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The derived functors of unramified cohomology

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To the memory of Vladimir Voevodsky

Abstract We study the first "derived functors of unramified cohomology" in the sense of Kahn and Sujatha (IMRN 2016. doi:10.1093/imrn/rnw184), applied to the sheaves $\mathbb{G}_m$ and $\mathcal{K}_2$. We find interesting connections with classical cycle-theoretic invariants of smooth projective varieties, involving notably a version of the Griffiths group and the group of indecomposable (2, 1)-cycles.

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Contents

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Introduction

To a perfect field F, Voevodsky associates in [25] a *triangulated category of (bounded above) effective motivic complexes* $\mathbf{DM}_-^{\mathrm{eff}}(F) = \mathbf{DM}_-^{\mathrm{eff}}$. In [15], we rather worked with the unbounded version $\mathbf{DM}^{\mathrm{eff}}$. We introduced a *triangulated category of birational motivic complexes* $\mathbf{DM}^{\mathrm{o}}$, and constructed a triple of adjoint functors

$$\mathbf{DM}^{\mathrm{eff}} \begin{array}{c} \xrightarrow{R_{\mathrm{nr}}} \\ \xleftarrow{i^{\mathrm{o}}} \\ \xrightarrow{\nu_{\leq 0}} \end{array} \mathbf{DM}^{\mathrm{o}}$$

with i^{o} fully faithful. Via i^{o}, the homotopy t-structure of $\mathbf{DM}^{\mathrm{eff}}$ induces a t-structure on $\mathbf{DM}^{\mathrm{o}}$ (also called the homotopy t-structure), and the functors $\nu_{\leq 0}$, i^{o} and R_{nr} are respectively right exact, exact and left exact.

The heart of $\mathbf{DM}^{\mathrm{eff}}$ is the abelian category $\mathbf{HI}$ of *homotopy invariant Nisnevich sheaves with transfers* (see [25]). The heart of $\mathbf{DM}^{\mathrm{o}}$ is the thick subcategory $\mathbf{HI}^{\mathrm{o}} \subset \mathbf{HI}$ of *birational sheaves*: an object $\mathcal{F} \in \mathbf{HI}$ lies in $\mathbf{HI}^{\mathrm{o}}$ if and only if it is locally constant for the Zariski topology.

In [15] we also started to study the right adjoint R_{nr}. Let $R^0_{\mathrm{nr}} = \mathcal{H}^0 \circ R_{\mathrm{nr}} : \mathbf{HI} \to \mathbf{HI}^{\mathrm{o}}$ be the induced functor. We proved that R^0_{nr} is given by the formula $R^0_{\mathrm{nr}}\mathcal{F} = \mathcal{F}_{\mathrm{nr}}$, where for a homotopy invariant sheaf $\mathcal{F} \in \mathbf{HI}$, $\mathcal{F}_{\mathrm{nr}}$ is defined by

$$\mathcal{F}_{\mathrm{nr}}(X) = \operatorname{Ker}\left(\mathcal{F}(K) \to \prod_v \mathcal{F}_{-1}(F(v))\right). \tag{1}$$

Here X is a smooth connected F-variety, K is its function field, v runs through all divisorial discrete valuations on K trivial on F, with residue field $F(v)$, and $\mathcal{F}_{-1}$ denotes the contraction of $\mathcal{F}$ (see [24] or [18, Lect. 23]). Thus $R^0_{\mathrm{nr}}\mathcal{F}$ is the *unramified part* of $\mathcal{F}$.

Here is the example which connects the above to the classical situation of unramified cohomology. Let $i \geq 0$, $n \in \mathbf{Z}$ and let m be an integer invertible in F. Then the Nisnevich sheaf $\mathcal{F} = \mathcal{H}^i_{\text{ét}}(\mu_m^{\otimes n})$ associated to the presheaf

$$U \mapsto H^i_{\text{ét}}\left(U, \mu_m^{\otimes n}\right)$$

defines an object of $\mathbf{HI}$, and $R^0_{\mathrm{nr}}\mathcal{F}$ is the usual unramified cohomology [6].

But the functor R_{nr} contains more information: for a general sheaf $\mathcal{F} \in \mathbf{HI}$, the birational sheaves

$$R^q_{\mathrm{nr}}\mathcal{F} = \mathcal{H}^q(R_{\mathrm{nr}}\mathcal{F})$$

need not be 0 for $q > 0$. Can we compute them, at least in some cases?

In this paper, we try our hand at the simplest examples: $\mathcal{F} = \mathbb{G}_m (= \mathcal{K}_1)$ and $\mathcal{F} = \mathcal{K}_2$. We cannot compute explicitly further than $q = 2$, except for varieties of dimension ≤ 2; but this already yields interesting connections with other birational invariants. For simplicity, *we restrict to the case where F is algebraically closed*;

throughout this paper, the cohomology we use is Nisnevich cohomology. The main results are the following:

Theorem 1 *Let X be a connected smooth projective F-variety. Then*

(i) $R^0_{\mathrm{nr}}\mathbb{G}_m(X) = F^*$.
(ii) $R^1_{\mathrm{nr}}\mathbb{G}_m(X) \xrightarrow{\sim} \mathrm{Pic}^\tau(X)$.
(iii) *There is a short exact sequence*

$$0 \to D^1(X) \to R^2_{\mathrm{nr}}\mathbb{G}_m(X) \to \mathrm{Hom}(\mathrm{Griff}_1(X), \mathbf{Z}) \to 0.$$

(iv) *For $q \geq 3$, we have short exact sequences*

$$0 \to \mathrm{Ext}_{\mathbf{Z}}(\mathrm{NS}_1(X, q-3), \mathbf{Z}) \to R^q_{\mathrm{nr}}\mathbb{G}_m(X) \to \mathrm{Hom}_{\mathbf{Z}}(\mathrm{NS}_1(X, q-2), \mathbf{Z}) \to 0. \quad (0.1)$$

Here the notation is as follows: $\mathrm{Pic}^\tau(X)$ *is the group of cycle classes in* $\mathrm{Pic}(X) = CH^1(X)$ *which are numerically equivalent to* 0. *We write* $\mathrm{Griff}_1(X) = \mathrm{Ker}\left(A_1^{\mathrm{alg}}(X) \to N_1(X)\right)$, *where $A_1^{\mathrm{alg}}(X)$ (resp. $N_1(X)$) denotes the group of* 1*-cycles on X modulo algebraic (resp. numerical) equivalence, and*

$$D^1(X) = \mathrm{Coker}\left(N^1(X) \to \mathrm{Hom}(N_1(X), \mathbf{Z})\right)$$

where $N^1(X) = \mathrm{Pic}(X)/\mathrm{Pic}^\tau(X)$ *and the map is induced by the intersection pairing. Finally, the groups* $\mathrm{NS}_1(X, r)$ *are those defined by Ayoub and Barbieri-Viale in* [1, 3.25].

Note that $D^1(X)$ is a finite group since $N_1(X)$ is finitely generated.

After Colliot-Thélène complained that there was no unramified Brauer group in sight, we tried to invoke it by considering

$$\mathbb{G}_m^{\text{ét}} = R\alpha_*\alpha^*\mathbb{G}_m$$

where α is the projection of the étale site on smooth k-varieties onto the corresponding Nisnevich site. There is a natural map $\mathbb{G}_m \to \mathbb{G}_m^{\text{ét}}$, and

Theorem 2 *The map $R^q_{\mathrm{nr}}\mathbb{G}_m \to R^q_{\mathrm{nr}}\mathbb{G}_m^{\text{ét}}$ is an isomorphism for $q \leq 1$, and for $q = 2$ there is an exact sequence for any smooth projective X:*

$$0 \to R^2_{\mathrm{nr}}\mathbb{G}_m(X) \to R^2_{\mathrm{nr}}\mathbb{G}_m^{\text{ét}}(X) \to \mathrm{Br}(X).$$

Considering now $\mathcal{F} = \mathcal{K}_2$:

Theorem 3 *We have an exact sequence*

$$0 \to \operatorname{Pic}^\tau(X)F^* \to R^1_{\mathrm{nr}}\mathcal{K}_2(X) \to H^1_{\mathrm{ind}}(X, \mathcal{K}_2) \xrightarrow{\bar\delta} \operatorname{Hom}(\operatorname{Griff}_1(X), F^*) \to R^2_{\mathrm{nr}}\mathcal{K}_2(X) \to CH^2(X)$$

for any smooth projective variety X. Here

$$H^1_{\mathrm{ind}}(X, \mathcal{K}_2) = \operatorname{Coker}\left(\operatorname{Pic}(X) \otimes F^* \to H^1(X, \mathcal{K}_2)\right)$$

and

$$\operatorname{Pic}^\tau(X)F^* = \operatorname{Im}\left(\operatorname{Pic}^\tau(X) \otimes F^* \to H^1(X, \mathcal{K}_2)\right).$$

The group $H^1(X, \mathcal{K}_2)$ appears in other guises, as the higher Chow group $CH^2(X, 1)$ or as the motivic cohomology group $H^3(X, \mathbf{Z}(2))$; its quotient $H^1_{\mathrm{ind}}(X, \mathcal{K}_2)$ has been much studied and is known to be often nonzero. Note that, while it is not clear from the literature whether there exist smooth projective varieties X such that $\operatorname{Hom}(\operatorname{Griff}_1(X), \mathbf{Z}) \neq 0$, no such issue arises for $\operatorname{Hom}(\operatorname{Griff}_1(X), F^*)$ since F^* is divisible.

The following theorem was suggested by James Lewis. For a prime $l \neq \operatorname{char} F$, write e_l for the exponent of the torsion subgroup of the l-adic cohomology group $H^3(X, \mathbf{Z}_l)$. Then $e_l = 1$ for almost all l: in characteristic 0 this follows from comparison with Betti cohomology, and in characteristic > 0 it is a famous theorem of Gabber [9]. Set $e = \operatorname{lcm}(e_l)$: in characteristic 0, e is the exponent of $H^3_B(X, \mathbf{Z})_{\mathrm{tors}}$, where H^*_B denotes Betti cohomology.

Theorem 4 *Assume that homological equivalence equals numerical equivalence on $CH_1(X) \otimes \mathbf{Q}$. Then, $e\bar\delta = 0$ in Theorem 3.*

0.1 Remarks (1) This hypothesis holds if $\operatorname{char} F = 0$ by Lieberman [17, Cor. 1]. His argument shows that, in characteristic p, it holds for l-adic cohomology if and only if the Tate conjecture holds for divisors on X – more correctly, for divisors on a model of X over a finitely generated field. In particular, it holds if X is an abelian variety; in this case, $e = 1$.

