K-THEORY OF SEMI-LOCAL RINGS WITH FINITE COEFFICIENTS AND ÉTALE COHOMOLOGY

BRUNO KAHN

Abstract. Let A be a commutative semi-local ring containing $1/2$. We construct natural isomorphisms

$$\coprod_{0\le i\le n} H^{2i-n}_{\text{ét}}(A,\mu_{2^\nu}^{\otimes i}) \xrightarrow{\sim} K_n(A,\mathbf{Z}/2^\nu) \qquad (\nu\ge 2)$$

if A is "non-exceptional". We deduce that, for a non-exceptional scheme X quasiprojective or regular over $\mathbf{Z}[1/2]$, the groups $K_n(X,\mathbf{Z}/2^\nu)$ and $K'_n(X,\mathbf{Z}/2^\nu)$ are finite for $n\ge \dim(X)-1$. When X is a variety over $\mathbf{F}_p$ or $\mathbf{Q}_p$ with p odd, we also obtain finiteness results for $K_*(X)$ and $K'_*(X)$. Finally, using higher Chern classes with values in truncated étale cohomology, we show that, for X over $\mathbf{Z}[1/2]$, of Krull dimension d, quasiprojective over an affine base (*resp.* smooth over a field or a discrete valuation ring), $K_n(X,\mathbf{Z}/2^\nu)$ is isomorphic for $n\ge 3$ (*resp.* for $n\ge 2$) to $\coprod_{i\ge 1} H^{2i-n}_{\text{Zar}}(X,\tau_{\le i}R\alpha_*\mu_{2^\nu}^{\otimes i})$, up to controlled torsion depending only on n and d (not on ν). Here α is the projection from the étale site of X to its Zariski site and τ denotes truncation in the derived category.

Contents

Date: January 10, 2002.

1. Statement of results

Let X be a $\mathbf{Z}[1/2]$-scheme. We say that X is *non-exceptional* if, for any connected component Y of X, the image of the cyclotomic character

$$\kappa_2 : \pi_1(Y)^{ab} \to \mathbf{Z}_2^*$$

does not contain -1 (*cf.* [34], for example). If $X = \operatorname{Spec} A$ for a ring A, we simply say that A is non-exceptional. This notion goes back to the article of Harris and Segal [28].

In this article, we draw consequences from Voevodsky's affirmation of the Milnor conjecture [63]. We also use the spectral sequence constructed by Bloch and Lichtenbaum [6] and developed by Friedlander-Suslin and Levine [17, 41, 43]. We prove the following theorems and corollaries:

Theorem 1. *Let A be a commutative semi-local ring such that 2 is invertible in A. Suppose that A is non-exceptional. Let $\nu \geq 2$. Then,*
a) For all $n \geq 0$, there exists an isomorphism

$$\coprod_{0\leq i\leq n} H^{2i-n}_{\text{ét}}(A, \mu_{2^\nu}^{\otimes i}) \xrightarrow[\sim]{B^n_A} K_n(A, \mathbf{Z}/2^\nu) \tag{1.1}$$

which is natural in A. These isomorphisms are compatible with change of coefficients, products and transfer (for finite morphisms).
b) The ring $K_*(A, \mathbf{Z}/2^\nu)$ is multiplicatively generated by units and the Bott element, up to transfer.
c) If A is a field F, then for all n, the natural map $K_n^M(F)/2^\nu \to K_n(F)/2^\nu$ is split injective.
d) If A is a field F and $\mu_{2^\nu} \subset F$, then the natural ring homomorphism

$$K_*^M(F)/2^\nu[t] \longrightarrow K_*(F, \mathbf{Z}/2^\nu)$$

mapping t to a Bott element in $K_2(F, \mathbf{Z}/2^\nu)$ is an isomorphism.

1.1. **Remarks.** a) Here as everywhere else in this paper, K-theory with finite coefficients is defined as homotopy with finite coefficients of the corresponding K-theory spectrum, *cf.* A.3. In particular, $K_0(A, \mathbf{Z}/2^\nu) = K_0(A)/2^\nu$.
b) The assumptions on A imply that the product on $K_*(A, \mathbf{Z}/2^\nu)$ is defined, associative and commutative for all $\nu \geq 2$, *cf.* [34, B.7] and Lemma B.3. The case $\nu = 1$ is trickier and will be discussed in another paper [36].

Part a) of Theorem 1 is a confirmation of the main conjecture in [31] for $l = 2$ and the case given. It even removes the regularity assumption of *loc. cit.* It was earlier obtained in [33, th. 4.1] for A a higher local field in the sense of Kato (and 2^ν replaced by l^ν, l any prime number different from the "essential" residue characteristic of A). Part b) is a variant of a conjecture of Suslin [55, conj. 4.1], and part c) extends [31, th. 3] to all values of n.

The next theorem settles the Dwyer-Friedlander-Snaith-Thomason version of the Quillen-Lichtenbaum conjecture [12, top p. 482] in the positive for 2-primary coefficients, improving on earlier results of Thomason ([60]; see [47, §6] for the original formulation of this conjecture). For any scheme X, define

$$d_2(X) = sup\{cd_2(\eta)\}$$

where η runs through the generic points of X and cd_2 denotes the étale 2-cohomological dimension.

Theorem 2. *Let X be a finite-dimensional Noetherian non-exceptional $\mathbf{Z}[1/2]$-scheme. Then*
a) The natural map

$$K_n^{TT}(X,\mathbf{Z}/2^\nu) \to K_n^{TT}(X,\mathbf{Z}/2^\nu)[\beta^{-1}]$$

is injective for $n \geq sup(d_2(X)-2,1)$ and bijective for $n \geq sup(d_2(X)-1,1)$. The 1 in the sup is not necessary if X is regular.
b) The natural map

$$K'_n(X,\mathbf{Z}/2^\nu) \to K'_n(X,\mathbf{Z}/2^\nu)[\beta^{-1}]$$

is injective for $n \geq d_2(X)-2$ and bijective for $n \geq d_2(X)-1$.
c) If $cd_2(X) < +\infty$, there are isomorphisms

$$K_n^{TT}(X,\mathbf{Z}/2^\nu)[\beta^{-1}] \xrightarrow{\sim} K_n^{\text{ét}}(X,\mathbf{Z}/2^\nu)$$

for all $n \in \mathbf{Z}$.
Here $K_n^{TT}(X,\mathbf{Z}/2^\nu)$ denotes the Thomason-Trobaugh algebraic K-theory of perfect complexes on X with finite coefficients [62, §6]*, $K_*^{TT}(X,\mathbf{Z}/2^\nu)[\beta^{-1}]$ denotes the corresponding K-theory with the Bott element inverted (ibid., 11.4), $K'_n(X,\mathbf{Z}/2^\nu)$ denotes Quillen's K'-theory of coherent sheaves and $K_n^{\text{ét}}(X,\mathbf{Z}/2^\nu)$ denotes the non-connective version of the étale K-theory of Dwyer-Friedlander* [11].

Recall that Thomason-Trobaugh's K-theory coincides with Quillen's algebraic K-theory when X has an ample family of line bundles (e.g. X quasi-projective over an affine base or regular) by [62, th. 7.6]. c) generalises a theorem of Thomason [59, th. 4.11] which holds under much more restrictive hypotheses (in particular, that X is regular and that its residue fields have a "Tate-Tsen filtration").

In [10], Colliot-Thélène, Sansuc and Soulé deduced finiteness results for torsion in the second Chow group from the Merkurjev-Suslin theorem, developing an earlier idea of Bloch. The two corollaries which follow are pendants to this work when "inputting" the Milnor conjecture rather than the Merkurjev-Suslin theorem.

Corollary 1. *Let S be a $\mathbf{Z}[1/2]$ scheme and X a non-exceptional separated S-scheme of finite type. Assume that S is of one of the following types:*

- (i) $\operatorname{Spec} R[1/2]$*, where R is the ring of integers of a non-exceptional number field.*
- (ii) $\operatorname{Spec} \mathbf{F}_p$*, $p > 2$.*
- (iii) $\operatorname{Spec} k$*, k a separably closed field of characteristic $\neq 2$.*
- (iv) $\operatorname{Spec} k$*, k a higher local field in the sense of Kato.*

Then $K_n^{TT}(X, \mathbf{Z}/2^\nu)$ is finite for $n \geq \dim(X/S) + d_2(S) - 2$, except perhaps in the case (iii), $n = 0$ and X a singular surface. The group $K'_n(X, \mathbf{Z}/2^\nu)$ is finite for $n \geq \dim(X/S) + d_2(S) - 2$.

1.2. **Remarks.** a) Actually, $K_0^{TT}(X, \mathbf{Z}/2^\nu)$ is also finite in the remaining case, *i.e.* X a singular surface over a separably closed field [67].
b) Note that in cases (i), (ii) and (iii), S itself is non-exceptional, so any S-scheme is automatically non-exceptional. In case (iv), S is certainly non-exceptional if its "essential" residue characteristic is $\neq 2$. On the other hand, a local field like $\mathbf{Q}_2$ is exceptional so one needs some care in the statement. (I am indebted to the referee for pointing out this issue.)

Corollary 2. *Let X be a variety of dimension d over a field k. Then,*
a) If $k = \mathbf{F}_p$ ($p > 2$), then the 2-primary torsion group $K_n^{TT}(X)\{2\}$ is finite and $\dfrac{K_{n+1}^{TT}(X)}{K_{n+1}^{TT}(X)\{2\}}$ is (uniquely) 2-divisible for $n \geq d$. The same holds when replacing $K^{TT}(X)$ by $K'(X)$.
b) If $k = \mathbf{Q}_p$ ($p > 2$), the conclusion of a) holds for $n \geq d + 1$.

In the last result, we consider higher Chern classes $c_{i,j}$ with values in truncated étale cohomology (compare [4, 5.10 D (vi)]); these Chern classes are constructed in Section 5.

Theorem 3. *Let $d \geq 0$ and $n \geq 3$. Then there exists an effectively computable integer $N = N(d, n) > 0$ such that, for any Noetherian $\mathbf{Z}[1/2]$-scheme X separated of Krull dimension $\leq d$ and for all $\nu \geq 2$,* *the kernel and cokernel of the map*

$$K_n^{TT}(X,\mathbf{Z}/2^\nu) \xrightarrow{(ic_{i,2i-n})} \prod_{i\geq 1} H^{2i-n}_{\mathrm{Zar}}(X,B/2^\nu(i)) \tag{1.2}$$

are killed by N. If X is smooth over a field or a discrete valuation ring, this holds also for $n=2$.

Here $B/2^\nu(i)=\tau_{\leq i}R\alpha_*\mu_{2^\nu}^{\otimes i}$, where α is the projection of the big étale site of $\operatorname{Spec}\mathbf{Z}[1/2]$ onto its big Zariski site, as in [57]. Using Newton polynomials in the Chern classes, one can expect to extend this result to $n=0$, although we haven't done it; we don't know what happens for $n=1$.

Theorem 3 is related to a conjecture of Beilinson [4, 5.10 B and C (vi)], asserting that there should exist a spectral sequence of cohomological type, at least for regular X

$$E_2^{p,q}=H^{p-q}_{\mathrm{Zar}}(X,B/2^\nu(-q))\Rightarrow K_{-p-q}(X,\mathbf{Z}/2^\nu) \tag{1.3}$$

which would be split, up to small torsion (another variant of the Quillen-Lichtenbaum conjecture). This spectral sequence is now constructed at least for X regular essentially of finite type over a regular base of dimension ≤ 1 [40] (see section 3 for more details), and one can expect that the Chern classes of Theorem 3 actually split it with the usual small factorials.

To prove Theorems 1, 2 and 3, we use (1.3) only for fields. As a by-product, we get that (1.3) degenerates completely and canonically for non-exceptional fields (see Theorem 3.1 for a slightly more general statement).

What we need as a crucial tool is the existence of a product structure on this spectral sequence. A construction of this product structure was finally performed by Marc Levine [41] and also by Friedlander-Suslin [17], thanks to the latter's reinterpretation of the Bloch-Lichtenbaum spectral sequence. To the best of our efforts, we haven't been able to find an argument avoiding it.

On the other hand, in the first version of this paper, the results were obtained under more restrictive hypotheses than now: rings and schemes had either to be of nonzero characteristic or to "contain" a square root of -1. Now, the assumption "non-exceptional" is sufficient. This improvement is made possible by a recent result of Hinda Hamraoui [27].

All this explains the long delay between the first version of this paper and the current revision.

Some further comments are appropriate:

(1) The condition A non-exceptional in Theorem 1 cannot be removed, as the example $A = \mathbf{R}$ shows [54].
(2) It is not clear how to extend Theorem 1 a) to Bass' negative K-groups with 2-primary coefficients. As stated, the result is certainly wrong for $n < 0$ as the left hand side is 0 and there are examples where the right hand side is not [65, ex. 8.5][1].
(3) One could naïvely hope that the isomorphisms (1.1) globalize into similar isomorphisms involving truncated étale cohomology groups, letting (1.3) degenerate on the nose, as in the proof of Theorem 1 a) for fields. However this is unreasonable. A special case of such isomorphisms would be a canonical decomposition $K_0(X)/2^\nu \simeq \coprod_{i\geq 0} CH^i(X)/2^\nu$ for any smooth variety X over a field, a fact for which counterexamples exist. Over $\mathbf{C}$, comparison with the Atiyah-Hirzebruch spectral sequence for topological K-theory also yields examples where the Thomason descent spectral sequence of [59] does not degenerate, which strongly suggests that (1.3) should not degenerate in general (compare [61, §8]). Theorem 1 a) does contain nontrivial information on (1.3), but exactly what information is not clear at this stage.
(4) The reason why we can avoid Thomason's restrictive hypotheses in Theorem 2 c) is partly that, unlike him, we use the Milnor conjecture. (I am indebted to Marc Levine for a discussion which clarified this point.)
(5) It may seem singular that our results hold even for non-smooth schemes. This is thanks to both Hoobler's Henselian pair trick [29] and the Thomason-Trobaugh descent theorem for the Bass extension of algebraic K-theory [62]. Theorem 3 suggests that a form of (1.3) exists even for singular schemes (I write a form because of negative K-groups, which remain mysterious at this stage), and also that Voevodsky's cdh cohomology (e.g. [57]) will not be necessary in the end.
(6) The Bass conjecture implies the finiteness of $K_n(X, \mathbf{Z}/2^\nu)$ for all n when X is regular of finite type over $\mathbf{Z}$; Corollary 1 suggests this should hold more generally without a regularity assumption, provided X is over $\mathbf{Z}[1/2]$. For X regular, this would

[1]More precisely, for all $d \geq 1$ L. Reid gives in [48] an example of a d-dimensional affine normal scheme X with exactly one singular point P, such that $K_{-d}(X)$ admits a nontrivial homomorphism to $\mathbf{Z}$. For $d > 1$, C. Weibel proves in [64, 1.6] that $K_{-d}(X) \xrightarrow{\sim} K_{-d}(X^h)$, where X^h is the Henselisation of X at P. Therefore $K_{-d}(X^h)$ admits a nontrivial homomorphism to $\mathbf{Z}$, hence $K_{-d}(X^h)/2^\nu \neq 0$ and $K_{-d}(X^h, \mathbf{Z}/2^\nu) \neq 0$.

be a consequence of (1.3) plus the finiteness of the groups $H^{p-q}_{\mathrm{Zar}}(X, B/2^\nu(-q))$, which at this point is known only for $p \leq 1$ by reduction to étale cohomology. The latter finiteness for all p, in turn, is essentially equivalent to the finiteness of the groups $H^p_{\mathrm{Zar}}(X, R^q\alpha_*\mu_{2^\nu}^{\otimes i})$. It is probably *wrong* for X of finite type over an algebraically closed field in general: C. Schoen has produced an example of a 3-dimensional variety X over $\overline{\mathbf{Q}}$ such that $CH^2(X)/l$ is infinite for a suitable prime l.

(7) The reader should beware that most of the above is wrong if we work with integral algebraic K-theory:
- Theorem 3 implies that the Beilinson-Soulé vanishing conjecture holds for the K-theory of singular schemes with 2-primary coefficients, in a sense that could be made more precise using Adams operations. However, Feigin-Tsygan [16] and independently Geller-Weibel [21] have produced examples of singular varieties whose integral K-theory fails to verify the Beilinson-Soulé conjecture.
- The integral analogue of Corollary 1 is expected to hold for regular schemes of finite type over $\mathbf{Z}$ (the Bass conjecture) but is known to be false for singular ones, as the famous example $\mathbf{Z}[T, U]/U^2$ shows for K_1.
- As shown by Rob de Jeu [30], the integral analogue of Theorem 1 b) is false for fields of characteristic 0 in general, already for K_4. (However we shall show in [37] that, for fields of characteristic > 0, it follows from the Bass conjecture.)

(8) Needless to say, the arguments in this paper will apply by replacing 2 by an odd prime l once the Bloch-Kato conjecture is proven for l, yielding the same results (without the non-exceptionality restriction).

