A MOTIVIC FORMULA FOR THE $L$-FUNCTION OF AN ABELIAN VARIETY OVER A FUNCTION FIELD

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ABSTRACT. Let $A$ be an abelian variety over the function field of a smooth projective curve $C$ over an algebraically closed field $k$. We compute the $l$-adic cohomology groups

$$H^i(C, j_* H^1(\bar{A}, \mathbb{Q}_l)), \quad j : \eta \hookrightarrow C$$

in terms of arithmetico-geometric invariants of $A$. We apply this, when $k$ is the algebraic closure of a finite field, to a motivic computation of the $L$-function of $A$.

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Introduction

Let $K$ be a global field, and let $A$ be an abelian variety over $K$. Its $L$-function is classically defined as

$$L(A, s) = \prod_{v \in \Sigma_K^f} \det(1 - \pi_v N(v)^{-s} | H^1_{l_v}(A))^{-1}$$

where $\Sigma_K^f$ is the set of non-archimedean places of $K$, $H^1_{l_v}(A) = H^1_{\text{ét}}(\bar{A}, \mathbb{Q}_{l_v})$ is geometric $l_v$-adic cohomology of $A$ (alternately, the dual of the Tate module $V_{l_v}(A)$) for some prime $l_v$ different from the residue characteristic at finite $v$, $I_v$ is the absolute inertia group at $v$ and $\pi_v$ is the geometric Frobenius at $v$, well-defined modulo $I_v$ as a conjugacy class.

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This function is independent of the choice of the \( l_i \)'s, as a consequence of Weil’s Riemann hypothesis for curves and the “weight-monodromy conjecture”, which is known in this case by [SGA7, exp. IX, Th. 4.3 and Cor. 4.4]. In positive characteristic we have the more precise

**Theorem 1.** Suppose \( \text{char } K > 0 \); let \( k = \mathbb{F}_q \) be the field of constants of \( K \). Then

a) (Grothendieck [GS], see also [Dc, §10].) One has the formula

\[
L(A, s) = \frac{P_1(q^{-s})}{P_0(q^{-s})P_2(q^{-s})}
\]

where \( P_i \in \mathbb{Q}[t] \) with \( P_i(0) = 1 \). Moreover, \( L(A, s) \) has a functional equation of the form

\[
L(A, 2 - s) = ab^s L(A, s)
\]

for suitable integers \( a, b \).

b) (Deligne [W.II].) The polynomials \( P_i \) have integer coefficients; the inverse roots of \( P_i \) are Weil \( q \)-numbers of weight \( i + 1 \).

In this note, we give a formula for the polynomial \( P_i \) in terms of pure motives over \( k \). To express the result, let us take some notation:

- \( B = \text{Tr}_{K/k} A \) is the \( K/k \)-trace of \( A \).
- \( \text{LN}(A, K\bar{k}/\bar{k}) = A(K\bar{k})/B(\bar{k}) \) is the geometric Lang-Néron group of \( A \), where \( \bar{k} \) is an algebraic closure of \( k \): it is finitely generated by the Lang-Néron theorem (e.g. [K1, App. B] or [C]). We view it as a Galois representation of \( k \).

**Theorem 2.** Let \( \mathcal{M} = \mathcal{M}_{\text{rat}}(k, \mathbb{Q}) \) be the category of pure motives over \( k \) with rational coefficients, modulo rational equivalence.\(^{1}\)

a) We have

\[
P_0(t) = Z(h^1(B), t), \quad P_2(t) = Z(h^1(B), qt)
\]

where \( h^1(B) \in \mathcal{M} \) is the degree 1 part of the Künneth decomposition of the motive of \( B \).

b) We have

\[
P_1(t) = Z(\ln(A, K/k), qt)^{-1} \cdot Z(\mathfrak{m}(A, K/k), t)^{-1}
\]

(a product of two polynomials), where \( \ln(A, K/k) \) is the Artin motive associated to \( \text{LN}(A, K\bar{k}/\bar{k}) \) and \( \mathfrak{m}(A, K/k) \in \mathcal{M} \) is an effective Chow motive of weight 2 whose \( l \)-adic realization is \( V_l(\text{III}(A, K\bar{k})) \), where \( \text{III}(A, K\bar{k}) \) is the geometric Tate-Šafarevič group of \( A \).