(2) The prime to the characteristic part of the unramified Brauer group also appears in the exact sequence of Theorem 3 as a Tate twist of the torsion of $H^1_{\mathrm{ind}}(X, \mathcal{K}_2)$ [13, Th. 1].

Theorem 5 *Suppose* $\dim X \le 2$ *in Theorems 1 and 3. Then there exists an integer $t > 0$ such that*

(i) $R^2_{\mathrm{nr}}\mathbb{G}_m(X) \simeq D^1(X)$, $R^3_{\mathrm{nr}}\mathbb{G}_m(X) \simeq \operatorname{Ext}_{\mathbf{Z}}(A_1^{\mathrm{alg}}(X), \mathbf{Z}) \simeq \operatorname{Hom}_{\mathbf{Z}}(\operatorname{NS}(X)_{\mathrm{tors}}, \mathbf{Q}/\mathbf{Z})$ *and* $tR^q_{\mathrm{nr}}\mathbb{G}_m(X) = 0$ *for* $q > 3$.

(ii) $tR^q_{\mathrm{nr}}\mathcal{K}_2(X) = 0$ *for* $q > 3$ *and* $R^3_{\mathrm{nr}}\mathcal{K}_2(X) = 0$. *Moreover, if* $\dim X = 2$, *the last map of Theorem 3 identifies $R^2_{\mathrm{nr}}\mathcal{K}_2(X)$ with an extension of the Albanese kernel by a finite group.*

We have $t = 1$ *if* $\dim X < 2$, *and* t *only depends on the Picard variety* $\mathrm{Pic}^0_{X/F}$ *if* $\dim X = 2$.

Let $CH^2(X)_{\mathrm{alg}}$ denote the subgroup of $CH^2(X)$ consisting of cycle classes algebraically equivalent to 0. Recall Murre's higher Abel–Jacobi map

$$AJ^3 : CH^2(X)_{\mathrm{alg}} \to J^3(X)$$

where $J^3(X)$ is an algebraic intermediate Jacobian of X [20]. Theorem 5 (ii) suggests that in general, $\mathrm{Im}(R^2_{\mathrm{nr}}(\mathcal{K}_2)(X) \to CH^2(X))$ should be contained in $\mathrm{Ker}\, AJ^3$.

A key ingredient in the proofs of Theorems 1 and 3 is the work of Ayoub and Barbieri-Viale [1], which identifies the "maximal 0-dimensional quotient" of the Nisnevich sheaf (with transfers) associated to the presheaf $U \mapsto CH^n(X \times U)$ with the group $A^n_{\mathrm{alg}}(X)$ of cycles modulo algebraic equivalence (see (5.3)).

The example $\mathcal{F} = \mathcal{H}^i_{\text{ét}}(\mu_m^{\otimes i})$ considered at the beginning relates to the sheaves studied in Theorems 1 and 3 through the Bloch–Kato conjecture: Kummer theory for $\mathcal{K}_1$ and the Merkurjev–Suslin theorem for $\mathcal{K}_2$. Unfortunately, Theorem 1 barely suffices to compute $R^q_{\mathrm{nr}}(\mathbb{G}_m/m)$ for $q \leq 1$ and we have not been able to deduce from Theorem 3 any meaningful information on $R^*_{\mathrm{nr}}(\mathcal{K}_2/m)$. We give the result for $R^1_{\mathrm{nr}}(\mathbb{G}_m/m)$ without proof; there is an exact sequence, where $\mathrm{NS}(X)$ is the Néron-Severi group of X:

$$0 \to (\mathrm{NS}(X)_{\mathrm{tors}})/m \to R^1_{\mathrm{nr}}(\mathbb{G}_m/m) \to {}_mD^1(X) \to 0$$

and encourage the reader to test his or her insight on this issue.

Let us end this introduction by a comment on the content of the statement "the assignment $X \mapsto \mathcal{F}(X)$ makes $\mathcal{F}$ an object of $\mathbf{HI}^{\mathrm{o}}$", which applies to the objects appearing in Theorems 1 and 3. It implies of course that $\mathcal{F}(X)$ is a (stable) birational invariant of smooth projective varieties, which was already known in most cases; but it also implies some non-trivial functoriality, due to the additional structure of presheaf with transfers on $\mathcal{F}$. For example, it yields a contravariant map $i^* : \mathcal{F}(X) \to \mathcal{F}(Y)$ for any closed immersion $i : Y \to X$. This does not seem easy to prove a priori, say for $\mathcal{F}(X) = D^1(X) = R^2_{\mathrm{nr}}\mathbb{G}_m(X)_{\mathrm{tors}}$ in Theorem 1 (iii).

1 Some results on birational motives

We recall here some results from [15].

1.1 Lemma *For any birational sheaf* $\mathcal{F} \in \mathbf{HI}^{\mathrm{o}}$ *and any smooth variety* X, $H^q(X, \mathcal{F}) = 0$ *for* $q > 0$.

Proof See [15, Prop. 1.3.3 b)]. □

For the next proposition, let us write

$$\nu^{\geq 1}M := \underline{\mathrm{Hom}}(\mathbf{Z}(1), M)(1) \tag{1.1}$$

for $M \in \mathbf{DM}^{\mathrm{eff}}$, where $\underline{\mathrm{Hom}}$ is the internal Hom [25, Prop. 3.2.8].

1.2 Proposition *For M as above, we have a functorial exact triangle*

$$\nu^{\geq 1} M \to M \to i^{\mathrm{o}} \nu_{\leq 0} M \xrightarrow{+1} .$$

Moreover, $M \in \operatorname{Im} i^{\mathrm{o}}$ if and only if $\underline{\mathrm{Hom}}(\mathbf{Z}(1), M) = 0$.

Proof See [15, Prop. 3.6.2 and Lemma 3.5.4]. □

1.3 Proposition *For any $\mathcal{F} \in$ **HI**, the counit map*

$$i^{\mathrm{o}} R^0_{\mathrm{nr}} \mathcal{F} \to \mathcal{F}$$

is a monomorphism.

Proof See [15, Prop. 1.6.3]. □

1.4 Proposition *Let $\mathcal{F} \in$ **HI**. Then $\mathcal{F} \in \mathbf{HI}^0$ if and only if $\mathcal{F}_{-1} = 0$, where $\mathcal{F}_{-1}$ is the contraction of $\mathcal{F}$ (*[24] *or* [18, Lect. 23]*).*

Proof This is [15, Prop. 1.5.2]. □

1.5 Proposition *Let $C \in \mathbf{DM}^{\mathrm{eff}}$, and let $D = \underline{\mathrm{Hom}}(\mathbf{Z}(1)[1], C)$. Then*

$$\mathcal{H}^i(D) = \mathcal{H}^i(C)_{-1}$$

for any $i \in \mathbf{Z}$.

Proof This is [15, (4.1)]. □

2 Computational tools

For $q \geq 0$, the R^q_{nr}'s define a cohomological δ-functor from **HI** to $\mathbf{HI}^{\mathrm{o}}$. Since **HI** is a Grothendieck category (it has a set of generators and exact filtering direct limits), it has enough injectives, so it makes sense to wonder if R^q_{nr} is the q-th derived functor of R^0_{nr}. However, if $\mathcal{I} \in$ **HI** is injective, while $R^0_{\mathrm{nr}}\mathcal{I}$ is clearly injective in $\mathbf{HI}^{\mathrm{o}}$, it is not clear whether $R^q_{\mathrm{nr}}\mathcal{I} = 0$ for $q > 0$: the problem is similar to the one raised in [25, Rk. 1 after Prop. 3.1.8]. (In particular, the title of this paper should be taken with a pinch of salt.) Thus one cannot a priori compute the higher R^q_{nr}'s via injective resolutions; we give here another approach.

2.1 Lemma *Let $\mathcal{F} \in$ **HI**, and let X be a smooth variety. Then the hypercohomology spectral sequence*

$$E_2^{p,q} = H^p\left(X, R^q_{\mathrm{nr}}\mathcal{F}\right) \Rightarrow H^{p+q}(X, R_{\mathrm{nr}}\mathcal{F})$$

degenerates, yielding isomorphisms

$$H^n(X, R_{\mathrm{nr}}\mathcal{F}) \simeq H^0\left(X, R^n_{\mathrm{nr}}\mathcal{F}\right).$$

Proof Indeed, $E_2^{p,q} = 0$ for $p > 0$ by Lemma 1.1. □

2.2 Proposition *Let $C \in \mathbf{DM}^{\mathrm{eff}}$, and let X be a smooth variety. Then we have a long exact sequence*

$$\begin{aligned} \cdots \to H^n(X, R_{\mathrm{nr}} C) \to H^n(X, C) \\ \to \mathbf{DM}^{\mathrm{eff}}(\nu^{\geq 1} M(X), C[n]) \to H^{n+1}(X, R_{\mathrm{nr}} C) \to \dots \end{aligned}$$

In particular, if $C = \mathcal{F}[0]$ for $\mathcal{F} \in \mathbf{HI}$, we get a long exact sequence

$$\begin{aligned} 0 \to R^0_{\mathrm{nr}} \mathcal{F}(X) \to \mathcal{F}(X) \to \mathbf{DM}^{\mathrm{eff}}(\nu^{\geq 1} M(X), \mathcal{F}[0]) \to \dots \\ \to R^n_{\mathrm{nr}} \mathcal{F}(X) \to H^n(X, \mathcal{F}) \to \mathbf{DM}^{\mathrm{eff}}(\nu^{\geq 1} M(X), \mathcal{F}[n]) \to \dots \end{aligned}$$

Proof By iterated adjunction, we have

$$\begin{aligned} H^n(X, R_{\mathrm{nr}} C) \simeq \mathbf{DM}^{\mathrm{eff}}(M(X), i^{\mathrm{o}} R_{\mathrm{nr}} C[n]) \\ \simeq \mathbf{DM}^{\mathrm{o}}(\nu_{\leq 0} M(X), R_{\mathrm{nr}} C[n]) \simeq \mathbf{DM}^{\mathrm{eff}}(i^{\mathrm{o}} \nu_{\leq 0} M(X), C[n]). \end{aligned}$$

The first exact sequence then follows from Proposition 1.2. The second follows from the first, Lemma 2.1 and Proposition 1.3 (b). □