This paper is organised as follows. In Section 2 we start the proof of Theorem 1 by constructing the maps B^n_A which appear in its statement; this proof is completed in Section 3. In Section 4, we deduce Theorem 2 from Theorem 1 and Corollaries 1 and 2 from Theorem 2. In Section 5, we define the higher Chern classes appearing in the statement of Theorem 3. In Section 6, we prove Theorem 3.

There are 3 appendices. In Appendix A, we discuss S-duality for Moore spectra a little more precisely than what is found in the literature: this is used in Subsection 5.1.2. In Appendix B, we recall a few things on Bott elements. In Appendix C, we extend the constructions of [34] to the non-exceptional case.

2. Construction of B^n_A

2.1. **Proposition.** *Let $\nu \geq 2$. For any commutative $\mathbf{Z}[1/2]$-algebra A, let*

$$H_n(A) = \coprod_{0\leq i\leq n} H^{2i-n}_{\text{ét}}(\operatorname{Spec} A, \mu_{2^\nu}^{\otimes i}).$$

Then there exists a unique collection of natural transformations

$$B^n_A : H_n(A) \to K_n(A, \mathbf{Z}/2^\nu)$$

defined on the category of non-exceptional commutative semi-local $\mathbf{Z}[1/2]$-algebras and having the following properties:

(i) *They are compatible when ν varies.*
(ii) *They are multiplicative.*
(iii) *They commute with transfer.*
(iv) *If A is a discrete valuation ring with quotient field E and residue field F, then the diagram*

$$\begin{tikzcd} H_n(E) \arrow[r, "\partial"] \arrow[d, "B^n_E"'] & H_{n-1}(F) \arrow[d, "B^{n-1}_F"'] \\ K_n(E, \mathbf{Z}/2^\nu) \arrow[r, "\partial"] & K_{n-1}(F, \mathbf{Z}/2^\nu) \end{tikzcd}$$

commutes, where the horizontal maps are residue homomorphisms in étale cohomology and algebraic K-theory.

(v) *(Normalization) If A is a field F, then B^0_F is the map $1 \mapsto [F]$, B^1_F is induced by the inclusion $GL_1(F) \hookrightarrow GL(F)$ via Kummer theory and B^2_F restricted to $H^0(F, \mu_{2^\nu})$ is given by the Bott element construction (see appendix B).*

There is a similar collection of natural transformations

$$B^{n,\text{ét}}_A : H'_n(A) \to K_n(A, \mathbf{Z}/2^\nu)[\beta^{-1}]$$

where $H'_n(A) = \coprod_{0\leq i} H^{2i-n}_{\text{ét}}(\operatorname{Spec} A, \mu_{2^\nu}^{\otimes i})$, enjoying the same properties; the diagrams

$$\begin{tikzcd} H_n(A) \arrow[r, "B^n_A"] \arrow[d] & K_n(A, \mathbf{Z}/2^\nu) \arrow[d] \\ H'_n(A) \arrow[r, "B^{n,\text{ét}}_A"] & K_n(A, \mathbf{Z}/2^\nu)[\beta^{-1}] \end{tikzcd}$$

commute.

Proof. We proceed in four steps:

I) A is a (non-exceptional) field F;

II) A is a (semi-)localisation of a smooth R-scheme of finite type, where R is a field or a discrete valuation ring;

III) A is a (semi-)localisation of a not necessarily smooth R-scheme of finite type, where R is as in II);

IV) the general case.

I) *Fields.* We first recall from [31] and [33] the construction of "anti-Chern classes"

$$\begin{aligned} H^j_{\text{ét}}(F,\mu_{2^\nu}^{\otimes i}) &\xrightarrow{\beta_F^{i,j}} \quad K_{2i-j}(F,\mathbf{Z}/2^\nu) \quad (i \geq j) \\ H^j_{\text{ét}}(F,\mu_{2^\nu}^{\otimes i}) &\xrightarrow{\tilde\beta_F^{i,j}} K_{2i-j}(F,\mathbf{Z}/2^\nu)[\beta^{-1}](i \in \mathbf{Z}) \end{aligned}$$

from étale cohomology to algebraic K-theory and Bott-localised algebraic K-theory; this construction uses the Milnor conjecture, the main results of [32] and [34] and was already given in [31] and [33] (in the former, assuming [32] and [34]), to which we refer for more details. Let

$$F(\mu_{2^\nu}^{\otimes i})$$

be the smallest algebraic extension E of F such that the Galois module $\mu_{2^\nu}^{\otimes i}$ becomes trivial over E: this is a finite, abelian extension of F. Recall the twisted Milnor K-groups of [33, §1]

$$K_j^M(i)(F,\mathbf{Z}/2^\nu) = (K_j^M(F(\mu_{2^\nu}^{\otimes i})) \otimes \mu_{2^\nu}^{\otimes i})_{G_i} \quad (j \geq 0, i \in \mathbf{Z})$$

where $G_i = Gal(F(\mu_{2^\nu}^{\otimes i})/F)$. In particular, $K_j^M(0)(F,\mathbf{Z}/2^\nu) = K_j^M(F)/2^\nu$. Set $E = F(\mu_{2^\nu}^{\otimes i-j})$ and $G = G_{i-j}$. We define composite homomorphisms

$$\text{(2.1)} \quad \begin{aligned} u_F^{i-j,j} : K_j^M(i-j)(F,\mathbf{Z}/2^\nu) &\xrightarrow{u_E^{0,j}\otimes 1} (H^j_{\text{ét}}(E,\mu_{2^\nu}^{\otimes j}) \otimes \mu_{2^\nu}^{\otimes (i-j)})_G \\ &= H^j_{\text{ét}}(E,\mu_{2^\nu}^{\otimes i})_G \xrightarrow{\text{Cor}} H^j_{\text{ét}}(F,\mu_{2^\nu}^{\otimes i}) \end{aligned}$$

where $u_E^{0,j}$ is the Galois symbol of degree j. (Note that the restriction $i \geq 2j$ from [33, (2.2)] is irrelevant.) We have the following proposition, which follows from the Milnor conjecture in degree j and the main result of [32]:

2.2. **Proposition** (*cf.* [32, th. 1(2)])**.** *If F is non-exceptional, the last map in* (2.1) *is an isomorphism.*

This proposition (together with the Milnor conjecture) implies:

2.3. **Corollary.** *The map $u_F^{i-j,j}$ of* (2.1) *is an isomorphism for all integers i.*

Next, we have homomorphisms

$$\text{(2.2)}\qquad \begin{aligned} K_j^M(i-j)(F,\mathbf{Z}/2^\nu) &\xrightarrow{\eta_F^{i,2i-j}} K_{2i-j}(F,\mathbf{Z}/2^\nu)\\ K_j^M(i-j)(F,\mathbf{Z}/2^\nu) &\xrightarrow{\tilde\eta_F^{i,2i-j}} K_{2i-j}(F,\mathbf{Z}/2^\nu)[\beta^{-1}]\end{aligned}$$

where $\eta_F^{i,2i-j}$ is defined for $i\geq j$, while $\tilde\eta_F^{i,2i-j}$ is defined for all $i\in\mathbf{Z}$. They are constructed as compositions

(2.3)

$$\begin{aligned} K_j^M(i-j)(F,\mathbf{Z}/2^\nu) &\xrightarrow{\eta_E^{j,j}\otimes\eta_E^{i-j,2(i-j)}} (K_j(E,\mathbf{Z}/2^\nu)\otimes K_{2(i-j)}(E,\mathbf{Z}/2^\nu))_G\\ &\xrightarrow{\mu} K_{2i-j}(E,\mathbf{Z}/2^\nu)_G \xrightarrow{N} K_{2i-j}(F,\mathbf{Z}/2^\nu)\end{aligned}$$

and similarly for étale K-theory. Here $\eta_E^{j,j}$ is the natural mapping of Milnor's K-theory to Quillen's K-theory, $\eta_E^{i-j,2(i-j)}$ is the canonical Bott element mapping of [34], μ is product and N is the norm map (compare [33, prop. 1.2 and its proof]). In [34], $\eta_E^{i-j,2(i-j)}$ is only defined when $\sqrt{-1}\in F$ or when F is of positive characteristic: see Appendix C for the general case.

For $i\geq j$, $\eta_F^{i,2i-j}$ and $\tilde\eta_F^{i,2i-j}$ commute with the canonical map $K_{2i-j}(F,\mathbf{Z}/2^\nu)\to K_{2i-j}(F,\mathbf{Z}/2^\nu)[\beta^{-1}]$.

The anti-Chern classes $\beta_F^{i,j}$ and $\tilde\beta_F^{i,j}$ are then defined as $\eta_F^{i,2i-j}\circ(u_F^{i-j,j})^{-1}$ and $\tilde\eta_F^{i,2i-j}\circ(u_F^{i-j,j})^{-1}$. The maps B_F^n and $B_F^{n,\text{ét}}$ described in Proposition 2.1 are defined from the $\beta_F^{i,j}$ and $\tilde\beta_F^{i,j}$ componentwise. They are natural in F, since the $\beta_F^{i,j}$ and $\tilde\beta_F^{i,j}$ are.

For the next step, we need:

2.4. **Lemma.** *The diagram of Proposition 2.1 (iv) commutes for A, E, F as in loc. cit.*

Proof. Let us first define residue maps

$$\text{(2.4)}\qquad K_j^M(i)(E,\mathbf{Z}/2^\nu)\xrightarrow{\partial} K_{j-1}^M(i)(F,\mathbf{Z}/2^\nu).$$

The extension $E(\mu_{2^\nu}^{\otimes i})/E$ is unramified, with Galois group (say) G_i. Let B be the integral closure of A in $E(\mu_{2^\nu}^{\otimes i})$ and $F'=B\otimes_A F$: then F' is of the form $\prod_g F(\mu_{2^\nu}^{\otimes i})$ for some integer g. The group G_i permutes these g copies transitively (via its action on B), and the isotropy subgroup of a given copy is $Gal(F(\mu_{2^\nu}^{\otimes i})/F)$. In [3, p. 370]), a residue map

$$K_j^M(E(\mu_{2^\nu}^{\otimes i}))\xrightarrow{\partial_v} K_{j-1}^M(F(\mu_{2^\nu}^{\otimes i}))$$

is defined for every discrete valuation v of B above the discrete valuation of A. Instead of ∂_v, however, we shall use $\partial'_v=(-1)^{j-1}\partial_v$. This map is characterised by the identity:

$$\partial'_v(\{y, x_1, \dots, x_{j-1}) = v(y)\{\bar{x}_1, \dots, \bar{x}_{j-1}\}$$

for $y \in E(\mu_{2^\nu}^{\otimes i})^*$, $x_1, \dots, x_{j-1} \in B^*$ and $\bar{x}_1, \dots, \bar{x}_{j-1}$ the residue images at v in $F(\mu_{2^\nu}^{\otimes i})^*$.

This yields a G_i-equivariant map

$$K_j^M(E(\mu_{2^\nu}^{\otimes i})) \otimes \mu_{2^\nu}^{\otimes i} \to \prod_g K_{j-1}^M(F(\mu_{2^\nu}^{\otimes i})) \otimes \mu_{2^\nu}^{\otimes i}$$

and we get (2.4) by taking its coinvariants.

This being done, it suffices by Corollary 2.3 to check that the two diagrams

$$\begin{array}{ccc}
K_j^M(i-j)(E, \mathbf{Z}/2^\nu) & \xrightarrow{\partial} & K_{j-1}^M(i-j)(F, \mathbf{Z}/2^\nu) \\
{\scriptstyle u_E^{i-j,j}}\downarrow & & {\scriptstyle u_E^{i-j,j-1}}\downarrow \\
H^j_{\text{ét}}(E, \mu_{2^\nu}^{\otimes i}) & \xrightarrow{\partial} & H^{j-1}_{\text{ét}}(F, \mu_{2^\nu}^{\otimes (i-1)})
\end{array}$$

$$\begin{array}{ccc}
K_j^M(i-j)(E, \mathbf{Z}/2^\nu) & \xrightarrow{\partial} & K_{j-1}^M(i-j)(F, \mathbf{Z}/2^\nu) \\
{\scriptstyle \eta_E^{i,2i-j}}\downarrow & & {\scriptstyle \eta_F^{i,2i-j-1}}\downarrow \\
K_{2i-j}(E, \mathbf{Z}/2^\nu) & \xrightarrow{\partial} & K_{2i-j-1}(F, \mathbf{Z}/2^\nu)
\end{array}$$

commute.

1) For the first one, we reduce to the case where the Galois module $\mu_{2^\nu}^{\otimes(i-j)}$ is trivial over E, and then to $i = j$. Commutativity is then classical [38, Lemma 1].

(Note that there is a sign in the diagram of loc. cit.; however, our sign modification of the Bass-Tate residue homomorphism corrects this sign. We are forced to do this: otherwise, the various signs corresponding to the summands of the maps B_E^n and B_F^{n-1} would not be compatible and we would get an awkward statement.)

2) For the second one, we first reduce again to the case where the Galois module $\mu_{2^\nu}^{\otimes(i-j)}$ is trivial over E. Then the vertical maps can be decomposed as follows:

$$\begin{array}{ccc}
K_j^M(E)/2^\nu \otimes H^0(E, \mu_{2^\nu}^{\otimes(i-j)}) & \xrightarrow{\partial\otimes\lambda} & K_{j-1}^M(F)/2^\nu \otimes H^0(F, \mu_{2^\nu}^{\otimes(i-j)}) \\
{\scriptstyle \eta_E^{j,j}\otimes\eta_E^{i-j,2(i-j)}}\downarrow & & {\scriptstyle \eta_F^{j-1,j-1}\otimes\eta_F^{i-j,2(i-j)}}\downarrow \\
K_j(E, \mathbf{Z}/2^\nu) \otimes K_{2(i-j)}(E, \mathbf{Z}/2^\nu) & & K_{j-1}(F, \mathbf{Z}/2^\nu) \otimes K_{2(i-j)}(F, \mathbf{Z}/2^\nu) \\
{\scriptstyle \mu}\downarrow & & {\scriptstyle \mu}\downarrow \\
K_{2i-j}(E, \mathbf{Z}/2^\nu) & \xrightarrow{\partial} & K_{2i-j-1}(F, \mathbf{Z}/2^\nu)
\end{array}$$

where μ is the product in K-theory with coefficients $\mathbf{Z}/2^\nu$ and $\lambda : H^0(E, \mu_{2^\nu}^{\otimes(i-j)}) \to H^0(F, \mu_{2^\nu}^{\otimes(i-j)})$ is the obvious isomorphism.

Note that $\eta_E^{i-j,2(i-j)}$ factors through $K_{2(i-j)}(A, \mathbf{Z}/2^\nu)$, since $H^0(A, \mu_{2^\nu}^{\otimes(i-j)}) \to H^0(E, \mu_{2^\nu}^{\otimes(i-j)})$ is an isomorphism.

Let $\iota : A \to E$ and $\pi : A \to F$ be respectively the inclusion and the projection. Recall the formula[2]

$$\partial(e \cdot \iota_* a) = \partial(e) \cdot \pi_* a \tag{2.5}$$

for $(x, y) \in K_*(E, \mathbf{Z}/2^\nu) \times K_*(A, \mathbf{Z}/2^\nu)$.

Applying it with $e = \eta_E^{j,j}(x)$ and $a = \eta_A^{i-j,2(i-j)}(y)$ for $(x, y) \in K_j^M(E)/2^\nu \times H^0(A, \mu_{2^\nu}^{\otimes(i-j)})$, using the compatibility of the residues for Milnor and Quillen's K-theory (which actually follows from (2.5)), and remarking that $\lambda = \pi_* \circ (\iota_*)^{-1}$, we get the desired commutativity thanks to the naturality of $\eta^{i-j,2(i-j)}$. (Here we have used the fact that $\eta_X^{i-j,2(i-j)}$ is already defined generally for non-exceptional schemes X, in particular for $X = \operatorname{Spec} A$, in [34] and Appendix C.)

The proof of commutativity for the $B^{n,\text{ét}}$ is exactly the same. $\square$

In the sequel, we only construct the maps B_A^n: the case of $B_A^{n,\text{ét}}$ is completely similar.

II) *Smooth (semi-)local rings.* Let E be the field of fractions of A. There is a commutative diagram of exact sequences:

$$\begin{array}{ccccccc}
0 \to & H_n(A) & \longrightarrow & H_n(E) & \longrightarrow & \coprod_{x \in X^{(1)}} H_{n-1}(\kappa(x)) \\
& & & \downarrow{\scriptstyle B_E^n} & & \downarrow{\scriptstyle (B_{\kappa(x)}^{n-1})} \\
0 \to & K_n(A, \mathbf{Z}/2^\nu) & \longrightarrow & K_n(E, \mathbf{Z}/2^\nu) & \longrightarrow & \coprod_{x \in X^{(1)}} K_{n-1}(\kappa(x), \mathbf{Z}/2^\nu)
\end{array}$$

where $X = \operatorname{Spec} A$, $\kappa(x)$ is the residue field at $x \in X$ and and $X^{(1)}$ denotes the set of points of codimension 1. Recall that, by assumption, A is essentially smooth over a ring R which is either a field or a discrete valuation ring.

