\Gamma_\lambda(2-j)_{|P}[-2j]\xrightarrow{[\mathcal{O}(1)]^j}\Gamma_\lambda(2)_{|P}\quad(j\geq 0)$$

hence, by adjunction, a morphism

$$\prod_{j=0}^{n}\Gamma_\lambda(2-j)_{|X}[-2j]\xrightarrow{\rho_\lambda}R\pi_*(\Gamma_\lambda(2)_{|P}).\tag{37}$$

We are now ready to state the result:

5.1. Theorem. *The morphism ρ_λ is a quasi-isomorphism for $\lambda=\mathrm{Zar}$ or ét (for $\lambda=\mathrm{Zar}$, assume F has more than two elements).*

Proof. We proceed as in the last section, first proving the Zariski case. Let A be a local ring of X, and K be its field of fractions. The restriction of E to $\operatorname{Spec}A$ is trivial, hence $P_{|\operatorname{Spec}A}\simeq\mathbf{P}^n_A$. Looking at the maps induced by ρ_{Zar} on cohomology sheaves and using theorem 1.6, we can identify them to:

$$\begin{aligned}K_3(K)_{\mathrm{ind}}&\to K_3(K(T_1,\dots,T_n))_{\mathrm{ind}}\\K_2(A)&\to H^0_{\mathrm{Zar}}(\mathbf{P}^n_A,\mathcal{K}_2)\\A^*&\to H^1_{\mathrm{Zar}}(\mathbf{P}^n_A,\mathcal{K}_2)\\\mathbf{Z}&\to H^2_{\mathrm{Zar}}(\mathbf{P}^n_A,\mathcal{K}_2).\end{aligned}$$

We have to show that all these maps are isomorphisms. The first one is an isomorphism because $K_{3,\mathrm{ind}}$ is invariant under rational extensions. The other ones follow from [9, lemma 8.11].

In the étale case, it is enough to check that ρ is a quasi-isomorphism after tensoring by $\mathbf{Q}$ and by $\mathbf{Z}/l$ for all prime l. In the case of $\mathbf{Q}$, we reduce to the Zariski case as above, by applying $R\alpha_*$ and using (14).

For $\mathbf{Z}/l$, we first need a lemma. Note that there are products:

$$\mathbf{Z}/l(i-1)\overset{L}{\otimes}\mathbf{Z}/l(1)\to\mathbf{Z}/l(i). \tag{38}$$

For $l\neq\operatorname{char}F$, they are nothing else than Tate twists of the natural product in $\mathcal{D}(Ab)$. For $l=\operatorname{char}F$ (and $i>0$), they come from the products

$$\Omega^{i-1}_{\log}\otimes\Omega^1_{\log}\to\Omega^i_{\log}.$$

5.2. LEMMA. *For any prime l and any $i\leq 2$, the diagram*

$$\begin{array}{ccc}
\Gamma_{\text{ét}}(i-1)\overset{L}{\otimes}\Gamma_{\text{ét}}(1)\overset{L}{\otimes}\mathbf{Z}/l & \longrightarrow & \Gamma_{\text{ét}}(i)\overset{L}{\otimes}\mathbf{Z}/l\\
\downarrow & & \downarrow\\
\mathbf{Z}/l(i-1)\overset{L}{\otimes}\mathbf{Z}/l(1) & \longrightarrow & \mathbf{Z}/l(i)
\end{array}$$

commutes, where the top horizontal map is (36)$\overset{L}{\otimes}\mathbf{Z}/l$*, the bottom horizontal map is* (38) *and the vertical maps are deduced from* (24)*.*

PROOF. For $l\neq\operatorname{char}F$, this follows from [22]. For $l=\operatorname{char}F$, it follows from the definition of the logarithmic symbol, since (for $i=1$) the étale sheaf $\mathbf{K}_{3,\text{ind}}$ is uniquely l-divisible. □

