A SPECIALISATION THEOREM FOR LANG-NÉRON GROUPS

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ABSTRACT. We show that, for a polarised smooth projective variety $B \hookrightarrow \mathbb{P}^n_k$ of dimension ≥ 2 over an infinite field k and an abelian variety A over the function field of B, there exists a dense Zariski open set of smooth geometrically connected hyperplane sections h of B such that A has good reduction at h and the specialisation homomorphism of Lang-Néron groups at h is injective (up to a finite p-group in positive characteristic p). This gives a positive answer to a conjecture of the first author, which is used to deduce a negative definiteness result on his refined height pairing. This also sheds a new light on Néron's specialisation theorem.

CONTENTS

1. INTRODUCTION

Let K/k be a finitely generated regular extension of fields, and let A be an abelian variety over K. Then A has a K/k-trace $T = \mathrm{Tr}_{K/k} A$, and a celebrated theorem of Lang and Néron says that the *Lang-Néron group*

$$\mathrm{LN}(K/k, A) = A(K)/T(k)$$

is finitely generated ([LN59], see also [Con06] and [Kah09]).

Let B be a smooth model of K/k, and let $h \in B$ be a point of codimension 1 whose residue field E is also regular over k. If A has good reduction at h, there is a commutative diagram of specialisation maps [Kah24, § 6B]

$$\begin{tikzcd} A(K) \arrow[r] \arrow[d, "\varphi"] & \mathrm{LN}(A, K/k) \arrow[d, "\psi"] \\ A_h(E) \arrow[r] & \mathrm{LN}(A_h, E/k) \end{tikzcd} \tag{1}$$

where A_h is the special fibre of A.

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Theorem 1.1. *Assume that B is smooth projective of dimension $d \geq 2$. For any projective embedding $B \hookrightarrow \mathbb{P}^n_k$, there exists a dense open subset $\mathcal{U}$ of the dual projective space $\mathcal{P}$ of $\mathbb{P}^n_k$ such that if H lies in $\mathcal{U}(k)$, then*

(a) *the hyperplane section $h := H \cap B$ is smooth geometrically connected of dimension $d-1$,*
(b) *A has good reduction at h,*
(c) *the maps φ and ψ of Diagram (1) are injective and have the same cokernel, up to finite p-groups in positive characteristic p.*

(If k is infinite, so is $\mathcal{U}(k)$. When k is finite, $\mathcal{U}(k)$ may be empty because in general there are no smooth hyperplane sections in B defined over k; this issue can presumably be solved by composing the given projective embedding with a suitable Veronese embedding (see [Gab01, Corollary 1.6] and [Poo04, Theorem 3.1]).)

Besides Bertini's theorem, our main tool is a form of the weak Lefschetz theorem due to Deligne [Kat93, A.5], which renders the proof almost trivial.

The first application is to a negative definiteness result for the height pairing introduced in [Kah24]. For a smooth projective variety X of dimension d over K and $i \in [0,d]$, the first author defined a subgroup $\mathrm{CH}^i(X)^{(0)}$ of the i-th Chow group of X and a pairing

$$\mathrm{CH}^i(X)^{(0)} \times \mathrm{CH}^{d+1-i}(X)^{(0)} \to \mathrm{CH}^1(B) \otimes \mathbb{Q}.$$

For $i = 1$, this pairing induces a quadratic form on the Lang-Néron group of the Picard variety of X. In [Kah24, Theorem 6.6], it is proven that this quadratic form is negative definite if B is a curve, and that one can reduce to this case when $\dim B > 1$ if ψ has finite kernel in (1) for a suitable h [Kah24, Conjecture 6.3]. Thus Theorem 1.1 proves this conjecture[1] (in a stronger form, and without the hypothesis of semi-stable reduction appearing in loc. cit.).

The second application is to Néron's specialisation theorem: if $B = \mathbb{P}^n_k$ and U is an open subset of B over which A extends to an abelian scheme $\mathscr{A}$, then the set of rational points $t \in U(k)$ such that the specialisation map $A(K) \to \mathscr{A}_t(k)$ is not injective is thin ([Ser97, 11.1, theorem], see [CT20] for generalisations). The injectivity of φ in Theorem 1.1 gives a version of this specialisation result which does not involve Hilbert's irreducibility theorem, but of course requires $\dim B > 1$; see Remark 3.1 for the case $\dim B = 1$.