2.3 Proposition *Let X be smooth and proper, and let $n \geq 0$. Then*

$$\underline{\mathrm{Hom}}(\mathbf{Z}(n)[2n], M(X)) \in (\mathbf{DM}^{\mathrm{eff}})^{\leq 0}.$$

Moreover,

$$\mathcal{H}^0 \left(\underline{\mathrm{Hom}}(\mathbf{Z}(n)[2n], M(X))\right) = \underline{CH}_n(X)$$

with

$$\underline{CH}_n(X)(U) = CH_n(X_{F(U)})$$

for any smooth connected variety U. Similarly, we have

$$\underline{\mathrm{Hom}}(M(X), \mathbf{Z}(n)[2n]) \in (\mathbf{DM}^{\mathrm{eff}})^{\leq 0}$$

and

$$\mathcal{H}^0 \left(\underline{\mathrm{Hom}}(M(X), \mathbf{Z}(n)[2n])\right) = \underline{CH}^n(X)$$

with

$$\underline{CH}^n(X)(U) = CH^n(X_{F(U)}).$$

Proof The first statement is [11, Th. 2.2]. The second is proven similarly. □

2.4 Lemma *Let $\mathcal{F} \in \mathbf{HI}^{\mathrm{o}}$. Then $R^q_{\mathrm{nr}} i^{\mathrm{o}} \mathcal{F} = 0$ for $q > 0$.*

Proof This is obvious from the adjunction isomorphism (due to the full faithfulness of i^{o}) $\mathcal{F}[0] \xrightarrow{\sim} R_{\mathrm{nr}} i^{\mathrm{o}} \mathcal{F}[0]$. □

3 Varieties of dimension ≤ 2

As in [25, §3.4], let $d_{\leq 0}\, \mathbf{DM}^{\mathrm{eff}}$ be the localising subcategory of $\mathbf{DM}^{\mathrm{eff}}$ generated by motives of varieties of dimension 0: since F is algebraically closed, this category is equivalent to the derived category $D(\mathbf{Ab})$ of abelian groups [25, Prop. 3.4.1]. In [1, Cor. 2.3.3], Ayoub and Barbieri-Viale show that the inclusion functor

$$j : d_{\leq 0}\, \mathbf{DM}^{\mathrm{eff}} \hookrightarrow \mathbf{DM}^{\mathrm{eff}}$$

has a left adjoint $L\pi_0$.

3.1 Lemma (a) *For any smooth connected variety X, the structural map $X \to$ Spec F induces an isomorphism $L\pi_0 M(X) \xrightarrow{\sim} L\pi_0 \mathbf{Z} = \mathbf{Z}$.*

(b) *We have $L\pi_0 \mathbb{G}_m = 0$.*

(c) *If C is a smooth projective irreducible curve with Jacobian J (viewed as an object of* **HI**), *then $L\pi_0 J = 0$.*

(d) *If A is an abelian variety, viewed as an object of* **HI** *(cf.* [23, Lemma 3.2] *or* [2, Lemma 1.4.4]*), then there exists an integer $t > 0$ such that $tL\pi_0 A = 0$. Moreover, $L_0\pi_0(A) := H_0(L\pi_0(A)) = 0$.*

Proof (a) By adjunction and Yoneda's lemma, we have to show that for any object $C \in D(\mathbf{Ab})$, the map

$$H^*_{\mathrm{Nis}}(F, C) \to H^*_{\mathrm{Nis}}(X, C)$$

is an isomorphism. This is well-known: by a hypercohomology spectral sequence, reduce to C being a single abelian group; then C is flasque (see [21, Lemma 1.40]).

(b) follows from (a), applied to $X = \mathbf{P}^1$ (note that $M(\mathbf{P}^1) \simeq \mathbf{Z} \oplus \mathbb{G}_m[1]$).

(c) Let $M^0(C)$ be the fibre of the map $M(C) \to \mathbf{Z}$. By (a), $L\pi M^0(C) = 0$. By [25, Th. 3.4.2], we have an exact triangle

$$\mathbb{G}_m[1] \to M^0(C) \to J[0] \xrightarrow{+1} \tag{3.1}$$

so the claim follows from (a) and (b).

(d) As is well-known, there exists a curve C with Jacobian J and an epimorphism $J \to A$, which is split up to some integer t by complete reducibility. The first claim then follows from (c).

Let **NST** be the category of Nisnevich sheaves with transfers [25]. To see that $L_0\pi_0(A) = 0$, it is equivalent by adjunction to see that $\mathrm{Hom}_{\mathbf{NST}}(A, \mathcal{F}) = 0$ for any constant $\mathcal{F} \in \mathbf{NST}$. We may identify $\mathcal{F}$ with its value on any connected $X \in \mathbf{Sm}$. Let $f : A \to \mathcal{F}$ be a morphism in **NST**. Evaluating it on $1_A \in A(A)$, we get an element $f(1_A) \in \mathcal{F}(A) = \mathcal{F}$. If $X \in \mathbf{Sm}$ is connected and $a \in A(X) = \mathrm{Hom}_F(X, A)$, then $f(a) = a^* f(1_A) = f(1_A)$. So f is constant, and since it is additive it must send 0 to 0. This proves that $f = 0$, and thus $\mathrm{Hom}_{\mathbf{NST}}(A, \mathcal{F}) = 0$. □

3.2 Proposition *Let X/F be a smooth projective variety of dimension ≤ 2. Then there exists an integer $t = t(X) > 0$ such that $t\,\mathrm{NS}_1(X, i) = 0$ for $i > 0$. We have $t = 1$ for* $\dim X \leq 1$*, and we may take for t the integer associated to* $\mathrm{Pic}^0(X)$ *in Lemma 3.1* (d) *for* $\dim X = 2$.

Proof Recall that $\mathrm{NS}_1(X, i) := H_i(L\pi_0 \underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], M(X)))$ [1, Def. 3.2.5]. For simplicity, write $C_X = \underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], M(X))$. We go case by case, using Poincaré duality as in [11, Lemma B.1]:

If $\dim X = 0$, then $X = \mathrm{Spec}\, F$ and hence $M(X) \simeq \mathbf{Z}$ is a birational motive; therefore $C_X = 0$ (Proposition 1.2) and $L\pi_0 C_X = 0$.

If $\dim X$=1, then Poincaré duality produces an isomorphism

$$C_X \simeq \underline{\mathrm{Hom}}(M(X), \mathbf{Z}) \simeq \mathbf{Z}[0].$$

Hence $L\pi_0 C_X = \mathbf{Z}[0]$.

Now suppose that X is a smooth projective surface. By Poincaré duality, we get an isomorphism

$$C_X \simeq \underline{\mathrm{Hom}}(M(X), \mathbf{Z}(1)[2]). \tag{3.2}$$

By evaluating the latter complex against a varying smooth variety, one computes its homology sheaves as $\mathrm{Pic}_{X/F}$ and $\mathbb{G}_m$ in degrees 0 and 1 respectively and zero elsewhere. Hence we have an exact triangle[1]

$$\mathbb{G}_m[1] \to C_X \to \mathrm{Pic}_{X/F}[0] \xrightarrow{+1} . \tag{3.3}$$

We have $L\pi_0 \mathbb{G}_m[1] = 0$ by Lemma 3.1 (b). On the other hand, the representability of $\mathrm{Pic}_{X/F}$ yields an exact sequence

$$0 \to \mathrm{Pic}^0_{X/F} \to \mathrm{Pic}_{X/F} \to \mathrm{NS}_{X/F} \to 0 \tag{3.4}$$

where $\mathrm{Pic}^0_{X/F}$ is the Picard variety of X and $\mathrm{NS}_{X/F}$ is the (constant) sheaf of connected components of the group scheme $\mathrm{Pic}_{X/F}$. Hence an exact triangle

$$L\pi_0\, \mathrm{Pic}^0_{X/F} \to L\pi_0\, \mathrm{Pic}_{X/F} \to L\pi_0\, \mathrm{NS}_{X/F} \xrightarrow{+1}$$

[1] It is split by the choice of a rational point of X, but this is useless for the proof.

where $L\pi_0\,\mathrm{NS}_{X/F} = \mathrm{NS}(X)$. By Lemma 3.1 (d), $L\pi_0\,\mathrm{Pic}^0_{X/F}$ is torsion, which concludes the proof. (The vanishing of $L_0\pi_0\,\mathrm{Pic}^0_{X/F}$ gives back the isomorphism $L_0\pi_0 C_X \xrightarrow{\sim} \mathrm{NS}(X)$ of [1], see (5.3) below.) □

4 Birational motives and indecomposable (2, 1)-cycles

In this section, we only assume F perfect; we give proofs of two results promised in [15, Rks 3.6.4 and 3.4.2]. These results are not used in the rest of the paper.

For the first one, let X be a smooth projective variety, and let $M = \underline{\mathrm{Hom}}(M(X), \mathbf{Z}(2)[4])$. Note that $M \simeq M(X)$ if $\dim X = 2$ by Poincaré duality (cf. proof of Proposition 3.2). The functor $\nu_{\le 0}$ is right t-exact as the left adjoint of the t-exact functor i^{o} [15, Th. 3.4.1], so $\nu_{\le 0}M \in (\mathbf{DM}^{\mathrm{o}})^{\le 0}$ since $M \in (\mathbf{DM}^{\mathrm{eff}})^{\le 0}$ by Proposition 2.3. We want to compute the last two non-zero cohomology sheaves of $\nu_{\le 0}M$. Here is the result:

4.1 Theorem *With the above notation, we have*

$$\mathcal{H}^i(\nu_{\le 0}M) = \begin{cases} \underline{CH}^2(X) & \text{for } i = 0 \\ \underline{H}^1_{\mathrm{ind}}(X, \mathcal{K}_2) & \text{for } i = -1 \end{cases}$$

where the sections of $\underline{H}^1_{\mathrm{ind}}(X, \mathcal{K}_2)$ over a smooth connected F-variety U with function field K are given by the formula

$$\underline{H}^1_{\mathrm{ind}}(X, \mathcal{K}_2)(U) = \mathrm{Coker}\left(\bigoplus_{[L:K]<\infty} \mathrm{Pic}(X_L) \otimes L^* \to H^1(X_K, \mathcal{K}_2)\right)$$

in which the map is given by products and transfers.

Proof We use the exact triangle of Proposition 1.2. From the cancellation theorem ([26], [11, Prop. A.1]), we get an isomorphism

$$\nu^{\ge 1}M \simeq \underline{\mathrm{Hom}}(M(X), \mathbf{Z}(1)[4])(1) \simeq C_X \otimes \mathbb{G}_m[1]$$

where $C_X = \underline{\mathrm{Hom}}(M(X), \mathbf{Z}(1)[2])$.