\(^{1}\)Throughout this paper we use the contravariant convention for pure motives, e.g. as in [Kl].
In Theorem 2, we used the Z-function of a motive $M \in \mathcal{M}$ [Kl]. It is known to be a rational function of $t$, with a functional equation; more precisely, if $M$ is homogeneous of weight $w$, then $Z(M, t)$ is a polynomial or the inverse of a polynomial according as $w$ is odd or even. That its inverse roots are Weil $q$-numbers of weight $w$ depends on [W.I] rather than [W.II]. Theorem 2 also provides a proof that $L(A, s)$ is independent of the $l_v$’s avoiding [SGA7, Exp. IX].

The motive $m(A, K/k)$ is really the new character in this story. We construct it “by hand” in Proposition 4.3; however, we will show in [K3] that it is actually canonical and functorial in $A$ (for homomorphisms of abelian varieties).

Theorem 2 “reduces” the Birch and Swinnerton-Dyer conjecture for $A$ to the non-vanishing of $Z(m(A, K/k), t)$ at $t = q^{-1}$. The existence of $m(A, K/k)$ actually yields a simple proof of the following theorem of Kato and Trihan by basically quoting the relevant literature [I, M, IR]:

**Corollary 1 ([KT]).** The following conditions are equivalent:

1. $\text{ord}_{s=1} L(A, s) = \text{rk} A(K)$.
2. $\text{III}(A, K)\{l\}$ is finite for some prime $l$.
3. $\text{III}(A, K)\{l\}$ is finite for all primes $l$.
4. $\text{III}(A, K)$ is finite.

(We almost don’t touch the special value at $s = 1$, see however §6.3.)

To prove Theorem 2, we start from Grothendieck’s formula for $P_i(t)$ (here we take $l_v = l \nmid q$ for all $v$):

$$P_i(t) = \det(1 - \pi_k t \mid H^i(\bar{C}, j_*H^1_l(A)))$$

where $\pi_k$ is the geometric Frobenius of $k$, $C$ is the smooth projective $k$-curve with function field $K$ and $j : \text{Spec } K \hookrightarrow C$ is the inclusion of the generic point. The issue is then to give an expression of the cohomology groups $H^i(\bar{C}, j_*H^1_l(A))$: this is done in Theorem 1.1 below when $A$ is the Jacobian of a curve, and in Corollary 1.2 in general.

When $A$ is the Jacobian $J$ of a curve $\Gamma$, we also get a precise relationship between $L(J, s)$ and the zeta function of a smooth projective $k$-surface spreading $\Gamma$, which was my original motivation for this work. More precisely, let $\Gamma$ be a regular, projective, geometrically irreducible curve over $K$ and $S$ a smooth projective surface over $k$, fibred over $C$ by a flat morphism $f$, with generic fibre $\Gamma$:

$$\Gamma \longrightarrow S$$

$$\downarrow f' \downarrow f$$

$$\Spec K \longrightarrow \Spec k$$

$$\Spec K \longrightarrow C$$

(0.2)
Define (cf. [S])
\[
L(h^i(\Gamma), s) = \prod_{v \in \Sigma_K} \det(1 - \pi_v N(v)^{-s} | H^i_1(\Gamma)_{I_v})^{-1})^{i+1}
\]
\[
L(h(\Gamma), s) = \prod_{i=0}^{2} L(h^i(\Gamma), s)
\]
so that
\[
L(h^0(\Gamma), s) = \zeta(C, s), \quad L(h^2(\Gamma), s) = \zeta(C, s-1), \\
L(h^1(\Gamma), s) = L(J, s)^{-1}
\]
(beware the exponent change!).

