Motivic cohomology of smooth geometrically cellular varieties

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Abstract. We construct spectral sequences converging to the motivic cohomology of a smooth variety X over a field F of characteristic 0. In case X is geometrically cellular, *i.e.* has a cellular decomposition over the algebraic closure of F, the spectral sequences take an especially simple form.

Introduction

Let F be a field and X a projective homogeneous variety over F, *i.e.* a smooth projective variety whose geometric fibre is isomorphic to the quotient G/P of a reductive group G by a parabolic subgroup P. Let K be the function field of X. A basic question is the study of the maps

$$\eta^n : H^{n+1}(F, \mathbf{Z}/m(n)) \to H^{n+1}_{\mathrm{nr}}(K/F, \mathbf{Z}/m(n)).$$

Here m is an integer prime to the characteristic of F, the cohomology is Galois cohomology and the index nr denotes unramified cohomology. The classical strategy to study η^n can be described as follows:

1. Over a separable closure F_s of F, the variety X admits a cellular decomposition. In particular, the cycle map from its Chow ring modulo m to its étale cohomology is an isomorphism. This implies that the Hochschild-Serre

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spectral sequence for the étale cohomology of X, after some renumbering, takes the form

$$H^{p-q}(F, CH^q(X_s) \otimes \mathbf{Z}/m(n-q)) \Rightarrow H^{p+q}_{\text{ét}}(X, \mathbf{Z}/m(n)). \tag{1}$$

2. On the other hand, we have the coniveau spectral sequence for the étale cohomology of X. Using these two spectral sequences jointly yields a relationship between the $\mathcal{H}$-cohomology of X and the Galois cohomology of its twisted geometric Chow groups.
3. Using the Kato conjecture, one can relate $\mathcal{H}$-cohomology with $\mathcal{K}$-cohomology. In practice, this works well only for $H^p(X, \mathcal{H}^q)$ and $H^p(X, \mathcal{K}_q)$ when $q-p \leq 1$.
4. Finally, the algebraic K-theory of X can be explicitly described in terms of the K-theory of semi-simple algebras attached to X (Quillen, Swan, Levine, Srinivas, Panin …). This description and the Quillen spectral sequence yield information on the $\mathcal{K}$-cohomology of X.

In favourable cases, this method yields an actual computation of the kernel and cokernel of η^n (*e.g.* [**8**]). However, the information given by the various spectral sequences becomes increasingly complicated and difficult to use as n gets bigger.

Replacing the finite coefficients $\mathbf{Z}/m(n)$ by divisible coefficients $\mathbf{Q}/\mathbf{Z}(n)$ turns out to simplify both the description of η^n and the proofs (one can then get back to finite coefficients, see [**11**] where this method is applied to quadrics). A greater simplification is obtained, however, by using *étale motivic cohomology* as in [**9**]. In [**9**], we consider only $n = 2$ and use the complex $\Gamma(2)$ of Lichtenbaum [**15**]; to deal with larger values of n, one should use the motivic complexes $\mathbf{Z}(n)$ of Suslin and Voevodsky [**25**] pulled back to the big étale site[1]. Note that, for $S = \operatorname{Spec} F$ and $i \geq n+1$, the map

$$H^i_{\text{ét}}(F, \mathbf{Q}/\mathbf{Z}(n)) \to H^{i+1}_{\text{ét}}(F, \mathbf{Z}(n))$$

is an isomorphism. In particular, the divisible version of η^n can be equally described in terms of motivic cohomology as

$$H^{n+2}_{\text{ét}}(F, \mathbf{Z}(n)) \xrightarrow{\eta^n} H^{n+2}_{\text{ét,nr}}(K/F, \mathbf{Z}(n)).$$

One therefore wishes to carry over the method outlined above with étale motivic cohomology instead of étale cohomology with finite coefficients. Unfortunately, the E_2-terms of the Hochschild-Serre spectral sequence for the étale motivic cohomology of X don't have the simple form of (1). This is due to the fact that $H^q_{\text{ét}}(F_s, \mathbf{Z}(n))$ is in general nontrivial for all $q \in [1, n]$.

The main aim of this note is to construct a spectral sequence converging to the étale motivic cohomology of X, whose E_2-terms resemble those of (1). This aim is only partially achieved, as we only get an approximation of such a spectral sequence, but this is sufficient for applications. Namely, given a smooth, equidimensional, geometrically cellular variety X over a field F of characteristic 0, we construct a spectral sequence $E(X, n)$ for all $n \geq 0$:

$$E_2^{p,q}(X, n) = H^{p-q}_{\text{ét}}(F, CH^q(X_s) \otimes \mathbf{Z}(n-q)) \Rightarrow H^{p+q}$$

[1]Recall that, in characteristic 0, $\Gamma(2) \simeq \tau_{>0}\alpha^*\mathbf{Z}(2)$, where α is the projection of the big étale site on the big Zariski site; under the Beilinson-Soulé conjecture, the truncation is unnecessary.

with maps $H^{p+q} \to H^{p+q}_{\text{ét}}(X, \mathbf{Z}(n))$ which are bijective for $p+q \le 2n$ and injective for $p+q = 2n+1$ (see theorem 4.4).

Restriction to characteristic 0 is the price we have to pay because the main theorems of Voevodsky need at the moment resolution of singularities. However, it is our feeling that all results deduced from this spectral sequence and not involving motivic cohomology groups can be recovered in all characteristics by using more pedestrian (and perhaps more inextricable) methods.

Using the coniveau spectral sequence converging to étale motivic cohomology, we then carry over the second step of the program above in low weights. In weight 2, we recover results of Merkurjev, Peyre and the author ([**17**], [**22**], [**9**]); in weight 3, we generalise results of Jacob-Rost and Rost ([**5**], [**16**, prop. 1]) from quadrics to arbitrary projective homogeneous varieties. One advantage of this approach is that the third step of the programme above is, so to say, "swallowed" by motivic cohomology (under the Kato conjecture, of course). We don't go through the last step, which is the most technical and demands to look at a specific variety X.

The present method can be seen as a refinement of the one used in [**7**] and [**9**]. There we worked with the cone of a morphism of complexes

$$\Gamma(2)_Y \to Rf_*\Gamma(2)_X \tag{2}$$

for certain morphisms $f : X \to Y$, thereby getting rid of unwanted contribution by $K_{3,ind}$. Here we replace this coarse "filtration" on $Rf_*\Gamma(2)_X$, when $Y = \operatorname{Spec} F$, by a finer one on $Rf_*\alpha^*\mathbf{Z}(n)$, where α is the projection of the big étale site of $\operatorname{Spec} F$ on its big Zariski site. The long cohomology exact sequence associated to the exact triangle stemming from (2) is then replaced by the spectral sequence of theorem 4.4.

The main technical difficulty in the use of this spectral sequence is the computation of its differentials. We give a general reduction for this in section 6, before computing all d_2 differentials for Severi-Brauer varieties in section 7 and for quadrics in section 8.

Markus Rost first suggested the existence of a spectral sequence with E_2-terms as in theorem 4.4 for quadrics (with the correct differentials d_2 in low weights and degrees) in e-mail correspondence with Sujatha and the author in September 1995, although it was clear, as explained above, that a simple descent spectral sequence would not give the right answer. On the other hand, I had been toying for some time with the idea to refine the "filtration" (2) by inserting something like $\Gamma(1)_Y \overset{L}{\otimes} Rf_*\Gamma(1)_X$, and similarly for higher weights, although it was clear that the right tensor product would have to take transfer and homotopy invariance into account. It is Voevodsky who suggested the correct construction in the Oberwolfach algebraic K-theory conference of June 1996. It came as a surprise that the resulting spectral sequence has the E_2-terms predicted by Rost.

Our approach is elementary in the sense that it makes use only of the category $DM_-^{eff}(F)$ of [**26**], not of the homotopy category of F-schemes $\mathcal{H}(F)$ introduced by F. Morel and V. Voevodsky [**20**]. In collaboration with D. Orlov and A. Vishik, Voevodsky announces a complete computation of the kernel of η^n, $n \geq 0$ for X a Pfister quadric by using motivic Steenrod operations as in [**27**] (see [**21**]). Unlike their predecessors, Voevodsky *et al.* do not use in [**27**] and [**21**] anything like the computation of K-theory of projective homogeneous varieties. Therefore one may feel that the present method is to a certain extent already getting outdated. Being optimistic, one can expect to be able to completely get rid of the K-theoretical input and successfully develop the techniques of [**27**] and [**21**] in order to tackle arbitrary weights for all projective homogeneous varieties. We give a very modest starting point to this program in the appendix.

A large part of this work was done during a visit to the Tata Institute of Fundamental Research of Bombay in February-March 1997, at the invitation of Sujatha. It is a pleasure to thank her, as well as TIFR staff, for their hospitality and excellent working conditions.

Notation and conventions. Throughout, F denotes a field of characteristic 0. We denote by Sch/F (*resp.* Sm/F) the category of schemes of finite type over F (*resp.* the full subcategory of smooth F-schemes). We assume familiarity with the categories constructed by Voevodsky in [**26**]. We only recall the diagram of triangulated categories

$$\begin{tikzcd} & DM_{gm}(F) & \\ DM_{gm}^{eff}(F) \arrow[ur] \arrow[dr] & & \\ & DM_{-}^{eff}(F) \arrow[r] & DM_{-,\text{ét}}^{eff}(F) \end{tikzcd}$$

where $DM_{gm}^{eff}(F)$ is the category of effective geometrical motives, constructed out of finite correspondences, $DM_{gm}(F)$ the category of geometrical motives (obtained from the previous one by inverting the Tate object) and $DM_{-}^{eff}(F)$ (*resp.* $DM_{-,\text{ét}}^{eff}(F)$) the category of complexes of Nisnevich (*resp.* étale) sheaves with transfers with homotopy invariant cohomology sheaves. The functor $DM_{gm}^{eff}(F) \to DM_{-}^{eff}(F)$ is induced by the functor on smooth varieties

$$X \mapsto \underline{C}_*(X)$$

where $\underline{C}_*(X)$ is the Suslin complex of X [**26**, §3.2]; we shall denote this object simply by $M(X)$. The two oblique functors are full embeddings [**26**, th. 3.2.6 and 4.3.1].