2. Auxiliary results

We start with the following standard lemmas.

Lemma 2.1. *Let B be a integral noetherian scheme and let A be an abelian variety over the function field K of B. Then there exist a dense open subset U of B and an abelian scheme $\mathscr{A}$ over U such that $A \simeq \mathscr{A}_K$.*

Proof. See [Mil86, Remark 20.9]. □

The following is a consequence of the valuative criterion of properness and Weil's extension theorem ([Art86, Proposition 1.3] or [BLR90, §4.4, Theorem 1]).

[1] At least for k infinite, but this is sufficient for the application: see [Kah24, part (d) of the proof of Theorem 6.6].

Lemma 2.2. *Let U be an integral normal noetherian scheme with function field K. Let $\mathscr{A}$ be an abelian scheme over U with generic fibre A. Then the pull-back map*

$$\mathscr{A}(U) \to A(K)$$

is an isomorphism. □

Lemma 2.3. *Let U be a scheme and let $\mathscr{A}$ be an abelian scheme over U. If n is invertible on U, i.e., n is prime to* $\mathrm{char}(k(x))$ *for all $x \in U$, then we have an injection*

$$\mathscr{A}(U)/n \hookrightarrow H^1_{\text{ét}}(U, {}_n\mathscr{A}),$$

where ${}_n\mathscr{A}$ is the kernel of the multiplication by n on $\mathscr{A}$.

Proof. Use the short exact sequence of étale sheaves

$$0 \to {}_n\mathscr{A} \to \mathscr{A} \xrightarrow{n} \mathscr{A} \to 0.$$ □

By the above lemmas, we can use cohomology to study the specialisation of $A(K)$. We shall rely on the following version of the weak Lefschetz theorem.

Theorem 2.4 (Deligne; see [Kat93, Corollary A.5])**.** *Let k be a separably closed field and let $\ell \neq \mathrm{char}(k)$ be a prime. Let $f : U \to \mathbb{P}^n_k$ be a separated quasi-finite morphism and let $\mathscr{F}$ be a lisse $\overline{\mathbb{Q}_\ell}$-sheaf. Assume that U is smooth over k and is of pure dimension d. Then there exists a dense open subset $\mathcal{U}$ of the dual projective space $\mathcal{P}$ of $\mathbb{P}^n_k$ such that if H lies in $\mathcal{U}$, then the restriction map*

$$H^i(U, \mathscr{F}) \longrightarrow H^i(f^{-1}(H), \mathscr{F}|_{f^{-1}(H)})$$

is an isomorphism for $i < d-1$ and injective for $i = d-1$.

Proof. In fact, in loc. cit., this theorem is proven when k is algebraically closed for general perverse sheaves without assuming that U is smooth and is of pure dimension d. In our case $\mathscr{F}[d]$ is a perverse sheaf, see [KW01, p. 139]. Moreover, the algebraically closed case immediately implies the separably closed case. □

Finally, we shall use the following easy result:

Lemma 2.5. *Let $\bar{A} = \mathrm{Coker}(T_K \to A)$ (an abelian variety). Then $\bar{A}(K)$ is finitely generated.*

Proof. Let $\bar{T} = \mathrm{Tr}_{K/k}\bar{A}$ and $\pi' : T \to \bar{T}$ be the homomorphism induced by $\pi : A \to \bar{A}$. By complete reducibility, there exists $\sigma : \bar{A} \to A$ such that $\pi\sigma$ is multiplication by some integer $N > 0$; the corresponding homomorphism $\sigma' : \bar{T} \to T$ then also verifies $\pi'\sigma' = N1_{\bar{T}}$. Since the composition $T_K \to A \to \bar{A}$ is 0, we get by the universal property of $\bar{T}$ that $\pi' = 0$. It implies that $N\bar{T} = 0$; hence $\bar{T} = 0$ and we conclude by the Lang-Néron theorem. □

3. Proof of Theorem 1.1

Choose U as in Lemma 2.1. Applying Bertini's theorem [Jou83, Corollary 6.11(2)] to B and U, we get a dense open subset $\mathcal{U}_1$ of the dual projective space $\mathcal{P}$ of $\mathbb{P}^n_k$ such that if H lies in $\mathcal{U}_1(k)$ then $B \cap H$ (hence $U \cap H$) is smooth and geometrically connected of dimension $d-1$, and $U \cap H \neq \emptyset$. In particular ,A has good reduction at $B \cap H$ if $H \in \mathcal{U}_1(k)$.