**Theorem 3.** We have
\[
\frac{\zeta(S, s)}{L(K, h(\Gamma), s)} = Z(a(D), q^{1-s})
\]
where \(a(D)\) is the Artin motive associated to the “divisor of multiple fibres”
\[
D = \bigoplus_{c \in C(0)} \bar{D}_c, \quad \bar{D}_c = \text{Coker}(\mathbb{Z} \xrightarrow{f^*} \bigoplus_{x \in \text{Supp}(f^{-1}(c))} \mathbb{Z}).
\]

Theorem 2, Corollary 1 and Theorem 3 are results on abelian varieties over a global field of positive characteristic. More intriguing for me is that Theorem 2 leads to a definition of the \(L\)-function of an abelian variety over a finitely generated field of Kronecker dimension 2, see Definition 7.1. This might be viewed a step towards answering the awkwardness of [T2, §4]: meanwhile, it raises more questions than it answers.

The main technical part of this work is to prove Theorem 1.1 below. The method is to “\(l\)-adify” Grothendieck’s computations with \(\mathbb{G}_m\) coefficients in [Br.III, §4]². In a forthcoming work with Amilcar Pacheco [KP], we shall extend these results to a general fibration of smooth projective \(k\)-varieties, with a different and (hopefully) less unpleasant proof.

**Contents of this paper.** Theorem 1.1 and Corollary 1.2 are stated in Section 1. The first is proven in Section 2 and the second in Section 3: as explained above, Corollary 1.2 implies Theorem 2. In section 4, we show how Theorem 1.1 yields an identity in \(K_0\) of a category of \(l\)-adic representations or pure motives, see Theorem 4.1: it implies

²These computations also appear with less generality in two other exposés of the volume *Dix exposés sur la cohomologie des schémas*: [Ra, §3] and [T1, Th. 3.1].
Theorem 3. In Section 5, we recall well-known facts on the crystalline realisation and present them in a convenient way. In Section 6, we examine what Theorem 2 teaches us on the functional equation and special values of \( L(A, s) \); in particular, we prove Corollary 1 in §6.2. Finally, in Section 7, we get a formula for the total \( L \)-function of a surface \( S \) over a global field \( k \) in terms of \( L \)-functions of motives over \( k \) associated to a fibration of \( S \) over a curve (Theorem 7.4).

Acknowledgements. This work was partly inspired by the papers of Hindry-Pacheco [HP] and Hindry-Pacheco-Wazir [HPW]; I would also like to acknowledge several discussions with Amilcar Pacheco around it, which eventually led to [KP]. For this, I thank the Réseau franco-brésilien de mathématiques (RFBM) for its support for two visits to Rio de Janeiro in 2008 and 2010.

Theorems 2 (for the Jacobian of a curve), 3, 1.1 and 4.1 were obtained in the fall 2008 at the Tata Institute of Fundamental Research of Mumbai during its \( p \)-adic semester; I thank this institution for its hospitality and R. Sujatha for having invited me. These results were initially part of a more ambitious project on adjunctions in categories of motives [K3], from which I extracted them. The rest of the present article was obtained more recently.

1. Cohomological results

Consider the situation of (0.2), with \( k \) separably closed. Take a prime number \( l \) invertible in \( k \). We write

\[
H^2_{tr}(S, \mathbb{Q}_l(1)) = \text{Coker} \left( \text{NS}(S) \otimes \mathbb{Q}_l \to H^2(S, \mathbb{Q}_l(1)) \right)
\]

where \( \text{NS}(S) \) is the Néron-Severi group of \( S \). Here are two other descriptions of this group:

\[
H^2_{tr}(S, \mathbb{Q}_l(1)) \cong V_l(\text{Br}(S)) \quad \text{(Kummer exact sequence)}.
\]

\[
H^2_{tr}(S, \mathbb{Q}_l(1)) \cong V_l(\text{III}(J, K)) \quad \text{[Br.III, pp. 120/121]}
\]

where \( J \) is the Jacobian variety of \( \Gamma \) and \( \text{III}(J, K) \) denotes its Tate-Šafarevič group.