Let T be a triangulated category and X an object of T. Suppose given a sequence of maps

$$X_0 \to X_1 \to \cdots \to X_n = X.$$

We call this a *filtration* on X. For all $i \in [1, n]$, denote by $X_{i/i-1}$ the cone of $X_{i-1} \to X_i$; for $i = 0$, set $X_{i/i-1} = X_0$, and set $X_{i/i-1} = 0$ for $i \notin [0, n]$. For any object $Y \in T$, there are long exact sequences of abelian groups

$$\cdots \to Hom(Y, X_{q-1}[n]) \to Hom(Y, X_q[n]) \to Hom(Y, X_{q/q-1}[n]) \\ \to Hom(Y, X_{q-1}[n+1]) \to \dots$$

These exact sequences yield an exact couple, hence a spectral sequence

$$E_2^{p,q} = Hom(Y, X_{q/q-1}[p+q]) \Rightarrow Hom(Y, X[p+q]).$$

We shall use this construction without further mention in the sequel.

1. A filtration on motivic cohomology

Let X be an F-variety. For any integer $n \geq 1$, we construct a chain of objects in $DM_-^{eff}(F)$

$$\mathbf{Z}(n,0,X) \to \mathbf{Z}(n,1,X) \to \cdots \to \mathbf{Z}(n,n,X) \tag{3}$$

by

$$\mathbf{Z}(n,i,X) = \underline{Hom}(M(X), \mathbf{Z}(i))(n-i).$$

Here $\underline{Hom}$ is the internal Hom object of $DM_-^{eff}(F)$ [**26**, prop. 3.2.8].

By *loc. cit.*, if X is smooth we can also describe $\underline{Hom}(M(X), \mathbf{Z}(i))$ as

$$Rf_* f^* \mathbf{Z}(i)$$

where $f : X \to \operatorname{Spec} F$ is the natural projection and f^*, Rf_* are relative to big Nisnevich sites.

In view of the equality $\mathbf{Z}(n-i) = \mathbf{Z}(1)^{\otimes(n-i)}$, it is enough to construct the map $\mathbf{Z}(n,i,X) \to \mathbf{Z}(n,i+1,X)$ for $n = i+1$. It is then defined as the composition

$$\underline{Hom}(M(X), \mathbf{Z}(i)) \otimes \mathbf{Z}(1) \to \underline{Hom}(M(X), \mathbf{Z}(i)) \otimes \underline{Hom}(M(X), \mathbf{Z}(1)) \\ \to \underline{Hom}(M(X) \otimes M(X), \mathbf{Z}(i) \otimes \mathbf{Z}(1)) = \underline{Hom}(M(X \times_F X), \mathbf{Z}(i+1)) \\ \to \underline{Hom}(M(X), \mathbf{Z}(i+1))$$

where the first map is induced by the structural morphism $M(X) \to \mathbf{Z}$, and the last one by the diagonal $M(X) \to M(X \times_F X)$.

Let $\mathbf{Z}(n, i/i-1, X)$ be the cone of $\mathbf{Z}(n, i-1, X) \to \mathbf{Z}(n,i,X)$, where we set $\mathbf{Z}(n,-1,X) = 0$ and $\mathbf{Z}(n,n+1,X) = \mathbf{Z}(n,n,X)$. We get a spectral sequence

$$E_2^{p,q} = Hom_{DM_-^{eff}(F)}(\mathbf{Z}, \mathbf{Z}(n, q/q-1, X)[p+q]) \\ \Rightarrow Hom_{DM_-^{eff}(F)}(M(X), \mathbf{Z}(n)[p+q]) =: H^{p+q}(X, \mathbf{Z}(n)). \tag{4}$$

Note that $\mathbf{Z}(n,i-1,X) \to \mathbf{Z}(n,i,X)$ is an isomorphism as soon as $i-1 \geq \dim X$. This follows from [**26**, prop. 4.3.3 and th. 3.2.6 1.]. In particular, $E_2^{p,q} = 0$ for $q > \dim X$.

2. Motivic computations

Lemma 2.1. *a) For any smooth connected scheme $X \in Sm/F$, $\underline{Hom}(M(X), \mathbf{Z}) = \mathbf{Z}$.*
b) For any object $A \in DM_{gm}^{eff}(F)$, $\underline{Hom}(A(1), \mathbf{Z}) = 0$.
c) For any integer $n > 0$, $\underline{Hom}(\mathbf{Z}(n), \mathbf{Z}) = 0$.

Proof. a) It is sufficient to see that $Hom(M(X), \mathbf{Z}[i]) = \begin{cases} \mathbf{Z} & \text{if } i = 0 \\ 0 & \text{if } i \neq 0. \end{cases}$ This follows readily from [**26**, cor. 3.2.7].

b) We may assume that $A = M(X)$ for some $X \in Sm/F$. In view of the definition of $\mathbf{Z}(1)$ in $DM_{gm}^{eff}(F)$, we have an exact triangle in $DM_{-}^{eff}(F)$:

$$\underline{Hom}(M(X), \mathbf{Z})[2] \to \underline{Hom}(M(X \times \mathbf{P}^1), \mathbf{Z})[2] \\ \to \underline{Hom}(M(X)(1), \mathbf{Z}) \to \underline{Hom}(M(X), \mathbf{Z})[3]$$

so it suffices to show that $\underline{Hom}(M(X), \mathbf{Z}) \to \underline{Hom}(M(X \times \mathbf{P}^1), \mathbf{Z})$ is an isomorphism, which follows from a).

c) This is a special case of b). □

Lemma 2.2. *For $m, n \in \mathbf{N}$, we have*

$$\underline{Hom}(\mathbf{Z}(m), \mathbf{Z}(n)) = \begin{cases} \mathbf{Z}(n-m) & \text{if } m \leq n \\ 0 & \text{if } m > n. \end{cases}$$

Proof. The case $m \leq n$ follows from the quasi-invertibility of the Tate object in $DM_{gm}^{eff}(F)$ [**26**, th. 4.3.1], hence in $DM_{-}^{eff}(F)$ since the functor from the former category to the latter is a full embedding [**26**, th. 3.2.6]. The case $m > n$ follows from the same reasons, plus lemma 2.1 c). □

Lemma 2.3. *Let $A \in DM_{gm}^{eff}(F)$ and $B \in DM_{-,\text{ét}}^{eff}(F)$. Then:*
a) $\underline{Hom}_{DM_{-,\text{ét}}^{eff}(F)}(A, B) \xrightarrow{\sim} \underline{Hom}_{DM_{-,\text{ét}}^{eff}(F)}(A(1), B(1))$.
b) $\underline{Hom}_{DM_{-,\text{ét}}^{eff}(F)}(A(1), B) \xrightarrow{\sim} \underline{Hom}_{DM_{-,\text{ét}}^{eff}(F)}(A, B \otimes (\mathbf{Q}/\mathbf{Z})'(-1)[-1])$. Here, $(\mathbf{Q}/\mathbf{Z})'(-1) = \varinjlim_{(n, \operatorname{char} F)=1} Hom(\mu_n, \mathbf{Q}/\mathbf{Z})$.

Proof. We may assume $A = M(X)$ for $X \in Sm/F$. It is enough to show the statements after tensoring by $\mathbf{Q}$ and by $\mathbf{Z}/p$, p prime (in the derived sense). When tensoring by $\mathbf{Q}$, the claims follow from lemma 2.2 and [**26**, prop. 3.3.2]. When tensoring by $\mathbf{Z}/p$, observe the isomorphism

$$\underline{Hom}_{DM_{-,\text{ét}}^{eff}(F)}(M(X), B) \otimes \mathbf{Z}/p \xrightarrow{\sim} \underline{Hom}_{DM_{-,\text{ét}}^{eff}(F, \mathbf{Z}/p)}(M(X) \otimes \mathbf{Z}/p, B \otimes \mathbf{Z}/p)$$

following from the étale analogue of [**26**, prop. 3.2.8] and the isomorphism

$$(Rf_* f^* B) \overset{L}{\otimes} \mathbf{Z}/p \xrightarrow{\sim} Rf_* f^* (B \overset{L}{\otimes} \mathbf{Z}/p).$$

For $p \neq \operatorname{char} F$, the claims follow from [**26**, prop. 3.3.3 1.] and the fact that $\mathbf{Z}/p(1) \simeq \mu_p$ [**27**, th. 2.6]; for $p = \operatorname{char} F$, they follow from [**26**, prop. 3.3.3 2]. □

LEMMA 2.4. *For $m, n \in \mathbf{N}$, we have*

$$\underline{Hom}_{DM^{eff}_{-,\text{ét}}(F)}(\mathbf{Z}(m), \mathbf{Z}(n)) = \begin{cases} \mathbf{Z}(n-m) & \text{if } m \leq n \\ (\mathbf{Q}/\mathbf{Z})'(n-m)[-1] & \text{if } m > n. \end{cases}$$

3. Varieties with cellular decomposition

REMARK 3.1. Let (Fr) be the category of finitely generated free abelian groups. For any additive category A, there is a biadditive functor

$$(Fr) \times A \xrightarrow{\otimes} A$$

such that $\mathbf{Z} \otimes -$ is the identity functor on A and, for any $(a, b, L) \in A \times A \times (Fr)$,

$$Hom_{(Ab)}(L, Hom_A(a, b)) \simeq Hom_A(L \otimes a, b).$$

(Construct $\otimes$ on the canonical skeletal subcategory $(Fr)_0$ of (Fr), and then compose with an equivalence of categories inverse to $(Fr)_0 \hookrightarrow (Fr)$.)

Let Sch_F be the category of schemes of finite type over F.

DEFINITION 3.2. A variety $X \in Sch_F$ has a *cellular decomposition* (briefly: is cellular) if there exists a proper closed subset $Z \subset X$ such that:

- $X - Z$ is isomorphic to an affine space;
- Z has a cellular decomposition.

This definition makes sense recursively, by Noetherian induction.

We recall from [**26**, §4] a few properties of motives with compact supports. For any $X \in Sch_F$, the object $\underline{C}^c_*(X)$ of loc. cit. defines an object of $DM^{eff}_{gm}(F)$ [**26**, cor. 4.1.6] that we denote by $M^c(X)$. The assignment $X \mapsto M^c(X)$ is covariant for proper morphisms and contravariant for flat equidimensional morphisms in the following sense: if $f : X \to Y$ is proper, there is (by construction) an associated morphism $f_* : M^c(X) \to M^c(Y)$; if f is flat and equidimensional of relative dimension d, there is an associated morphism $f^* : M^c(Y)(d)[2d] \to M^c(X)$ [**26**, cor. 4.2.4]. Proper covariance and flat equidimensional contravariance commute in cartesian squares in the obvious sense (this is not explicitly stated in [**26**], but follows easily from the construction of f^*). Moreover, M^c is homotopy invariant in the following sense: For any $d \geq 0$, $f^* : \mathbf{Z}(d)[2d] \to M^c(\mathbf{A}^n_F)$ is an isomorphism, where $f : \mathbf{A}^n_F \to \operatorname{Spec} F$ is the structural morphism [**26**, cor. 4.1.8]. If $X, Y \in Sch_F$, there is a canonical isomorphism $M^c(X \times Y) = M^c(X) \otimes M^c(Y)$ [**26**, prop. 4.1.7]. Finally, if $Z \xrightarrow{i} X$ is a closed immersion with complementary open immersion $U \xrightarrow{j} X$, there is an exact triangle [**26**, prop. 4.1.5]

$$M^c(Z) \xrightarrow{i_*} M^c(X) \xrightarrow{j^*} M^c(U) \to M^c(Z)[1].$$

Fulton's homology Chow groups $CH_p(X)$ [**4**] have the same functoriality.