Let us insert Diagram (1) in the larger commutative diagram with exact rows:

$$
\begin{tikzcd}
0 \arrow[r] & T(k) \arrow[r] \arrow[d, "\varphi_0"] & A(K) \arrow[r] \arrow[d, "\varphi"] & \mathrm{LN}(A, K/k) \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & T_h(k) \arrow[r] & A_h(E) \arrow[r] & \mathrm{LN}(A, E/k) \arrow[r] & 0
\end{tikzcd}
\tag{2}
$$

where T_h is the E/k-trace of A_h. We now proceed in three steps:

3.1. $\operatorname{Ker}\psi$ **is finite.** The immersion $f\colon U \hookrightarrow \mathbb{P}^n_k$ induced by the projective embedding $B \hookrightarrow \mathbb{P}^n_k$ is separated quasi-finite. Let ℓ be a prime not divisible by the characteristic of k. Then by [BLR90, §7.3, Lemma 2], the kernel ${}_{\ell^m}\mathscr{A}$ of multiplication by ℓ^m on $\mathscr{A}$ is finite and étale. Thus it represents a locally constant constructible étale sheaf on U. Denote by $T_\ell\mathscr{A}$ the lisse ℓ-adic sheaf $({}_{\ell^m}\mathscr{A})$.

Let k_s be a separable closure of k. We denote base change from k to k_s by an index s. By Theorem 2.4, there exists a dense open subset $\mathcal{U}_2$ of the dual projective space $\mathcal{P}_s$ such that if H lies in $\mathcal{U}_2$, then the restriction map

$$H^i(U_s, T_\ell\mathscr{A}) \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}_\ell} \longrightarrow H^i(U_s \cap H, T_\ell\mathscr{A}) \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}_\ell}$$

is an isomorphism for $i < d-1$ and injective for $i = d-1$. Therefore the restriction map

$$H^i(U_s, T_\ell\mathscr{A}) \longrightarrow H^i(U_s \cap H, T_\ell\mathscr{A})$$

has finite kernel and cokernel for $i < d-1$ and finite kernel for $i = d-1$. (Recall that $H^i_{\text{ét}}(U_s, {}_{\ell^m}\mathscr{A})$ is finite for all m by [SGA $4\frac{1}{2}$, Th. finitude], hence $H^i(U_s, T_\ell\mathscr{A})$ is a finitely generated $\mathbb{Z}_l$-module.)

The open subset $\mathcal{U}_2$ is defined over a finite Galois extension of k; taking the intersection of its conjugates, we may assume that it is defined over k. Take $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2$. For $H \in \mathcal{U}(k)$, we write $h = B \cap H$. Since the groups $T(k_s)$ and $T_h(k_s)$ are ℓ-divisible, we have the isomorphisms

$$\mathscr{A}(U_s)/\ell^m \simeq (\mathscr{A}(U_s)/T(k_s)) \otimes_{\mathbb{Z}} \mathbb{Z}/\ell^m\mathbb{Z} \simeq \mathrm{LN}(A, Kk_s/k_s)/\ell^m,$$

where the second one holds by Lemma 2.2. Similarly, we have such isomorphisms for $\mathrm{LN}(A_h, Ek_s/k_s)$. Taking the inverse limit of the following commutative diagrams

$$
\begin{tikzcd}
H^1_{\text{ét}}(U_s, {}_{\ell^m}\mathscr{A}) \arrow[d] & \mathscr{A}(U_s)/\ell^m \arrow[l, hook'] \arrow[r, "\sim"] \arrow[d] & \mathrm{LN}(A, Kk_s/k_s)/\ell^m \arrow[d] \\
H^1_{\text{ét}}((U \cap h)_s, {}_{\ell^m}\mathscr{A}) & \mathscr{A}((U \cap h)_s)/\ell^m \arrow[l, hook'] \arrow[r, "\sim"] & \mathrm{LN}(A_h, Ek_s/k_s)/\ell^m
\end{tikzcd}
$$