By Proposition 2.3, $C_X \in (\mathbf{DM}^{\mathrm{eff}})^{\le 0}$. On the other hand, $\otimes$ is right t-exact because it is induced by a right t-exact $\otimes$-functor on $D(\mathbf{NST})$ via the right t-exact functor $LC : D(\mathbf{NST}) \to \mathbf{DM}^{\mathrm{eff}}$. Hence $\nu^{\ge 1}M \in (\mathbf{DM}^{\mathrm{eff}})^{\le -1}$.

Using Proposition 2.3 again, this shows the assertion in the case $i = 0$ (compare [11, Th. 2.2 and its proof]). For the case $i = -1$, the long exact sequence of cohomology sheaves yields an exact sequence:

$$\cdots \to \mathcal{H}^0(C_X \otimes \mathbb{G}_m) \to \mathcal{H}^{-1}(M) \to \mathcal{H}^{-1}(i^{\mathrm{o}}\nu_{\le 0}M) \to 0.$$

Let $\mathcal{F} = \mathcal{H}^0(C_X) = \underline{CH}^1(X)$; then $\mathcal{H}^0(C_X \otimes \mathbb{G}_m) = \mathcal{F} \otimes_{\mathbf{HI}} \mathbb{G}_m$ by right t-exactness of $\otimes$; here $\otimes_{\mathbf{HI}}$ is the tensor structure induced by $\otimes$ on $\mathbf{HI}$. For any function field K/F, the map induced by transfers

$$\bigoplus_{[L:K]<\infty} \mathcal{F}(L) \otimes \mathbb{G}_m(L) \to (\mathcal{F} \otimes_{\mathbf{HI}} \mathbb{G}_m)(K)$$

is surjective [16, 2.14], which concludes the proof. □

The second result which was promised in [15, Rk. 4.3.2] is:

4.2 Proposition *Let E be an elliptic curve over F. Then the sheaf*

$$\operatorname{Tor}_1^{\mathbf{DM}}(E, E) := \mathcal{H}^{-1}(E[0] \otimes E[0])$$

is not birational. Here E is viewed as an object of **HI** [2, Lemma 1.4.4].

(This contrasts with the fact that the tensor product of two birational sheaves is birational, [15, Th. 4.3.1].)

Proof Up to extending scalars, we may and do assume that $\operatorname{End}(E) = \operatorname{End}(E_{\bar{F}})$. Consider the surface $X = E \times E$. The choice of the rational point $0 \in E$ yields a Chow–Künneth decomposition of the Chow motive of E, hence by [25, Prop. 2.1.4] an isomorphism

$$M(E) \simeq \mathbf{Z}[0] \oplus E[0] \oplus \mathbf{Z}(1)[2]$$

(compare also [25, Th. 3.4.2]). Therefore

$$M(X) \simeq \mathbf{Z}[0] \oplus 2E[0] \oplus 2\mathbf{Z}(1)[2] \oplus E[0] \otimes E[0] \oplus 2E(1)[2] \oplus \mathbf{Z}(2)[4].$$

This allows us to compute $\underline{\operatorname{Hom}}(\mathbf{Z}(1)[2], E[0] \otimes E[0])$ as a direct summand of $\underline{\operatorname{Hom}}(\mathbf{Z}(1)[2], M(X)) = C_X$. First we have

$$\underline{\operatorname{Hom}}(\mathbf{Z}(1)[2], \mathbf{Z}[0]) = \underline{\operatorname{Hom}}(\mathbf{Z}(1)[2], E[0]) = 0.$$

The first vanishing is [11, Lemma A.2], while the second one follows from the Poincaré duality isomorphism $\underline{\operatorname{Hom}}(\mathbf{Z}(1)[2], M(E)) \simeq \underline{\operatorname{Hom}}(M(E), \mathbf{Z}) = \mathbf{Z}$ [12, Lemma 2.1 a)]. Hence, using the cancellation theorem:

$$C_X \simeq 2\mathbf{Z}[0] \oplus \underline{\operatorname{Hom}}(\mathbf{Z}(1)[2], E[0] \otimes E[0]) \oplus 2E[0] \oplus \mathbf{Z}(1)[2]$$

and

$$\operatorname{Pic}_{X/F} = \mathcal{H}^0(C_X) \simeq 2\mathbf{Z} \oplus \mathcal{H}^0(\underline{\operatorname{Hom}}(\mathbf{Z}(1)[2], E[0] \otimes E[0])) \oplus 2E.$$

On the other hand, using Weil's formula for the Picard group of a product, we have a canonical decomposition

$$\operatorname{Pic}_{E\times E/F} \simeq \operatorname{Pic}^0_{E\times E/F} \oplus \operatorname{NS}(E) \oplus \operatorname{NS}(E) \oplus \operatorname{Hom}(E, E) = 2E \oplus 2\mathbf{Z} \oplus \operatorname{End}(E).$$

One checks that the idempotents involved in the two decompositions of $\mathrm{Pic}_{X/F}$ match to yield an isomorphism

$$\mathcal{H}^0(\underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], E[0] \otimes E[0])) \simeq \mathrm{End}(E)$$

where $\mathrm{End}(E)$ is viewed as a constant sheaf. By the t-exactness of Voevodsky's contraction functor $(-)_{-1} = \underline{\mathrm{Hom}}(\mathbb{G}_m, -)$ [15, Prop. 4.1.1], this yields an isomorphism $\mathrm{End}(E) \xrightarrow{\sim} \mathrm{Tor}_1^{\mathbf{DM}}(E, E)_{-1}$, which proves that $\mathrm{Tor}_1^{\mathbf{DM}}(E, E)$ is not birational (see Proposition 1.4). ◻

5 The case of $\mathbb{G}_m$: proof of Theorems 1, 2 and 5 (i)

5.1 Proof of Theorem 1

We apply Proposition 2.2 to $\mathcal{F} = \mathbb{G}_m$. The Nisnevich cohomology of $\mathbb{G}_m$ is well-known: we have

$$H^n(X, \mathbb{G}_m) = \begin{cases} F^* & \text{if } n = 0 \\ \mathrm{Pic}(X) & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Noting that $\mathbb{G}_m[0] = \mathbf{Z}(1)[1]$ in $\mathbf{DM}^{\mathrm{eff}}$, we get

$$\begin{aligned} &\mathbf{DM}^{\mathrm{eff}}(\nu^{\geq 1} M(X), \mathbb{G}_m[n]) = \\ &\quad \mathbf{DM}^{\mathrm{eff}}(\underline{\mathrm{Hom}}(\mathbf{Z}(1), M(X))(1), \mathbf{Z}(1)[n+1]) \\ &\quad = \mathbf{DM}^{\mathrm{eff}}(\underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], M(X)), \mathbf{Z}[n-1]) \end{aligned}$$

by using the cancellation theorem. Thus

$$\begin{aligned} &\mathbf{DM}^{\mathrm{eff}}(\underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], M(X)), \mathbf{Z}[n-1]) \\ &\quad \simeq d_{\leq 0}\, \mathbf{DM}^{\mathrm{eff}}(L\pi_0\, \underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], M(X)), \mathbf{Z}[n-1]) \\ &\quad = D(\mathbf{Ab})(L\pi_0\, \underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], M(X)), \mathbf{Z}[n-1]) =: F_n(X). \end{aligned} \tag{5.1}$$

The homology group $H_s(L\pi_0\, \underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], M(X)))$ is denoted by $\mathrm{NS}_1(X, s)$ in [1, 3.25]. The universal coefficients theorem then gives an exact sequence

$$\begin{aligned} 0 \to \mathrm{Ext}_{\mathbf{Ab}}(\mathrm{NS}_1(X, n-2), \mathbf{Z}) \to F_n(X) \\ \to \mathbf{Ab}(\mathrm{NS}_1(X, n-1), \mathbf{Z}) \to 0. \end{aligned} \tag{5.2}$$

By Proposition 2.3, $\underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], M(X)) \in (\mathbf{DM}^{\mathrm{eff}})^{\leq 0}$. Since the inclusion functor j is t-exact, $L\pi_0$ is right t-exact by a general result on triangulated categories [3, Prop. 1.3.17], hence $\mathrm{NS}_1(X, n) = 0$ for $n < 0$. For $n = 0$, Ayoub and Barbieri-Viale find

$$\mathrm{NS}_1(X, 0) = A_1^{\mathrm{alg}}(X) \tag{5.3}$$

in [1, Th. 3.1.4][2].

Gathering all this, we get (i) (which also follows from (1)), an exact sequence

$$0 \to R^1_{\mathrm{nr}}\mathbb{G}_m(X) \to \mathrm{Pic}(X) \xrightarrow{\delta} \mathrm{Hom}\left(A_1^{\mathrm{alg}}(X), \mathbf{Z}\right) \to R^2_{\mathrm{nr}}\mathbb{G}_m(X) \to 0 \tag{5.4}$$

and isomorphisms

$$F_n(X) \xrightarrow{\sim} R^{n+1}_{\mathrm{nr}}\mathbb{G}_m(X) \tag{5.5}$$

for $n \geq 2$, which yield (iv) thanks to (5.4).

In Lemma 5.1 below, we shall show that δ is induced by the intersection pairing. Granting this for the moment, (ii) is immediate and we get a cross of exact sequences

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \mathrm{Hom}(N_1(X), \mathbf{Z}) \arrow[d] & & \\
0 \arrow[r] & N^1(X) \arrow[r] \arrow[ur] & \mathrm{Hom}(A_1^{\mathrm{alg}}(X), \mathbf{Z}) \arrow[r] \arrow[d] & R^2_{\mathrm{nr}}\mathbb{G}_m(X) \arrow[r] & 0 \\
 & & \mathrm{Hom}(\mathrm{Griff}_1(X), \mathbf{Z}) \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

in which the triangle commutes, and where we used that $N_1(X)$ is a free finitely generated abelian group. The exact sequence of (iii) then follows from a diagram chase.