**Theorem 1.1.** Suppose \( k \) separably closed. There are isomorphisms

\[
H^0(C, j_* R^1 f'_* \mathbb{Q}_l(1)) \cong H^1(\text{Tr}_{K/k} J, \mathbb{Q}_l(1))
\]

\[
H^2(C, j_* R^1 f'_* \mathbb{Q}_l(1)) \cong H^1(\text{Tr}_{K/k} J, \mathbb{Q}_l)
\]

and an exact sequence

\[
0 \to \text{LN}(J, K/k) \otimes \mathbb{Q}_l \to H^1(C, j_* R^1 f'_* \mathbb{Q}_l(1)) \to H^2_{tr}(S, \mathbb{Q}_l(1)) \to 0.
\]
Corollary 1.2. Let $f' : A \to \text{Spec} K$ be an abelian variety over $K$. Suppose $k$ separably closed. There are isomorphisms

$$H^0(C, j_* R^1 f'_* \mathbb{Q}_l(1)) \simeq H^1(\text{Tr}_{K/k} A, \mathbb{Q}_l(1))$$

and

$$H^2(C, j_* R^1 f'_* \mathbb{Q}_l(1)) \simeq H^1(\text{Tr}_{K/k} A, \mathbb{Q}_l)$$

and an exact sequence

$$0 \to \text{LN}(A, K/k) \otimes \mathbb{Q}_l \to H^1(C, j_* R^1 f'_* \mathbb{Q}_l(1)) \to V_l(\text{III}(A, K)) \to 0.$$

If $k = \mathbb{C}$, it seems likely that similar results hold for the analytic cohomology of $R^1 f_* \mathbb{Q}(1)$ with similar techniques as those used in the next section (replacing Kummer sequences by exponential sequences), but I haven’t tried to prove them.

2. Proof of Theorem 1.1

2.1. Reduction to the cohomology of the Néron model.

Lemma 2.1. Let $J = j_* J$ be the Néron model of $J$ over $C$. There are short exact sequences

$$0 \to (\lim_{\leftarrow} H^{p-1}(C, J)/l^\nu) \otimes \mathbb{Q} \to H^p(C, j_* R^1 f'_* \mathbb{Q}_l(1)) \to V_l(H^p(C, J)) \to 0.$$

Proof. Given $c \in C$, write $i_c : c \hookrightarrow C$ for the corresponding closed immersion and let $\Phi_c$ be the group of connected components of the special fibre of $J$ at $c$. Then $\Phi_c$ is finite for any $c$ and is 0 except for a finite number of $c$’s. Write $J^0 = \text{Ker}(J \to \bigoplus_{c \in C} (i_c)_* \Phi_c)$ for the connected component of $J$. Since $k$ is separably closed, we have isomorphisms $H^p(C, J^0) \sim \to H^p(C, J)$ for $p > 0$ and an injection with finite cokernel $H^0(C, J^0) \hookrightarrow H^0(C, J)$. So,

$$(\lim_{\leftarrow} H^*(C, J^0)/l^\nu) \otimes \mathbb{Q} \sim \to (\lim_{\leftarrow} H^*(C, J)/l^\nu) \otimes \mathbb{Q},$$

$$V_l(H^*(C, J^0)) \sim \to V_l(H^*(C, J)).$$

To handle the cohomology of $J^0$, we may use the Kummer exact sequences

$$0 \to \nu_* J^0 \to J^0 \xrightarrow{l^\nu} J^0 \to 0$$

which yield exact sequences with finite central terms

$$0 \to H^{p-1}(C, J^0)/l^\nu \to H^p(C, \nu_* J^0) \to \nu_* H^p(C, J^0) \to 0$$

hence other exact sequences

$$0 \to (\lim_{\leftarrow} H^{p-1}(C, J^0)/l^\nu) \otimes \mathbb{Q} \to H^p(C, V_l(J^0)) \to V_l(H^p(C, J^0)) \to 0.$$

But $V_l(J^0) \sim \to V_l(J)$; as $R^1 f'_* \mu_{l^\nu} \sim \to \nu_* J$ and $j_* \nu_* J = \nu_* J$, the lemma follows. □
2.2. **Cohomology of** \( B := j_* \text{Pic}_{\Gamma/K} \). For \( c \in C \), view \( f^{-1}(c) \) as a divisor on \( S \). Let \( D_c = \bigoplus_{x \in \text{Supp}(f^{-1}(c))} \mathbb{Z} \) and \( \bar{D}_c = D_c / \langle f^{-1}(c) \rangle \); thus \( \bar{D}_c = 0 \) whenever \( f \) is smooth over \( c \). Write

\[
D = \bigoplus_{c \in C} D_c
\]

(a finite sum).