If $l\neq\operatorname{char}F$, using lemma 5.2, $\rho\overset{L}{\otimes}\mathbf{Z}/l$ becomes the map γ of [15, th. 2.2.1], which is a quasi-isomorphism, Tate-twisted twice. If $p=\operatorname{char}F$, still using lemma 5.2, $\rho\overset{L}{\otimes}\mathbf{Z}/p$ becomes the map

$$\mathbf{Z}/p[-2]\oplus(\Omega^1_{\log})_{|X}[-1]\oplus(\Omega^2_{\log})_{|X}\to R\pi_*(\Omega^2_{\log})_{|P}$$

shifted, which is an isomorphism by [10, cor. I.2.1.12]. □

5.3. COROLLARY. $H^0_{\text{Zar}}(X,\mathcal{H}^3(2))\xrightarrow{\sim}H^0_{\text{Zar}}(P,\mathcal{H}^3(2))$.

PROOF. (We don't really need $\Gamma(2)$ for this.) Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & CH^2X & \longrightarrow & \mathbb{H}^4_{\text{ét}}(X,\Gamma(2)) & \longrightarrow & H^0_{\text{Zar}}(X,\mathcal{H}^3(2)) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & CH^2P & \longrightarrow & \mathbb{H}^4_{\text{ét}}(P,\Gamma(2)) & \longrightarrow & H^0_{\text{Zar}}(P,\mathcal{H}^3(2)) & \longrightarrow & 0
\end{array}$$

where the rows come from (9). Using theorem 5.1 and the analogous result for Chow groups, the bottom left horizontal map can be rewritten

$$CH^0(X)\oplus CH^1(X)\oplus CH^2(X)\to\mathbb{H}^0_{\text{ét}}(X,\Gamma(0))\oplus\mathbb{H}^2_{\text{ét}}(X,\Gamma(1))\oplus\mathbb{H}^4_{\text{ét}}(X,\Gamma(2)).$$

The result now comes from the fact that $CH^i(X)\to\mathbb{H}^{2i}_{\text{ét}}(X,\Gamma(i))$ is an isomorphism for $i=0,1$. □

6. The coniveau spectral sequence and Gersten's conjecture

By the standard procedure, we can construct a coniveau spectral sequence ([3], [5])

$$E_1^{p,q} = \coprod_{x\in X^{(p)}} \mathbb{H}_x^{q+p}(X_{\text{ét}}, \Gamma(2)) \Longrightarrow \mathbb{H}_{\text{ét}}^{p+q}(X, \Gamma(2))$$

where $\mathbb{H}_x^{q+p}(X_{\text{ét}}, \Gamma(2)) = \varinjlim_{U\ni x} \mathbb{H}^{q+p}_{\overline{\{x\}}\cap U}(U_{\text{ét}}, \Gamma(2))$.

Applying theorem 4.2, we get, for $x \in X^{(p)}$:

$$\mathbb{H}_x^{q+p}(X_{\text{ét}}, \Gamma(2)) = \begin{cases} \mathbb{H}^q(F(X), \Gamma(2)) & \text{for } p = 0 \\ H^{q-2}(F(x), \mathbb{G}_m) & \text{for } p = 1 \text{ and } q \neq 4, 5 \\ H^{q-2}(F(x), \mathbf{Z}) & \text{for } p = 2 \text{ and } q \neq 4, 5 \\ H^{q-p-1}(F(x), \mathbf{Q}/\mathbf{Z}(-p)) & \text{for } p > 2 \text{ and } q \neq 4, 5. \end{cases}$$

Moreover, we have exact sequences:

$$0 \to H^{3-p}(F(x), \mathbf{Q}/\mathbf{Z}(2-p)) \to \mathbb{H}_x^{p+4}(X_{\text{ét}}, \Gamma(2)) \to H^0(F(x), \mathcal{F})$$
$$\to H^{4-p}(F(x), \mathbf{Q}/\mathbf{Z}(2-p)) \to \mathbb{H}_x^{p+5}(X_{\text{ét}}, \Gamma(2)) \to 0$$

where $\mathcal{F}$ is an l-primary torsion sheaf if $\operatorname{char} F = l > 0$ (and is 0 if $\operatorname{char} F = 0$). For $p > 2$, the map $H^0(F(x), \mathcal{F}) \to H^{4-p}(F(x), \mathbf{Q}/\mathbf{Z}(2-p))$ has to be 0, so the sequence splits into

$$0 \to H^{3-p}(F(x), \mathbf{Q}/\mathbf{Z}(2-p)) \to \mathbb{H}_x^{p+4}(X_{\text{ét}}, \Gamma(2)) \to H^0(F(x), \mathcal{F}) \to 0$$
$$H^{4-p}(F(x), \mathbf{Q}/\mathbf{Z}(2-p)) \xrightarrow{\sim} \mathbb{H}_x^{p+5}(X_{\text{ét}}, \Gamma(2)).$$