5.1 Lemma *The map δ of (5.4) is induced by the intersection pairing.*

Proof This map comes from the composition

$$\begin{aligned}
&\mathbf{DM}^{\mathrm{eff}}(M(X), \mathbf{Z}(1)[2]) \\
&\quad \to \mathbf{DM}^{\mathrm{eff}}(\underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], M(X))(1)[2], \mathbf{Z}(1)[2]) \\
&\quad = \mathbf{DM}^{\mathrm{eff}}(\underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], M(X)), \mathbf{Z}) \\
&\quad \to \mathrm{Hom}_{\mathbf{Z}}(\mathrm{Hom}(\mathbf{Z}(1)[2], M(X)), \mathbf{Z})
\end{aligned} \tag{5.6}$$

in which the first map is induced by the canonical morphism $\nu^{\geq 1}M(X) \to M(X)$, the equality follows from the cancellation theorem [26] and the third is by taking global sections at $\mathrm{Spec}\, k$.

[2] The hypothesis F algebraically closed is sufficient for their proof.

Consider the natural pairing

$$\begin{aligned}&\underline{\mathrm{Hom}}(M(X),\mathbf{Z}(1)[2])\otimes\underline{\mathrm{Hom}}(\mathbf{Z}(1)[2],M(X))\\&\quad\to\underline{\mathrm{Hom}}(\mathbf{Z}(1)[2],\mathbf{Z}(1)[2])=\mathbf{Z}[0].\end{aligned}$$

By Proposition 2.3, this pairing factors through a pairing

$$\underline{CH}^1(X)[0]\otimes\underline{CH}_1(X)[0]\to\mathbf{Z}[0].$$

Taking global sections, we clearly get the intersection pairing.

From the above, we get a commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(M(X),\mathbf{Z}(1)[2]) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(\mathbf{Z}(1)[2],M(X)),\mathbf{Z}[0]) \\
\underline{CH}^1(X)[0] \arrow[r] & \underline{\mathrm{Hom}}(\underline{CH}_1(X)[0],\mathbf{Z}[0]). \arrow[u]
\end{tikzcd}$$

Applying the functor $\mathbf{DM}^{\mathrm{eff}}(\mathbf{Z},-)$ to this diagram, we get a commutative diagram of abelian groups

$$\begin{tikzcd}
\mathbf{DM}^{\mathrm{eff}}(M(X),\mathbf{Z}(1)[2]) \arrow[r, "a"] \arrow[d] & \mathbf{DM}^{\mathrm{eff}}(\underline{\mathrm{Hom}}(\mathbf{Z}(1)[2],M(X)),\mathbf{Z}[0]) \\
CH^1(X) \arrow[r] & \mathbf{DM}^{\mathrm{eff}}(\underline{CH}_1(X)[0],\mathbf{Z}[0]). \arrow[u, "b"]
\end{tikzcd}$$

In this diagram, one checks easily that a corresponds to (5.6) via the cancellation theorem. On the other hand, b is an isomorphism. Now the evaluation functor at $\operatorname{Spec} F$, $\mathcal{F}\mapsto\mathcal{F}(F)$, yields a commutative triangle

$$\begin{tikzcd}
CH^1(X) \arrow[r] \arrow[dr, "\cap"'] & \mathbf{DM}^{\mathrm{eff}}(\underline{CH}_1(X)[0],\mathbf{Z}[0]) \arrow[d, "ev_F"] \\
& \mathrm{Hom}(CH_1(X),\mathbf{Z}).
\end{tikzcd}$$

where $\cap$ is the intersection pairing (see above). But we saw that $\mathbf{DM}^{\mathrm{eff}}(\underline{CH}_1(X)[0],\mathbf{Z}[0])\simeq\mathrm{Hom}(A_1^{\mathrm{alg}}(X),\mathbf{Z})$ ((5.1), (5.2) and (5.3)); via this isomorphism, ev_F is induced by the surjection $CH^1(X)\twoheadrightarrow A^1_{\mathrm{alg}}(X)$, hence is injective. This concludes the proof. □

5.2 Proof of Theorem 2

We use the following lemma:

5.2 Lemma *In* $\mathbf{DM}^{\mathrm{eff}}$*, the map* $\mathbb{G}_m \to \mathbb{G}_m^{\text{ét}}$ *is an isomorphism on* H^0*; moreover,* $R^1\alpha_*\alpha^*\mathbb{G}_m = 0$ *and* $R^2\alpha_*\alpha^*\mathbb{G}_m$ *is the Nisnevich sheaf* Br *associated to the presheaf* $U \mapsto \mathrm{Br}(U)$[3]*. Here,* $\alpha : \mathbf{Sm}_{\text{ét}} \to \mathbf{Sm}_{\mathrm{Nis}}$ *is the change of topology morphism.*

Proof The first statement is obvious, the second one follows from the local vanishing of Pic and the third one is tautological. □

To compute $R_{\mathrm{nr}}\mathbb{G}_m^{\text{ét}}$, we may use the "hypercohomology" spectral sequence

$$E_2^{p,q} = R^p_{\mathrm{nr}} R^q \alpha_*\alpha^*\mathbb{G}_m \Rightarrow R^{p+q}_{\mathrm{nr}}\mathbb{G}_m^{\text{ét}}.$$

From Lemma 5.2, we find an isomorphism

$$R^1_{\mathrm{nr}}\mathbb{G}_m \xrightarrow{\sim} R^1_{\mathrm{nr}}\mathbb{G}_m^{\text{ét}}$$

and a five term exact sequence

$$0 \to R^2_{\mathrm{nr}}\mathbb{G}_m \to R^2_{\mathrm{nr}}\mathbb{G}_m^{\text{ét}} \to R^0_{\mathrm{nr}}\,\mathrm{Br} \to R^3_{\mathrm{nr}}\mathbb{G}_m \to R^3_{\mathrm{nr}}\mathbb{G}_m^{\text{ét}}$$

which yields (a more precise form of) Theorem 2 in view of the obvious isomorphism $R^0_{\mathrm{nr}}\,\mathrm{Br} = \mathrm{Br}_{\mathrm{nr}}$, where $\mathrm{Br}_{\mathrm{nr}}$ is the unramified Brauer group.

5.3 Proof of Theorem 5 (i)

Since $\dim X \le 2$, $\mathrm{Griff}_1(X)$ is torsion hence $\mathrm{Hom}(\mathrm{Griff}_1(X), \mathbf{Z}) = 0$, which gives the first statement. Then, Theorem 1 (iv) and Proposition 3.2 yield isomorphisms

$$\mathrm{Ext}_{\mathbf{Z}}(\mathrm{NS}_1(X, q-3), \mathbf{Z}) \xrightarrow{\sim} R^q_{\mathrm{nr}}\mathbb{G}_m(X), \quad q \ge 3.$$

For $q > 3$, the left hand group is killed by the integer t of Proposition 3.2. Suppose $q = 3$; then $\mathrm{NS}_1(X, q-3) = A_1^{\mathrm{alg}}(X)$, which proves Theorem 5 (i) except for the isomorphism involving $\mathrm{NS}(X)_{\mathrm{tors}}$. For this we distinguish 3 cases:

(1) If $\dim X = 0$, $A_1^{\mathrm{alg}}(X) = \mathrm{NS}(X) = 0$ and the statement is true.
(2) If $\dim X = 1$, $A_1^{\mathrm{alg}}(X) \simeq \mathbf{Z} \simeq \mathrm{NS}(X)$ and the statement is still true.
(3) If $\dim X = 2$, $A_1^{\mathrm{alg}}(X) = \mathrm{NS}(X)$. But for any finitely generated abelian group A, there is a string of canonical isomorphisms

$$\mathrm{Ext}_{\mathbf{Z}}(A, \mathbf{Z}) \xrightarrow{\sim} \mathrm{Ext}_{\mathbf{Z}}(A_{\mathrm{tors}}, \mathbf{Z}) \xleftarrow{\sim} \mathrm{Hom}_{\mathbf{Z}}(A_{\mathrm{tors}}, \mathbf{Q}/\mathbf{Z}).$$

This concludes the proof.

[3] This presheaf is in fact already a Nisnevich sheaf.

6 The case of $\mathcal{K}_2$: proof of Theorems 3 and 5 (ii)

6.1 Preparations

6.1 Lemma (a) *The natural map*

$$\mathbf{Z}(2)[2] \to \mathcal{K}_2[0] \tag{6.1}$$

induces an isomorphism

$$\operatorname{cone}\left(i^{\mathrm{o}} R_{\mathrm{nr}}\mathbf{Z}(2)[2] \to \mathbf{Z}(2)[2]\right) \xrightarrow{\sim} \operatorname{cone}\left(i^{\mathrm{o}} R_{\mathrm{nr}}\mathcal{K}_2[0] \to \mathcal{K}_2[0]\right).$$

(b) *The map* (6.1) *induces an isomorphism*

$$\mathbf{DM}^{\mathrm{eff}}(\nu^{\geq 1}C, \mathbf{Z}(2)[2]) \xrightarrow{\sim} \mathbf{DM}^{\mathrm{eff}}(\nu^{\geq 1}C, \mathcal{K}_2[0])$$

for any $C \in \mathbf{DM}^{\mathrm{eff}}$. (See (1.1) *for the definition of $\nu^{\geq 1}C$.)*

Proof By the cancellation theorem, we have

$$\underline{\mathrm{Hom}}(\mathbf{Z}(1)[1], \mathbf{Z}(2)[2]) \simeq \mathbf{Z}(1)[1] \simeq \mathbb{G}_m[0]$$

in $\mathbf{DM}^{\mathrm{eff}}$.

Let $\mathcal{H}^i(C)$ denote the i-th cohomology sheaf of an object $C \in \mathbf{DM}^{\mathrm{eff}}$. By Proposition 1.5, the i-th cohomology sheaf of the left hand side is $\mathcal{H}^i(\mathbf{Z}(2)[2])_{-1}$. Thus the latter sheaf is 0 for $i \neq 0$. By Proposition 1.4, $\mathcal{H}^i(\mathbf{Z}(2)[2]) \in \mathbf{HI}^{\mathrm{o}}$ for $i \neq 0$, hence $\tau_{<0}(\mathbf{Z}(2)[2]) \in \mathbf{DM}^{\mathrm{o}}$. By adjunction, we deduce

$$\operatorname{cone}\left(i^{\mathrm{o}} R_{\mathrm{nr}}\tau_{<0}(\mathbf{Z}(2)[2]) \to \tau_{<0}(\mathbf{Z}(2)[2])\right) = 0$$

which in turn implies (a).