In case R is a field, the bottom exact sequence follows from [46, th. 7.5.11] and the top one from [7]. In case R is a discrete valuation ring, the bottom exact sequence is Gersten's conjecture for K-theory with finite coefficients for essentially smooth local rings over a discrete

[2]Apply for example [26, Th. 2.5] with $F_1 = F = \Omega^\infty(K'(F) \wedge M(2^\nu))$, $E_1 = E = \Omega^\infty(K'(A) \wedge M(2^\nu))$, $B_1 = B = \Omega^\infty(K'(E) \wedge M(2^\nu))$ with $E_1 \to B_1$ and $E \to B$ given by ι_*, and $F_2 = *$, $E_2 = B_2 = \Omega^\infty(K(A) \wedge M(2^\nu))$ with $E_2 \to B_2$ given by the identity map, and notice that the product $F \wedge X \to F$ factors through $F \wedge F$.

valuation ring, due to Gillet and Levine [25]; the top exact sequence is due to Gillet [24]. The diagram commutes by Lemma 2.4. Note that, since A is non-exceptional, all its residue fields are non-exceptional. The diagram defines in a unique way a homomorphism $B_A^n : H_n(A) \to K_n(A, \mathbf{Z}/2^\nu)$. We also note that, by construction, B_A^n is natural in A when A varies in the category of essentially smooth (semi-)local R-algebras.

III) *(Semi-)local rings of "geometric" origin.* We use a method of Hoobler [29]: there exists a henselian pair (B, I) such that

- $B/I = A$;
- B is a union of (semi-)localisations of smooth R-schemes.

(To see this, write A as the (semi-)localisation of some finitely generated R-algebra A_0 at a prime ideal $\mathfrak{p}$ (resp. at a finite set of prime ideals S), and write A_0 as a quotient of $B_0 = R[T_1, \dots, T_n]$ for n large enough. Then A is a quotient of the local ring $B_1 := (B_0)_{\mathfrak{p}\cap B_0}$ (resp. $B_1 := (B_0)_{S\cap B_0}$). We take for (B, I) the henselization of the pair $(B_1, \operatorname{Ker}(B_1 \to A))$.)

Since both algebraic K-theory and étale cohomology commute with filtering inverse limits of schemes with affine transition morphisms, and since the homomorphism B^n constructed in II) is natural, we deduce from II) a homomorphism $B_B^n : H_n(B) \to K_n(B, \mathbf{Z}/2^\nu)$. There is a diagram

$$\begin{array}{ccc} H_n(B) & \xrightarrow{\sim} & H_n(A) \\ {\scriptstyle B_B^n}\downarrow & & \\ K_n(B, \mathbf{Z}/2^\nu) & \xrightarrow{\sim} & K_n(A, \mathbf{Z}/2^\nu). \end{array}$$

The top horizontal map is an isomorphism by a theorem of Gabber [18] and independently Strano [53]. The bottom one is also an isomorphism, by another theorem of Gabber [19]. Note that B is non-exceptional, since A is non-exceptional and (B, I) is a henselian pair. Hence the diagram again defines a homomorphism $B_A^n : H_n(A) \to K_n(A, \mathbf{Z}/2^\nu)$.

Let us show that B_A^n does not depend on the choice of (B, I). Let (B', I') be another Henselian pair as above. Consider the diagonal embedding

$$\operatorname{Spec} A \to \operatorname{Spec} B \times_R \operatorname{Spec} B' = \operatorname{Spec}(B \otimes_R B').$$

This is a closed embedding, with corresponding ideal I''_0. The pair $(B \otimes_R B', I''_0)$ may not be henselian, but we can henselise it into (R'', I''): this henselian pair is still of the same type as above, with residue ring

A. We then get a diagram of Henselian pairs

$$\begin{array}{ccc}
(B,I) & & \\
& \searrow & \\
& & (B'',I'') \\
& \nearrow & \\
(B',I') & &
\end{array}$$

which reduces us to the case where $B \subset B'$. Then B^n_A (defined out of B^n_B) equals B^n_A (defined out of $B^n_{B'}$) thanks to the naturality of B^n for essentially smooth semi-local non-exceptional R-algebras.

If $f : A \to A'$ is a homomorphism of non-exceptional semi-local R-algebras essentially of finite type, we can cover it by a homomorphism of henselian pairs with residue rings A, A'. This, and the well-definedness of B^n, provides it with the structure of a natural transformation on the category of local R-algebras essentially of finite type.

IV) *The general case.* Write A as a direct limit of its finitely generated subalgebras. After localising the latter, we write A as a direct limit $\varinjlim A_i$, where the A_i are as in III), with R a suitable (semi-)localisation of $\mathbf{Z}[1/2]$. Note that, since A is non-exceptional, then A_i is non-exceptional for i large enough (by a compactness argument). By naturality, we then get the desired homomorphism B^n_A as the direct limit of the $B^n_{A_i}$s. And B^n_A is clearly natural in A.

Property (iv) of Proposition 2.1 has already been proven in Lemma 2.4. The other properties follow similarly from the definition of the anti-Chern classes and [34, cor. 9.5]. As for uniqueness, the method of construction shows that B^n is determined by its value on fields. By the Milnor conjecture and [32, th. 1(2)], the bigraded ring $H^*(F, \mu_{2^\nu}^{\otimes *})$ is generated by $H^0(F, \mu_{2^\nu})$ and $H^1(F, \mu_{2^\nu})$ modulo transfer. Hence B^n is determined by the value of B^0, B^1 and B^2 on fields.

The commutation of the diagrams at the end of Proposition 2.1 is clear for fields by construction; the general case is obtained by going through II), III) and IV) as before. $\square$

Protoglobalization. We have the following variant of Proposition 2.1:

2.5. **Proposition.** *Let $\mathcal{H}^j_{\text{ét}}(\mu_{2^\nu}^{\otimes i})$ (resp. $\mathcal{K}_n(\mathbf{Z}/2^\nu)$) be the sheaf associated to the presheaf $U \mapsto H^j_{\text{ét}}(U, \mu_{2^\nu}^{\otimes i})$ (resp. $U \mapsto K_n(U, \mathbf{Z}/2^\nu)$) over the big Zariski site of* $\operatorname{Spec} \mathbf{Z}[1/2]$ *restricted to non-exceptional schemes.*

Then, for all $n, \nu \geq 0$, there is a unique commutative diagram of homomorphism of sheaves:

$$\begin{array}{ccc}
\coprod_{0\le i\le n} \mathcal{H}^{2i-n}_{\text{ét}}(\mu_{2^\nu}^{\otimes i}) & \xrightarrow{\mathcal{B}^n} & \mathcal{K}_n(\mathbf{Z}/2^\nu) \\
\downarrow & & \downarrow \\
\coprod_{0\le i} \mathcal{H}^{2i-n}_{\text{ét}}(\mu_{2^\nu}^{\otimes i}) & \xrightarrow{\mathcal{B}^{n,\text{ét}}} & \mathcal{K}_n(\mathbf{Z}/2^\nu)[\beta^{-1}]
\end{array}$$

which coincides with B^n and $B^{n,\text{ét}}$ at the stalks. The homomorphisms of sheaves $\mathcal{B}^n$ and $\mathcal{B}^{n,\text{ét}}$ have properties similar to those of Proposition 2.1. The same holds when replacing the Zariski topology by the Nisnevich topology of [45].

Proof. It is enough to construct $\mathcal{B}^n$ over any non-exceptional affine scheme, hence over any non-exceptional affine scheme X of finite type over $\operatorname{Spec}\mathbf{Z}[1/2]$. We proceed exactly as above. If X is smooth over $\operatorname{Spec}\mathbf{Z}[1/2]$, the argument in step II) constructs $\mathcal{B}^n$ over X and shows that it is contravariant in X. In general, we write X as a closed subscheme of $\mathbf{A}^N_{\mathbf{Z}[1/2]}$ for N large, henselize the latter along X and mimic the argument in step III). □

3. Proof of Theorem 1

Proof of Theorem 1 a). The construction of section 2 reduces us to the case of a field. We shall use a spectral sequence

$$E_2^{p,q} \Rightarrow K_{-p-q}(F, \mathbf{Z}/2^\nu) \tag{3.1}$$

where

$$E_2^{p,q} = \begin{cases} H^{p-q}(F, \mu_{2^\nu}^{\otimes(-q)}) & \text{if } p, q \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Let us recall some facts on this spectral sequence. In [6], Bloch and Lichtenbaum construct a spectral sequence

$$E_2^{p,q} = H^{p-q}(F, \mathbf{Z}(-q)) \Rightarrow K_{-p-q}(F)$$

where the E_2-terms are motivic cohomology of F, defined as Bloch's higher Chow groups [5] renumbered (see also [43]). In [49], Rognes and Weibel construct a variant:

$$E_2^{p,q} = H^{p-q}(F, \mathbf{Z}/m(-q)) \Rightarrow K_{-p-q}(F, \mathbf{Z}/m)$$

where the E_2-terms are now motivic cohomology with finite coefficients. Now, for $m = 2^\nu$, there are isomorphisms

$$H^{p-q}(F, \mathbf{Z}/2^\nu(-q)) = \begin{cases} H^{p-q}(F, \mu_{2^\nu}^{\otimes(-q)}) & \text{if } p, q \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, this follows from Voevodsky's proof of the Milnor conjecture [63] and the main result of Geisser-Levine [22]. Before [22], this isomorphism was proven in characteristic 0 by Suslin-Voevodsky [57], modulo the identification of Bloch's motivic cohomology with Voevodsky's motivic cohomology [56].

Then Friedlander and Suslin found a fundamental reinterpretation of the construction in [6], allowing them and Levine to globalise the Bloch-Lichtenbaum spectral sequence to smooth schemes over a field (Friedlander-Suslin [17]) and even regular schemes of finite type over a regular base of dimension ≤ 1 (Levine [40]). This also gave another, direct, construction of the Rognes-Weibel variant with finite coefficients. Finally, it allowed Friedlander-Suslin [17] and Levine [41] to provide these spectral sequences with a product structure. (For the product structure with finite coefficients, see [17, Th. 15.1].)

The product structure on (3.1) and the easier existence of transfers[3] allow us to play the same game as in [31] and [33], using the anti-Chern classes to kill all differentials of the spectral sequence and show that the E_∞ filtration on the abutment is split by them (see [33, proof of th. 3.1]). Let us give some details of this "game":

1) Consider the composition

$$H^0(F, \mu_{2^\nu}^{\otimes i}) \xrightarrow{\beta^{i,0}} K_{2i}(F, \mathbf{Z}/2^\nu) \xrightarrow{e} H^0(F, \mu_{2^\nu}^{\otimes i})$$

where the second map is the edge homomorphism from the spectral sequence. We claim that this composition is an odd multiple of the identity. To see this, we immediately reduce to the case where F is separably closed; then $\beta^{i,0}$ and e are both isomorphisms (the second one because the spectral sequence collapses and the first one by [34, cor. 9.5 (v)] and [54, Cor. 3.13]).[4]

2) Consider the composition

$$H^1(F, \mu_{2^\nu}^{\otimes i}) \xrightarrow{\beta^{i,1}} K_{2i-1}(F, \mathbf{Z}/2^\nu) \xrightarrow{e'} H^1(F, \mu_{2^\nu}^{\otimes i})$$

[3]Such transfers are constructed in [41, §4] in a much more general situation. However, in the case of fields it is sufficient to notice that the construction of the Bloch-Lichtenbaum exact couple in [6] is compatible with transfers at every step.

[4]With a little more effort, one could probably prove that the composition is actually the identity. The issue is to show it for F separably closed and $i = 1$.

where e' is also an edge homomorphism from the spectral sequence. We claim that this composition is again an odd multiple of the identity. Using 1), the multiplicativity of the spectral sequence, its compatibility with transfers and Proposition 2.2, we reduce to $i = 1$. Then we have a commutative diagram

$$\begin{array}{ccccc}
H^1(F, \mu_{2^\nu}) & \longrightarrow & K_1(F)/2^\nu & \xrightarrow{\tilde{e}'} & H^1(F, \mathbf{Z}(1))/2^\nu \\
\| & & \downarrow & & \downarrow \\
H^1(F, \mu_{2^\nu}) & \xrightarrow{\beta^{1,1}} & K_1(F, \mathbf{Z}/2^\nu) & \xrightarrow{e'} & H^1(F, \mu_{2^\nu})
\end{array}$$

where $\tilde{e}'$ stems from an edge homomorphism of the integral spectral sequence and the vertical maps are induced by the "change of coefficients" morphism from the integral motivic spectral sequence to the same with $\mathbf{Z}/2^\nu$ coefficients. The top left horizontal map is the inverse of the Kummer theory isomorphism and $\tilde{e}'$ is also an isomorphism by the integral motivic spectral sequence. Since the right vertical map coincides with the Kummer theory isomorphism, the claim is proven.

We have split off one summand from the K-groups and also from (3.1). We can then go on, using the next anti-Chern classes, and split off higher and higher chunks from both $K_*(F, \mathbf{Z}/2^\nu)$ and the spectral sequence. More precisely, one shows inductively on j that

- all differentials leaving $E_r^{p,q}$ are 0 for $p - q < j$;
- $\operatorname{Im} \beta_F^{i,j} \subseteq F^{j-i} K_{2i-j}(F, \mathbf{Z}/2^\nu)$, where $F^* K_{2i-j}(F, \mathbf{Z}/2^\nu)$ is the filtration induced by the spectral sequence (3.1);
- The composition

$$H^j(F, \mu_{2^\nu}^{\otimes i}) \xrightarrow{\beta_F^{i,j}} F^{j-i} K_{2i-j}(F, \mathbf{Z}/2^\nu) \to E_\infty^{j-i,-i} \hookrightarrow E_2^{j-i,-i} = H^j(F, \mu_{2^\nu}^{\otimes i})$$

  is an odd multiple of the identity (a fact which reduces to 0) and 1) by the Milnor conjecture and Proposition 2.2, since all maps commute with products and transfers).

Since (3.1) is convergent, we find in the end that it canonically degenerates and that B_F^n is bijective.

Proof of Theorem 1 b). For A a semi-local ring, we define the Milnor K-groups $K_*^M(A)$ exactly as for a field, using the Steinberg presentation. If A is a non-exceptional $\mathbf{Z}[1/2]$-algebra, the groups $K_j^M(i)(A, \mathbf{Z}/2^\nu)$ are then defined in the same way as for a field. Note that the maps in (2.1) and (2.2) exist for any such A (see [33, §§1 and

2]). By a), they can now be completed into commutative triangles

$$
\text{(3.2)}\qquad
\begin{tikzcd}
 & H_n(A) \arrow[dd, "B_A^n", "\wr"'] \\
\coprod_{i\le n} K^M_{2i-n}(n-i)(A,\mathbf{Z}/2^\nu) \arrow[ur, "u_A^n"] \arrow[dr, "\eta_A^n"'] & \\
 & K_n(A,\mathbf{Z}/2^\nu)
\end{tikzcd}
$$

where u_A^n and η_A^n are sums of the $u_A^{i,j}$ and the $\eta_A^{i,j}$. Theorem 1 b) claims that η_A^n is surjective; to see this, it is enough to show that u_A^n is surjective.

We first show that the Galois symbol $K_j^M(A)/2^\nu \xrightarrow{u_A^{0,j}} H^j_{\text{ét}}(A,\mu_{2^\nu}^{\otimes j})$ is surjective (*cf.* [29]). To do this, we proceed as for a) along steps I) – IV) of the proof of Proposition 2.1. If A is a field, this is part of Voevodsky's theorem. Suppose A is as in step II). We have a commutative diagram of complexes

$$
\begin{tikzcd}
0 \arrow[r] & K_j^M(A)/2^\nu \arrow[r] \arrow[d, "u_A^{0,j}"] & K_j^M(E)/2^\nu \arrow[r] \arrow[d, "u_E^{0,j}", "\wr"'] & \coprod_{x\in X^{(1)}} K^M_{j-1}(\kappa(x))/2^\nu \arrow[d, "(u^{0,j-1}_{\kappa(x)})", "\wr"'] \\
0 \arrow[r] & H^j_{\text{ét}}(A,\mu_{2^\nu}^{\otimes j}) \arrow[r] & H^j_{\text{ét}}(E,\mu_{2^\nu}^{\otimes j}) \arrow[r] & \coprod_{x\in X^{(1)}} H^{j-1}_{\text{ét}}(\kappa(x),\mu_{2^\nu}^{\otimes (j-1)}).
\end{tikzcd}
$$

The bottom row is exact by the Bloch-Ogus-Gillet theorem, and the top row is exact at $K_j^M(E)/2^\nu$ by an unpublished theorem of Gabber [20] (see also [14]). Therefore $u_A^{0,j}$ is surjective.