we get the commutative diagram

$$
\begin{tikzcd}
H^1(U_s, T_\ell\mathscr{A}) \arrow[d] & \mathscr{A}(U_s)^\wedge \arrow[l, hook'] \arrow[r, "\sim"] \arrow[d] & \mathrm{LN}(A, Kk_s/k_s)^\wedge \arrow[d, "\mathrm{sp}_h^\wedge"] \\
H^1((U \cap h)_s, T_\ell\mathscr{A}) & \mathscr{A}((U \cap h)_s)^\wedge \arrow[l, hook'] \arrow[r, "\sim"] & \mathrm{LN}(A_h, Ek_s/k_s)^\wedge
\end{tikzcd}
$$

where $(-)^\wedge$ denotes ℓ-adic completion. Since the left vertical arrow has finite kernel, so do the others. By the Lang-Néron theorem, the abelian group $\mathrm{LN}(A, K/k)$ is finitely generated. Thus $\mathrm{sp}_h(-)^\wedge = \mathrm{sp}_h \otimes \mathbb{Z}_\ell$, which implies that sp_h has a finite kernel. But $\mathrm{LN}(A, K/k)$ injects into $\mathrm{LN}(A, Kk_s/k_s)$, so we are done.

3.2. $\operatorname{Ker}\varphi$ **is a finite p-group, where p is the exponential characteristic of k.** Since $A(K)$ injects into $A(Kk_s)$, we may assume k separably closed. First, φ is injective on n-torsion in (2) for any n invertible in k, cf. [Ser97, p. 153]. This implies that φ_0 is also injective on n-torsion, hence has finite kernel of p-primary order. Now the conclusion follows from the snake lemma and §3.1.

3.3. **End of proof.** The morphism $T_K \to A$ extends uniquely to a morphism of abelian schemes $T \times U \to \mathscr{A}$. For $m \geq 1$, let $\mathscr{B}_m = \operatorname{Coker}({}_{\ell^m}T_U \to {}_{\ell^m}\mathscr{A})$: this is a locally constant constructible sheaf over U, and we have an isomorphism

$$H^0(U_s, \mathscr{B}_m) \xrightarrow{\sim} {}_{\ell^m}\bar{A}(Kk_s)$$

where $\bar{A}$ is as in Lemma 2.5. This lemma then implies that $H^0(U_s, \mathscr{B}(\ell)) = 0$, where $\mathscr{B}(\ell)$ is the ℓ-adic sheaf $(\mathscr{B}_m)_{m\geq 1}$. For clarity, let $i : U \cap h \hookrightarrow U$ be the closed immersion. Applying Theorem 2.4 again, we get that $H^0((U\cap h)_s, i^*\mathscr{B}(\ell))$ is torsion, hence 0 since it is a priori torsion-free. But $i^*\mathscr{B}(\ell)$ contains the constant subsheaf $\operatorname{Coker}(T_\ell(T_h) \to T_\ell(T))$. Therefore this subsheaf is 0, which implies that $T \to T_h$ is an *isogeny* (an isomorphism in characteristic 0). A new diagram chase in (2) concludes the proof.

Remark 3.1. Suppose that B is a curve with a rational point h, and consider Diagram (2) in this situation. We have $E = k$ and $\operatorname{LN}(A, E/k) = 0$. If $k = k_s$, the argument in §3.2 then gives a short exact sequence, up to a finite p-group:

$$0 \to \operatorname{Ker}\varphi \to \operatorname{LN}(A, K/k) \to T(k)$$

where $T = \operatorname{Coker}\varphi_0$ (as an abelian variety). In particular, $\operatorname{Ker}\varphi$ is finitely generated and its rank is uniformly bounded when h varies.

It seems possible that this extends to $\dim B > 1$ in the following sense: for $U \subseteq B$ as in the beginning of Section 3, the kernel of $A(K) \to \mathscr{A}_t(k(t))$ is finitely generated of bounded rank when t runs through the closed points of U (perhaps after suitably shrinking U).

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