To pass from (a) to (b), use the fact that, for $C, D \in \mathbf{DM}^{\mathrm{eff}}$, adjunction transforms the exact sequence

$$\mathbf{DM}^{\mathrm{eff}}(i^{\mathrm{o}}\nu_{\leq 0}C, D) \to \mathbf{DM}^{\mathrm{eff}}(C, D) \to \mathbf{DM}^{\mathrm{eff}}(\nu^{\geq 1}C, D)$$

into the exact sequence

$$\begin{aligned}\mathbf{DM}^{\mathrm{eff}}(C, i^{\mathrm{o}} R_{\mathrm{nr}} D) &\to \mathbf{DM}^{\mathrm{eff}}(C, D)\\ &\to \mathbf{DM}^{\mathrm{eff}}(C, \operatorname{cone}(i^{\mathrm{o}} R_{\mathrm{nr}} D \to D)).\end{aligned}$$

□

Applying the exact sequence of Proposition 2.2 to $C = \mathcal{K}_2[0]$ and using Lemma 6.1 (b), we get a long exact sequence

$$\begin{aligned}\cdots &\to H^n(X, R_{\mathrm{nr}}\mathcal{K}_2) \to H^n(X, \mathcal{K}_2)\\ &\to \mathbf{DM}^{\mathrm{eff}}(\nu^{\geq 1}M(X), \mathbf{Z}(2)[n+2]) \to H^{n+1}(X, R_{\mathrm{nr}}\mathcal{K}_2) \to \dots\end{aligned}$$

Using the cancellation theorem, we get an isomorphism

$$\mathbf{DM}^{\mathrm{eff}}(\nu^{\geq 1}M(X), \mathbf{Z}(2)[n+2]) \simeq \mathbf{DM}^{\mathrm{eff}}(\underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], M(X)), \mathbf{Z}(1)[n]).$$

Since $\mathbf{Z}(1)[n] = \mathbb{G}_m[n-1]$, using Lemma 2.1 we get an exact sequence

$$\begin{aligned}
&0 \to (R^1_{\mathrm{nr}}\mathcal{K}_2)(X) \to H^1(X, \mathcal{K}_2)\\
&\xrightarrow{\delta} \mathbf{DM}^{\mathrm{eff}}(\underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], M(X)), \mathbb{G}_m[0]) \to (R^2_{\mathrm{nr}}\mathcal{K}_2)(X) \to CH^2(X)\\
&\xrightarrow{\varphi} \mathbf{DM}^{\mathrm{eff}}(\underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], M(X)), \mathbb{G}_m[1]) \to (R^3_{\mathrm{nr}}\mathcal{K}_2)(X) \to 0
\end{aligned} \tag{6.2}$$

and isomorphisms for $q > 3$

$$\mathbf{DM}^{\mathrm{eff}}(\underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], M(X)), \mathbb{G}_m[q-2]) \xrightarrow{\sim} (R^q_{\mathrm{nr}}\mathcal{K}_2)(X) \tag{6.3}$$

where we also used that $H^2(X, \mathcal{K}_2) \simeq CH^2(X)$ and $H^i(X, \mathcal{K}_2) = 0$ for $i > 2$.

6.2 Proof of Theorem 3

The group $\mathbf{DM}^{\mathrm{eff}}(\underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], M(X)), \mathbb{G}_m[0])$ may be computed as follows:

$$\begin{aligned}
&\mathbf{DM}^{\mathrm{eff}}(\underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], M(X)), \mathbb{G}_m[0])\\
&\overset{1}{\simeq} \mathbf{HI}(\mathcal{H}_0(\underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], M(X))), \mathbb{G}_m)\\
&\overset{2}{\simeq} \mathbf{HI}(\underline{CH}_1(X), \mathbb{G}_m) \overset{3}{\simeq} \mathbf{HI}^{\mathrm{o}}(\underline{CH}_1(X), R^0_{\mathrm{nr}}\mathbb{G}_m)\\
&\overset{4}{\simeq} \mathbf{HI}(\underline{CH}_1(X), jF^*) \overset{5}{\simeq} \mathbf{Ab}(L_0\pi_0\underline{CH}_1(X), F^*)\\
&\overset{6}{\simeq} \mathbf{Ab}(A_1^{\mathrm{alg}}(X), F^*).
\end{aligned} \tag{6.4}$$

Here, isomorphism 1 follows from the fact that $\underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], M(X)) \in (\mathbf{DM}^{\mathrm{eff}})^{\leq 0}$ (Proposition 2.3), 2 comes from the computation of $\mathcal{H}_0$ (ibid.), 3 follows from adjunction, knowing that $\underline{CH}_1(X)$ is a birational sheaf (ibid.), 4 follows from Theorem 1 (i), 5 comes from adjunction and 6 follows from (5.3).

Thus the homomorphism δ corresponds to a pairing

$$H^1(X, \mathcal{K}_2) \times A_1^{\mathrm{alg}}(X) \to F^*.$$

Let $d = \dim X$. An argument analogous to that in the proof of Lemma 5.1 shows that this pairing comes from the "intersection" pairing

$$\begin{aligned}
&H^3(X, \mathbf{Z}(2)) \times H^{2d-2}(X, \mathbf{Z}(d-1)) \xrightarrow{\cap} H^{2d+1}(X, \mathbf{Z}(d+1))\\
&\xrightarrow{\pi_*} H^1(F, \mathbf{Z}(1)) = F^*
\end{aligned} \tag{6.5}$$

where the last map is induced by the "Gysin" morphism ${}^t\pi : \mathbf{Z}(d)[2d] \to M(X)$. Here we used the isomorphisms

$$H^1(X, \mathcal{K}_2) \simeq H^3(X, \mathbf{Z}(2)), \quad CH_1(X) \simeq H^{2d-2}(X, \mathbf{Z}(d-1)).$$

In particular, (6.5) factors through algebraic equivalence. This was proven by Coombes [8, Cor. 2.14] in the special case of a surface; we shall give a different proof below, which avoids the use of (5.3).

Consider the product map

$$c : CH^1(X) \otimes F^* = H^1(X, \mathcal{K}_1) \otimes H^0(X, \mathcal{K}_1) \to H^1(X, \mathcal{K}_2).$$

By functoriality, we have a commutative diagram of pairings

$$\begin{tikzcd}
CH^1(X) \otimes F^* \times A_1^{\mathrm{alg}}(X) \arrow[r] \arrow[d, "c\times 1"'] & F^* \arrow[d, equal] \\
H^1(X, \mathcal{K}_2) \times A_1^{\mathrm{alg}}(X) \arrow[r] & F^*
\end{tikzcd}$$

where the top pairing is the intersection pairing $CH^1(X) \times A_1^{\mathrm{alg}}(X) \to \mathbf{Z}$, tensored with F^*. Since the latter is 0 when restricted to $\mathrm{Griff}_1(X)$, we get an induced pairing

$$H^1_{\mathrm{ind}}(X, \mathcal{K}_2) \times \mathrm{Griff}_1(X) \to F^*$$

yielding a commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] \\
0 \arrow[r] \mathrm{Pic}^\tau(X) \otimes F^* \arrow[r] & \mathrm{Pic}(X) \otimes F^* \arrow[r, "\alpha"] \arrow[d] & \mathrm{Hom}(A_1^{\mathrm{num}}(X), F^*) \arrow[d] \\
0 \arrow[r] (R^1_{\mathrm{nr}}\mathcal{K}_2)(X) \arrow[r] & H^1(X, \mathcal{K}_2) \arrow[r, "\delta"] \arrow[d] & \mathrm{Hom}(A_1^{\mathrm{alg}}(X), F^*) \arrow[d] \\
 & H^1_{\mathrm{ind}}(X, \mathcal{K}_2) \arrow[r, "\bar\delta"] \arrow[d] & \mathrm{Hom}(\mathrm{Griff}_1(X), F^*). \arrow[d] \\
 & 0 & 0
\end{tikzcd}$$

In this diagram, all rows and columns are complexes. The middle row and the two columns are exact; moreover, α is surjective as one sees by tensoring with F^* the exact sequence

$$0 \to \mathrm{Pic}^\tau(X) \to \mathrm{Pic}(X) \to \mathrm{Hom}(A_1^{\mathrm{num}}(X), \mathbf{Z}) \to D^1(X) \to 0.$$

Then a diagram chase yields an exact sequence

$$\mathrm{Pic}^\tau(X) \otimes F^* \to (R^1_{\mathrm{nr}}\mathcal{K}_2)(X) \to H^1_{\mathrm{ind}}(X, \mathcal{K}_2) \xrightarrow{\bar\delta} \mathrm{Hom}(\mathrm{Griff}_1(X), F^*)$$

and the surjectivity of α implies that the map $\mathrm{Hom}(A_1^{\mathrm{alg}}(X), F^*) \to (R^2_{\mathrm{nr}}\mathcal{K}_2)(X)$ given by (6.2) and (6.4) factors through $\mathrm{Hom}(\mathrm{Griff}_1(X), F^*)$. This concludes the proof.

6.3 Direct proof that (6.5) factors through algebraic equivalence

Consider classes $\alpha \in H^3(X, \mathbf{Z}(2))$ and $\beta \in CH^{d-1}(X)$: assuming that β is algebraically equivalent to 0, we must prove that $\pi_*(\alpha \cdot \beta) = 0$, where π is the projection $X \to \mathrm{Spec}\, F$.

By hypothesis, there exists a smooth projective curve C, two points $c_0, c_1 \in C$ and a cycle class $\gamma \in CH^{d-1}(X \times C)$ such that $\beta = c_0^*\gamma - c_1^*\gamma$. Let $\pi_X : X \times C \to X$ and $\pi_C : X \times C \to C$ be the two projections.

The Gysin morphism ${}^t\pi : \mathbf{Z}(d)[2d] \to M(X)$ used in the definition of (6.5) extends trivially to give morphisms $M(d)[2d] \to M \otimes M(X)$ for any $M \in \mathbf{DM}^{\mathrm{eff}}$, which are clearly natural in M: this applies in particular to $M = M(C)$, giving a Gysin morphism ${}^t\pi_C : M(C)(d)[2d] \to M(X \times C)$ which induces a map

$$(\pi_C)_* : H^{2d+1}(X \times C, \mathbf{Z}(d+1)) \to H^1(C, \mathbf{Z}(1)).$$

The naturality of these Gysin morphisms then gives

$$\begin{aligned}\pi_*(\alpha \cdot \beta) &= \pi_*(\alpha \cdot (c_0^*\gamma - c_1^*\gamma))) \\ &= \pi_*(c_0^*(\pi_X^*\alpha \cdot \gamma) - c_1^*(\pi_X^*\alpha \cdot \gamma)) = (c_0^* - c_1^*)(\pi_C)_*(\pi_X^*\alpha \cdot \gamma).\end{aligned}$$

But $c_i^* : H^1(C, \mathbf{Z}(1)) \to H^1(F, \mathbf{Z}(1))$ is left inverse to $\pi'^* : H^1(F, \mathbf{Z}(1)) \to H^1(C, \mathbf{Z}(1))$ (where $\pi' : C \to \mathrm{Spec}\, F$ is the structural map), which is an isomorphism since C is proper. Hence $c_0^* = c_1^*$ on $H^1(C, \mathbf{Z}(1))$, and the proof is complete.