Suppose A is as in step III). Using the same construction as in the proof of a), we get a commutative diagram

$$
\begin{tikzcd}
K_j^M(B)/2^\nu \arrow[r] \arrow[d, "u_B^{0,j}"] & K_j^M(A)/2^\nu \arrow[d, "u_A^{0,j}"] \\
H^j_{\text{ét}}(B,\mu_{2^\nu}^{\otimes j}) \arrow[r, "\sim"] & H^j_{\text{ét}}(A,\mu_{2^\nu}^{\otimes j})
\end{tikzcd}
$$

in which $u_B^{0,j}$ is surjective by step II). Since $K_j^M(A)$ and $K_j^M(B)$ are generated by units and B is semi-local, the top horizontal map is surjective and so is $u_A^{0,j}$. Finally, step IV) follows from step III) by a passage to the limit, just as in the proof of a).

We now deal with the general case and prove that

$$K_j^M(i-j)(A,\mathbf{Z}/2^\nu) \xrightarrow{u_A^{i-j,j}} H^j_{\text{ét}}(A,\mu_{2^\nu}^{\otimes i})$$

is surjective for all A, i, j with $i \geq j$. Let A' be the smallest finite étale extension of A such that the étale sheaf $\mu_{2^\nu}^{\otimes(i-j)}$ is constant over $\operatorname{Spec} A'$, and let $G = Gal(A'/A)$. Consider the commutative diagram

$$\begin{array}{ccc}
K_j^M(i-j)(A',\mathbf{Z}/2^\nu) := K_j^M(A')\otimes\mu_{2^\nu}^{\otimes(i-j)} & \xrightarrow{u_{A'}^{i-j,j}} & H^j_{\text{ét}}(A',\mu_{2^\nu}^{\otimes j})\otimes\mu_{2^\nu}^{\otimes(i-j)} = H^j_{\text{ét}}(A',\mu_{2^\nu}^{\otimes i}) \\
\downarrow & & \downarrow f_* \\
K_j^M(i-j)(A,\mathbf{Z}/2^\nu) & \xrightarrow{u_A^{i-j,j}} & H^j_{\text{ét}}(A,\mu_{2^\nu}^{\otimes i}).
\end{array}$$

Here the left vertical map is given by the definition of $K_j^M(i-j)(A, \mathbf{Z}/2^\nu) = K_j^M(i-j)(A', \mathbf{Z}/2^\nu)_G$, and the right vertical map is the direct image map associated to the finite morphism $f : \operatorname{Spec} A' \to \operatorname{Spec} A$. By the first part of the proof, $u_{A'}^{i-j,j}$ is surjective. By [32, th. 1], the right vertical map in the diagram is surjective, and therefore so is $u_A^{i-j,j}$.

We observe that the above allows the proof of degeneration of (3.1) to carry over mutatis mutandis for Levine's spectral sequence (1.3) in the case of more general semi-local rings, thanks to the existence of transfers on this spectral sequence for finite morphisms [41, §4]. We record this in the following theorem.

3.1. **Theorem.** *Let A be a semi-local regular $\mathbf{Z}[1/2]$-algebra, essentially of finite type over a field or a Dedekind domain. Suppose that A is non-exceptional. Then the spectral sequence* (1.3) *canonically degenerates for $X = \operatorname{Spec} A$.* □

Proof of Theorem 1 c) and d). These are special cases of a), in view of the construction of B_F^n. □

3.2. **Corollary.** *Let A be a semi-local non-exceptional $\mathbf{Z}[1/2]$-algebra. Then the natural map*

$$K_n(A, \mathbf{Z}/2^\nu) \to K_n(A, \mathbf{Z}/2^\nu)[\beta^{-1}]$$

is a split injection for all $n \geq 0$ and an isomorphism for $n \geq cd_2(A)-1$.

Proof. Arguing as in the proof of Theorem 1 a), we prove that the homomorphisms $B_A^{n,\text{ét}}$ of Proposition 2.1 are isomorphisms. Here we reduce to the case where A is a field and use a localisation of the Bloch-Lichtenbaum spectral sequence by inverting the Bott element [17, 42]. It remains to observe that, in the diagram of Proposition 2.1, the left vertical map is a split injection in general and an isomorphism for $n \geq cd_2(A)-1$. (For the nervous reader, if $cd_2(A) = 0$ and $n = -1$, the map is indeed an isomorphism as both sides are 0). □

3.3. **Remark.** This provides a negative answer to Problem 5, p. 540 in [68] (which was actually formulated for odd primes!).

3.4. **Corollary.** *If $cd_2(A) < +\infty$, the natural map*

$$K_n(A, \mathbf{Z}/2^\nu)[\beta^{-1}] \to K_n^{\text{ét}}(A, \mathbf{Z}/2^\nu)$$

is an isomorphism for all $n \in \mathbf{Z}$.

Proof. Use the Dwyer-Friedlander spectral sequence of [11, prop. 5.2] (or rather its non-connective analogue) and split it with the help of the $B_A^{n,\text{ét}}$ just as in the proof of Theorem 1 a). □

3.5. **Corollary.** *For all $n, \nu \geq 0$, the homomorphisms of sheaves $\mathcal{B}^n$ and $\mathcal{B}^{n,\text{ét}}$ of Proposition 2.5 are isomorphisms (for the Zariski or the Nisnevich topology).* □

3.6. **Corollary.** *a) For any non-exceptional $\mathbf{Z}[1/2]$-scheme X and any $r, s \geq 0$, there is a natural isomorphism of abelian groups:*

$$\coprod_{i \leq s} H^r_{\text{Zar}}(X, \mathcal{H}^{2i-s}_{\text{ét}}(\mu_{2^\nu}^{\otimes i})) \xrightarrow{\sim} H^r_{\text{Zar}}(X, \mathcal{K}_s(\mathbf{Z}/2^\nu)).$$

b) Suppose X is smooth over a field or a Dedekind domain. Then the above isomorphism refines to

$$\coprod_{r+s \leq 2i \leq 2s} H^r_{\text{Zar}}(X, \mathcal{H}^{2i-s}_{\text{ét}}(\mu_{2^\nu}^{\otimes i})) \xrightarrow{\sim} H^r_{\text{Zar}}(X, \mathcal{K}_s(\mathbf{Z}/2^\nu)).$$

a) and b) hold when replacing $\mathcal{K}_*(\mathbf{Z}/2^\nu)$ by $\mathcal{K}_*(\mathbf{Z}/2^\nu)[\beta^{-1}]$, and summing over all i in a) and $r + s \leq 2i$ in b). Finally, the same holds when replacing Zariski cohomology by Nisnevich cohomology.

Proof. This is simply evaluating Corollary 3.5 at X. b) follows from the fact that $H^r_{\text{Zar}}(X, \mathcal{H}^{2i-s}_{\text{ét}}(\mu_{2^\nu}^{\otimes i})) = 0$ for $r > 2i - s$ (Gersten's conjecture). □

4. Proofs of Theorem 2, Corollary 1 and Corollary 2

4.1. **Proposition.** *Let X be a Noetherian $\mathbf{Z}[1/2]$-scheme of finite Krull dimension. Assume that no residue field of X is formally real.*
a) If $x \in X$ is a point of codimension p, then $cd_2(x) \leq d_2(X) - p$.
b) If $X \to S$ is a dominant morphism essentially of finite type, with $d_2(S) < \infty$, then $d_2(X) = d_2(S) + \dim(X/S)$.

Proof. a) follows from [SGA4, exposé X, cor. 2.4]. For b), we reduce to the case where $X \to S$ is a field extension; then the result follows from [51, p. II-13, prop. 11]. □

4.2. **Corollary.** *For any finite-dimensional Noetherian non-exceptional* $\mathbf{Z}[1/2]$*-scheme* X *and any* $\nu \geq 1$*, the natural map*

$$H^r_{\mathrm{Nis}}(X, \mathcal{K}_s(\mathbf{Z}/2^\nu)) \to H^r_{\mathrm{Nis}}(X, \mathcal{K}_s(\mathbf{Z}/2^\nu)[\beta^{-1}])$$

is split injective for all r, s *and bijective for all* r *and* $s \geq d_2(X) - 1$.

Proof. Using Corollary 3.6, it is enough to check that the Nisnevich sheaf $\mathcal{H}^q_{\text{ét}}(\mu_{2^\nu}^{\otimes i})$ is 0 for $q > d_2(X)$. Its stalk at a point $x \in X$ is $H^q_{\text{ét}}(\mathcal{O}^h_{X,x}, \mu_{2^\nu}^{\otimes i})$, where $\mathcal{O}^h_{X,x}$ is the henselisation of $\mathcal{O}_{X,x}$. By [SGA4, exposé XII, cor. 5.5], this group coincides with $H^q_{\text{ét}}(\kappa(x), \mu_{2^\nu}^{\otimes i})$, which is 0 by Proposition 4.1 a) (note that X satisfies the assumptions of this proposition). $\square$

4.3. **Proposition.** *Let* X *be an excellent Noetherian scheme of dimension* ≤ 1*. Let* $K^B(X)$ *be Thomason's Bass extension of the* K*-theory spectrum of* X [62, def. 6.4]*. Then there is a canonical isomorphism* $K^B_{-1}(X) \simeq H^1_{\text{ét}}(X, \mathbf{Z})$.

Proof. [5] By [62, th. 10.3], for any quasi-compact, quasi-separated X, the natural map of spectra

$$K^B(X) \to \mathbb{H}^{\cdot}_{\mathrm{Nis}}(X, \mathcal{K}^B) \tag{4.1}$$

is a homotopy equivalence. Here, $\mathbb{H}^{\cdot}_{\mathrm{Nis}}(X, \mathcal{K}^B)$ is as in [59, def. 1.33]. Hence a spectral sequence

$$E_2^{p,q} = H^p_{\mathrm{Nis}}(X, \mathcal{K}^B_{-q}) \Rightarrow K^B_{-p-q}(X)$$

where $\mathcal{K}^B_{-q}$ is the Nisnevich sheaf associated to the presheaf $U \mapsto K^B_{-q}(U)$. Since $\dim X \leq 1$, this spectral sequence yields a short exact sequence

$$0 \to H^1_{\mathrm{Nis}}(X, \mathcal{K}^B_0) \to K^B_{-1}(X) \to H^0_{\mathrm{Nis}}(X, \mathcal{K}^B_{-1}) \to 0.$$

The rank canonically identifies $\mathcal{K}^B_0$ to the constant sheaf $\mathbf{Z}$, and the natural map $H^1_{\mathrm{Nis}}(X, \mathbf{Z}) \to H^1_{\text{ét}}(X, \mathbf{Z})$ is an isomorphism. It remains to see that $H^0_{\mathrm{Nis}}(X, \mathcal{K}^B_{-1}) = 0$, *i.e.* that $K^B_{-1}(X) = 0$ for $X = \operatorname{Spec} A$ with A local Henselian of dimension ≤ 1.

By [65, (1.7.1)], there is an exact sequence

$$0 \to LSK_0(X) \to K^B_{-1}(X) \to LPic(X) \to 0$$

and by *loc. cit.*, th. 5.5, $LPic(X) \simeq H^1_{\text{ét}}(X, \mathbf{Z}) = 0$. Therefore it suffices to show that $LSK_0(X) = 0$. By [2, th. 7.8], for any 1-dimensional Noetherian ring A whose integral closure is finite over A, the determinant map $\widetilde{K}_0(A[t, t^{-1}]) \xrightarrow{\det} \operatorname{Pic} A[t, t^{-1}]$ is an isomorphism, where

[5] I thank Chuck Weibel for his help in this proof.

$\widetilde{K}_0(A[t,t^{-1}]) = \operatorname{Ker}(K_0(A[t,t^{-1}]) \xrightarrow{rk} H^0(A[t,t^{-1}],\mathbf{Z})$. In particular, $LSK_0(A) := \operatorname{Coker}\det = 0$. This applies to $A = \Gamma(X,\mathcal{O}_X)$. □

4.4. **Corollary.** *For X as in Proposition 4.3, the map $K_0(X,\mathbf{Z}/n) \to K_0^B(X,\mathbf{Z}/n)$ is an isomorphism for any $n > 0$.*

Proof. By Proposition 4.3, $K_{-1}^B(X)$ is n-torsion-free, as follows from the obvious surjectivity of $H^0_{\text{ét}}(X,\mathbf{Z}) \to H^0_{\text{ét}}(X,\mathbf{Z}/n)$. Therefore, the map

$$K_0(X)/n = K_0^B(X)/n \to K_0^B(X,\mathbf{Z}/n)$$

is bijective. □

Proof of Theorem 2. The assumption implies that all Hensel local rings of X satisfy the assumptions of Theorem 1.

Smashing the equivalence (4.1) with the Moore spectrum $M(\mathbf{Z}/2^\nu)$ yields another homotopy equivalence

$$K^B(X,\mathbf{Z}/2^\nu) \xrightarrow{\approx} \mathbb{H}^{\cdot}_{\text{Nis}}(X,\mathcal{K}^B(\mathbf{Z}/2^\nu))$$

which in turn yields a strongly convergent spectral sequence

$$E_2^{p,q} = H^p_{\text{Nis}}(X,\mathcal{K}^B_{-q}(\mathbf{Z}/2^\nu)) \Rightarrow K^B_{-p-q}(X,\mathbf{Z}/2^\nu). \tag{4.2}$$

This spectral sequence maps to the analogous spectral sequence for Bott-localised K^B-theory

$$H^p_{\text{Nis}}(X,\mathcal{K}^B_{-q}(\mathbf{Z}/2^\nu)[\beta^{-1}]) \Rightarrow K^B_{-p-q}(X,\mathbf{Z}/2^\nu)[\beta^{-1}]$$

(compare [62, (11.8.1)]). There are natural transformations

$$K \to K^{TT} \to K^B. \tag{4.3}$$

For any quasi-compact quasi-separated X, the map $K_n^{TT}(X) \to K_n^B(X)$ is an isomorphism for $n \geq 0$ [62, th. 7.5 a)], hence the map $K_n^{TT}(X,\mathbf{Z}/2^\nu) \to K_n^B(X,\mathbf{Z}/2^\nu)$ is an isomorphism for $n \geq 1$, and also for $n = 0$ if $K_{-1}(X)$ is torsion-free, *e.g.* if X is regular or as in Proposition 4.3. The map $K_n^{TT}(X,\mathbf{Z}/2^\nu)[\beta^{-1}] \to K_n^B(X,\mathbf{Z}/2^\nu)[\beta^{-1}]$ is an isomorphism for all n [62, (11.4.2)]. Also, since X is locally quasi-projective for the Zariski topology, the map of Zariski sheaves $\mathcal{K}_n(\mathbf{Z}/2^\nu) \to \mathcal{K}_n^{TT}(\mathbf{Z}/2^\nu)$ is an isomorphism for all n [62, prop. 3.10], and the same holds with the Bott element inverted.

In particular, the group $H^p_{\text{Nis}}(X,\mathcal{K}^B_{-q}(\mathbf{Z}/2^\nu))$ coincides with $H^p_{\text{Nis}}(X,\mathcal{K}_{-q}(\mathbf{Z}/2^\nu))$ provided $-q \geq 1$ (and also for $q = 0$, provided the sheaf $\mathcal{K}_{-1}$ is torsion-free). Hence, by Corollary 4.2, the map of spectral sequences is an isomorphism on the $E_2^{p,q}$ terms for $-q \geq max(d_2(X)-1,1)$ and is split injective for all $q \leq -1$. The conclusion follows for Thomason-Trobaugh K-theory.

For K'-theory, we use the Quillen spectral sequence

$$E_1^{p,q} = \coprod_{x\in X^{(p)}} K_{-p-q}(\kappa(x), \mathbf{Z}/2^\nu) \Rightarrow K'_{-p-q}(X, \mathbf{Z}/2^\nu)$$

and its analogue for K'-theory with the Bott element inverted (compare [46, th. 5.5.4]). Here $X^{(p)}$ denotes the set of points of X of codimension p. By Corollary 3.2, the natural map between the two spectral sequences is split injective on the E_1-terms, and bijective on the $E_1^{p,q}$-terms provided $-p-q \geq sup_{x\in X^{(p)}} cd_2(x) - 1$. By Proposition 4.1 a), the right hand side is $\leq d_2(X) - p - 1$. Hence the natural map induces a bijection on $E_1^{p,q}$ as soon as $-q \geq d_2(X) - 1$, and the result follows.