LEMMA 3.3. *Let X be a cellular variety and $Z \subseteq X$ a closed subset such that the complementary open subset U is isomorphic to $\mathbf{A}^d_F$. Then:*
a) The triangle

$$M^c(Z) \to M^c(X) \to M^c(U) \to M^c(Z)[1]$$

is canonically split.
b) For all $p \geq 0$, the sequence

$$0 \to CH_p(Z) \to CH_p(X) \to CH_p(U) \to 0$$

is split exact. In particular, $CH_p(X)$ is free and finitely generated.

Proof. Since U is irreducible, there is an irreducible component Y of X containing U. For a), since Y is equidimensional of dimension d there is a natural map $\pi^* : \mathbf{Z}(d)[2d] \to M^c(Y)$, where $\pi : Y \to \operatorname{Spec} F$ denotes the structural morphism. Since $i : Y \to X$ is a closed immersion, we can follow it by $i_* : M^c(Y) \to M^c(X)$; finally we can follow this map by $j^* : M^c(X) \to M^c(U)$. The cartesian diagram

$$\begin{tikzcd} U \arrow[r, equal] \arrow[d, "k"'] & U \arrow[d, "j"'] \\ Y \arrow[r, "i"] & X \end{tikzcd}$$

(which defines k) shows that $j^* i_* = k^*$, hence $j^* i_* \pi^*$ is an isomorphism by homotopy invariance. This provides the desired canonical splitting.

For b), we argue exactly in the same way by considering the exact sequence (compare [**23**, p. 356])

$$A_p(X, K_1) \to A_p(U, K_1) \to CH_p(Z) \to CH_p(X) \to CH_p(U) \to 0.$$

Free finite generation of $CH_p(X)$ follows immediately by Noetherian induction.
□

Proposition 3.4. *Let $X \in Sch_F$ be a cellular variety.*
a) There is a canonical isomorphism:

$$\coprod_{p \geq 0} CH_p(X) \otimes \mathbf{Z}(p)[2p] \xrightarrow{\sim} M^c(X).$$

b) If $f : X \to Y$ is proper, the diagram

$$\begin{tikzcd} \coprod_{p\geq 0} CH_p(X) \otimes \mathbf{Z}(p)[2p] \arrow[r, "\sim"] \arrow[d, "f_*"'] & M^c(X) \arrow[d, "f_*"'] \\ \coprod_{p\geq 0} CH_p(Y) \otimes \mathbf{Z}(p)[2p] \arrow[r, "\sim"] & M^c(Y) \end{tikzcd}$$

commutes, where the left hand side morphism is given diagonally by the push-forward morphisms $f_* : CH_p(X) \to CH_p(Y)$.
c) If $f : X \to Y$ is flat and equidimensional of relative dimension d, the diagram

$$\begin{tikzcd} \coprod_{p\geq 0} CH_p(X) \otimes \mathbf{Z}(p)[2p] \arrow[r, "\sim"] & M^c(X) \\ \coprod_{p\geq 0} CH_{p-d}(Y) \otimes \mathbf{Z}(p)[2p] \arrow[r, "\sim"] \arrow[u, "f^*"] & M^c(Y)(d)[2d] \arrow[u, "f^*"] \end{tikzcd}$$

commutes, where the left hand side morphism is given diagonally by the flat pull-back morphisms $f^* : CH_{p-d}(Y) \to CH_p(X)$.
d) If X and Y are cellular, the isomorphisms for X, Y and $X \times Y$ are compatible in a similar sense.

Proof. We first construct the morphism. By [**26**, prop. 4.2.9], there is for all p a canonical isomorphism

$$CH_p(X) \xrightarrow{\sim} Hom_{DM_{gm}^{eff}(F)}(\mathbf{Z}(p)[2p], M^c(X)). \tag{5}$$

In fact we only need the existence of a canonical map from the left to the right hand side. Let us describe this map elementarily on the level of cycles: if $Z \xrightarrow{i} X$ is an irreducible subvariety of dimension p, the associated map $\mathbf{Z}(p)[2p] \to M^c(X)$ is defined as the composition

$$\mathbf{Z}(p)[2p] \xrightarrow{\pi^*} M^c(Z) \xrightarrow{i_*} M^c(X)$$

where π is the structural morphism of Z.

If X is cellular, this gives by adjunction a morphism (compare remark 3.1)

$$CH_p(X) \otimes \mathbf{Z}(p)[2p] \to M^c(X).$$

Collecting all these morphisms, we get the desired canonical morphism.

We now prove that this morphism is an isomorphism by Noetherian induction. Let Z be as in definition 3.2 and $U = X \setminus Z$. By lemma 3.3, we have a commutative diagram of split exact triangles

$$\begin{array}{ccccc}
\coprod_p CH_p(Z) \otimes \mathbf{Z}(p)[2p] & \to & \coprod_p CH_p(X) \otimes \mathbf{Z}(p)[2p] & \to & \coprod_p CH_p(U) \otimes \mathbf{Z}(p)[2p] \\
\downarrow & & \downarrow & & \downarrow \\
M^c(Z) & \to & M^c(X) & \to & M^c(U).
\end{array}$$

Since the left vertical map is an isomorphism by Noetherian induction and the right vertical map is an isomorphism by homotopy invariance, the middle vertical map is an isomorphism as well. Finally, the assertions on functoriality follow from the fact that the isomorphisms (5) verify this functoriality, which itself follows from the construction of (5) in the proof of [**26**, prop. 4.2.9]. $\square$

Corollary 3.5. *Let X be cellular, smooth and equidimensional. Then there is a natural isomorphism in $DM_{gm}^{eff}(F)$:*

$$\coprod_{p\geq 0} CH^p(X)^* \otimes \mathbf{Z}(p)[2p] \xleftarrow{\sim} M(X)$$

where $CH^p(X)^*$ denotes the $\mathbf{Z}$-dual of the Chow group of cycles of codimension p on X modulo linear equivalence. This isomorphism has the following properties:
a) If $f : X \to Y$ is flat and equidimensional, the diagram

$$\begin{array}{ccc}
\coprod_{p\geq 0} CH^p(X)^* \otimes \mathbf{Z}(p)[2p] & \xleftarrow{\sim} & M(X) \\
{\scriptstyle f_*}\downarrow & & {\scriptstyle f_*}\downarrow \\
\coprod_{p\geq 0} CH^p(Y)^* \otimes \mathbf{Z}(p)[2p] & \xleftarrow{\sim} & M(Y)
\end{array}$$

commutes, where the right hand side morphism is motivic covariance and the left hand side morphism is given diagonally by the transposes of the flat pull-back morphisms $f^* : CH^p(Y) \to CH^p(X)$.

b) If $f : X \to Y$ is a closed immersion of relative dimension d, with X, Y smooth, the diagram

$$\begin{array}{ccc}
\coprod_{p\geq 0} CH^p(X)^* \otimes \mathbf{Z}(p)[2p] & \xleftarrow{\sim} & M(X) \\
{\scriptstyle f^*}\uparrow & & {\scriptstyle f^*}\uparrow \\
\coprod_{p\geq 0} CH^{p+d}(Y)^* \otimes \mathbf{Z}(p)[2p] & \xleftarrow{\sim} & M(Y)(d)[2d]
\end{array}$$

commutes, where the right hand side morphism is the Gysin morphism [**26**, prop. 3.5.4] *and the left hand side morphism is given diagonally by the transposes of the push-forward morphisms $f_* : CH^p(X) \to CH^{p+d}(Y)$.*
c) If X and Y are smooth cellular, the isomorphisms for X, Y and $X \times Y$ are compatible in a similar sense.

Proof. For the isomorphism, we may assume X connected. Let $d = \dim X$. By [**26**, th. 4.3.7], we have a canonical isomorphism in $DM_{gm}(F)$:

$$M(X)^* = M^c(X)(-d)[-2d]$$

where $M(X)^*$ denotes the dual of $M(X)$. From this and proposition 3.4 we deduce immediately the formula of corollary 3.5 in $DM_{gm}(F)$. Since both sides of the isomorphism belong to $DM_{gm}^{eff}(F)$ and since $DM_{gm}^{eff}(F) \to DM_{gm}(F)$ is a full embedding [**26**, th. 4.3.1], the isomorphism already holds in $DM_{gm}^{eff}(F)$. Properties a), b) and c) follow from the analogous ones for M^c by duality, as one checks that duality transforms flat pull-backs for M^c into covariant morphisms for M and push-forward attached to smooth pairs for M^c into Gysin morphisms for M. □

Corollary 3.6. *With the assumptions of corollary 3.5, we have isomorphisms*
(i) $\underline{Hom}_{DM_-^{eff}}(M(X), \mathbf{Z}(n)) \simeq \coprod_{0\leq p\leq n} CH^p(X) \otimes \mathbf{Z}(n-p)[-2p]$*;*
(ii) $\mathbf{Z}(n, i, X) \simeq \coprod_{0\leq p\leq i} CH^p(X) \otimes \mathbf{Z}(n-p)[-2p]$*;*
(iii) $\mathbf{Z}(n, q/q-1, X) \simeq CH^q(X) \otimes \mathbf{Z}(n-q)[-2q]$*;*
(iv) $\underline{Hom}_{DM_{-,\text{ét}}^{eff}}(M(X), \mathbf{Z}(n)) \simeq \coprod_{0\leq p\leq n} CH^p(X)\otimes\mathbf{Z}(n-p)[-2p] \oplus \coprod_{p>n} CH^p(X) \otimes (\mathbf{Q}/\mathbf{Z})'(n-p)[-1-2p]$.

Proof. This follows from corollary 3.5, lemma 2.2 and lemma 2.4. □

Corollary 3.7. *For X smooth cellular, the spectral sequence* (4) *degenerates.* □

4. Geometrically cellular varieties

Definition 4.1. A variety X over a field F is *geometrically cellular* if $X_s := X \otimes_F F_s$ is cellular, where F_s is a separable closure of F. A geometrically cellular variety is *split* if it is already cellular over F.