This shows that $E_1^{p,5} = 0$ for $p \geq 5$ and $E_1^{p,4}$ is l-primary torsion for $p \geq 4$. For $q \neq 4, 5$, $E_1^{p,q} = 0$ for $p \geq q$, except for $E_1^{2,2} = Z^2(X)$ (codimension 2 cycles). Note also that

$$E_1^{p,3} = 0 \quad \text{for all } p.$$

Using theorem 5.1 for $P = \mathbf{P}^1_X$, the arguments of [8], [5] show that Gersten's conjecture holds for étale weight-two motivic cohomology. Therefore we get a Bloch-Ogus-type theorem:

6.1. Theorem. *The $E_2^{p,q}$ term of the coniveau spectral sequence for weight-two motivic cohomology coincides with $H^p(X_{\text{Zar}}, R^q\alpha_*\Gamma(2)) =: H^p_{\text{Zar}}(X, \mathcal{H}^q(\Gamma(2)))$.* □

6.2. Corollary. *For any $i \geq 0$,*
a) The functor $X \mapsto \mathbb{H}^i_{\text{ét}}(X, \Gamma(2))$ satisfies "codimension 1 purity" for regular local rings of a smooth variety in the sense of [4, def. 2.1.4 (b)].
b) $H^0_{\text{Zar}}(X, \mathcal{H}^i(\Gamma(2)))$ is a birational invariant of smooth, proper varieties X/F.
c) For any proper morphism $P \xrightarrow{f} X$ of smooth, integral F-varieties such that the generic fibre of f is $F(X)$-rational,

$$H^0_{\text{Zar}}(X, \mathcal{H}^i(\Gamma(2))) \xrightarrow{\sim} H^0_{\text{Zar}}(P, \mathcal{H}^i(\Gamma(2))).$$

Proof. a) follows from theorem 6.1. b) follows from theorem 6.1 and [4, prop. 2.1.8]. Finally, c) follows from b) and corollary 5.3. (In [5, §8], we give a general proof of these properties for suitable "cohomology theories with supports".) □

Remark. As for corollary 5.3, we could prove this without having recourse to $\Gamma(2)$, in view of lemma 1.4. More precisely, we could "merely" use Gersten's conjecture for K-theory (Quillen [31]), étale cohomology with coefficients in twisted roots of unity (Bloch-Ogus [3]) and logarithmic Hodge-Witt cohomology (Gros-Suwa [11]).

7. Projective homogeneous varieties

Let X be a projective homogeneous variety in the sense of [25] and [30]. In particular X is rational, so the assumptions of theorem 1 are satisfied, including $H^0_{\mathrm{Zar}}(\overline{X}, \mathcal{H}^3(2)) = 0$ by corollary 6.2 c). Moreover, we have $K_{j-i}(\overline{F}) \otimes CH^i\overline{X} \xrightarrow{\sim} H^i_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_j)$ for all $i \leq j$ (loc. cit.). Finally, the G_F-modules $CH^i\overline{X}$ are permutation modules, hence torsion-free [30]. In particular:

$$H^1(F, H^1_{\mathrm{Zar}}(\overline{X}, \mathcal{K}_2)) = 0$$
$$\operatorname{Ker}\xi = (CH^2X)_{\mathrm{torsion}}.$$

Let E be the étale F-algebra associated to X as in [25]. We get the following corollary of theorem 1, containing [25, Theorem] and [30, th. 1]:

7.1. Corollary. *If X is projective homogeneous, there is an exact sequence:*

$$0 \to H^1_{\mathrm{Zar}}(X, \mathcal{K}_2) \to E^* \xrightarrow{\rho} \operatorname{Ker}\eta \to (CH^2X)_{torsion} \to 0$$

and a complex

$$0 \to \operatorname{Coker}\eta \to \operatorname{Coker}\xi \to \operatorname{Br}(E)$$

which is exact, except perhaps at $\operatorname{Coker}\xi$*, where its homology is* $\operatorname{Ker}(H^4(F,2) \to \operatorname{Coker}\mathrm{cl}^2_X)$.

The map ρ in corollary 7.1 is described by Merkurjev [25]: there is an Azumaya E-algebra A associated to X, and ρ is cup-product by $[A]$ followed by transfer.

7.2. Corollary. $\operatorname{Coker}\eta$ *is finite.*

Indeed, $\operatorname{Coker}\xi$ is finite, as a torsion quotient of the finitely generated group $(CH^2\overline{X})^{G_F}$. □

In [19] we show that $\operatorname{Coker}\eta$ is isomorphic to $\operatorname{Ker}\mathrm{cl}^2_X$.

Acknowledgements. I wish to thank Jean-Louis Colliot-Thélène, Hélène Esnault, Marc Levine, Eckart Viehweg and the referee for useful comments.

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