Finally, let us prove c). By the non-connective analogue of [11, prop. 5.2], the natural map

$$K^{\text{ét}}(X, \mathbf{Z}/2^\nu) \to \mathbb{H}^{\cdot}_{\text{ét}}(X, \mathcal{K}^{\text{ét}}(\mathbf{Z}/2^\nu)_{\text{ét}})$$

is a weak equivalence. It follows by a simple argument (using the weak equivalence $\mathbb{H}^{\cdot}_{\text{ét}}(X, F_{\text{ét}}) \simeq \mathbb{H}^{\cdot}_{\text{Nis}}(X, \mathbb{H}^{\cdot}_{\text{ét}}(-, F_{\text{ét}})_{\text{Nis}})$ for a presheaf of fibrant spectra F) that the natural map

$$K^{\text{ét}}(X, \mathbf{Z}/2^\nu) \to \mathbb{H}^{\cdot}_{\text{Nis}}(X, \mathcal{K}^{\text{ét}}(\mathbf{Z}/2^\nu)_{\text{Nis}})$$

is a weak equivalence as well. We then have a commutative diagram

$$\begin{array}{ccc}
K^{TT}(X, \mathbf{Z}/2^\nu)[\beta^{-1}] & \xrightarrow{\approx} & \mathbb{H}^{\cdot}_{\text{Nis}}(X, \mathcal{K}(\mathbf{Z}/2^\nu)_{\text{Nis}}[\beta^{-1}]) \\
\downarrow & & \approx\downarrow \\
K^{\text{ét}}(X, \mathbf{Z}/2^\nu) & \xrightarrow{\approx} & \mathbb{H}^{\cdot}_{\text{Nis}}(X, \mathcal{K}^{\text{ét}}(\mathbf{Z}/2^\nu)_{\text{Nis}})
\end{array}$$

where the top row is a weak equivalence by the considerations at the beginning of the proof (compare [62, (11.8.1)]), and the right column is a weak equivalence by Corollary 3.4. □

Proof of Corollary 1. By Proposition 4.1 b), we have

$$d_2(X) = \dim(X/S) + d_2(S)$$

and moreover

$$d_2(S) = \begin{cases} 2 & \text{in case (i)} \\ 1 & \text{in case (ii)} \\ 0 & \text{in case (iii)} \\ d+1 & \text{in case (iv)} \end{cases}$$

where, in case (iv), d is the (Kato) dimension of k.

With the exception $\dim(X/S) + d_2(S) \leq 2$, the case of K^{TT}-theory follows from Theorem 2 and the fact that $K_n(X, \mathbf{Z}/2^\nu)[\beta^{-1}]$ is finite for all n, as the abutment of a spectral sequence with E_2-terms étale cohomology groups [62, th. 11.5] which are finite by the classical finiteness theorems for étale cohomology ([SGA4 1/2, Th. finitude] plus class field theory). In the remaining case, we are

(1) either in case (i), with $\dim(X/S) \leq 0$;
(2) either in case (ii), with $\dim(X/S) \leq 1$;
(3) either in case (iii), with $\dim(X/S) \leq 2$;
(4) or in case (iv), with $\dim X \leq 1$ (note that $cd_2(k) \geq 1$).

If $\dim X \leq 1$, by Corollary 4.4, the map of Brown-Gersten-Nisnevich spectral sequences in the proof of Theorem 2 is an isomorphism even for $q = 0$. The assertion then follows as above. The only case where $\dim X$ can equal 2 is case (iii). If X is regular, then $K_i^B(X) = 0$ for $i < 0$; this completes the proof of Corollary 1 for K^{TT}-theory.

For K'-theory, it is enough in view of Theorem 2 to show that $K'_n(X, \mathbf{Z}/2^\nu)[\beta^{-1}]$ is finite for all $n \in \mathbf{Z}$. This is true for X regular, by reduction to K-theory. In general, we can apply [EGA4, cor. 6.12.6] in each case of the corollary to find in X a regular dense open subset U. Using the localisation exact sequence for X, U and $X \setminus U$, we get the result by Noetherian induction. □

Proof of Corollary 2. We reduce the case of K'-theory to the case of Thomason-Trobaugh's K-theory just as in the proof of Corollary 1. Taking a direct limit of homotopy exact sequences for finite coefficients, it is then enough to prove that $K_{n+1}^{TT}(X, \mathbf{Q}_2/\mathbf{Z}_2)$ is finite or, by Theorem 2 a), that the isomorphic group $K_{n+1}^{TT}(X, \mathbf{Q}_2/\mathbf{Z}_2)[\beta^{-1}]$ is finite. The E_2-terms in the corresponding limit of Thomason-Trobaugh spectral sequences [62, (11.5.3)] are

$$H^{2m-n-1}_{\text{ét}}(X, \mathbf{Q}_2/\mathbf{Z}_2(m)) \quad (m \in \mathbf{Z}).$$

Suppose first that $k = \mathbf{F}_p$. By [35, Th. 2], the group above is finite for $m > d$ or for $2m - n - 1 < m$. Since by assumption $n \geq d$, this covers all values of m. If $k = \mathbf{Q}_p$, [35, Th. 5] shows that the group above is finite for $m > d + 1$ or for $2m - n - 1 < m$, which again covers all values of m since this time we assume $n \geq d + 1$. □

5. Higher Chern classes with values in truncated étale cohomology

In this section, we give a construction of higher Chern classes as indicated by Beilinson in [4, 5.10 D (vi)], and relate them to Soulé's étale Chern classes [52].

5.1. **Construction.** Let $S = \operatorname{Spec} R$, where R is a field or a Dedekind domain. Suppose 2 is invertible on S. Using Gillet's method [23], we construct higher Chern classes

$$c_{i,j} : K_{2i-j}(X, \mathbf{Z}/2^\nu) \longrightarrow H^j_{\mathrm{Zar}}(X, B/2^\nu(i)) \tag{5.1}$$

for any S-scheme X and $2i - j \geq 2$. Here $B/2^\nu(i) = \tau_{\leq i} R\alpha_* \mu_{2^\nu}^{\otimes i}$, where α is the projection of the big étale site of S onto its big Zariski site, as in [57].

5.1.1. *Equivariant Chern classes.* Looking at the arguments in [23, §2], one sees that to get Chern classes

$$C_i \in H^{2i}_{\mathrm{Zar}}(X, \mathcal{GL}(\mathcal{O}_X), B/2^\nu(i)) \tag{5.2}$$

[23, p. 225] for any S-scheme X, it is enough to prove a projective bundle theorem for vector bundles over X, where X is (essentially) smooth over S. (The other axioms of [23, def. 1.1 and 1.2], and in particular the existence of a Borel-Moore homology theory, are not needed.) For such a vector bundle E of rank r, with associated projective bundle $\mathbb{P}(E) \xrightarrow{\pi} X$, the projective bundle formula will take the form

$$\coprod_{i=0}^{r} B/2^\nu(n-i)_X[-2i] \xrightarrow{\sim} R\pi_* B/2^\nu(n)_{\mathbb{P}(E)} \tag{5.3}$$

for all $n \geq 0$, in the derived category of Zariski sheaves over X, where the i-th component of the isomorphism is given by pull-back under π followed by cup-product by the i-th power of a certain class $\xi \in H^2(\mathbb{P}(E), B/2^\nu(1))$.

We first construct the class ξ. The Kummer exact sequence in the étale topology

$$1 \to \mu_{2^\nu} \to \mathbb{G}_m \xrightarrow{2^\nu} \mathbb{G}_m \to 1$$

yields, by higher direct image and truncation, a morphism in the derived category

$$\tau_{\leq 1} R\alpha_* \mathbb{G}_m \to B/2^\nu(1)[1]$$

hence a map in cohomology

$$H^1_{\mathrm{Zar}}(\mathbb{P}(E), \tau_{\leq 1} R\alpha_* \mathbb{G}_m) \to H^2_{\mathrm{Zar}}(\mathbb{P}(E), B/2^\nu(1)). \tag{5.4}$$

But $H^1_{\mathrm{Zar}}(\mathbb{P}(E), \tau_{\leq 1} R\alpha_* \mathbb{G}_m) = H^1_{\mathrm{Zar}}(\mathbb{P}(E), R\alpha_* \mathbb{G}_m) = H^1_{\text{ét}}(\mathbb{P}(E), \mathbb{G}_m) = \operatorname{Pic} \mathbb{P}(E)$. The class ξ is the image under (5.4) of the class of the tautological bundle $\mathcal{O}(1)$ in $\operatorname{Pic} \mathbb{P}(E)$.

We note that the morphism (5.3) induces, for all $q \in [0,n]$, a morphism of truncated complexes (use the formula $\tau_{\leq n}(A[i]) = (\tau_{\leq n+i}A)[i])$

$$\coprod_{i=0}^{r} (\tau_{\leq q-i} B/2^\nu(n-i)_X)[-2i] \to R\pi_* \tau_{\leq q} B/2^\nu(n)_{\mathbb{P}(E)}. \tag{5.5}$$

Since the corresponding filtration by successive truncations on the left hand side of (5.3) is exhaustive, in order to prove that (5.3) is an isomorphism it suffices to prove that it induces an isomorphism on the successive cones of (5.5), that is, on

$$\coprod_{i=0}^{r} R^{q-i}\alpha_*\mu_{2^\nu}^{\otimes(n-i)}[-q-i] \to R\pi_* R^q\alpha_*\mu_{2^\nu}^{\otimes n}[-q]$$

or on

$$\coprod_{i=0}^{r} R^{q-i}\alpha_*\mu_{2^\nu}^{\otimes(n-i)}[-i] \to R\pi_* R^q\alpha_*\mu_{2^\nu}^{\otimes n}. \tag{5.6}$$

Note that the morphism in (5.6) is induced by the powers of the image of ξ in $H^1_{\mathrm{Zar}}(\mathbb{P}(E), R^1\alpha_*\mu_{2^\nu})$.

We now want to mimick the proof of [23, Lemma 8.11]. For this, it is convenient to introduce cycle homology groups as in [50]. Recall that, for any $i \in \mathbf{Z}$, the functor on fields

$$k \mapsto (H^q(k, \mu_{2^\nu}^{\otimes(q+i)}))_{q\geq 0}$$

is a cycle module over fields of characteristic $\neq 2$ in the sense of Rost (*loc. cit.*, Remark 2.5). For any $\mathbf{Z}[1/2]$-scheme X, we define

$$A_p(X, H^q(\mu_{2^\nu}^{\otimes n}))$$

as the p-th homology group of the complex

$$\cdots \to \coprod_{x\in X_{(p)}} H^{p+q}(k(x), \mu_{2^\nu}^{\otimes(p+n)}) \to \cdots \to \coprod_{x\in X_{(0)}} H^q(k(x), \mu_{2^\nu}^{\otimes n}) \to 0$$

defined by Kato [39]. Here $X_{(p)}$ denotes the set of points of dimension p in X. (If $q < 0$, the terms of this complex are of course 0 as soon as $p + q < 0$.) If X is equidimensional of dimension d, we set

$$A^p(X, H^q(\mu_{2^\nu}^{\otimes n})) = A_{d-p}(X, H^{d+q}(\mu_{2^\nu}^{\otimes d+n}))$$

and extend this definition by additivity if X is a disjoint union of equidimensional schemes.

If X is smooth over a field (*resp.* a discrete valuation ring), the Bloch-Ogus theorem (*resp.* the Bloch-Ogus-Gillet theorem of [24]) gives an isomorphism

$$A^p(X, H^q(\mu_{2^\nu}^{\otimes n})) \simeq H^p_{\mathrm{Zar}}(X, R^q\alpha_*\mu_{2^\nu}^{\otimes n}). \tag{5.7}$$

The interest of the cycle homology groups is that they tautologically satisfy a localisation theorem: if Z is a closed subset of X, with complementary open set U, there are long exact sequences [50, p. 356]

$$\begin{aligned}\cdots \to A_p(Z, H^q(\mu_{2^\nu}^{\otimes n})) \to A_p(X, H^q(\mu_{2^\nu}^{\otimes n})) \to A_p(U, H^q(\mu_{2^\nu}^{\otimes n}))\\ \to A_{p-1}(Z, H^q(\mu_{2^\nu}^{\otimes n})) \to \dots\end{aligned}$$

In particular, if X and Z are regular and Z is purely of codimension c in X, we get Gysin exact sequences

(5.8)

$$\begin{aligned}\cdots \to A^{p-c}(Z, H^{q-c}(\mu_{2^\nu}^{\otimes(n-c)})) \to A^p(X, H^q(\mu_{2^\nu}^{\otimes n})) \to A^p(U, H^q(\mu_{2^\nu}^{\otimes n}))\\ \to A^{p-c+1}(Z, H^{q-c}(\mu_{2^\nu}^{\otimes(n-c)})) \to \dots\end{aligned}$$

This allows us to use Noetherian induction. For example

5.1. **Lemma.** *For any Noetherian scheme X of finite Krull dimension and any $r \geq 0$, the map $[\mathcal{O}, f, r]$ of* [50, 3.5.3]*:*

$$A_p(X, H^q(\mu_{2^\nu}^{\otimes n})) \to A_{p+r}(\mathbb{A}^r_X, H^{q+r}(\mu_{2^\nu}^{\otimes(n+r)}))$$

is an isomorphism, where $\mathcal{O} = \mathcal{O}_{\mathbb{A}^r_X}$ and f is the projection $\mathbb{A}^r_X \to X$.

Proof. We do as in [23, proof of th. 8.3], with a slight simplification. By [50, prop. 4.1 (2)], we first reduce to $r = 1$. As in [23], we then reduce by Noetherian induction to the case where X is the spectrum of a field F. We now have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A^1(\mathbb{A}^1_F, H^{q-1}(\mu_{2^\nu}^{\otimes n})) \arrow[r] & H^q_{\text{ét}}(\mathbb{A}^1_F, \mu_{2^\nu}^{\otimes n}) \arrow[r] & A^0(\mathbb{A}^1_F, H^q(\mu_{2^\nu}^{\otimes n})) \arrow[r] & 0 \\ & & H^q_{\text{ét}}(F, \mu_{2^\nu}^{\otimes n}). \arrow[u, "\wr"] \arrow[ur] & & \end{tikzcd}$$

Here the top row is exact by the coniveau spectral sequence (which is also used below), while the vertical map is an isomorphism by homotopy invariance. On the other hand, the diagonal map is split injective, by (5.7) and the choice of a rational point of $\mathbb{A}^1_F$. It follows that it is bijective and that $A^1(\mathbb{A}^1_F, H^{q-1}(\mu_{2^\nu}^{\otimes n})) = 0$. □

To proceed, we need products

$$A^p(Y, H^q(\mu_{2^\nu}^{\otimes n})) \otimes H^{p'}_{\text{Zar}}(Y, R^{q'}\alpha_*\mu_{2^\nu}^{\otimes n'}) \to A^{p+p'}(Y, H^{q+q'}(\mu_{2^\nu}^{\otimes(n+n')}))$$

for Y regular over R. Such products are constructed in [50, §14] when R is a field, in view of Gersten's conjecture. When R is a discrete valuation ring, the only way I found to construct them is to mimick the arguments of [23, p. 281] and use Gabber's (unpublished) absolute cohomological purity theorem. More precisely, for any equidimensional

Noetherian scheme Y of finite Krull dimension, we have a coniveau spectral sequence [9, §1]

$$E_1^{p,q}(n) = \coprod_{x\in Y^{(p)}} H_x^{p+q}(Y,\mu_{2^\nu}^{\otimes n}) \Rightarrow H_{\text{ét}}^{p+q}(Y,\mu_{2^\nu}^{\otimes n}).$$

The construction of this spectral sequence shows that there are products

$$E_r^{p,q}(n)\otimes H^s_{\text{ét}}(Y,\mu_{2^\nu}^{\otimes t}) \to E_r^{p,q+s}(n+t)$$

cf. [23, p. 276]. Sheafifying these pairings for $r=1$ yields pairings of sheaves, hence on hypercohomology:

$$E_2^{p,q}(n)\otimes H^{p'}_{\text{Zar}}(Y,R^{q'}\alpha_*\mu_{2^\nu}^{\otimes n'}) \to E_2^{p+p',q+q'}(n+n').$$

Now, using Gabber's theorem

$$H^p_Z(X,\mu_n^{\otimes i})\simeq H^{p-2c}(Z,\mu_n^{\otimes(i-c)})$$

for a regular pair (X,Z) of codimension c, we can identify $E_2^{p,q}(n)$ with $A^p(Y,H^q(\mu_{2^\nu}^{\otimes n}))$ when Y is regular (use [EGA4, cor. 6.12.6] again to note that any closed integral subscheme of Y has a dense open regular subset). This gives the desired products.

For any regular scheme Y, we have the Leray spectral sequence

$$E_2^{p,q} = H^p_{\text{Zar}}(Y,R^q\alpha_*\mu_{2^\nu}) \Rightarrow H^{p+q}_{\text{ét}}(Y,\mu_{2^\nu}).$$

In this spectral sequence, $E_2^{2,0}=H^2_{\text{Zar}}(Y,\mu_{2^\nu})=0$, hence there is a canonical morphism

$$H^2_{\text{ét}}(Y,\mu_{2^\nu})\to H^1_{\text{Zar}}(Y,\mathcal{H}^1_{\text{ét}}(\mu_{2^\nu}))$$

hence any line bundle on Y has a class in the left hand side. For X regular and E a vector bundle of rank r, we define homomorphisms

$$\coprod_{i=0}^{r} A^{p-i}(X,H^{q-i}(\mu_{2^\nu}^{\otimes(n-i)}))\to A^p(\mathbb{P}(E),H^q(\mu_{2^\nu}^{\otimes n})) \tag{5.9}$$

by cupping with the powers of the class $\xi\in H^1_{\text{Zar}}(\mathbb{P}(E),\mathcal{H}^1_{\text{ét}}(\mu_{2^\nu}))$ of $\mathcal{O}(1)$. When X is smooth over a field or a discrete valuation ring, these homomorphisms are compatible with those of (5.6) via (5.7). The fact that (5.6) is an isomorphism will therefore follow from

5.2. **Proposition.** *For any regular X of finite Krull dimension and any (p,q,n,ν), (5.9) is an isomorphism.*

Proof. This is a variant of the proof of [23, Lemma 8.11]. Using Noetherian induction (this time with the help of (5.8)), we again reduce to the case where X is the spectrum of a field F. Then the vector bundle E is trivial. We can now finish as in *loc. cit.*, using Lemma 5.1 to argue by induction on r (see also [15, prop. 3.7]). $\square$

5.1.2. *Higher Chern classes.* To define the $c_{i,j}$, we could proceed directly along the lines of [23], using the Hurewicz homomorphism with finite coefficients as in [52]; however, for our purposes, we are forced to do something slightly more complicated.

