POINTWISE INTERSECTION PRODUCTS

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This is a development of [2, Rem. 2.3 b)].

Lemma 1. *Let k be a field, and*

$$\begin{array}{ccc} S' & \xrightarrow{f'} & T' \\ {\scriptstyle g'}\downarrow & & {\scriptstyle g}\downarrow \\ S & \xrightarrow{f} & T \end{array}$$

be a Cartesian square of k-schemes, where g is proper and f, f' are l.c.i. morphisms of same codimension. Then

$$f^! g_* = g'_*(f')^!$$

as homomorphisms from $CH_*(T')$ to $CH_*(S)$.

Proof. This follows from [1, Th. 6.6 (c)]. □

Suppose k perfect; let B be a smooth connected separated k-scheme of finite type with generic point $\eta = \operatorname{Spec} K$, and let $f : \mathcal{X} \to B$ be a dominant projective morphism, with $\mathcal{X}$ smooth; let $f' : X \to \eta$ be the generic fibre of f. We assume X irreducible of dimension d, and f' smooth.

Let $CH^i_{\text{num}}(X)$ denote the subgroup of $CH^i(X)$ formed of cycles numerically equivalent to 0; write j for the inclusion $X \hookrightarrow \mathcal{X}$.

Lemma 2. *For $\alpha \in CH^i(\mathcal{X})$, the following are equivalent:*

(1) *$j^*\alpha \in CH^i_{\text{num}}(X)$;*
(2) *for any $\beta \in CH^{d-i}(\mathcal{X})$, $f_*(\alpha \cdot \beta) = 0$.*

Proof. (2) ⇒ (1) because of the surjectivity of j^* and the formula

$$\jmath^* f_*(\alpha \cdot \beta) = f'_* j^*(\alpha \cdot \beta) = f'_*(j^*\alpha \cdot j^*\beta) \tag{1}$$

[1, Prop. 1.7 and 8.3 (a)] where $\jmath : \eta \hookrightarrow B$ is the inclusion, and (1) ⇒ (2) because of (1) and the injectivity of $\jmath^* : CH^0(B) \to CH^0(\eta)$. □

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Let $r \geq 0$ and $b \in B^{(r)}$; write $Z = \overline{\{b\}}$. Recall the cap-product [1, p. 131]

$$\cdot_\iota : CH^i(\mathcal{X}) \times CH_l(\mathcal{X}_Z) \to CH_{l-i}(\mathcal{X}_Z)$$
$$(\alpha, \beta) \mapsto \gamma_\iota^!(\beta \times \alpha)$$

where ι is the closed immersion $\mathcal{X}_Z \hookrightarrow \mathcal{X}$. We record two useful formulas:

$$\alpha \cdot \iota_*\beta = \iota_*(\alpha \cdot_\iota \beta) \in CH_{l-i}(\mathcal{X}) \tag{2}$$

which follows from Lemma 1 applied to the Cartesian diagram

$$\begin{array}{ccc} \mathcal{X}_Z & \xrightarrow{\gamma_\iota} & \mathcal{X}_Z \times \mathcal{X} \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \iota\times 1} \\ \mathcal{X} & \xrightarrow{\Delta_\mathcal{X}} & \mathcal{X} \times \mathcal{X} \end{array}$$

of regular embeddings of codimension $d + \delta$. Hence

$$f_*(\alpha \cdot \iota_*\beta) = f_*\iota_*(\alpha \cdot_\iota \beta) = \iota'_*(f_Z)_*(\alpha \cdot_\iota \beta), \tag{3}$$

where ι' is the closed immersion $Z \hookrightarrow B$.

Take $l = \delta + i - r$. Composing with $(f_Z)_*$, we get a pairing

$$CH^i(\mathcal{X}) \times CH_{\delta+i-r}(\mathcal{X}_Z) \to CH_{\delta-r}(Z) = CH^0(Z) = \mathbf{Z}.$$

Removing proper closed subsets from Z and passing to the limit, we see that this pairing factors through a pairing

$$\langle , \rangle_b : CH^i(\mathcal{X}) \times CH_{i-r}(\mathcal{X}_b) \to \mathbf{Z}. \tag{4}$$

(Note the drop on dimension of cycles from $CH_{\delta+i-r}(\mathcal{X}_Z)$ to $CH_{i-r}(\mathcal{X}_b)$; compare with the relative dimensions of [1, §20.1].)

Lemma 3. *If $\mathcal{X}_b$ is smooth over $k(b)$, then*

$$\langle \alpha, \beta \rangle_b = \langle \iota^*\alpha, \beta \rangle_{\mathcal{X}_b}$$

where $\langle\rangle_{\mathcal{X}_b}$ is the intersection product on $\mathcal{X}_b$.