**Lemma 2.2.** There is an isogeny

\[
\text{Pic}^0_{S/k} / \text{Pic}^0_{C/k} \rightarrow \text{Tr}_{K/k} J
\]

and a complex

\[
0 \rightarrow \text{NS}(C) \rightarrow \text{NS}(S) \rightarrow \text{Pic}(\Gamma)/\text{Tr}_{K/k} J(k) \rightarrow 0
\]

which, modulo finite groups, is acyclic except at \( \text{NS}(S) \), where its homology is \( D \).

**Proof.** This follows from [HP, prop. 3.3 et 3.8] or [K2, 3.2 a]). □

**Lemma 2.3.** a) There is an exact sequence

\[
0 \rightarrow D \rightarrow \text{Pic}(S/C) \rightarrow H^0(C, B) \rightarrow 0
\]

where \( \text{Pic}(S/C) = H^0(C, \text{Pic}_{S/C}) \) and \( B = j_* \text{Pic}_{\Gamma/K} \).

b) There is an exact sequence

\[
0 \rightarrow \text{Pic}(C) \rightarrow \text{Pic}(S) \rightarrow \text{Pic}(S/C) \rightarrow 0
\]

and isomorphisms

\[
H^n(S, \mathbb{G}_m) \rightarrow H^{n-1}(C, B) \text{ for } n > 1.
\]

In particular,

\[
\text{Br}(S) \rightarrow H^1(C, B)
\]

and \( H^n(C, B) = 0 \) for \( n > 3 \).

**Proof.** This follows from the computations in [Br.III, §4]. After [Br.III, (4.1)], we have a long cohomology exact sequence (a consequence of op. cit., (3.2)):

\[
\cdots \rightarrow H^n(C, \mathbb{G}_m) \rightarrow H^n(S, \mathbb{G}_m) \rightarrow H^{n-1}(C, P) \rightarrow \cdots
\]

where \( P = \text{Pic}_{S/C} \). Moreover, the homomorphism \( P \rightarrow B \) is epi and its kernel is a skyscraper sheaf whose global sections are \( D \) [Br.III, p. 114]: (2.2) follows, as well as isomorphisms \( H^n(C, P) \rightarrow H^n(C, B) \) for \( n > 0 \). As \( H^n(C, \mathbb{G}_m) = 0 \) for \( n > 1 \) and \( H^n(S, \mathbb{G}_m) = 0 \) for \( n > 4 \), one gets (2.3) and (2.4). □
2.3. $l$-adic conversion. Using (2.3), (2.4) and the structure of $\text{Pic}(C)$ and $\text{Pic}(S)$, we find exact sequences

\begin{align*}
0 & \to V_1(\text{Pic}^0(C)) \to V_1(\text{Pic}^0(S)) \to V_1(\text{Pic}(S/C)) \\
& \to \mathbb{Q}_l \to \text{NS}(S) \otimes \mathbb{Q}_l \to (\lim \text{Pic}(S/C)/l^\nu) \otimes \mathbb{Q} \to 0 \\
0 & \to V_1(\text{Pic}(S/C)) \to V_1(H^0(C, B)) \to D \otimes \mathbb{Q}_l \\
& \to (\lim \text{Pic}(S/C)/l^\nu) \otimes \mathbb{Q} \to (\lim H^0(C, B)/l^\nu) \otimes \mathbb{Q} \to 0 \\
H^2_{\text{et}}(S, \mathbb{Q}_l(1)) & \sim V_1(H^1(C, B)), \\
(\lim \text{Pic}(S/C)/l^\nu) \otimes \mathbb{Q} & = 0.
\end{align*}