As in [23, p. 225], we can interpret (5.2) as a map of simplicial sheaves over X_{Zar}

$$C_i : B\mathcal{GL}(\mathcal{O}_X) \to \mathcal{K}(B/2^\nu(i), 2i)$$

where the right hand side is a sheaf of generalised Eilenberg-Mac Lane spaces (the Dold-Kan construction). Applying the functor $\mathbf{Z}_\infty$ of Bousfield-Kan, noting that up to homotopy this transforms the left hand side into $\Omega B\mathcal{QP}(\mathcal{O}_X)$ (the sheaf associated to Quillen's Q-construction) and does not change the right hand side, we get, up to zig-zags of weak equivalences, a new map of simplicial sheaves, that we still denote by C_i:

$$C_i : \Omega B\mathcal{QP}(\mathcal{O}_X) \to \mathcal{K}(B/2^\nu(i), 2i). \tag{5.10}$$

For any $k \geq 2$, let $P^k(2^\nu)$ be a Moore space of level k (denoted by Y^k in [52, p. 259]), and let $M(2^\nu)$ be the Moore spectrum for $\mathbf{Z}/2^\nu$: we have $\Sigma P^k(2^\nu) = P^{k+1}(2^\nu)$ and $M(2^\nu) = \Sigma^{\infty-k+1} P^k(2^\nu)$ for any $k \geq 2$, see A.4.

We shall show that (5.10) refines into a *collection of maps*, for $2 \leq k \leq 2i$:

$$C_i^{(k)} : Hom(P^k(2^\nu), \Omega B\mathcal{QP}(\mathcal{O}_X)) \to \mathcal{K}(B/2^\nu(i), 2i-k). \tag{5.11}$$

For this, let $H(C)$ denote the Eilenberg-Mac Lane spectrum associated to a complex of abelian groups C; for $C = \mathbf{Z}/m[0]$, let us abbreviate this by $H(m)$. If $\mathcal{C}$ is a complex of sheaves of abelian groups over X_{Zar}, we denote by $\mathcal{H}(\mathcal{C})$ the corresponding sheaf of Eilenberg-Mac Lane spectra. We note that there is a canonical weak equivalence

$$H(\mathbf{Z}) \wedge M(2^\nu) \simeq H(2^\nu).$$

From the unit morphism $\Sigma^\infty \to H(\mathbf{Z})$ and this equivalence, we therefore deduce a morphism (mod 2^ν Hurewicz morphism):

$$M(2^\nu) \xrightarrow{hu} H(2^\nu).$$

Let $\mathcal{I}$ be an injective $\mathbf{Z}/2^\nu$-resolution of $B/2^\nu(i)$. Denote by $\mathcal{X}$ the simplicial sheaf $\Omega B\mathcal{QP}(\mathcal{O}_X)$. By adjunction in the Dold-Kan equivalence of categories (*e.g.* [59, 5.32]), we can interpret C_i as a homotopy class of maps

$$C_i : C_*(\mathcal{X}) \otimes \mathbf{Z}/2^\nu \to \mathcal{I}[2i]$$

where C_* denotes the standard chain complex. We have a formal adjunction computation, where H (*resp.* SH) denotes the homotopy category (*resp.* the stable homotopy category) and the Homs are internal Homs:

$$\begin{aligned}
&\Sigma Hom_{SH}(M(2^\nu), \Sigma^\infty \mathcal{X}) = \Sigma Hom_{SH}(\Sigma^{\infty-k+1} P^k(2^\nu), \Sigma^\infty \mathcal{X})\\
&\simeq \Sigma Hom_{SH}(\Sigma^\infty P^k(2^\nu), \Sigma^{\infty+k-1}\mathcal{X}) \simeq \Sigma^k Hom_{SH}(\Sigma^\infty P^k(2^\nu), \Sigma^\infty \mathcal{X})\\
&\qquad\qquad\simeq \Sigma^{\infty+k} Hom_H(P^k(2^\nu), \Omega^\infty\Sigma^\infty \mathcal{X})
\end{aligned}$$

from which we deduce a chain of maps, for any $k \in [2, 2i]$:

$$\begin{aligned}
&[C_*(\mathcal{X}) \otimes \mathbf{Z}/2^\nu, \mathcal{I}[2i]] \simeq [\mathcal{X} \wedge H(2^\nu), \mathcal{H}(\mathcal{I}[2i])]\\
&\xrightarrow{hu^*} [\mathcal{X} \wedge M(2^\nu), \mathcal{H}(\mathcal{I}[2i])] \xrightarrow{\delta} [\Sigma Hom_{SH}(M(2^\nu), \Sigma^\infty \mathcal{X}), \mathcal{H}(\mathcal{I}[2i])]\\
&\qquad \simeq [\Sigma^{\infty+k} Hom_H(P^k(2^\nu), \Omega^\infty\Sigma^\infty \mathcal{X}), \mathcal{H}(\mathcal{I}[2i])]\\
&\qquad \simeq [Hom_H(P^k(2^\nu), \Omega^\infty\Sigma^\infty \mathcal{X}), \mathcal{K}(B/2^\nu(i), 2i-k)]\\
&\qquad\qquad \xrightarrow{\theta} [Hom_H(P^k(2^\nu), \mathcal{X}), \mathcal{K}(B/2^\nu(i), 2i-k)].
\end{aligned}$$

Here hu^* is induced by the mod 2^ν Hurewicz homomorphism defined above, δ (an isomorphism) comes from one of the (fixed) two "good" S-duality isomorphisms $M(2^\nu)^\vee \simeq \Sigma^{-1}M(2^\nu)$ of A.2 and θ is induced by the stabilisation map $\mathcal{X} \to \Omega^\infty\Sigma^\infty\mathcal{X}$. The map $C_i^{(k)}$ is the image of C_i under this chain.

For $2 < k \le 2i$, there are tautological weak equivalences

$$\begin{aligned}
Hom(P^k(2^\nu), \mathcal{X}) &\xrightarrow{\sim} \Omega Hom(P^{k-1}(2^\nu), \mathcal{X})\\
\mathcal{K}(B/2^\nu(i), 2i-k) &\xrightarrow{\sim} \Omega\mathcal{K}(B/2^\nu(i), 2i-k+1)
\end{aligned}$$

and it is clear, by construction, that the diagram

$$\begin{array}{ccc}
Hom(P^k(2^\nu), \mathcal{X}) & \xrightarrow{\sim} & \Omega Hom(P^{k-1}(2^\nu), \mathcal{X})\\
{\scriptstyle C_i^{(k)}}\downarrow & & \downarrow{\scriptstyle \Omega C_i^{(k-1)}}\\
\mathcal{K}(B/2^\nu(i), 2i-k) & \xrightarrow{\sim} & \Omega\mathcal{K}(B/2^\nu(i), 2i-k+1)
\end{array} \tag{5.12}$$

commutes.

Taking homotopy groups of global sections, we get composite maps

$$\begin{aligned}
c_{i,j}^{(k)} &: \pi_j(Hom(P^k(2^\nu), \Omega BQ\mathcal{P}(X)))\\
&\qquad \to \pi_j(\mathbb{H}^{\cdot}_{\mathrm{Zar}}(X, Hom(P^k(2^\nu), \Omega BQ\mathcal{P}(\mathcal{O}_X)))\\
&\qquad\qquad \to \pi_j(\mathbb{H}^{\cdot}_{\mathrm{Zar}}(X, \mathcal{K}(B/2^\nu(i), 2i-k)))
\end{aligned}$$

or

$$c_{i,j}^{(k)} : K_{j+k}(X, \mathbf{Z}/2^\nu) \to H^{2i-j-k}_{\mathrm{Zar}}(X, B/2^\nu(i))$$

for $k \geq 2$ and $j+k \leq 2i$. Here we denote by $\mathbb{H}_{\mathrm{Zar}}(X, \mathcal{Y})$ the global sections of a functorial simplicial cofibrant or flasque resolution of a simplicial sheaf $\mathcal{Y}$, lifting the functor $R\Gamma(X, \mathcal{Y})$ of [8]. The most convenient is to use the Godement resolution of $\mathcal{Y}$, as in [59, def. 1.33], where this is done for a (pre)sheaf of fibrant spectra. Diagram (5.12) shows that

$$c_{i,j}^{(k)} = c_{i,j+1}^{(k-1)} \quad \text{for } j+k \leq 2i \text{ and } 2 < k \leq 2i.$$

5.3. **Definition.** We denote by $c_{i,j} : K_{2i-j}(X, \mathbf{Z}/2^\nu) \to H^j_{\mathrm{Zar}}(X, B/2^\nu(i))$ the homomorphism $c_{i,l}^{(k)}$ for any pair (l,k) such that $l+k = 2i-j$ and $2 \leq k \leq 2i$.

This homomorphism is therefore defined only for $2i-j \geq 2$. For $2i-j=1$, Gillet's Chern classes from integral K-theory exist and are group homomorphisms; they therefore induce homomorphisms

$$c_{i,2i-1} : K_1(X)/2^\nu \to H^{2i-1}_{\mathrm{Zar}}(X, B/2^\nu(i)). \tag{5.13}$$

Finally, for $2i-j=0$, the Chern classes defined in A) induce non-additive maps

$$c_i : K_0(X) \to H^{2i}_{\mathrm{Zar}}(X, B/2^\nu(i)).$$

It is not true in general that the c_i factor through $K_0(X)/2^\nu$ (compare [52, II.2.3]). For example, if k is a field, $X = \mathbb{P}^n_k$ and $x = [\mathcal{O}(1)] \in K_0(X)$, then $c_{2^\nu}(2^\nu x) = c_1(x)^{2^\nu}$, which is nonzero for $n > 2^\nu$. Using Newton polynomials in the c_i rather than the c_i themselves, we get however homomorphisms $s_i : K_0(X)/2^\nu \to H^{2i}_{\mathrm{Zar}}(X, B/2^\nu(i))$. Note that, if X is smooth over a field, the latter group coincides with $CH^i(X)/2^\nu$ by the Bloch-Ogus theorem.

5.2. **Relationship with Soulé's étale Chern classes.** Pushing truncated étale cohomology to étale cohomology, we get composite classes

$$c_{i,j}^{\text{ét}} : K_{2i-j}(X, \mathbf{Z}/2^\nu) \to H^j_{\mathrm{Zar}}(X, B/2^\nu(i)) \to H^j_{\text{ét}}(X, \mu_{2^\nu}^{\otimes i}).$$

5.4. **Lemma.** *If X is affine, $c_{i,j}^{\text{ét}}$ coincide with Soulé's étale Chern class $\bar{c}_{i,j}$* [52, II.2.3].

Proof. Soulé's Chern classes are defined via the mod 2^ν Hurewicz homomorphisms

$$K_i(A, \mathbf{Z}/2^\nu) \to H_i(GL(A), \mathbf{Z}/2^\nu)$$

which are themselves defined by means of the canonical generators of $H_{i-1}(P^i(2^\nu), \mathbf{Z}/2^\nu)$. Our version of the mod 2^ν Hurewicz homomorphisms rather uses the canonical generators of $H^{i-1}(P^i(2^\nu), \mathbf{Z}/2^\nu)$ and S-duality. It is easy to check that these two definitions are compatible. $\square$

If $X = \operatorname{Spec} A$ for A a local ring, the group $H^i(X_{\mathrm{Zar}}, B/2^\nu(n))$ reduces to

- $H^i_{\text{ét}}(X, \mu_{2^\nu}^{\otimes n})$ for $i \le n$
- 0 for $i > n$.

5.5. **Lemma.** *Let A be a local algebra over $\mathbf{Z}[1/2]$, and B/A a finite étale extension. Then, for all i, j,*

$$ic_{i,j} \circ f_* = i f_* \circ c_{i,j}$$

where f is the morphism $\operatorname{Spec} B \to \operatorname{Spec} A$.

Proof. We proceed in 4 steps, as usual:

I) A is a field. We may assume B is a field as well. The claim follows from Lemma 5.4 and [52, proof of th. 2 ii)].

II) A is a local ring of a scheme smooth over a Dedekind domain. We reduce to I) by Gillet's Bloch-Ogus theorem, which implies that the étale cohomology of A injects into that of its field of fractions.

III) A is a local ring of a scheme of finite type over a Dedekind domain. We use Hoobler's henselian couple trick [29], noting that étale coverings of $\operatorname{Spec} A$ lift, plus Gabber's rigidity result [18].

IV) The general case. Passage to the limit. □

6. Proof of Theorem 3

6.1. **Effect of the Chern classes on hypercohomology.** We now examine the effect of applying total global sections to (5.11) in more detail. For any sheaf $\mathcal{Y}$ of simplicial sets on X, there is by [8, Th. 3 and Remark p. 285] a "fringed" spectral sequence

$$E_2^{p,q} = H^p_{\mathrm{Zar}}(X, \pi_{-q}(\mathcal{Y})) \Rightarrow \pi_{-p-q}(\mathbb{H}^{\cdot}_{\mathrm{Zar}}(X, \mathcal{Y}))$$

with fringe effect concentrated on the line $p + q = 0$.

By functoriality of $\mathbb{H}^{\cdot}$, any morphism of simplicial sheaves induces a corresponding morphism of spectral sequences. Applying this to $C_i^{(k)}$, we get a morphism from the spectral sequence

$$(6.1) \quad A_2^{p,q} = H^p_{\mathrm{Zar}}(X, \mathcal{K}_{k-q}(\mathbf{Z}/2^\nu)) \Rightarrow \pi_{-p-q}(\mathbb{H}^{\cdot}_{\mathrm{Zar}}(X, \Omega B\mathcal{QP}(\mathcal{O}_X)))$$

to the spectral sequence

$$(6.2) \quad B_2^{p,q}(i) = H^p_{\mathrm{Zar}}(X, \mathcal{H}^{2i+q-k}(B/2^\nu(i))) \Rightarrow H^{2i+p+q-k}_{\mathrm{Zar}}(X, B/2^\nu(i)).$$

The corresponding morphism $A_2^{p,q} \to B_2^{p,q}$ is $H^p_{\mathrm{Zar}}(X, c_{i,-q}^{(k)}) = H^p_{\mathrm{Zar}}(X, c_{i,2i+q-k})$ (Definition 5.3).

For both spectral sequences, the simplicial sheaf in question is an infinite loop space. One could therefore use the technique of [8, §3, Remark 1] to deloop them and get rid of the fringe effect. However,

$C_i^{(k)}$ is not an infinite loop map, hence does not yield a morphism of the corresponding delooped spectral sequences. In other words, we cannot get rid of the fringe effects as long as we work with the $C_i^{(k)}$.