Lemma 4.2. *Let X be geometrically cellular. Then X becomes cellular over a suitable finite separable extension of F.*

Proof. Noetherian induction as usual. Pick a $Z \subset X_s$ as in definition 3.2. Then Z and hence $X_s - Z$ are defined over some finite, separable extension E_1 of F. There is a finite separable extension E_2 of E_1 such that $X_s - Z$ becomes isomorphic to an affine space over E_2. By Noetherian induction, Z becomes cellular over a suitable finite, separable extension of E_2. □

LEMMA 4.3. *Let X be smooth, geometrically cellular over F and let $p : X \to \operatorname{Spec} F$ be the structural morphism. Let α be the projection of the big étale site of $\operatorname{Spec} F$ onto its big Zariski site. Then, for all $n \geq 0$, the natural map*

$$\alpha^* Rp_*^{\mathrm{Zar}} \mathbf{Z}(n)_X \to Rp_*^{\text{ét}} \alpha^* \mathbf{Z}(n)_X$$

induces an isomorphism

$$\alpha^* Rp_*^{\mathrm{Zar}} \mathbf{Z}(n)_X \xrightarrow{\sim} \tau_{\leq 2n} Rp_*^{\text{ét}} \alpha^* \mathbf{Z}(n)_X.$$

Proof. This follows immediately from corollary 3.6 (i) and (iv). □

THEOREM 4.4. *Let X be a smooth, equidimensional, geometrically cellular variety over a field F of characteristic 0. For all $n \geq 0$, there is a spectral sequence $E(X, n)$:*

$$E_2^{p,q}(X, n) = H_{\text{ét}}^{p-q}(F, CH^q(X_s) \otimes \mathbf{Z}(n - q)) \Rightarrow H^{p+q} \tag{6}$$

with maps $H^{p+q} \to H_{\text{ét}}^{p+q}(X, \mathbf{Z}(n))$ which are bijective for $p + q \leq 2n$ and injective for $p + q = 2n + 1$. These spectral sequences have the following properties:

(i) **Naturality**. (6) *is covariant in F and contravariant in X (varying among smooth, equidimensional, geometrically cellular varieties) under flat equidimensional maps.*

(ii) **Products**. *There are pairings of spectral sequences*

$$E_r^{p,q}(X, m) \times E_r^{p',q'}(X, n) \to E_r^{p+p',q+q'}(X, m + n)$$

which coincide with the usual cup-product on the E_2-terms and the abutments.

(iii) **Transfer**. *For any finite extension E/F and any $n \geq 0$, there is a morphism of spectral sequences*

$$E_r^{p,q}(X_E, n) \to E_r^{p,q}(X, n)$$

which coincides with the usual transfer on the E_2-terms and the abutment.

(iv) **Covariance for closed equidimensional immersions**. *For any closed immersion $i : Y \hookrightarrow X$ of pure codimension c, where X and Y are smooth, geometrically cellular, there is a morphism of spectral sequences*

$$E_r^{p-c,q-c}(Y, n - c) \xrightarrow{i_*} E_r^{p,q}(X, n)$$

"abutting" to the Gysin homomorphisms

$$H_{\text{ét}}^{p+q-2c}(Y, \mathbf{Z}(n - c)) \xrightarrow{i_*} H_{\text{ét}}^{p+q}(X, \mathbf{Z}(n)).$$

If X is split, then (6) *degenerates at E_2.*

Proof. The spectral sequence is the one associated to the pull-back of (3) to the big étale site of $\operatorname{Spec} F$ (compare end of introduction). Corollary 3.6 (iii) identifies the E_2-terms, while lemma 4.3 identifies the abutment. (iv) follows from the fact that (3) for Z (shifted and with different weights) maps to (3) for X, by the Gysin exact triangle of [**26**, prop. 3.5.4] and the quasi-invertibility of the Tate object (*loc. cit.*, th. 4.3.1). The last claim follows, as corollary 3.7, from corollary 3.6, which shows that the exact triangles of (3) are split. □

Remark 4.5. If we use the étale analogue of the filtration $\mathbf{Z}(n,i,X)$ (*i.e.* $\mathbf{Z}(n,i,X)_{\text{ét}} := \underline{Hom}_{DM^{eff}_{-,\text{ét}}(F)}(M(X),\mathbf{Z}(i))\otimes\mathbf{Z}(n-i))$, we get a strange answer: lemma 2.4 and corollary 3.6 identify this complex of étale sheaves with

$$\coprod_{0\le p\le i} CH^p(X)\otimes\mathbf{Z}(n-p)\oplus\coprod_{p>i} CH^p(X)\otimes\mathbf{Q}/\mathbf{Z}(n-p)[-1-2p].$$

Hence, for $q>0$, $\mathbf{Z}(n,q/q-1,X)_{\text{ét}}\simeq CH^q(X)\otimes\mathbf{Q}(q)[-2q]$ and the spectral sequence is not very interesting ...

One may hope that there is a spectral sequence converging to $H^{p+q}_{\text{ét}}(X,\mathbf{Z}(n))$, with E_2-terms

$$E_2^{p,q}=\begin{cases}H^{p-q}_{\text{ét}}(F,CH^q(X_s)\otimes\mathbf{Z}(n-q)) & \text{if } q\le n\\ H^{p-q-1}_{\text{ét}}(F,CH^q(X_s)\otimes\mathbf{Q}/\mathbf{Z}(n-q)) & \text{if } q>n\end{cases}$$

but at the moment I don't know how to construct such a spectral sequence. This would amount to showing that the "filtration" on $\underline{Hom}_{DM^{eff}_{-,\text{ét}}(F_s)}(X_s,\mathbf{Z}(n))$ given by corollary 3.6 (iv) descends to a filtration on $\underline{Hom}_{DM^{eff}_{-,\text{ét}}(F)}(X,\mathbf{Z}(n))$.

Proposition 4.6. *Let X be a smooth, geometrically cellular variety over a field F of characteristic* 0. *Then all differentials in the spectral sequence* (6) *are torsion.*

Proof. By lemma 4.2 and theorem 4.4, the spectral sequence (6) degenerates over some finite extension of F. The conclusion follows by a transfer argument. □

Corollary 4.7. *Let X be as in proposition 4.6. In the spectral sequence* (6) *tensored by $\mathbf{Z}_{(2)}$,*
a) all differentials starting from $E_r^{p,q}$, $p\le q<n$, are 0.
b) $E_2^{n+1,q}=0$ for all q.

Proof. a) In view of the exact triangle

$$\mathbf{Z}_{(2)}(n)\xrightarrow{2}\mathbf{Z}_{(2)}(n)\to\mu_2^{\otimes n}\to\mathbf{Z}_{(2)}(n)[1]$$

[**27**, proof of th. 2.6], $H^i(F,\mathbf{Z}_{(2)}(n))$ is uniquely divisible for $i<0$, and also for $i=0$ if $n>0$ by the same argument as [**10**, proof of th. 3.1 a)]. Therefore $E_2^{p,q}$ is uniquely divisible for $p\le q$. The claim now follows from proposition 4.6.

b) This follows from [**27**, th. 4.1] (Hilbert 90). □

5. Examples: weights 1, 2, 3 and 4

In this section and the following ones, all motivic cohomology groups are étale.

5.1. Generalities. Let X be a projective homogeneous variety over F. Then X is geometrically cellular; moreover, Schubert cycles provide canonical bases of the groups $CH^i(X_s)$, which are permuted under Galois action (see Peyre [**22**, prop. 1]). In particular, the Galois module $CH^i(X_s)$ is a permutation module. Denote by E_i the étale algebra corresponding to the canonical basis of $CH^i(X_s)$. Then, by Shapiro's lemma, we have

$$H^{p-q}(F,CH^q(X_s)\otimes\mathbf{Z}(n-q))=H^{p-q}(E_q,\mathbf{Z}(n-q)).$$

Using the spectral sequence (6) tensored by $\mathbf{Z}_{(2)}$, we get for small n some exact sequences; using the coniveau spectral sequence for motivic cohomology, we get other exact sequences; putting them together we obtain some information on the unramified cohomology of X.

We start by recalling the coniveau spectral sequence for étale motivic cohomology. The following lemma is an analogue of lemma 2.4, and is proved in exactly the same way, using [**25**, prop. 2.3], [**27**, th. 2.5] and purity for étale cohomology with finite coefficients.

Lemma 5.1. *Let $Z \subset X$ be a smooth F-pair of pure codimension c. Then, for all i, n, there is a canonical isomorphism*

$$H^{i-2c}(Z, \mathbf{Z}(n-c)) \xrightarrow{\sim} H^i_Z(X, \mathbf{Z}(n))$$

where, for $i < 0$, $\mathbf{Z}(i)$ is to be interpreted as $\mathbf{Q}/\mathbf{Z}(i)[-1]$. □

Filtering étale motivic cohomology by codimension of support gives a spectral sequence

$$E_1^{p,q} = \coprod_{x \in X^{(p)}} H^{p+q}_x(X, \mathbf{Z}(n)) \Rightarrow H^{p+q}(X, \mathbf{Z}(n)).$$

By lemma 5.1, the E_1-terms can be rewritten as

$$E_1^{p,q} = \coprod_{x \in X^{(p)}} H^{q-p}(F(x), \mathbf{Z}(n-p)).$$

In particular,

- $E_1^{p,q} = \coprod_{x \in X^{(p)}} H^{q-p-1}(F(x), \mathbf{Q}/\mathbf{Z}(n-p))$ for $p > n$ or $q > n+1$
- $E_1^{p,q} = 0$ for $p \geq q$ and $p > n$
- after localisation at 2, $E_1^{p,q}$ is uniquely divisible for $p \geq q$ and $p < n$ and 0 for $p = q = n-1$
- after localisation at 2, $E_1^{p,q} = 0$ for $q = n+1$ (Hilbert 90).

For convenience, we denote the term $E_2^{p,q}$ by $H^p(X, \mathcal{H}^q(\mathbf{Z}(n)))$; we won't use the fact that this is Zariski cohomology of the sheaf associated with the presheaf $U \mapsto H^q_{\text{ét}}(U, \mathbf{Z}(n))$. However we shall use without further mention the fact that

$$H^p(X, \mathcal{H}^q(\mathbf{Z}(n))) \simeq \begin{cases} H^p(X, \mathcal{K}^M_n) & \text{for } q = n \\ H^p(X, \mathcal{H}^{q-1}(\mathbf{Q}/\mathbf{Z}(n))) & \text{for } q > n+1 \end{cases}$$

where the right hand side is the cohomology of the Gersten complex for Milnor's K-theory [**14**], [**23**] (*resp.* the E_2-term of the coniveau spectral sequence for divisible coefficients).