Using Lemma 2.2, we derive new exact sequences

\begin{align*}
0 & \to V_1(\text{Pic}^0(C)) \to V_1(\text{Pic}^0(S)) \to V_1(\text{Tr}_{K/k} J) \to 0 \\
0 & \to D \otimes \mathbb{Q}_l \to (\text{NS}(S)/\mathbb{Z}) \otimes \mathbb{Q}_l \to (\text{Pic}(\Gamma)/(\text{Tr}_{K/k} J)(k)) \otimes \mathbb{Q}_l \to 0.
\end{align*}

Hence

\begin{align*}
V_1(\text{Tr}_{K/k} J) & \sim V_1(\text{Pic}(S/C)) \\
(\text{NS}(S)/\mathbb{Z}) \otimes \mathbb{Q}_l & \sim (\lim \text{Pic}(S/C)/l^\nu) \otimes \mathbb{Q}
\end{align*}

then

\begin{align*}
V_1(\text{Tr}_{K/k} J) & \sim V_1(H^0(C, B)) \\
(\text{Pic}(\Gamma)/(\text{Tr}_{K/k} J)(k)) \otimes \mathbb{Q}_l & \sim (\lim H^0(C, B)/l^\nu) \otimes \mathbb{Q}.
\end{align*}

2.4. From $B$ to $J$. To pass from $B$ to $J$, we work in the category $C$ of abelian groups modulo the thick sub-category of finite groups, which does not affect the functor $V_1$.

Lemma 2.4. In $C$, we have

1. A split exact sequence $0 \to H^0(C, J) \to H^0(C, B) \to \mathbb{Z} \to 0$.
2. An isomorphism $H^1(C, J) \simeq \text{Br}(S)$.
3. An isomorphism $H^2(C, J) \{1\} \simeq \text{Im}_{K/k} J\{1\}(-1)$; $H^2(C, J)$ is torsion.
4. $H^p(C, J) = 0$ for $p \geq 3$.

Proof. We have an exact sequence, split in $C$

\begin{equation}
0 \to J \to B \to \mathbb{Z} \to 0
\end{equation}

which gives (still in $C$) split exact sequences

\begin{equation}
0 \to H^p(C, J) \to H^p(C, B) \to H^p(C, \mathbb{Z}) \to 0.
\end{equation}

For $p = 0$, we get (1). For $p = 1$, we get (2) in view of $H^1(C, \mathbb{Z}) = 0$ and (2.5).

For $p > 1$, this gives in view of (2.4)

\begin{equation}
0 \to H^p(C, J) \to H^{p+1}(S, \mathbb{G}_m) \overset{\tau}{\to} H^p(C, \mathbb{Z}) \to 0
\end{equation}
Still for \( p > 1 \), we have isomorphisms
\[
H^{p-1}(C, \mathbb{Q}_l/\mathbb{Z}_l) \cong H^p(C, \mathbb{Z})\{l\}, \quad H^{p+1}(S, \mathbb{Q}_l/(1)) \cong H^{p+1}(S, \mathbb{G}_m)\{l\}.
\]

The morphism \( \tau \) then gets identified to the trace morphism, which is an isomorphism for \( p \geq 3 \) (hence (4)), while for \( p = 2 \) it is the dual of
\[
f^* : H^1(C, \mathbb{Z}(1)) \rightarrow H^1(S, \mathbb{Z}(1)).
\]

We can then identify
\[
H^2(C, J) \cong \ker(\text{Alb}(S)\{l\}(-1) \rightarrow \text{Alb}(C)\{l\}(-1), i.e., to \text{Im}_K/k J\{l\}(-1).\]

2.5. Conclusion. From Lemma 2.4 and the computations in §2.3, we derive
\[
V_l(\text{Tr}_{K/k} J) \cong V_l(H^0(C, J))
\]
\[
V_l(H^1(C, J)) \cong V_l(\text{Br}(S)) \cong H^2_{tr}(S, \mathbb{Q}_l(1)),
\]
\[
\text{LN}(J, K/k) \otimes \mathbb{Q}_l \cong (\varprojlim H^0(C, J)/l^\nu) \otimes \mathbb{Q}
\]
\[
H^2_{tr}(S, \mathbb{Q}_l(1)) \cong V_l(H^1(C, J)), \quad (\varprojlim H^1(C, J)/l^\nu) \otimes \mathbb{Q} = 0
\]
\[
V_l(H^2(C, J)) \cong V_l(\text{Im}_{K/k} J)(-1) \cong V_l(\text{Tr}_{K/k} J)(-1)
\]
\[
(\varprojlim H^2(C, J)/l^\nu) \otimes \mathbb{Q} = 0
\]
\[
V_l(H^p(C, J)) = (\varprojlim H^p(C, J)/l^\nu) \otimes \mathbb{Q} = 0 \text{ for } p > 2
\]
and finally the isomorphisms and the exact sequence of Theorem 1.1, using Lemma 2.1 and the isomorphisms
\[
V_l(A)(-1) \cong V_l(A)^* \cong H^1(A, \mathbb{Q}_l)
\]
valid for any abelian variety \( A \) over a separably closed field. \( \square \)