6.1. **Lemma.** *Assume X quasi-compact and quasi-separated. Then a) The Chern classes of Definition 5.3 extend to Chern classes from Thomason-Trobaugh K-theory*

$$c_{i,j} : K^{TT}_{2i-j}(X, \mathbf{Z}/2^\nu) \to H^j_{\mathrm{Zar}}(X, B/2^\nu(i)).$$

b) The spectral sequence (6.1) *"abuts" to $K^{TT}_{k-p-q}(X, \mathbf{Z}/2^\nu)$ (with fringe effect, of course).*

Proof. Consider the morphism of simplicial sheaves

$$\Omega B\mathcal{G}\mathcal{Q}(\mathcal{O}_X) \to \Omega^\infty \mathcal{K}^{TT}(\mathcal{O}_X) \xrightarrow{\approx} \Omega^\infty \mathcal{K}^B(\mathcal{O}_X)$$

stemming from (4.3). Note that the right map is a weak equivalence by [62, th. 7.5 a)]. Taking total global sections, we get a commutative diagram of simplicial sets

$$\begin{array}{ccccc}
\Omega BGQ(X) & \longrightarrow & \Omega^\infty K^{TT}(X) & \xrightarrow{\approx} & \Omega^\infty K^B(X) \\
\downarrow & & \downarrow & & \approx\downarrow \\
\mathbb{H}^{\cdot}_{\mathrm{Zar}}(X, \Omega B\mathcal{G}\mathcal{Q}(\mathcal{O}_X)) & \xrightarrow{\approx} & \mathbb{H}^{\cdot}_{\mathrm{Zar}}(X, \Omega^\infty \mathcal{K}^{TT}(\mathcal{O}_X)) & \xrightarrow{\approx} & \mathbb{H}^{\cdot}_{\mathrm{Zar}}(X, \Omega^\infty \mathcal{K}^B(\mathcal{O}_X))
\end{array}$$

where the right vertical map is a weak equivalence by the Thomason-Trobaugh descent theorem [62, th. 10.3] (or rather its analogue for the Zariski topology) and the fact that homotopy limits commute with desuspension (compare [59, (5.3)]), the top and bottom right horizontal maps are weak equivalences again by [62, th. 7.5 a)] and the bottom left horizontal map is a weak equivalence as well by [62, prop. 3.10]. Hence there is a weak equivalence

$$\Omega^\infty K^{TT}(X) \approx \mathbb{H}^{\cdot}_{\mathrm{Zar}}(X, \Omega B\mathcal{G}\mathcal{Q}(\mathcal{O}_X))$$

and a weak equivalence

$$Hom(P^k(2^\nu), \Omega^\infty K^{TT}(X)) \approx \mathbb{H}^{\cdot}_{\mathrm{Zar}}(X, Hom(P^k(2^\nu), \Omega B\mathcal{G}\mathcal{Q}(\mathcal{O}_X)))$$

for $k \geq 2$. $\square$

6.2. Proof of Theorem 3.

6.2. **Proposition.** *Let A be a non-exceptional local $\mathbf{Z}[1/2]$-algebra. Let $(i,j), (i',j')$ be such that $j \leq i$, $j' \leq i'$ and $2i-j = 2i'-j' \geq 2$. Suppose*

$\nu \geq 2$. *Then the composite*

$$H^j_{\mathrm{Zar}}(A, B/2^\nu(i)) = H^j_{\text{ét}}(A, \mu_{2^\nu}^{\otimes i}) \xrightarrow{\beta^{i,j}} K_{2i-j}(A, \mathbf{Z}/2^\nu)$$
$$\xrightarrow{i' c_{i',j'}} H^{j'}_{\mathrm{Zar}}(A, B/2^\nu(i'))$$

is 0 *if* $(i',j') \neq (i,j)$ *and is multiplication by* $(-1)^{i-1} i!$ *if* $(i,j) = (i',j')$. *Here* $\beta^{i,j}$ *is the* (i,j)*-th component of the isomorphism* B_A^{2i-j} *of Theorem 1.*

Proof. By the proof of Theorem 1 b), the map u_A^{2i-j} of diagram (3.2) is surjective; hence the étale cohomology ring $H^*_{\text{ét}}(A, \mu_{2^\nu}^{\otimes *})$ is generated by $H^0_{\text{ét}}(A, \mu_{2^\nu})$ and $H^1_{\text{ét}}(A, \mu_{2^\nu})$ up to transfer. By construction, the $\beta^{i,j}$ commute with product and transfer. By Lemma 5.5, $i' c_{i',j'}$ commutes with transfer and, by [66, th. 3.2 (iv)], Soulé's product formula [52, th. 1]

$$c_{i',j'}(x \cdot y) = \sum_{\substack{i_1 + i_2 = i' \\ i_1 \geq m/2, i_2 \geq n/2}} -\frac{(i'-1)!}{(i_1-1)!(i_2-1)!} c_{i_1,j_1}(x) \cdot c_{i_2,j_2}(y)$$

holds for any $(x,y) \in K_m(A, \mathbf{Z}/2^\nu) \times K_n(A, \mathbf{Z}/2^\nu)$ with $m + n = 2i - j$ since A is local (in the formula, $j_1 = 2i_1 - m$, $j_2 = 2i_2 - n$). Here, if m or n is 1, we use the Chern class of (5.13), noting that $K_1(A)/2^\nu \xrightarrow{\sim} K_1(A, \mathbf{Z}/2^\nu)$. Thus we are reduced to the cases $(i,j) = (1,0)$ and $(i,j) = (1,1)$, which are trivial. $\square$

6.3. **Corollary.** *Let* A, ν *be as in Proposition 6.2. Then, for all* $n \geq 2$, *the kernel and cokernel of the map*

$$K_n(A, \mathbf{Z}/2^\nu) \xrightarrow{(i c_{i,2i-n})} \prod_{i \geq 1} H^{2i-n}_{\mathrm{Zar}}(A, B/2^\nu(i))$$

are killed by $n!$.

Proof. This follows from Theorem 1 a) and Proposition 6.2. $\square$

6.4. **Corollary.** *For any* X *and any* $p \geq 0$, $n \geq 2$, *the map*

$$H^p_{\mathrm{Zar}}(X, \prod_{i \geq 1} i c_{i,2i-n})$$

has kernel and cokernel killed by $n!$. $\square$

Proof of Theorem 3. We may assume that -1 is a square on X; the general case follows from this one by a transfer argument. Consider the morphism of spectral sequences from (6.1) to the direct product of all (6.2) ($k = 2, i \geq 1$) induced by all $iC_i^{(2)}$, $i \geq 1$[6]. Note that, in this direct product, each $E_r^{p,q}$-term actually contains only a finite number of nonzero factors, and the same is true for the "abutment". Therefore, this direct product operation does not make convergence problems worse. By Corollary 6.4, the induced map

$$A_2^{p,q} \to \prod_{i\geq 1} B_2^{p,q}(i)$$

has kernel and cokernel killed by $(2-q)!$. The conclusion for $K_n^{TT}(X, \mathbf{Z}/2^\nu)$, $n \geq 3$ now follows from the following lemma, which is an easy consequence of computing two hypercohomology spectral sequences:

6.5. **Lemma.** *Let $C^\cdot \to D^\cdot$ be a morphism of complexes. Assume that, for some q, the map $C^i \to D^i$ has kernel killed by m_i and cokernel killed by n_i for $i \in \{q-1, q, q+1\}$, where m_i, n_i are nonzero integers, except that n_{q+1} is possibly 0. Then the map $H^q(C^\cdot) \to H^q(D^\cdot)$ has kernel killed by $m_q n_{q-1}$ and cokernel killed by $m_{q+1} n_q$.* □

Note that this lemma takes care of the fringe effect as, by definition, $A_{r+1}^{p,-p}$ is a subgroup of $H^{p,-p}(A_r)$, and similarly for the $B_r(n)$.

If X is smooth over a field or a discrete valuation ring, then $H^p_{\mathrm{Zar}}(X, \mathcal{H}^q(B/2^\nu(n))) = 0$ for $p > q$ by Gersten's conjecture, hence the fringe effect disappears (compare [8, Th. 3 and Remark 1 p. 290]) and we can extend the conclusion to $n = 2$. □

Appendix A. Moore spectra and S-duality

A.1. Let $\mathbb{S}$ denote the sphere spectrum. Let $M(n)$ be the mod n Moore spectrum, defined as the homotopy cofibre of multiplication by n on $\mathbb{S}$, so that we have a homotopy fibre sequence (in the stable homotopy category)

$$\mathbb{S} \xrightarrow{\rho_n} M(n) \xrightarrow{\delta_n} \Sigma\mathbb{S} \xrightarrow{n} \Sigma\mathbb{S}.$$

Here we take up the same notation as in [34, Appendix B].

A.2. **S-duality.** Taking the S-dual of this sequence, we get a (non-canonical) isomorphism in the stable homotopy category:

$$\varphi : M(n)^\vee \simeq \Sigma^{-1} M(n)$$

[6]As all simplicial sheaves are sheaves of H-spaces, it makes sense to multiply $C_i^{(2)}$ by an integer.

such that the following diagram of exact triangles commutes:

$$\begin{CD}
\Sigma^{-1}\mathbb{S} @>{\delta_n^\vee}>> M(n)^\vee @>{\rho_n^\vee}>> \mathbb{S} @>{n}>> \mathbb{S}\\
@V{||}VV @V{\varphi}VV @V{-1}VV @V{||}VV\\
\Sigma^{-1}\mathbb{S} @>{-\Sigma^{-1}\rho_n}>> \Sigma^{-1}M(n) @>{-\Sigma^{-1}\delta_n}>> \mathbb{S} @>{-n}>> \mathbb{S}
\end{CD}$$

(recall that shifting exact triangles in an triangulated category changes signs; on the other hand, the sign in the S-duality is $+$).

An easy double dual computation shows that, if φ is such an isomorphism, then $-\Sigma^{-1}\varphi^\vee$ is another one.

Two choices of φ differ by an element of the form $\rho_n^\vee \circ \lambda$ for $\lambda \in [\mathbb{S}, \Sigma^{-1}M(n)] = \pi_1(\mathbb{S}, \mathbf{Z}/n)$. This group is 0 for n odd: in this case, φ is unique and $\varphi = -\Sigma^{-1}\varphi^\vee$. If $n = 2^r$ with $r \geq 2$, then $\varphi + \Sigma^{-1}\varphi^\vee = \rho_n^\vee \circ \xi$ for $\xi \in \pi_1(\mathbb{S}, \mathbf{Z}/n) \simeq \mathbf{Z}/2$. If $\varphi_1 = \varphi + \rho_n^\vee \circ \lambda$ for λ as above, then $\rho_n^\vee \circ \lambda$ and $\Sigma^{-1}\lambda^\vee \circ \rho_n$ define two elements of order ≤ 2 in $[M(n)^\vee, \Sigma^{-1}M(n)] \simeq End(M(n)) \simeq \mathbf{Z}/n$. Since they are obviously zero or nonzero together, their sum is always zero, which shows that we also have $\varphi_1 + \Sigma^{-1}\varphi_1^\vee = \rho_n^\vee \circ \xi$. So $\rho_n^\vee \circ \xi$, and hence ξ, does not depend on the choice of φ.

For $r = 1$, the same argument works since $End(M(2)) \simeq \mathbf{Z}/4$ is still cyclic. So ξ is still well-defined in this case. I don't know whether this element is 0 or not.

A.3. For E a spectrum, we write $\pi_i(E, \mathbf{Z}/n) := \pi_i(E \wedge M(n))$: this is the *homotopy of E with coefficients* $\mathbf{Z}/n$. By S-duality (see A.2), this is also $[\Sigma^{i-1}M(n), E]$. If n is odd, there is exactly one choice for this identification to be compatible with ρ_n and δ_n in a suitable sense; if n is even there are two such choices.

A.4. To define homotopy of a space with coefficients $\mathbf{Z}/n$, we need an unstable version of $M(n)$. For $i \geq 2$, let $P^i(n)$ denote a Moore space of level i, so that there is a homotopy cofibre sequence $S^{i-1} \xrightarrow{n} S^{i-1} \longrightarrow P^i(n)$: we have $\Sigma P^i(n) \simeq P^{i+1}(n)$ and $M(n) \simeq \Sigma^{\infty-i+1}P^i(n)$ for any $i \geq 2$. For a pointed space X, we then define $\pi_i(X, \mathbf{Z}/n)$ as $[P^i(n), X]$ for $i \geq 2$ (we won't need to discuss a definition for $i = 0, 1$ here). If $X = \Omega^\infty E$ for some spectrum E, then $\pi_i(X, \mathbf{Z}/n) = \pi_i(E, \mathbf{Z}/n)$ by A.3.

Appendix B. The Bott element construction

B.1. We shall need the following assumption on a unital ring spectrum E:

B.2. **Assumption.** *$(\rho_E)_*\eta^2 = 0$ in $\pi_2(E)/2$, where $\eta \in \pi_1^S$ is the Hopf map and $\rho_E : \mathbb{S} \to E$ is the unit map of E.*

If $E = KX$ for a scheme X, the assumption is that $\{-1,-1\}$ is divisible by 2 in $K_2(X)$.

B.3. **Lemma.** *a) Assumption B.2 is satisfied when E is of the form $\mathrm{j}(2,\Delta)$, where Δ is a closed, torsion-free subgroup of $\mathbf{Z}_2^*$ and $\mathrm{j}(2,\Delta)$ is the unital ring spectrum defined in* [34, def. 4.9 and C.3].
b) Assumption B.2 is satisfied for $E = KX$ where X is a non-exceptional scheme.

Proof. In fact, we even have $\rho_*\eta^2 = 0$ in $\pi_2(E)$ in cases a) and b):

a) The group $\pi_2(\mathrm{j}(2,\Delta))$ is 0 since, if Δ is torsion-free, $\mathrm{j}(2,\Delta)$ is the Bousfield-Kan localisation at 2 of the spectrum of the algebraic K-theory of a suitable finite field, or of the algebraic closure of a finite field if $\Delta = \{1\}$ [34, prop. C.13].

b) We may first assume X connected, and then $X = \operatorname{Spec} R$, where R is a suitable (ind-)ring of algebraic integers localised away from 2 (Let $\Delta_X \subset \mathbf{Z}_2^*$ be the image of the dyadic cyclotomic character of X: consider the largest R such that the map $X \to \operatorname{Spec} \mathbf{Z}[1/2]$ factors through $\operatorname{Spec} R$, so that $\Delta_X = \Delta_{\operatorname{Spec} R}$). The following proof was kindly communicated by the referee: let $S = R[t]/(t^2+1)$. Since R is not exceptional, $\mu_{2^\infty}(S)$ is finite, say of order 2^n with $n \geq 2$, and the norm of a primitive root ζ is -1. Then

$$\begin{aligned}\{-1,-1\} = \{N(\zeta),-1\} = N(\{\zeta,-1\}) &= 2^{n-1}N(\{\zeta,\zeta\}) \\ &= 2^{n-1}N(\{-1,\zeta\}) = 0 \in K_2(R)\end{aligned}$$

hence $\{-1,-1\} = 0$ in $K_2(X)$.

Another proof is to use a) and the results of Appendix C. □

B.4. **Remark.** Assumption B.2 is verified for KX for more schemes X than indicated in Lemma B.3. Namely, if $X = \operatorname{Spec} R$ where $R = O_S$ is a ring of S-integers of a number field F such that the class group of R has odd order, then the map

$$K_2(R)/2 \xrightarrow{h_v} \coprod_{\substack{v \in S \\ v \text{ not complex}}} \mu_2$$

is injective, where h_v is the collection of 2-power norm residue symbols [58, th. 6.2]. Then $\{-1,-1\}$ is 0 in $K_2(R)/2$ if and only if $(-1,-1)_v = 0$ for any dyadic or real place v of F. This happens if and only if F is totally imaginary and, for any dyadic v, the local degree $[F_v : \mathbf{Q}_2]$ is even.

Therefore, assumption B.2 is verified for any R-scheme X, for R as above.

B.5**.** The Bott element construction is described inter alia in [59, A.7], [44, p. 827] and [34, prop. 6.3]. We recall a version of this construction:

Let $n > 1$ and X be a scheme on which n is invertible. We denote by $\mu_n(X)$ the group of n-th roots of unity in $R = \Gamma(X, \mathcal{O}_X)$. The Bott element construction

$$\beta : \mu_n(X) \to K_2(X, \mathbf{Z}/n)$$

is given by the following sequence of maps:

$$\begin{tikzcd}
\mu_n(X) = {}_n\pi_1(BR^*) \\
\pi_2(BR^*, \mathbf{Z}/n) \arrow[u, "\wr"] \arrow[r, equal] & \pi_2(BGL_1(R), \mathbf{Z}/n) \arrow[d, "\alpha"] \\
& \pi_2(BGL(R)^+, \mathbf{Z}/n) \arrow[r, equal] & K_2(R, \mathbf{Z}/n) \arrow[d] \\
& & K_2(X, \mathbf{Z}/n).
\end{tikzcd}$$

B.6**.** The spaces BR^* and $BGL(R)^+$ are both commutative H-groups, hence $\pi_2(BR^*, \mathbf{Z}/n)$ and $K_2(R, \mathbf{Z}/n)$ are both abelian groups. Moreover, the Bockstein maps

$$\pi_2(BR^*, \mathbf{Z}/n) \to {}_n\pi_1(BR^*)$$
$$K_2(R, \mathbf{Z}/n) \to {}_nK_1(R)$$

are group homomorphisms (the first one is the isomorphism used to define the Bott element map). However, the map α is induced by the inclusion $BGL_1(R) \to BGL(R)^+$, which is a map between H-spaces but not an H-space map; therefore, it is not clear when α is a group homomorphism. This issue is tackled in none of the above references.

In fact, let $\zeta_1, \zeta_2 \in \mu_n(R)$, Then, by the above remarks, the element

$$\varphi(\zeta_1, \zeta_2) = \alpha(\zeta_1) + \alpha(\zeta_2) - \alpha(\zeta_1\zeta_2)$$

lies in $K_2(R)/n$.

B.7. **Proposition.** $\varphi(\zeta_1, \zeta_2) = \binom{n}{2}\{\zeta_1, \zeta_2\}$.