6.4 Proof of Theorem 5 (ii)

Note that $\mathrm{Griff}_1(X)$ is finite if $\dim X \le 2$. In view of (6.2) and (6.3), it therefore suffices to prove

6.2 Proposition (a) *If* $\dim X \le 2$, *we have*

$$t\,\mathbf{DM}^{\mathrm{eff}}(\underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], M(X)), \mathbb{G}_m[i]) = 0$$

for $i > 1$, *and also for* $i = 1$ *if* $\dim X < 2$.

(b) *Suppose* $\dim X = 2$. *Then the map* φ *of* (6.2) *is the Albanese map from* [14, (8.1.1)].

(a) is a dévissage similar to the one for Proposition 3.2 (using (3.3) and (3.4) for $\dim X = 2$); we leave details to the reader. As for (b), we have a diagram in $\mathbf{DM}^{\text{eff}}$

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(M(X), \mathbf{Z}(2)[4]) \arrow[r] & \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], M(X)), \mathbf{Z}(1)[2]) \\
M(X) \arrow[u, "\Delta", "\wr"'] \arrow[r, "\varepsilon_X"] & \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(M(X), \mathbf{Z}(1)[2]), \mathbf{Z}(1)[2]) \arrow[u, "\Delta^*", "\wr"']
\end{tikzcd} \tag{6.6}$$

defined as follows. The top row is obtained by applying $\underline{\mathrm{Hom}}(-, \mathbf{Z}(2)[4])$ to the map $\nu^{\geq 1}M(X) \to M(X)$ of Proposition 1.2, and using the cancellation theorem. The bottom row is obtained by adjunction from the evaluation morphism $M(X) \otimes \underline{\mathrm{Hom}}(M(X), \mathbf{Z}(1)) \to \mathbf{Z}(1)$. The Poincaré duality isomorphism Δ is induced by adjunction by the map

$$M(X \times X) \simeq M(X) \otimes M(X) \to \mathbf{Z}(2)[4]$$

defined by the class of the diagonal $\Delta_X \in CH^2(X \times X) = \mathbf{DM}^{\text{eff}}(M(X \times X), \mathbf{Z}(2)[4])$ (see [2, Prop. 2.5.4]). The isomorphism Δ^* is induced by the isomorphism $\underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], M(X)) \xrightarrow{\sim} \underline{\mathrm{Hom}}(M(X), \mathbf{Z}(1)[2])$ of (3.2), deduced by adjunction from the composition

$$\begin{aligned}
&\underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], M(X)) \otimes M(X) \to \underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], M(X) \otimes M(X)) \\
&\xrightarrow{(\Delta_X)_*} \underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], \mathbf{Z}(2)[4]) \simeq \mathbf{Z}(1)[2]
\end{aligned}$$

where the last isomorphism follows again from the cancellation theorem.[4] A tedious but trivial bookkeeping yields:

6.3 Lemma *The diagram* (6.6) *commutes.* □

We are therefore left to identify $\mathbf{DM}^{\text{eff}}(\mathbf{Z}, \varepsilon_X)$ (where ε_X is as in (6.6)) with the Albanese map. For simplicity, let us write in the sequel $\mathcal{F}$ rather than $\mathcal{F}[0]$ for a sheaf $\mathcal{F} \in \mathbf{HI}$ placed in degree 0 in $\mathbf{DM}^{\text{eff}}$. Let $\mathcal{A}_X$ be the Albanese scheme of X in the sense of Serre–Ramachandran, and let $a_X : M(X) \to \mathcal{A}_X$ be the map defined by [23, (7)]. On the other hand, write D for the (contravariant) endofunctor $M \mapsto \underline{\mathrm{Hom}}(M, \mathbb{G}_m[1])$ of $\mathbf{DM}^{\text{eff}}$, and $\varepsilon : Id_{\mathbf{DM}^{\text{eff}}} \Rightarrow D^2$ for the biduality morphism, so that $\varepsilon_X = \varepsilon_{M(X)}$. We get a commutative diagram:

$$\begin{tikzcd}
M(X) \arrow[r, "\varepsilon_{M(X)}"] \arrow[d, "a_X"'] & D^2 M(X) \arrow[d, "D^2(a_X)"] \\
\mathcal{A}_X \arrow[r, "\varepsilon_{\mathcal{A}_X}"] & D^2 \mathcal{A}_X
\end{tikzcd} \tag{6.7}$$

It is sufficient to show:

[4] Note that evaluation and adjunction yield a tautological morphism $\underline{\mathrm{Hom}}(A, B) \otimes C \to \underline{\mathrm{Hom}}(A, B \otimes C)$ for $A, B, C \in \mathbf{DM}^{\text{eff}}$.

6.4 Proposition *After application of* $\mathbf{DM}^{\mathrm{eff}}(\mathbf{Z}, -) = H^0_{\mathrm{Nis}}(k, -)$ *to* (6.7), *we get a commutative diagram*

$$\begin{array}{ccc} CH_0(X) & \longrightarrow & \mathcal{A}_X(k) \\ {\scriptstyle a_X(k)}\downarrow & & {\scriptstyle u}\downarrow \\ \mathcal{A}_X(k) & \xrightarrow{v} & \mathcal{A}_X(k) \oplus Q \end{array}$$

where $a_X(k)$ *is the Albanese map,* Q *is some abelian group and* u, v *are the canonical injections.*

The main lemma is:

6.5 Lemma *Let* A *be an abelian* F*-variety. Then there is a canonical isomorphism*

$$DA \simeq A^* \oplus \tau_{\geq 2} DA$$

where A^* *is the dual abelian variety of* A.

Proof Note that (3.3) holds for any smooth projective variety Y, if we replace C_Y by $D(M(Y))$. We shall take $Y = A$ and $Y = A \times A$. Let $p_1, p_2, m : A \times A \to A$ be respectively the first and second projection and the multiplication map. The composition

$$M(A \times A) \xrightarrow{(p_1)_* + (p_2)_* - m_*} M(A) \xrightarrow{a_A} \mathcal{A}_A$$

is 0. One characterisation of $\mathrm{Pic}^0_{A/F} \subset \mathrm{Pic}_{A/F}$ is as the kernel of $(p_1)^* + (p_2)^* - m^*$ (e.g. [19, § before Rk. 9.3]). Therefore, the composition

$$DA \to D\mathcal{A}_A \xrightarrow{D(a_A)} D(M(A)) \xrightarrow{(3.3)} \mathrm{Pic}_{A/F}$$

induces a morphism

$$DA \to \mathrm{Pic}^0_{A/F} = A^*. \tag{6.8}$$

Here we used the canonical splitting of the extension

$$0 \to A \to \mathcal{A}_A \to \mathbf{Z} \to 0$$

given by the choice of the origin $0 \in A$. In view of the exact triangle

$$\tau_{\leq 1} DA \to DA \to \tau_{\geq 2} DA \xrightarrow{+1},$$

to prove the lemma we have to show that (6.8) becomes an isomorphism after applying the truncation $\tau_{\leq 1}$ to its left hand side.

For this, we may evaluate on smooth k-varieties, or even on their function fields K by "Gersten's principle" [2, §2.4]. For such K, we have to show that the homomorphism

$$\mathrm{Ext}^{1+i}_{\mathbf{NST}}(A_K, \mathbb{G}_m) \to H^i_{\mathrm{Nis}}(K, A^*)$$

is an isomorphism for $i \leq 1$. This is clear for $i < -1$. For $i = -1, 0, 1$, let **EST** be the category of étale sheaves with transfers of [18, Lect. 6], and **ES** the category of sheaves of abelian groups on $\mathbf{Sm}_{\text{ét}}$, so that we have exact functors

$$\mathbf{NST} \xrightarrow{\alpha^*} \mathbf{EST} \xrightarrow{\omega} \mathbf{ES}$$

where α^* is étale sheafification and ω is "forgetting transfers". If α_* denotes the right adjoint of α^*, the hyperext spectral sequence

$$E_2^{p,q} = \operatorname{Ext}^p_{\mathbf{NST}}(A_K, R^q\alpha_*\alpha^*\mathbb{G}_m) \Rightarrow \operatorname{Ext}^{p+q}_{\mathbf{EST}}(\alpha^* A_K, \alpha^*\mathbb{G}_m)$$

and the vanishing of $R^1\alpha_*\alpha^*\mathbb{G}_m$ (Hilbert 90!) yield isomorphisms

$$\operatorname{Ext}^{1+i}_{\mathbf{NST}}(A_K, \mathbb{G}_m) \xrightarrow{\sim} \operatorname{Ext}^{1+i}_{\mathbf{EST}}(\alpha^* A_K, \alpha^*\mathbb{G}_m), \quad i \leq 0$$

and an injection

$$\operatorname{Ext}^2_{\mathbf{NST}}(A_K, \mathbb{G}_m) \hookrightarrow \operatorname{Ext}^2_{\mathbf{EST}}(\alpha^* A_K, \alpha^*\mathbb{G}_m).$$

Finally, by [2, Th. 3.14.2 a)], we have an isomorphism

$$\operatorname{Ext}^i_{\mathbf{EST}}(\mathcal{F}_K, \mathcal{G}) \xrightarrow{\sim} \operatorname{Ext}^i_{\mathbf{ES}}(\omega\mathcal{F}_K, \omega\mathcal{G})$$

when $\mathcal{F}, \mathcal{G} \in \mathbf{EST}$ are "1-motivic", e.g. $\mathcal{F} = \alpha^* A, \mathcal{G} = \alpha^*\mathbb{G}_m$; moreover, these groups vanish for $i \geq 2$. Lemma 6.5 now follows from the obvious vanishing of $H^1_{\text{Nis}}(K, A^*)$, the vanishing of $\operatorname{Hom}_{\mathbf{ES}}(A_K, \mathbb{G}_m)$ and the isomorphism

$$\operatorname{Ext}^1_{\mathbf{ES}}(A_K, \mathbb{G}_m) \xrightarrow{\sim} A^*(K)$$

deduced from the Weil–Barsotti formula. $\square$

Proof of Proposition 6.4 Let $A = \mathcal{A}^0_{X/F}$ be the Albanese variety of X. Lemma 6.5 yields an isomorphism

$$D^2 A \simeq A \oplus \tau_{\geq 2} DA^* \oplus D(\tau_{\geq 2} DA)$$

hence a split exact triangle

$$\mathcal{A}_X \xrightarrow{\varepsilon_{\mathcal{A}_X}} D^2\mathcal{A}_X \to \tau_{\geq 2} DA^* \oplus D(\tau_{\geq 2} DA) \xrightarrow{+1} .$$