Proof. Up to removing a proper closed subset from Z, we may assume that $\mathcal{X}_Z$ itself is smooth. Then this is clear since $\alpha \cdot_\iota \beta = \iota^*\alpha \cdot \beta$, as follows from [1, Th. 6.4] applied to the Cartesian square

$$\begin{array}{ccc} \mathcal{X}_Z & \xrightarrow{\delta} & \mathcal{X}_Z \times_k \mathcal{X}_Z \\ {\scriptstyle ||}\downarrow & & \downarrow{\scriptstyle 1\times\iota} \\ \mathcal{X}_Z & \xrightarrow{\gamma_\iota} & \mathcal{X}_Z \times \mathcal{X}. \end{array}$$

□

Let $b' \in Z^{(1)} \subset B^{(r+1)}$ be a specialisation of b; we define a specialisation homomorphism

$$\sigma_{b \to b'} : CH_*(\mathcal{X}_b) \to \ CH_{*-1}(\mathcal{X}_{b'})$$

as follows (compare [1, Ex. 1.2.3]). Let R be the integral closure of $\mathcal{O}_{Z,b'}$ in $k(b)$, $\pi : \operatorname{Spec} R \to \operatorname{Spec} \mathcal{O}_{Z,b'}$ the projection and write the cycle $\pi^*(b')$ as $\sum n_i b'_i$ where the b'_i are closed points and $n_i > 0$. For each i, let $\mathcal{X}_{b'_i}$ be the pull-back of $\mathcal{X}_{b'}$, and let $\varphi_i : \mathcal{X}_{b'_i} \to \mathcal{X}_{b'}$ be the projection. Then

$$\sigma_{b \to b'}(x) = \sum_i n_i (\varphi_i)_* \sigma_{b \to b'_i}(x)$$

where $\sigma_{b \to b'_i}$ is the specialisation homomorphism of [1, §20.3]. We set

$$e_{b \to b'} = \sum_i n_i.$$

Proposition 4. *For* $(x, y) \in CH^i(\mathcal{X}) \times CH_{i-r}(\mathcal{X}_b)$, *one has*

$$e_{b \to b'} \langle x, y \rangle_b = \langle x, \operatorname{sp}_{b \to b'}(y) \rangle_{b'}.$$

Proof. Let $\lambda_i : b'_i \hookrightarrow \operatorname{Spec} R$ be the closed immersion. By definition $\operatorname{sp}_{b \to b'_i}(y) = \lambda_i^! \tilde{y}$ where $\tilde{y}$ is a lift of y in $CH_*(\mathcal{X}_{\operatorname{Spec} R})$. Thus:

$$\langle x, \operatorname{sp}_{b \to b'}(y) \rangle_{b'} = \sum_i n_i \langle x, (\varphi_i)_* \lambda_i^! \tilde{y} \rangle_{b'}. \tag{5}$$

We need to spread this situation. Consider the Cartesian diagrams

$$\begin{array}{ccccccc}
\tilde{Z}' & \xrightarrow{\lambda} & \tilde{Z} & \quad & \mathcal{X}_{\tilde{Z}'} & \longrightarrow & \mathcal{X}_{\tilde{Z}} \\
{\scriptstyle \varphi}\downarrow & & {\scriptstyle \psi'}\downarrow & & \downarrow & & {\scriptstyle \psi}\downarrow \\
Z' & \longrightarrow & Z & & \mathcal{X}_{Z'} & \longrightarrow & \mathcal{X}_Z
\end{array}$$

where $Z' = \overline{\{b'\}}$ and $\tilde{Z}$ is the normalisation of Z in $Z - Z'$ (the second is the pull-back of the first under the projection $f_Z : \mathcal{X}_Z \to Z$). Up to removing a large enough proper closed subset from Z', we may and do assume that $\tilde{Z}$ is regular (hence smooth over k) and that $\tilde{Z}' = \coprod_i Z'_i$ where each Z'_i is irreducible with generic point b'_i. Then $\mathcal{X}_{\tilde{Z}'}$ splits accordingly. We keep the notations $\varphi_i : Z'_i \to Z'$, $\lambda_i : Z'_i \hookrightarrow \tilde{Z}$ in this spread context.

We can lift $\tilde{y}$ further to $z \in CH_*(\mathcal{X}_{\tilde{Z}})$. First, the diagram of Cartesian squares

$$\begin{array}{ccccc} \tilde{Z} & \xleftarrow{f_{\tilde{Z}}} & \mathcal{X}_{\tilde{Z}} & \xrightarrow{\gamma_{\tilde{\iota}}} & \mathcal{X}_{\tilde{Z}} \times \mathcal{X} \\ {\scriptstyle \psi'}\downarrow & & {\scriptstyle \psi}\downarrow & & {\scriptstyle \psi\times 1}\downarrow \\ Z & \xleftarrow{f_Z} & \mathcal{X}_Z & \xrightarrow{\gamma_\iota} & \mathcal{X}_Z \times \mathcal{X}, \end{array}$$

where $\tilde{\iota}$ is the composition $\mathcal{X}_{\tilde{Z}} \xrightarrow{\psi} \mathcal{X}_Z \xrightarrow{\iota} \mathcal{X}$, yields

$$\begin{aligned} \langle x, y\rangle_b &= (f_Z)_*\gamma_\iota^!(\psi_* z \times x) = (f_Z)_*\gamma_\iota^!(\psi \times 1)_*(z \times x) \\ &\overset{(a)}{=} (f_Z)_*\psi_*\gamma_{\tilde{\iota}}^!(z \times x) = \psi'_*(f_{\tilde{Z}})_*\gamma_{\tilde{\iota}}^!(z \times x) \overset{(b)}{=} (f_{\tilde{Z}})_*\gamma_{\tilde{\iota}}^!(z \times x) \end{aligned}$$

where (a) follows from Lemma 1 and (b) uses the isomorphism $\psi_* : CH^0(\tilde{Z}) \xrightarrow{\sim} CH^0(Z) = \mathbf{Z}$.