Proof. By [66, p. 254, formula before Lemma 1.4.1], the expression $\varphi(\zeta_1, \zeta_2) - \binom{n}{2}\{\zeta_1, \zeta_2\}$ lies in the kernel of the mod n Hurewicz homomorphism h_n. (Actually, the said formula is stated there only for n

even, but one sees that it also holds for n odd by reading the two previous pages.) On the other hand, h_2 is split injective by [1, cor. 3.4 (a)]. $\square$

B.8. **Corollary.** *a) If $n \not\equiv 2 \pmod 4$, β is a group homomorphism.*
b) If $n = 2$, β is quadratic; it is a group homomorphism if and only if KR verifies assumption B.2.

Proof. If n is odd, then $\binom{n}{2} \equiv 0 \pmod n$. If n is even, then $2\{\zeta_1, \zeta_2\} = 0$ for any $\zeta_1, \zeta_2 \in \mu_n(R)$ (compare [66, Lemma 1.4.1]). This takes care of a). In case b), we reduce to the case R connected and then to $n = 2$ by using a); then the result is clear since $\mu_2(R) = \{\pm 1\}$. $\square$

Appendix C. Bott elements for arbitrary non exceptional schemes

The aim of [33] was to construct a theory of Bott elements on the homotopy level for the algebraic K-theory of non-exceptional schemes. This was achieved in two special cases: for $\mathbf{Z}[1/2, i]$-schemes and for schemes over $\mathbf{F}_p$ (p odd). Using a recent result of Hinda Hamraoui [27], we shall extend the construction to arbitrary non-exceptional schemes (over $\mathbf{Z}[1/2]$).

Recall that, to any closed subgroup $\Delta \subseteq \mathbf{Z}_2^*$, we associated in [34, §§3, 4 and Appendix C] two 2-local spectra and a map between them:

$$\Sigma(2, \Delta) \xrightarrow{\ell^\Delta} \mathrm{j}(2, \Delta).$$

In case $\Delta = 1 + 2^n\mathbf{Z}_2$, $\Sigma(2, \Delta)$ is just the localisation at 2 of the suspension spectrum $\Sigma^\infty(B(\mathbf{Z}/2^n)_+)$, *cf.* [34, Remark 3.5 (1)][7], and the spectrum $\mathrm{j}(2, \Delta)$ is obtained from $\Sigma(2, \Delta)$ by inverting the Bott element and then truncating above 0, *cf.* [34, Def. 4.9]. In general, see [34, Appendix C].

Here we shall only be concerned by the case where Δ is *torsion-free.* In this case, let R^Δ be the (ind-)ring of integers of the 2-cyclotomic extension of $\mathbf{Q}$ which corresponds to Δ. Pick a residue field E of R^Δ of odd characteristic such that the image of the dyadic cyclotomic character of E in $\mathbf{Z}_2^*$ is Δ (there are infinitely many of them by the Dirichlet's theorem on the arithmetic progression). Then there is a

[7]There is an obvious misprint in this remark: the + should be in index as an added disjoint base point, not in exponent as a plus construction.

homotopy commutative diagram

$$\begin{tikzcd}
\Sigma(2,\Delta) \arrow[r, "\Phi^\Delta"] \arrow[d, "\ell^\Delta"'] \arrow[dr, "\Phi_E"] & \mathbf{L}KR^\Delta \arrow[d] \\
\mathrm{j}(2,\Delta) \arrow[r, "\beta_E", "\sim"'] & \mathbf{L}KE
\end{tikzcd}$$

where $\mathbf{L}$ denotes localisation at 2 and β_E is a weak equivalence [34, (C2) p. 1002].

The main theorem of Harris-Segal [28] is equivalent to the fact that $\Omega^\infty\Phi_E$ has homotopy sections, at least on the level of connected components of 0 [8]. Hence the same holds for the map $\Omega^\infty\ell^\Delta$.

Let s be a homotopy section of $\Omega^\infty\Phi_E$, on the level of connected components of 0. Assume that $\Delta \subseteq 1+4\mathbf{Z}_2$. By the main result of Dwyer-Friedlander-Mitchell [13], the maps $\Omega^\infty\Phi^\Delta$ and $\Omega^\infty\Phi^\Delta\circ s\circ\Omega^\infty\Phi_E$ are homotopic. By the main result of Hamraoui [27], this extends to the case where Δ is not contained in $1+4\mathbf{Z}_2$. Therefore, the same holds for a section of $\Omega_0^\infty\ell^\Delta$, the restriction of $\Omega^\infty\ell^\Delta$ to the connected components of 0.

As in [34, def. 5.4], we may now define a map

$$\bar\beta^\Delta : \Omega^\infty\mathrm{j}(2,\Delta) \to \Omega^\infty\mathbf{L}KR^\Delta$$

also in the case where $\Delta \not\subseteq 1+4\mathbf{Z}_2$. Because of the problem on π_0, we define this map separately on the connected components of 0 and on π_0. Picking a section s of $\Omega_0^\infty\ell^\Delta$, we check as in [34, th. 5.3] that the map $\Omega_0^\infty\Phi^\Delta\circ s$ does not depend on the choice of s: this is the restriction of $\bar\beta^\Delta$ to the connected components of 0. We then map $\pi_0(\mathrm{j}(2,\Delta)) = \mathbf{Z}_{(2)}$ (*cf.* [34, Prop. C.13 (a)]) to $\pi_0(\mathbf{L}KR^\Delta) = K_0(R^\Delta)\otimes\mathbf{Z}_{(2)}$ by sending 1 to 1. The map $\bar\beta^\Delta$ is now defined as the product of these two maps.

To any non-exceptional scheme X, we now associate a spectrum $\mathrm{j}_2(X)$ as the wedge of the $\mathrm{j}(2,\Delta_i)$ where the X_i are the connected components of X and Δ_i is the image of $\pi_1(X_i)$ in $\mathbf{Z}_2^*$ by the cyclotomic character. as in [34, §9], we define a map

$$\bar\beta_X : \Omega^\infty\mathrm{j}_2(X) \to \Omega^\infty\mathbf{L}KX$$

by simply pulling back the previous maps via the canonical projections $X_i \to \mathrm{Spec}(R^{\Delta_i})$. As in [34, th. 9.3], one proves that $\bar\beta_X$ commutes with base change, transfers in the semi-local case, and products when X is essentially of finite type over $\mathbf{Z}$.

[8] In case $\Delta \subseteq 1+4\mathbf{Z}_2$, this restriction is unnecessary. However, in the other case, $\mathrm{Im}\,\pi_0(\Phi_E) = 2K_0(E)\otimes\mathbf{Z}_{(2)}$, so the correct statement needs a little care.

Acknowledgements.I am indebted to the referee for a very careful reading of this paper and a number of comments which helped improve the exposition.

References

[1] D. Arlettaz *On the k-invariants of iterated loop spaces*, Proc. Royal Soc. Edinburgh **110A** (1988), 343–350.

[2] H. Bass, M. P. Murthy *Grothendieck groups and Picard groups of abelian group rings*, Ann. of Math. **86** (1967), 16–73.

[3] H. Bass, J. Tate *The Milnor ring of a global field*, Lect. Notes in Math. **342**, 349–428.

[4] A. A. Beilinson *Height pairings between algebraic cycles*, Lect. Notes in Math. **1289**, Springer, 1987, 1–26.

[5] S. Bloch *Algebraic cycles and higher K-theory*, Adv. Math. **61** (1986), 267–304.

[6] S. Bloch, S. Lichtenbaum *A spectral sequence for motivic cohomology*, preprint, 1995.

[7] S. Bloch, A. Ogus *Gersten's conjecture and the homology of schemes*, Ann. Sci. Ec. Norm. Sup., 4. sér. **7** (1974), 181–202.

[8] K. Brown, S. Gersten *Algebraic K-theory as generalized sheaf cohomology*, Lect. Notes **341**, Springer, 1973, 266–292.

[9] J.-L. Colliot-Thélène, R. T. Hoobler, B. Kahn *The Bloch-Ogus–Gabber theorem*, Fields Inst. Comm. **16**, AMS, 1997, 31–94.

[10] J.-L. Colliot-Thélène, J.-J. Sansuc, C. Soulé *Torsion dans le groupe de Chow de codimension* 2, Duke Math. J. **50** (1983), 763–801.

[11] W. Dwyer, E. Friedlander *Algebraic and étale K-theory*, Trans. A.M.S. **292** (1985), 247–280.

[12] W. Dwyer, E. Friedlander, V. Snaith, R. Thomason *Algebraic K-theory eventually surjects onto topological K-theory*, Invent. Math. **66** (1982), 481–491.

[13] W. G. Dwyer, E. M. Friedlander, S. A. Mitchell, *the generalised Burnside ring and the K-theory of ring with roots of unity*, K-theory **6** (1992), 285–300.

[14] P. Elbaz-Vincent, S. Müller-Stach *Milnor K-theory of rings, higher Chow groups and applications*, to appear in Invent. Math.

[15] H. Esnault, B. Kahn, M. Levine, E. Viehweg *The Arason invariant and mod 2 algebraic cycles*, J. A.M.S. **11** (1998), 73–118.

[16] B. Feigin, B. Tsygan *Additive K-theory*, Lect. Notes in Math. **1289**, Springer, New York, 1987, 67–209.

[17] E. Friedlander, A. Suslin *The spectral sequence relating algebraic K-theory and motivic cohomology*, preprint, 1999.

[18] O. Gabber *Affine analog of the proper base change theorem*, Israel J. Math. **87** (1994), 325–335.

[19] O. Gabber *K-theory of henselian local rings and henselian pairs*, Contemp. Math. **126**, 59–70, AMS, 1992.

[20] O. Gabber, letter to the author, February 18, 1998.

[21] S. Geller, C. Weibel *Hodge decompositions of Loday symbols in K-theory and cyclic homology*, K-theory **8** (1994), 587–632.

[22] T. Geisser. M. Levine *The Bloch-Kato conjecture and a theorem of Suslin-Voevodsky*, J. reine angew. Math. **530** (2001), 55–103.

[23] H. Gillet *Riemann-Roch theorems for higher algebraic K-theory*, Adv. Math. **40** (1981), 203–289.

[24] H. Gillet, *Bloch-Ogus for the étale cohomology of certain arithmetic schemes*, manuscript notes distributed at the 1997 Seattle algebraic K-theory conference.

[25] H. Gillet, M. Levine *The relative form of Gersten's conjecture over a discrete valuation ring: the smooth case*, J. Pure Appl. Algebra **46** (1987), 59–71.

[26] D. Grayson *Products in K-theory and intersecting algebraic cycles*, Invent. Math. **47** (1978), 71–83.

[27] H. Hamraoui *Un facteur direct canonique de la K-théorie d'anneaux d'entiers algébriques non exceptionnels*, C. R. Acad. Sci. Paris **332** (2001), 957–962.

[28] B. Harris, G. Segal *K_i of rings of algebraic integers*, Ann. of Math. **101** (1975), 20–33.

[29] R. Hoobler *The Merkuriev-Suslin theorem for arbitrary semi-local rings*, preprint, 1996.

[30] R. de Jeu *On $K_4^{(3)}$ of curves over number fields*, Invent. Math. **125** (1996), 523–556.

[31] B. Kahn *Some conjectures in the algebraic K-theory of fields, I: K-theory with coefficients and étale K-theory*, NATO ASI Series, Ser. C. **279**, Kluwer, 1989, 117–176.

[32] B. Kahn *Deux théorèmes de comparaison en cohomologie étale; applications*, Duke Math. J. **69** (1993), 137–165.

[33] B. Kahn *On the Lichtenbaum-Quillen conjecture*, NATO ASI Series, Ser. C. **407**, Kluwer, 1993, 147–166.

[34] B. Kahn *Bott elements in algebraic K-theory*, Topology **36** (1997), 963–1006.

[35] B. Kahn *Calculations in étale cohomology*, preprint, 2001.

[36] B. Kahn *On algebraic K-theory with coefficients* $\mathbf{Z}/2$, in preparation.

[37] B. Kahn *Some consequences of the Bass conjecture*, in preparation.

[38] K. Kato *A generalisation of local class field theory by using K-groups, II*, J. Fac. Sci., Univ. Tokyo **27** (1980), 603–683.

[39] K. Kato *A Hasse principle for two-dimensional global fields*, J. reine angew. Math. **366** (1986), 142–183.

[40] M. Levine *Techniques of localization in the theory of algebraic cycles*, J. Alg. Geom. **10** (2001), 299–363.

[41] M. Levine *K-theory and motivic cohomology of schemes, I*, preprint, 2001.

[42] M. Levine *K-theory and motivic cohomology of schemes, II*, in preparation.

[43] M. Levine *Simplicial topology for schemes*, preprint, 2000.

[44] S. Mitchell *Harmonic localization of algebraic K-theory spectra*, Trans. AMS **332** (1992), 823–837.

[45] Y. Nisnevich, *The completely decomposed topology on schemes and the associated descent spectral sequences in algebraic K-theory*, *in* Algebraic K-theory: connections with geometry and topology, J.F. Jardine, V.P. Snaith, eds., NATO ASI Series, Sec. C **279** (1989), 241–342.

[46] D. Quillen *Higher algebraic K-theory, I*, Lecture Notes in Mathematics **341**, Springer, Berlin, 1973, 77–139.

[47] D. Quillen *Higher algebraic K-theory*, Proc. Intern. Congress Math. (I), Vancouver, 1974, 171–176.

[48] L. Reid *N-dimensional rings with an isolated singular point having nonzero K_{-N}*, K-theory **1** (1987), 197–206.
[49] J. Rognes, C. Weibel, *Two-primary algebraic K-theory of rings of integers in number fields*, J. Amer. Math. Soc. **13** (2000), 1–54.
[50] M. Rost *Chow groups with coefficients*, Doc. Math. **1** (1996), 319–393.
[51] J.-P. Serre Cohomologie galoisienne (new edition), Lect. Notes in Math. **5**, Springer, 1994.
[52] C. Soulé *K-théorie des anneaux d'entiers de corps de nombres et cohomologie étale*, Invent. Math. **55** (1979), 251–295.
[53] R. Strano *On the étale cohomology of Hensel rings*, Comm. in Alg. **12** (1984), 2195–2211.
[54] A. Suslin *On the K-theory of local fields*, J. Pure Appl. Algebra **34** (1984), 301–318.
[55] A. Suslin *Algebraic K-theory of fields*, Proceedings of the International Congress of Mathematicians, Berkeley, 1986, 222–244.
[56] A. Suslin *Higher Chow groups and étale cohomology*, *in* Cycles, Transfers, and Motivic Cohomology Theories, Annals of Math Studies **143**, 239–254, Princeton Univ. Press, Princeton, 2000.
[57] A. Suslin, V. Voevodsky *Bloch-Kato conjecture and motivic cohomology with finite coefficients*, The arithmetic and geometry of algebraic cycles (Banff, 1998), 117–189, NATO Sci. Ser. C Math. Phys. Sci., **548**, Kluwer, 2000.
[58] J. Tate *Relations between K_2 and Galois cohomology*, Invent. Math. **36** (1976), 257–274.
[59] R. Thomason *Algebraic K-theory and étale cohomology*, Ann. Sci. Ec. Norm. Sup. 4e sér. **13** (1985), 437–452.
[60] R. Thomason *Bott stability in algebraic K-theory*, Contemp. Math. **55** (I), AMS, 1986, 389–406.
[61] R. Thomason *Survey of algebraic vs. topological K-theory*, Contemp. Math. **83**, AMS, 1989, 393–443.
[62] R. Thomason, T. Trobaugh *Higher algebraic K-theory of schemes and of derived categories*, The Grothendicek Festschrift II, Progress in Math., Birkhäuser, 1990, 247–435.
[63] V. Voevodsky *The Milnor conjecture*, preprint, 1996.
[64] C. Weibel *K-theory and analytic isomorphisms*, Invent. Math. **61** (1980), 177–197.
[65] C. Weibel Pic *is a contracted functor*, Invent. Math. **103** (1991), 351–377.
[66] C. Weibel *Etale Chern classes at the prime* 2, NATO ASI Series, Ser. C **407**, Kluwer, 1993, 249–286.
[67] C. Weibel *The negative K-theory of normal surfaces*, Duke Math. J. **108** (2001), 1–35.
[EGA4] A. Grothendieck, J. Dieudonné, Éléments de géométrie algébrique, chapitre IV: *Étude locale des schémas et des morphismes de schémas* (2ème partie), Publ. Math. IHES **24**, 1965.
[SGA4] A. Grothendieck, M. Artin, J.-L. Verdier *Théorie des topos et cohomologie étale des schémas* (SGA4), Vol. 3, Lect Notes in Math. **305**, Springer, 1971.
[SGA4 1/2] P. Deligne, Séminaire de Géométrie algébrique du Bois-Marie (SGA 4 1/2), Lect. Notes in Math. **569**, Springer Verlag, 1977.

[68] *The Lake Louise problem session*, *in* Algebraic K-theory: connections with geometry and topology (J.F. Jardine and V.P. Snaith, ed.), NATO ASI Series, Ser. C. **279**, 517–550, Kluwer, 1989.

Institut de Mathématiques de Jussieu, Case 7012, 75251 Paris Cedex 05, France

E-mail address: kahn@math.jussieu.fr