In the sequel, we write for simplicity

$$H^i_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(n)) =: H^i(X, n)$$
$$H^p(X, \mathcal{H}^q_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(n))) =: H^p(X, \mathcal{H}^q(n))$$

for any variety X. We now describe the general results for small weights.

5.2. $n = 1$**.** From theorem 4.4, we get an exact sequence:

$$0 \to H^2(X, \mathbf{Z}(1)) \to CH^1(X_s)^{G_F} \to H^3(F, \mathbf{Z}(1)) \to H^3(X, \mathbf{Z}(1))$$

which can be identified with the familiar exact sequence (from the Hochschild-Serre spectral sequence)

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(X_s)^{G_F} \to Br(F) \to Br(X). \tag{7}$$

On the other hand, from the coniveau spectral sequence, we get an isomorphism

$$Br(X) \xrightarrow{\sim} H^0(X, \mathcal{H}^2(\mathbf{Q}/\mathbf{Z}(1)))$$

the well-known equality between the Brauer group and the unramified Brauer group.

Note that the range of application of the spectral sequence from theorem 4.4 is not sufficient here to get something on the cokernel of $Br(F) \to Br(X)$. However, using the Hochschild-Serre spectral sequence, one can easily see that it injects into $(\mathrm{Pic}(X_s) \otimes \mathbf{Q}/\mathbf{Z})^{G_F}$.

5.3. $n = 2$**.** We get the following diagram:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & H^1(X, \mathcal{K}_2) \arrow[r, "\sim"] & H^3(X, \mathbf{Z}(2)) \arrow[d] & \\
 & & & E_1^* \arrow[d, "d_2^{2,1}"] & \\
 & & & H^3(F,2) \arrow[d] \arrow[dr, "\eta^3"] & \\
0 \arrow[r] & & CH^2(X) \arrow[r] \arrow[dr, "\xi^3"] & H^4(X, \mathbf{Z}(2)) \arrow[r] \arrow[d] & H^0(X, \mathcal{H}^3(2)) \to 0 \\
 & & & CH^2(X_s)^{G_F} \arrow[d, "d_2^{2,2}"] \arrow[dl, "d_3^{2,2}"] & \\
 & & H^4(F,2) \arrow[dl] & Br(E_1) & \\
 & H^5(X, \mathbf{Z}(2)) & & &
\end{tikzcd}$$

(Recall that E_1 is the étale F-algebra corresponding to the distinguished basis of $\mathrm{Pic}(X_s)$.) In this diagram, the differential $d_3^{2,2}$ is of course defined only on the kernel of $d_2^{2,2}$. The horizontal sequence is exact (compare [**9**, th. 1.1], where this is proven using the Lichtenbaum complex $\Gamma(2)$ and the Leray rather than coniveau spectral sequence). The vertical sequence is exact, except at $CH^2(X_s)^{G_F}$. Its homology H at this point sits in an exact sequence

$$0 \to H \xrightarrow{d_3^{2,2}} H^4(F,2) \to H^5(X,\mathbf{Z}(2)).$$

In this way, we recover the sequences of [**9**, cor. 7.1]:

(8)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(X,\mathcal{K}_2) & \longrightarrow & E_1^* & \xrightarrow{d_2^{2,1}} & \operatorname{Ker}\eta^3 & \longrightarrow & CH^2(X)_{tors} \to 0 \\ 0 & \longrightarrow & \operatorname{Coker}\eta^3 & \longrightarrow & \operatorname{Coker}\xi^3 & \xrightarrow{d_2^{2,2}} & Br(E_1) & & \end{array}$$

which are exact, except perhaps for the second one at Coker ξ^3 (note that $H^5(X,\mathbf{Z}(2))$ is isomorphic to the cokernel of the divisible cycle map by [**9**, th. 1.1 (v)]).

We take this opportunity to prove a result which was announced in [**9**] (see [**11**, cor. 10.2] in the case of quadrics). At the price of more complicated arguments, we could use $\Gamma(2)$ and the results of [**9**] instead of $\mathbf{Z}(2)$ and the results of the present paper, hence making the proposition below valid without restriction on the characteristic of F.

Proposition 5.2. *For any projective homogeneous variety X, the d_2 differential of the Bloch-Ogus spectral sequence*

$$H^0(X,\mathcal{H}^3(2)) \xrightarrow{d_2} CH^2(X)\otimes\mathbf{Q}/\mathbf{Z}$$

induces an isomorphism

$$\operatorname{Coker}\eta^3 \xrightarrow{\sim} \operatorname{Ker} cl_X^2$$

where $cl_X^2 : CH^2(X)\otimes\mathbf{Q}/\mathbf{Z} \to H^4(X,2)$ is the divisible cycle map.

Proof. Consider the commutative diagram with exact rows:

$$\begin{tikzcd}
 & & H^3(F,2) \arrow[d] \arrow[dr, "\eta^3"] & & \\
0 \arrow[r] & CH^2X \arrow[r] \arrow[d] & H^4(X,\mathbf{Z}(2)) \arrow[r] \arrow[d] & H^0(X,\mathcal{H}^3(2)) \arrow[r] & 0 \\
0 \arrow[r] & CH^2X\otimes\mathbf{Q} \arrow[r] \arrow[d] & H^4(X,\mathbf{Z}(2))\otimes\mathbf{Q} \arrow[r] \arrow[d] & 0 & \\
 & CH^2X\otimes\mathbf{Q}/\mathbf{Z} \arrow[r, "cl_X^2"] \arrow[d] & H^4(X,\mathbf{Q}/\mathbf{Z}(2)) & & \\
 & 0 & & &
\end{tikzcd}$$

By [**3**, th. 7.1], the homomorphism $H^0(X,\mathcal{H}^3(2)) \to CH^2(X)\otimes\mathbf{Q}/\mathbf{Z}$ obtained by applying the snake lemma to the first two rows in this diagram coincides with the differential of the Bloch-Ogus spectral sequence, up to sign. The conclusion now follows from

Lemma 5.3. *The sequence*

$$H^3(F,2) \to H^4(X,\mathbf{Z}(2)) \to H^4(X,\mathbf{Z}(2)) \otimes \mathbf{Q}$$

is exact.

This is clear from the exact sequence

$$H^3(F,2) \to H^4(X,\mathbf{Z}(2)) \to CH^2(X_s)^{G_F}$$

from the cross above, since $CH^2(X_s)$ is torsion-free. □

5.4. $n = 3$**.** From now on, all cohomology groups are supposed to be localised at 2, unless otherwise specified. We get a diagram

$$\begin{tikzcd}
& & 0 \arrow[d] & & \\
& H^1(X,\mathcal{K}_3^M) \arrow[r, "\sim"] & H^4(X,\mathbf{Z}(3)) \arrow[d] & & \\
& & K_2(E_1) \arrow[d, "d_2^{3,1}"] & & \\
& & H^4(F,3) \arrow[d] \arrow[dr, "\eta^4"] & & \\
0 \arrow[r] & H^2(X,\mathcal{K}_3^M) \arrow[r] \arrow[dr, "\xi^4"] & H^5(X,\mathbf{Z}(3)) \arrow[r] \arrow[d] & H^0(X,\mathcal{H}^4(3)) \arrow[r] & CH^3(X) \arrow[d] \\
& & E_2^* \arrow[dl, "d_3^{3,2}"'] \arrow[d, "d_2^{3,2}"] & & H^6(X,\mathbf{Z}(3)) \\
& H^5(F,3) \arrow[d] & H^3(E_1,2) & & \\
& H^6(X,\mathbf{Z}(3)). & & &
\end{tikzcd}$$

To see that the top horizontal map is indeed an isomorphism, note that, in the coniveau spectral sequence, $E_1^{2,2} = E_1^{0,4} = 0$ (see subsection 5.1). The horizontal sequence forks downwards to form a long exact sequence. The vertical sequence is exact, except at E_2^*, where $d_3^{3,2}$ is only defined on its homology H and yields an exact sequence

$$0 \to H \xrightarrow{d_3^{3,2}} H^5(F,3) \to H^6(X,\mathbf{Z}(3)).$$

Proposition 5.4. $H^i(X,\mathcal{K}_3^M) = H^i(X,\mathcal{K}_3)$ *for* $i > 0$*.*

Proof. This follows from [**18**, prop. 11.11]. □

The following corollary extends the results of [**17**], [**22**] and [**9**] one degree higher, and also generalises a theorem of Jacob-Rost [**5**] and Rost [**16**, prop. 1] for quadrics of dimension ≥ 3. It attempts to be as exhaustive a description of $\operatorname{Ker} \eta^4$

and $\operatorname{Coker}\eta^4$ as possible; in practice, one will clearly have to look more closely at the two spectral sequences involved (see also corollary 6.7 below).

Corollary 5.5. *For any projective homogeneous variety X, there is an exact sequence after localisation at 2:*

$$0 \to H^1(X, \mathcal{K}_3) \to K_2(E_1) \xrightarrow{d_2^{3,1}} \operatorname{Ker}\eta^4 \to \operatorname{Ker}\xi^4 \to 0.$$

There is another complex, still after localisation at 2:

$$0 \to \operatorname{Coker}\eta^4 \to CH^3(X)_{\mathrm{tors}} \to \operatorname{Coker} d_2^{3,2}$$

whose homology at $\operatorname{Coker}\eta^4$ is isomorphic to the kernel of the map

$$\operatorname{Ker}\left(\operatorname{Coker}\xi^4 \xrightarrow{d} H^3(E_1, 2)\right) \xrightarrow{d_3^{3,2}} H^5(F, 3)$$

and whose homology at $CH^3(X)_{\mathrm{tors}}$ injects into $\operatorname{Coker} d_3^{3,2}$ (a quotient of $H^5(F, 3)$). Here all differentials are those of the spectral sequence (6), and d is the map induced by $d_2^{3,2}$ on $\operatorname{Coker}\xi^4$.
These complexes are natural with respect to extension of scalars and transfer.

Proof. Everything follows from the diagram above if we replace the second complex by

$$0 \to \operatorname{Coker}\eta^4 \to CH^3(X) \to H^6(X, \mathbf{Z}(3)) \tag{9}$$

which is exact at $CH^3(X)$.