Let now $M^0(X)$ be the reduced motive of X, sitting in the (split) exact triangle $M^0(X) \to M(X) \to \mathbf{Z} \xrightarrow{+1}$, as in the proof of Lemma 3.1 (c). The map a_X induces a map $a^0_X : M^0(X) \to A$, hence a dual map

$$D(a^0_X) : A^* \oplus \tau_{\geq 2} DA \simeq DA \to DM^0(X) \simeq \operatorname{Pic}_{X/F}$$

where the left (resp. right) hand isomorphism follows from Lemma 6.5 (resp. from (3.3)). By construction, $D(a_X^0)$ restricts to the isomorphism $A^* \xrightarrow{\sim} \mathrm{Pic}^0_{X/F}$. Dualising the resulting exact triangle $A^* \to DM^0(X) \to \mathrm{NS}_X \xrightarrow{+1}$ and reusing Lemma 6.5, we get an exact triangle

$$\mathrm{NS}_X^*[1] \to D^2M^0(X) \to A \oplus \tau_{\geq 2} DA^* \xrightarrow{+1}$$

where NS_X^* is the Cartier dual of NS_X. It follows that

$$H^0(k, D^2M^0(X)) = A(k)$$

and therefore that $H^0(k, D^2M(X)) = \mathcal{A}_X(k)$, the map induced by $D^2(a_X)$ being the canonical injection. We thus get the requested diagram, with $Q = H^0(k, D(\tau_{\geq 2}DA))$. □

7 Proof of Theorem 4

Instead of Lewis' idea to use the complex Abel–Jacobi map, we use the l-adic Abel–Jacobi map in order to cover the case of arbitrary characteristic.

We may find a regular $\mathbf{Z}$-algebra R of finite type, a homomorphism $R \to F$, and a smooth projective scheme $p : \mathcal{X} \to \operatorname{Spec} R$, such that $X = \mathcal{X} \otimes_R F$. By a direct limit argument, it suffices to show the theorem when F is the algebraic closure of the quotient field of R and, moreover, to show that the composition

$$H^1(\mathcal{X}, \mathcal{K}_2) \to H^1_{\mathrm{ind}}(X, \mathcal{K}_2) \xrightarrow{\bar{\delta}} \mathrm{Hom}(\mathrm{Griff}_1(X), F^*)$$

has image killed by e.

Let l be a prime number different from char F. We may assume that l is invertible in R. We have l-adic regulator maps

$$H^1(\mathcal{X}, \mathcal{K}_2) \xrightarrow{c} H^3_{\text{ét}}(\mathcal{X}, \mathbf{Z}_l(2)), \quad H^{d-1}(\mathcal{X}, \mathcal{K}_{d-1}) \xrightarrow{c'} H^{2d-2}_{\text{ét}}(\mathcal{X}, \mathbf{Z}_l(d-1))$$

and two compatible pairings

$$\begin{aligned} &H^1(\mathcal{X}, \mathcal{K}_2) \times H^{d-1}(\mathcal{X}, \mathcal{K}_{d-1}) \to H^d(\mathcal{X}, \mathcal{K}_{d+1}) \\ &\quad \xrightarrow{p_*} H^0(R, \mathcal{K}_1) = R^* \end{aligned} \tag{7.1}$$

$$\begin{aligned} &H^3_{\text{ét}}(\mathcal{X}, \mathbf{Z}_l(2)) \times H^{2d-2}_{\text{ét}}(\mathcal{X}, \mathbf{Z}_l(d-1)) \to H^{2d+1}_{\text{ét}}(\mathcal{X}, \mathbf{Z}_l(d+1)) \\ &\quad \xrightarrow{p_*} H^1_{\text{ét}}(R, \mathbf{Z}_l(1)). \end{aligned} \tag{7.2}$$

The Leray spectral sequence for the projection p yields a filtration $F^r H^*_{\text{ét}}(\mathcal{X}, \mathbf{Z}_l(\bullet))$ on the l-adic cohomology of $\mathcal{X}$.

Let $H^{d-1}(\mathcal{X}, \mathcal{K}_{d-1})_0 = c'^{-1}(F^1 H^{2d-2}_{\text{ét}}(\mathcal{X}, \mathbf{Z}_l(d-1)))$ and $H^1(\mathcal{X}, \mathcal{K}_2)_0 = c^{-1}(F^1 H^3_{\text{ét}}(\mathcal{X}, \mathbf{Z}_l(2)))$.

7.1 Lemma *The restriction of* (7.1) *to* $H^1(\mathcal{X}, \mathcal{K}_2)_0 \times H^{d-1}(\mathcal{X}, \mathcal{K}_{d-1})_0$ *has image in* $R^*\{l'\}$*, the subgroup of* R^* *of torsion prime to* l.

Proof Since R is a finitely generated $\mathbf{Z}$-algebra, its group of units R^* is a finitely generated $\mathbf{Z}$-module, hence the map $R^* \otimes \mathbf{Z}_l \to H^1_{\text{ét}}(R, \mathbf{Z}_l(1))$ from Kummer theory is injective; therefore the induced map $R^* \to H^1_{\text{ét}}(R, \mathbf{Z}_l(1))$ has finite kernel of cardinality prime to l. It therefore suffices to show that the restriction of (7.2) to

$$F^1 H^3_{\text{ét}}(\mathcal{X}, \mathbf{Z}_l(2)) \times F^1 H^{2d-2}_{\text{ét}}(\mathcal{X}, \mathbf{Z}_l(d-1))$$

is 0. By multiplicativity of the Leray spectral sequences, it suffices to show that $p_*(F^2 H^{2d+1}_{\text{ét}}(\mathcal{X}, \mathbf{Z}_l(d+1))) = 0$.

Since $\dim X = d$, we have $H^0_{\text{ét}}(R, H^{2d+1}_{\text{ét}}(X, \mathbf{Z}_l(d+1)) = 0$ and hence $H^{2d+1}_{\text{ét}}(\mathcal{X}, \mathbf{Z}_l(d+1))) = F^1 H^{2d+1}_{\text{ét}}(\mathcal{X}, \mathbf{Z}_l(d+1)))$. The edge map

$$F^1 H^{2d+1}_{\text{ét}}(\mathcal{X}, \mathbf{Z}_l(d+1))) \to H^1_{\text{ét}}(R, H^{2d}_{\text{ét}}(X, \mathbf{Z}_l(d+1)))$$

coincides with the map p_* of (7.2) via the isomorphism

$$H^{2d}_{\text{ét}}(X, \mathbf{Z}_l(d+1)) \xrightarrow{p_*} H^0_{\text{ét}}(F, \mathbf{Z}_l(1)) = \mathbf{Z}_l(1).$$

This concludes the proof. □

Passing to the $\varinjlim$ in Lemma 7.1, we find that the pairing

$$H^1(X, \mathcal{K}_2)_0 \times CH^{d-1}(X)_0 \to F^*$$

has image in $F^*\{l'\}$.

7.2 Lemma *The group* $H^1(X, \mathcal{K}_2)/H^1(X, \mathcal{K}_2)_0$ *is finite of exponent dividing* e_l.

Proof It suffices to observe that the regulator map

$$H^1(X, \mathcal{K}_2) \to H^3_{\text{ét}}(X, \mathbf{Z}_l(2))$$

has finite image [7, Th. 2.2]. □

Lemmas 7.1 and 7.2 show that the pairing $H^1(X, \mathcal{K}_2) \times CH^{d-1}(X) \to F^*$ has image in a group of roots of unity whose l-primary component is finite of exponent e_l for all primes $l \neq \operatorname{char} F$. This completes the proof of Theorem 4.

8 Questions and remarks

(1) Does the conclusion of Proposition 3.2 remain true when $\dim X > 2$?
(2) Can one give an a priori, concrete, description of the extension in Theorem 1 (iii)?

(3) It is known that $\mathrm{Griff}_1(X) \otimes \mathbf{Q}$ (*resp* $\mathrm{Griff}_1(X)/l$ for some primes l) may be nonzero for some threefolds X [4,10]; these groups may not even be finite dimensional, *e.g.* [5,22]. Can one find examples for which $\mathrm{Hom}(\mathrm{Griff}_1(X), \mathbf{Z}) \neq 0$?

(4) To put the previous question in a wider context, let A be a torsion-free abelian group. Replacing $\mathbb{G}_m$ by $\mathcal{F} = \mathbb{G}_m \otimes A$ in Theorem 1 yields the following computation (with same proofs):
   (i) $R^0_{\mathrm{nr}}\mathcal{F}(X) = F^* \otimes A$.
   (ii) $R^1_{\mathrm{nr}}\mathcal{F}(X) \xrightarrow{\sim} \mathrm{Pic}^\tau(X) \otimes A$.
   (iii) There is a short exact sequence

$$0 \to D^1(X) \otimes A \to R^2_{\mathrm{nr}}\mathcal{F}(X) \to \mathrm{Hom}(\mathrm{Griff}_1(X), A) \to 0. \tag{8.1}$$

Taking $A = \mathbf{Q}$ we get examples, from the nontriviality of $\mathrm{Griff}_1(X) \otimes \mathbf{Q}$, where $R^2_{\mathrm{nr}}\mathcal{F}(X)$ is not reduced to $D^1(X) \otimes A$. But, choosing X such that $\mathrm{Griff}_1(X) \otimes \mathbf{Q}$ is not finite dimensional and varying A among $\mathbf{Q}$-vector spaces, (8.1) also shows that *the functor* $\mathcal{F} \mapsto R^2_{\mathrm{nr}}\mathcal{F}$ *does not commute with infinite direct sums*. (Therefore R_{nr} cannot have a right adjoint.) This is all the more striking as R^0_{nr} does commute with infinite direct sums, which is clear from formula (1) in the introduction.

We don't know whether R^1_{nr} commutes with infinite direct sums or not.

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