Then we note that $\varphi_*\lambda^! : \mathbf{Z} = CH^0(\tilde{Z}) \to CH^0(Z) = \mathbf{Z}$ is multiplication by $e_{b\to b'}$, hence

$$\begin{aligned} e_{b\to b'}(f_{\tilde{Z}})_*\gamma_{\tilde{\iota}}^!(z \times x) &= \varphi_*\lambda^!(f_{\tilde{Z}})_*\gamma_{\tilde{\iota}}^!(z \times x) \overset{(a)}{=} \varphi_*(f_{\tilde{Z}'})_*\lambda^!\gamma_{\tilde{\iota}}^!(z \times x) \\ &\overset{(b)}{=} \varphi_*(f_{\tilde{Z}'})_*\gamma_{\tilde{\iota}}^!\lambda^!(z \times x) \overset{(c)}{=} \varphi_*(f_{\tilde{Z}'})_*\gamma_{\tilde{\iota}'}^!\lambda^!(z \times x) \\ &= \varphi_*(f_{\tilde{Z}'})_*\gamma_{\tilde{\iota}'}^!(\lambda^! z \times x). \end{aligned}$$

Here (a) follows as usual from Lemma 1, (b) follows from [1, Th. 6.4] applied to the diagram of Cartesian squares

$$\begin{array}{ccc} \mathcal{X}_{\tilde{Z}'} & \longrightarrow & \mathcal{X}_{\tilde{Z}} \\ {\scriptstyle \gamma_{\tilde{\iota}'}}\downarrow & & {\scriptstyle \gamma_{\tilde{\iota}}}\downarrow \\ \mathcal{X}_{\tilde{Z}'} \times \mathcal{X} & \longrightarrow & \mathcal{X}_{\tilde{Z}} \times \mathcal{X} \\ \downarrow & & \downarrow \\ \tilde{Z}' & \xrightarrow{\lambda} & \tilde{Z}, \end{array}$$

and (c) from [1, Th. 6.2 (c)] applied to the same diagram. Now, using the diagram

$$\begin{array}{ccccc} \tilde{Z}' & \xleftarrow{f_{\tilde{Z}''}} & \mathcal{X}_{\tilde{Z}''} & \xrightarrow{\gamma_{\tilde{\iota}'}} & \mathcal{X}_{\tilde{Z}'} \times \mathcal{X} \\ {\scriptstyle \varphi}\downarrow & & {\scriptstyle \Phi}\downarrow & & {\scriptstyle \Phi\times 1}\downarrow \\ Z' & \xleftarrow{f_{Z'}} & \mathcal{X}_{Z'} & \xrightarrow{\gamma_{\iota'}} & \mathcal{X}_{Z'} \times \mathcal{X} \end{array}$$

we get

$$\varphi_*(f_{\tilde{Z}'})_*\gamma^!_{\tilde{\iota}'}(\lambda^! z \times x) = (f_{\tilde{Z}})_*\Phi_*\gamma^!_{\tilde{\iota}'}(\lambda^! z \times x) \overset{(a)}{=} (f_{\tilde{Z}})_*\gamma^!_{\iota'}(\Phi_*\lambda^! z \times x)$$
$$= \langle x, \mathrm{sp}_{b\to b'}(y)\rangle_{b'}$$

(see (5)), where (a) uses Lemma 1 again. □

Definition 5. Let $\alpha \in CH^i(\mathcal{X})$ and $b \in B^{(r)}$. We write $\alpha \equiv_b 0$ if $\langle \alpha, \beta\rangle_b = 0$ for all $\beta \in CH_{i-r}(\mathcal{X}_b)$.

Lemma 6. *Let η be the generic point of B. Then $\alpha \equiv_\eta 0 \iff j^*\alpha \in CH^i_{\mathrm{num}}(X)$.*

Proof. This follows from Lemma 2. □

We have the following corollary to Proposition 4:

Corollary 7. *Let b' be a specialisation of b. Then $\alpha \equiv_{b'} 0 \Rightarrow \alpha \equiv_b 0$.* □

Proof. By catenarity, we reduce to the case where $\mathrm{codim}_B(b') = \mathrm{codim}_B(b) + 1$. □

References

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[2] B. Kahn *Refined height pairing* (with an appendix by Qing Liu), to appear in Alg & Number theory.

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