Note that the image of $\operatorname{Coker}\eta^4$ sits into $CH^3(X)_{\mathrm{tors}}$, which is the kernel of the natural map

$$CH^3(X) \to CH^3(X_s)^{G_F}.$$

By naturality of (6), this map factors as

$$CH^3(X) \to H^6(X, \mathbf{Z}(3)) \to E_2^{3,3}(X, 3)$$

where the last map is the edge homomorphism of the motivic spectral sequence. Therefore $CH^3(X)_{\mathrm{tors}}$ maps to $F^1H^6(X, \mathbf{Z}(3)) = F^2H^6(X, \mathbf{Z}(3))$. Hence (9) has same homology as

$$0 \to \operatorname{Coker}\eta^4 \to CH^3(X)_{\mathrm{tors}} \to F^2H^6(X, \mathbf{Z}(3)).$$

The latter map induces a homomorphism

$$CH^3(X)_{\mathrm{tors}} \to E_3^{5,1}(X, 3) \subseteq \operatorname{Coker} d_2^{3,2}$$

whose kernel maps to $E_3^{6,0}(X, 3) = \operatorname{Coker} d_3^{3,2}$. The claim on the homology of the second complex of corollary 5.5 at $CH^3(X)_{\mathrm{tors}}$ follows. □

5.5. $n = 4$**.** Here things become even more complicated. We note a complex

$$K_3^M(E_1) \to \operatorname{Ker}\eta^5 \to \operatorname{Ker}(H^2(X, \mathcal{K}_4^M) \to K_2(E_2)) \to 0$$

which is exact except perhaps at $\operatorname{Ker}\eta^5$, where its homology is a quotient of $K_3(E_2)_{\mathrm{ind}}$. Details are left to the reader. We shall go back to this case in section 6 for special X (see corollary 6.8).

6. Differentials

In this section, we partially describe the $d_2^{p,q}$-differentials of the spectral sequences $E_r(X,n)$ of theorem 4.4. We start with the case $p=q=n$. The relevant differential then has the form

$$d_2^{n,n} : \mathbf{Z}^{\pi_0(E_n)} \to Br(E_{n-1})$$

where $\pi_0(E_n)$ denotes the set of connected components of $\operatorname{Spec} E_n$. Write $E_n = \prod_\alpha E_n^\alpha$, $E_{n-1} = \prod_\beta E_{n-1}^\beta$, where the E_n^α and E_{n-1}^β are fields. Let e_α be the basis vector of $\mathbf{Z}^{\pi_0(E_n)}$ which corresponds to E_n^α. Then $d_2^{n,n}$ is determined by the collection

$$(a_{\alpha,\beta})$$

where $a_{\alpha,\beta}$ is the component in $Br(E_{n-1}^\beta)$ of $d_2^{n,n}(e_\alpha)$.

In general, it seems rather intricate to determine these Brauer classes exactly, although it is pretty clear that they involve the algebras appearing in *e.g.* [**19**]. We shall determine them exactly in the next sections when X is a Severi-Brauer variety or a quadric. Meanwhile, let us show that the knowledge of these algebras determine all other d_2 differentials:

Lemma 6.1. *For any a,b,c, the composition*

$$\coprod_\alpha E_2^{a,a}(X_{E_a^\alpha},a)\otimes E_2^{b,0}(X_{E_a^\alpha},c) \xrightarrow{\cup} \coprod_\alpha E_2^{a+b,a}(X_{E_a^\alpha},a+c) \xrightarrow{\sum \operatorname{Cor}_{E_a^\alpha/F}} E_2^{a+b,a}(X,a+c)$$

is surjective.

Proof. Let us write the terms explicitly:

$$\begin{aligned} E_2^{a,a}(X_{E_a^\alpha},a)\otimes E_2^{b,0}(X_{E_a^\alpha},c) &= \mathbf{Z}^{\pi_0(E_a\otimes_F E_a^\alpha)} \otimes H^b(E_a^\alpha,\mathbf{Z}(c))\\ E_2^{a+b,a}(X_{E_a^\alpha},a+c) &= H^b(E_a\otimes_F E_a^\alpha,\mathbf{Z}(c))\\ E_2^{a+b,a}(X,a+c) &= H^b(E_a,\mathbf{Z}(c)) = \coprod_\alpha H^b(E_a^\alpha,\mathbf{Z}(c)).\end{aligned}$$

Lemma 6.1 is now obvious, since the composition maps an element $\sum n_\gamma e_\gamma \otimes x \in \mathbf{Z}^{\pi_0(E_a\otimes_F E_a^\alpha)} \otimes H^b(E_a^\alpha,\mathbf{Z}(c))$ to $\sum n_\gamma x \in H^b(E_a^\alpha,\mathbf{Z}(c)) \subseteq H^b(E_a,\mathbf{Z}(c))$, where the e_γ's are the generators corresponding to the elements of $\pi_0(E_a\otimes_F E_a^\alpha)$. □

In view of theorem 4.4 (ii) and (iii), lemma 6.1 gives us a recipe to compute $d_2^{a+b,c}$ on $E_2^{a+b,a}(X,a+c)$: let $x \in E_2^{a+b,a}(X,a+c)$ and choose elements $\sum n_\gamma^\alpha e_\gamma^\alpha \otimes y^\alpha \in E_2^{a,a}(X_{E_a^\alpha},a)\otimes E_2^{b,0}(X_{E_a^\alpha},c)$ such that $x = \sum \operatorname{Cor}_{E_a^\alpha/F} n_\gamma^\alpha e_\gamma^\alpha \cup y^\alpha$. Then $d_2^{a+b,a}(x) = \sum \operatorname{Cor}_{E_a^\alpha/F} n_\gamma^\alpha a_\gamma^\alpha \cup y^\alpha$, where a_γ^α is the Brauer class corresponding to e_γ^α.

In order to smoothen up later computations, we include the following compatibility between cup-products in étale motivic cohomology and étale cohomology with finite coefficients.

Proposition 6.2. *Let a, b, p, q, m be integers. Then, for any smooth F-variety X, the diagram of Zariski or étale motivic cohomology groups*

$$\begin{tikzcd}
H^a(X,\mathbf{Z}(p)) \times H^b(X,\mathbf{Z}(q)) \arrow[r, "\cup"] & H^{a+b}(X,\mathbf{Z}(p+q)) \\
H^{a-1}(X,\mathbf{Z}/m(p)) \times H^b(X,\mathbf{Z}(q)) \arrow[u, "\partial\times Id"] \arrow[d] & \\
H^{a-1}(X,\mathbf{Z}/m(p)) \times H^b(X,\mathbf{Z}/m(q)) \arrow[r, "\cup"] & H^{a+b-1}(X,\mathbf{Z}/m(p+q)) \arrow[uu, "\partial"']
\end{tikzcd}$$

commutes up to sign, where ∂ denotes boundary morphisms.

Proof. This follows from the obvious commutative diagram in the derived category (say, of big Zariski sheaves)

$$\begin{tikzcd}
\mathbf{Z}(p) \overset{L}{\otimes} \mathbf{Z}(q) \arrow[r] & \mathbf{Z}(p+q) \\
\mathbf{Z}/m(p)[-1] \overset{L}{\otimes} \mathbf{Z}(q) \arrow[u] \arrow[d, "\wr"'] & \\
\mathbf{Z}/m(p)[-1] \overset{L}{\otimes} \mathbf{Z}/m(q) \arrow[r, "\sim"] & \mathbf{Z}/m(p+q)[-1] \arrow[uu]
\end{tikzcd}$$

(the sign comes from the bottom horizontal isomorphism). □

In the next corollary, we assume for simplicity that $E_q = E_{q-1} = F$; the general case is similar but more technical to state.

Corollary 6.3. *Assume that $E_q = E_{q-1} = F$. Let $x = \{x_1, \dots, x_{n-q}\} \in K^M_{n-q}(F) = E_2^{n,q}(X, n)$. Let $a = d_2^{q,q}(1)$, m its order in $Br(F)$ and view a as an element of $H^2(F, \mathbf{Z}/m(1)$ accordingly. Then $d_2^{n,q}(x)$ is the image of*

$$a \cdot (x_1, \dots, x_q) \in H^{n-q+2}(F, \mathbf{Z}/m(n-q+1))$$

by the boundary map

$$H^{n-q+2}(F, \mathbf{Z}/m(n-q+1)) \xrightarrow{\partial} H^{n-q+3}(F, \mathbf{Z}(n-q+1)).$$

□

We conclude this section by giving some important cases when the differentials are 0.

Lemma 6.4. *If the map*

$$CH^n(X) \to CH^n(X_s)^{G_F}$$

is surjective, then $d_r^{n,n}(X, n) = 0$ for all $r \geq 2$. □

Proof. Indeed, this map can be factored as

$$CH^n(X) = H^{2n}_{\text{Zar}}(X, \mathbf{Z}(n)) \to H^{2n}_{\text{ét}}(X, \mathbf{Z}(n)) \to E_2^{n,n}(X, n) = CH^n(X_s)^{G_F}.$$

□

Proposition 6.5. *a) If $CH^q(X) \to CH^q(X_s)$ is surjective, then all differentials $d_r^{p,q}(X,n)$ are 0. In particular, if $CH^q(X) \to CH^q(X_s)$ is surjective for all $q \le n$, then all differentials in the spectral sequence $E(X,n)$ are 0.*
b) If $CH^q(X) \to CH^q(X_s)$ is surjective for all $q \le i/2$, then, for all n, the natural map

$$\bigoplus_{0\le q\le i/2} H^{i-2q}_{\text{ét}}(F,\mathbf{Z}(n-q)) \otimes CH^q(X) \to H^i_{\text{ét}}(X,\mathbf{Z}(n))$$

given by cup-product is surjective.

Proof. a) follows from lemma 6.4 and lemma 6.1. To see b), denote by

$$F^p H^i_{\text{ét}}(X,\mathbf{Z}(n))$$

the p-th step of the (decreasing) filtration induced on $H^i_{\text{ét}}(X,\mathbf{Z}(n))$ by the spectral sequence. We prove by induction on p that the map

$$\bigoplus_{0\le q\le i/2} H^{i-2q}_{\text{ét}}(F,\mathbf{Z}(n-q)) \otimes CH^q(X) \to H^i_{\text{ét}}(X,\mathbf{Z}(n))/F^p H^i_{\text{ét}}(X,\mathbf{Z}(n))$$

is surjective. For $p=0$, this is trivial. For $p>0$, we have by a) a short exact sequence

$$\begin{aligned}0 \to F^p H^i_{\text{ét}}(X,\mathbf{Z}(n)) &\to F^{p-1} H^i_{\text{ét}}(X,\mathbf{Z}(n))\\ &\to E_2^{p,i-p}(X,n) = H^{2p-i}_{\text{ét}}(F, CH^{i-p}(X_s)\otimes \mathbf{Z}(n-i+p)) \to 0.\end{aligned}$$

It remains to see that the map

$$H^{2p-i}_{\text{ét}}(F,\mathbf{Z}(n-i+p))\otimes CH^{i-p}(X) \to H^{2p-i}_{\text{ét}}(F, CH^{i-p}(X_s)\otimes \mathbf{Z}(n-i+p))$$

is surjective. This is clear, since we can factor it as follows:

$$\begin{aligned}H^{2p-i}_{\text{ét}}(F,\mathbf{Z}(n-i+p))\otimes CH^{i-p}(X) &\to H^{2p-i}_{\text{ét}}(F,\mathbf{Z}(n-i+p))\otimes CH^{i-p}(X_s)\\ &\to H^{2p-i}_{\text{ét}}(F, CH^{i-p}(X_s)\otimes \mathbf{Z}(n-i+p))\end{aligned}$$

where the first map is surjective because $CH^{i-p}(X) \to CH^{i-p}(X_s)$ is surjective, and the second one is an isomorphism because $CH^{i-p}(X_s)$ is free. □

Remark 6.6. By [**12**], [**11**, prop. 1.1], the assumption in proposition 6.5 a) is satisfied when X is a quadric of dimension $> 2q$.

As an application, we generalise [**11**, th. 6 (1)] to other projective homogeneous varieties and get a first estimate of $\operatorname{Ker}\eta^5$ and $\operatorname{Coker}\eta^5$ when X has trivial CH^1/tors and CH^2/tors. Both cases apply to quadrics of dimension > 4.

Corollary 6.7. *a) Suppose that $CH^1(X) \to CH^1(X_s)$ is surjective. Then the first exact sequence of corollary 5.5 simplifies to an isomorphism*

$$\operatorname{Ker}\eta^4 \xrightarrow{\sim} \operatorname{Ker}\xi^4.$$

b) Suppose moreover that $CH^2(X) \to CH^2(X_s)$ is surjective. Then there is an exact sequence

$$0 \to \operatorname{Coker}\eta^4 \to CH^3(X)_{\text{tors}} \to H^3(F,2)\otimes CH^1(X)/\text{tors} \oplus H^5(F,3).$$

Proof. By proposition 6.5 a), the differentials $d_r^{p,q}$ are all 0 for $q \leq 2$. This takes care of a). To see b), we first observe that ξ^4 is surjective, by an argument similar to that in the proof of proposition 6.5 b). Taking notation as in the proof of corollary 5.5, we have two short exact sequences (the latter because of the vanishing of differentials)

$$0 \to \operatorname{Coker} \eta^4 \to CH^3(X)_{\mathrm{tors}} \to F^2 H^6(X, \mathbf{Z}(3))$$

$$0 \to H^5(F,3) \to F^2 H^6(X, \mathbf{Z}(3)) \to H^3(E_1, 2) \to 0.$$

But, as usual, the compatibility of the motivic spectral sequences with cup-product provides a canonical splitting of the second exact sequence. Hence the claim. □

Corollary 6.8. *Suppose that $CH^1(X) \to CH^1(X_s)$ and $CH^2(X) \to CH^2(X_s)$ are surjective. Then there are exact sequences*

$$0 \to \operatorname{Ker} \eta^5 \to H^2(X, \mathcal{K}_4^M) \xrightarrow{\xi^5} K_2(F) \otimes CH^2(X)/\mathrm{tors}$$

$$0 \to \operatorname{Coker} \eta^5 \to H^3(X, \mathcal{K}_4^M)_0 \to H^4(F,3) \otimes CH^1(X)/\mathrm{tors} \oplus H^6(F,4)$$

where

$$H^3(X, \mathcal{K}_4^M)_0 := \operatorname{Ker}(H^3(X, \mathcal{K}_4^M) \to H^3(X_s, \mathcal{K}_4^M)^{G_F} = E_3^*).$$

Proof. We argue as above. By proposition 6.5 a), we have a cross of exact sequences (the vertical one from (6), the forking one from the coniveau spectral sequence for étale motivic cohomology)

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
 & & H^5(F,4) \arrow[d] \arrow[dr, "\eta^5"] & \\
0 \arrow[r] & H^2(X, \mathcal{K}_4^M) \arrow[r] \arrow[dr, "\xi^5"'] & H^6(X, \mathbf{Z}(4)) \arrow[r] \arrow[d] & H^0(X, \mathcal{H}^5(4)) \arrow[d] \\
 & & K_2(F) \otimes CH^2(X)/\mathrm{tors} \arrow[d] & H^3(X, K_4^M) \arrow[d] \\
 & & 0 & H^7(X, \mathbf{Z}(4))
\end{tikzcd}$$

This diagram takes care of the first exact sequence. To get the second, it suffices to notice that ξ^5 is surjective, by the argument in the proof of proposition 6.5 b), and to conclude as in the proof of corollary 6.7. □

7. Example: Severi-Brauer varieties

Let A be a central simple algebra of degree d over F and X its Severi-Brauer variety. Then $X_s \simeq \mathbf{P}^{d-1}$, hence $CH^*(X_s)$ is multiplicatively generated by the class h of a hyperplane section, with trivial Galois action. It follows that $E_n = F$ for all n.

Theorem 7.1. *Let A, X be as above and $n \geq 0$. Then, for any $x \in E_2^{p,q}(X,n) = H^{p-q}(F, \mathbf{Z}(n-q))$, we have*

$$d_2^{p,q}(X,n)(x) = q[A]\cdot x \in E_2^{p+2,q-1}(X,n) = H^{p-q+3}(F, \mathbf{Z}(n-q+1))$$

where $[A]$ is the class of A in $Br(F) = H^3(F, \mathbf{Z}(1))$.

Proof. By lemma 6.1 and theorem 4.4 (ii) and (iii), we can reduce to $p = q = n$. Since the algebra $CH^*(X_s)$ is generated in degree 1, we even reduce to $n = 1$. Then the result follows from the exact sequence (7) and Amitsur's theorem that $\operatorname{Ker}(Br(F) \to Br(F(X))$ is generated by $[A]$ [**1**] (note that $Br(X) \to Br(F(X))$ is injective since X is smooth). □

Suppose that A has exponent 2. Then we have

$$d_2^{2,2}(X,2)(1) = 2[A] = 0.$$

We would like to compute $d_3^{2,2}(X,2)(1)$, but this seems out of range at the moment. At least, it is (trivially) 0 in two cases:

- $cd_2(F) \leq 3$;
- $\operatorname{ind}(A) \leq 2$. In this case, X is a conic, hence $CH^2(X_s) = 0\dots$

Moreover, we can show that it is always divisible by 2 in $H^5(F, \mathbf{Z}(2))$. Note that in this case

$$\operatorname{Coker}\xi^3 \simeq \begin{cases} \mathbf{Z}/2 & \text{if } \operatorname{ind} A \leq 4\\ \mathbf{Z}/4 & \text{if } \operatorname{ind} A \geq 8\end{cases}$$

cf. [**13**] for $\operatorname{ind} A \leq 4$ and [**3**, lemma 9.4] for $\operatorname{ind} A \geq 8$.

8. Example: quadrics

Proposition 8.1. *Let X be an isotropic quadric of dimension d over F. Let q be a quadratic form defining X; write $q = q' \perp x_d x_{d+1}$ and let Y be the quadric of equation $q' = 0$. Then:*
a) $M(X) \simeq M(Y)(1)[2] \oplus \mathbf{Z} \oplus \mathbf{Z}(d)[2d]$.
b) For any $n \geq 0$ and $i \in [0,n]$, we have

$$\mathbf{Z}(n,i,X) \simeq \begin{cases} \mathbf{Z}(n) & \text{if } i = 0\\ \mathbf{Z}(n-1,i-1,Y)[-2] \oplus \mathbf{Z}(n) & \text{if } 0 < i < d\\ \mathbf{Z}(n-1,i-1,Y)[-2] \oplus \mathbf{Z}(n) \oplus \mathbf{Z}(n-d)[-2d] & \text{if } i \geq d\end{cases}$$

and

$$\mathbf{Z}(n,i/i-1,X) \simeq \begin{cases} \mathbf{Z}(n) & \text{if } i = 0\\ \mathbf{Z}(n-1,d-1/d-2,Y)[-2] \oplus \mathbf{Z}(n-d)[-2d] & \text{if } i = d\\ \mathbf{Z}(n-1,i-1/i-2,Y)[-2] & \text{if } i \neq 0,d.\end{cases}$$

c) The spectral sequence (6) splits as a direct sum of three spectral sequences

$$E_r^{p,q}(X,n) \simeq E_r^{p-1,q-1}(Y,n-1) \oplus {}'E_r^{p,q} \oplus {}''E_r^{p,q}$$

where $${}'E_2^{p,q} = \begin{cases} H^p_{\text{ét}}(F,\mathbf{Z}(n)) & \text{if } q = 0\\ 0 & \text{otherwise}\end{cases}, \quad {}''E_2^{p,q} = \begin{cases} H^{p-d}_{\text{ét}}(F,\mathbf{Z}(n-d)) & \text{if } q = d\\ 0 & \text{otherwise.}\end{cases}$$

Proof. a) The proof is a variant of that of corollaries 3.5 and 3.6 (see also [**11**, §2]). Let Z be the hyperplane section of equation $x_{d+1} = 0$ and let $P = (0 : \cdots : 1 : 0) \in Z$. We have exact triangles in $DM_{gm}^{eff}(F)$:

$$M^c(Z) \to M^c(X) \to M^c(X \setminus Z) \to M^c(Z)[1]$$
$$M^c(P) \to M^c(Z) \to M^c(Z \setminus P) \to M^c(P)[1].$$

These triangles are split: the first one by homotopy invariance, as $X \setminus Z \simeq \mathbf{A}^n$, and the second via the rational point P (note that Z is proper). Moreover, $Z \setminus P$ is fibred over Y, with affine line fibres. This gives the claim with M^c instead of M; finally we can replace M^c by M since X and Y are both proper.

b) and c) follow immediately from a). □

We are now all set to compute the differentials $d_2(X, n)$ for quadrics. For a quadric X, we denote its dimension by $\dim X$. In case $\dim X$ is even, we denote by $d(X)$ the signed discriminant of an equation q of X; in case $\dim X$ is odd or $\dim X$ is even and $d(X) = 1$, we denote by $c(X)$ the Clifford invariant of q. Let us start with an easy observation:

Lemma 8.2. *For a quadric X and $n \geq 1$, we have*

$$E_n = \begin{cases} F[t]/(t^2 - d(X)) & \text{if } \dim X = 2n \\ F & \text{otherwise.} \end{cases}$$

Here E_n is the n-th étale algebra associated to X in the spectral sequence (6) *(see beginning of section 5).*

This is clear. □

Theorem 8.3. *Let X be a quadric. Then*

- *If $\dim X = 2n$, $d(X) = 1$ or $\dim X = 2n - 1$, then $d_2^{n,n}(X, n)(l) = c(X)$, where l is a plane section of X_s not rational over F.*
- *If $\dim X = 2n - 2$, then $d_2^{n,n}(X, n)(l) = c(X \times_F E)$, where $E = F[t]/(t^2 - d(X))$ and l is a plane section of X_s not rational over F.*
- *Otherwise, $d_2^{n,n}(X, n) = 0$.*

To prove theorem 8.3, we shall need two lemmas:

Lemma 8.4. *Theorem 8.3 is true for $n = 0, 1$.*

Proof. For $n = 0$ this is trivial; for $n = 1$, it follows easily from the exact sequence (7). □

Lemma 8.5. *Theorem 8.3 is true for $\dim X \leq 2$.*

Proof. By lemma 8.4, we may assume $n \geq 2$. If $\dim X < n$, the claim is trivial since then $CH^n(X_s) = 0$. Assume $\dim X = n = 2$. Let Y be a hyperplane section of X. Using theorem 4.4 (iv), we get a commutative diagram

$$\begin{tikzcd} CH^1(Y_s)^{G_F} \arrow[r, "\sim"] \arrow[d, "d_2^{1,1}(Y,1)"'] & CH^2(X_s)^{G_F} \arrow[d, "d_2^{2,2}(X,2)"'] \\ Br(F) \arrow[r] & Br(E_1) \end{tikzcd}$$

where the bottom horizontal map is induced by extension of scalars. The claim now follows from lemma 8.4. □

Proof of theorem 8.3. By proposition 6.4 and, say, [**11**, prop. 1.1], we may assume $\dim X < 2n$ or $\dim X = 2n, d(X) = 1$. We use induction on n, the cases $n = 0, 1$ being dealt with by lemma 8.4. Assume $n > 1$ and the theorem proven in weight $< n$. Let $K = F(X)$ be the function field of X. Then X_K is isotropic. Let Y be a codimension 2 subquadric of X as in proposition 8.1. By that proposition, we have

$$E_2^{n,n}(X_K, n) = \begin{cases} E_2^{n-1,n-1}(Y, n-1) & \text{if } \dim X \neq n \\ E_2^{n-1,n-1}(Y, n-1) \oplus \mathbf{Z} & \text{if } \dim X = n \end{cases}$$

and the differential d_2 vanishes on the summand $\mathbf{Z}$.

For simplicity, let us distinguish two cases:

A) $\dim X < 2n - 2$.
B) $\dim X = 2n - 2$, $\dim X = 2n - 1$ or $\dim X = 2n$, $d(X) = 1$.

We note that if we are in case A) for (X, n), then we are in case A) for $(Y, n-1)$ as well, and similarly for case B).

By lemma 8.5, we may assume $\dim X > 2$. Then $Br(E_{n-1}) \to Br(E_{n-1}(X))$ is injective (see [**2**, p. 269]). In case A), $d_2^{n-1,n-1}(Y, n-1) = 0$ by induction, so $d_2^{n,n}(X, n) = 0$ as well. In case B), let l be a plane section of X of codimension n, not rational over F. Then, still by induction, $d_2^{n,n}(l)_K = c(Y) = c(X)_K$. In other terms, $(d_2^{n,n}(l) - c(X))_K = 0$, hence $d_2^{n,n}(l) = c(X)$. The proof is complete. □

Corollary 8.6. *Let X be a quadric and $E = F[t]/(t^2 - d(X))$.*
a) If $\dim X \neq 2q-2, 2q-1, 2q$, then $d_2^{p,q}(X, n) = 0$.
b) If $\dim X = 2q$, then, for all $x \in E_2^{p,q}(X, n) = H^{p-q}(E, \mathbf{Z}(n-q))$, $d_2^{p,q}(X, n)(x) = \mathrm{Cor}_{E/F}(x \cdot c(X_E)) \in H^{p-q+3}(F, \mathbf{Z}(n-q+1))$.
c) If $\dim X = 2q-1$, then, for all $x \in E_2^{p,q}(X, n) = H^{p-q}(F, \mathbf{Z}(n-q))$, $d_2^{p,q}(X, n)(x) = x \cdot c(X) \in H^{p-q+3}(F, \mathbf{Z}(n-q+1))$.
c) If $\dim X = 2q-2$, then, for all $x \in E_2^{p,q}(X, n) = H^{p-q}(F, \mathbf{Z}(n-q))$, $d_2^{p,q}(X, n)(x) = x_E \cdot c(X_E) \in H^{p-q+3}(E, \mathbf{Z}(n-q+1))$.
Here the cup-product (e.g. by $c(X_E)$) is computed e.g. by identifying $Br(E)$ with $H^3(E, \mathbf{Z}(1))$.

Proof. This follows immediately from lemma 6.1 and theorem 8.3. □

Appendix A. The Čech construction

We take the notation of [**27**].

Theorem A.1. *Let X be a smooth proper integral F-variety, $K = F(X)$, and suppose that X_K is retract-rational in the sense of [**24**]. Then, for all $n \geq 1$, there* *is an exact sequence*

$$0 \to H_B^{n+2}(\check{C}(X), \mathbf{Z}_{(2)}(n)) \to H_{\text{ét}}^{n+1}(F, \mathbf{Q}_2/\mathbf{Z}_2(n)) \xrightarrow{\eta^n} H_{\text{ét,nr}}^{n+1}(K/F, \mathbf{Q}_2/\mathbf{Z}_2(n))$$
$$\to H_B^{n+3}(\check{C}(X), \mathbf{Z}_{(2)}(n)) \to H_{\text{ét}}^{n+2}(F, \mathbf{Q}_2/\mathbf{Z}_2(n)).$$

In other words, there is an isomorphism

$$H_B^{n+2}(\check{C}(X), \mathbf{Z}_{(2)}(n)) \xrightarrow{\sim} \operatorname{Ker} \eta^n$$

and an exact sequence

$$0 \to \operatorname{Coker} \eta^n \to H_B^{n+3}(\check{C}(X), \mathbf{Z}_{(2)}(n)) \to H_{\text{ét}}^{n+2}(F, \mathbf{Q}_2/\mathbf{Z}_2(n)).$$

Proof. By [**27**, th. 2.11], the natural morphism

$$\mathbf{Z}_{(2)}(n) \to \tau_{\leq n+1} R\alpha_* \alpha^* \mathbf{Z}_{(2)}(n)$$

is a quasi-isomorphism for all $n \geq 1$. On the other hand, the morphism

$$\tau_{>n+1} R\alpha_* \alpha^* \mathbf{Q}_2/\mathbf{Z}_2(n) \to \tau_{>n+1}(R\alpha_* \alpha^* \mathbf{Z}_{(2)}(n)[1])$$

is a quasi-isomorphism as well. Putting this together, we get an exact triangle

$$\mathbf{Z}_{(2)}(n) \to R\alpha_* \alpha^* \mathbf{Z}_{(2)}(n) \to (\tau_{>n} R\alpha_* \alpha^* \mathbf{Q}_2/\mathbf{Z}_2(n))[-1] \to \mathbf{Z}_{(2)}(n)[1]$$

and, taking its cohomology, an exact sequence

$$0 \to H_B^{n+2}(\check{C}(X), \mathbf{Z}_{(2)}(n)) \to H_L^{n+2}(\check{C}(X), \mathbf{Z}_{(2)}(n))$$
$$\to H^0_{\text{Zar}}(\check{C}(X), R^{n+1}\alpha_* \alpha^* \mathbf{Q}_2/\mathbf{Z}_2(n))$$
$$\to H_B^{n+3}(\check{C}(X), \mathbf{Z}_{(2)}(n)) \to H_L^{n+3}(\check{C}(X), \mathbf{Z}_{(2)}(n)) \to \dots$$

The second (*resp.* last) group from the left is

$$H_L^{n+2}(F, \mathbf{Z}_{(2)}(n)) \xleftarrow{\sim} H_{\text{ét}}^{n+1}(F, \mathbf{Q}_2/\mathbf{Z}_2(n))$$

(*resp.* $H_L^{n+3}(F, \mathbf{Z}_{(2)}(n)) \xleftarrow{\sim} H_{\text{ét}}^{n+2}(F, \mathbf{Q}_2/\mathbf{Z}_2(n))$) after [**27**]. We compute the third one by means of the simplicial spectral sequence

$$E_1^{p,q} = H^q_{\text{Zar}}(X^{p+1}, R^{n+1}\alpha_* \alpha^* \mathbf{Q}_2/\mathbf{Z}_2(n)) \Rightarrow H^{p+q}_{\text{Zar}}(\check{C}(X), R^{n+1}\alpha_* \alpha^* \mathbf{Q}_2/\mathbf{Z}_2(n))$$

which yields an exact sequence

$$0 \to H^0_{\text{Zar}}(\check{C}(X), R^{n+1}\alpha_* \alpha^* \mathbf{Q}_2/\mathbf{Z}_2(n)) \to H^0_{\text{Zar}}(X, R^{n+1}\alpha_* \alpha^* \mathbf{Q}_2/\mathbf{Z}_2(n))$$
$$\xrightarrow{p_1^* - p_2^*} H^0_{\text{Zar}}(X \times X, R^{n+1}\alpha_* \alpha^* \mathbf{Q}_2/\mathbf{Z}_2(n)),$$

where p_1, p_2 are the first and second projection.

But by assumption on X and the rational invariance of unramified cohomology, p_1^* and p_2^* are isomorphisms with Δ^* as inverse, where Δ is the diagonal. Therefore $p_1^* = p_2^*$ and we get an isomorphism

$$H^0_{\text{Zar}}(\check{C}(X), R^{n+1}\alpha_* \alpha^* \mathbf{Q}_2/\mathbf{Z}_2(n)) \xrightarrow{\sim} H^0_{\text{Zar}}(X, R^{n+1}\alpha_* \alpha^* \mathbf{Q}_2/\mathbf{Z}_2(n))$$
$$= H_{\text{ét,nr}}^{n+1}(K/F, \mathbf{Q}_2/\mathbf{Z}_2(n)).$$

□

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