3. Proofs of Corollary 1.2 and Theorem 2

If \( X \) is a smooth projective variety of dimension \( d \) over a field \( F \), we write \( CH^d_K(X \times_F X) \) for the quotient of the ring of Chow correspondences on \( X \) by the ideal generated by those \( Z \subset X \times X \) such that \( p_1(Z) \neq X \) or \( p_2(Z) \neq X \), where \( p_1, p_2 \) are the two projections \( X \times X \rightarrow X \) (cf. [F, ex. 16.1.2 (b)].)

Proposition 3.1. a) In the situation of (0.2), there is a ring isomorphism \( CH^1_{\equiv}(\Gamma \times_K \Gamma) \cong \text{End}_K(J) \), and a ring homomorphism
\[
r : CH^1_{\equiv}(\Gamma \times_K \Gamma) \rightarrow CH^2_{\equiv}(S \times_k S).
\]

b) The rings \( \text{End}_K(J) \otimes \mathbb{Q} \) and \( CH^2_{\equiv}(S \times_k S) \otimes \mathbb{Q} \) act compatibly on the isomorphisms and the exact sequence of Theorem 1.1, as well as on (1.1) and (1.2).
Proof. a) The first isomorphism is due to Weil [W, ch. 6, th. 22]. We have a homomorphism
\[ R : Z^1(C \times_K C) \to Z^2(S \times_k S) \]
defined as follows: let \( Z \subset \Gamma \times_K \Gamma \) be an irreducible cycle of codimension 1. Write \( Z \) for its closure in \( S \times C \). We set \( R(Z) = \text{image of } Z \) in \( Z^2(S \times_k S) \). One checks that \( R \) passes to rational equivalence and to the equivalences \( \equiv \), and that the induced map \( r \) is compatible with composition of correspondences.

A more functorial construction of \( r \) will be given in [K3].

b) This is a long but eventless verification. □

In view of Theorem 1, Theorem 2 immediately follows from (0.1) and Corollary 1.2.

4. Comparing classes in \( K_0 \); proof of Theorem 3

In (0.2), let us come back to the case of an arbitrary base field \( k \). Let \( k_s \) be a separable closure of \( k \) and \( G = \text{Gal}(k_s/k) \). Write \( C_s, \Gamma_s \ldots \) for the objects of (0.2) after base change to \( k_s \). Then Theorem 1.1 “over \( k_s \)” is \( G \)-equivariant; moreover, the \( Kk_s/k_s \)-trace of \( J_s \) is \((\text{Tr}_{k_s/k} J)_s \) ([K1, Prop. 6] or [C, Th. 6.8]). One might want to compare
\[ R(pf)_* Q_l \]
and
\[ Rp_* j_* Rf'_* Q_l \]
in the derived category of \( Q_l[[G]] \)-modules. Unfortunately this has no meaning, because \( j_* \) has no meaning in the derived category.

On the other hand, let \( K_l \) be the Grothendieck group of continuous, finite dimensional \( Q_l \)-representations \( G \). We may consider in \( K_l \):

1. \( H_l = [H^*_l(S)] \), the alternating sum of the \( l \)-adic cohomology groups of \( S \);
2. \( H'_l = [R^* p_* j_* R^* f'_* Q_l] \) (9 terms).

We may also consider \( D \) as a discrete topological \( G \)-module.

Theorem 4.1. \( H_l - H'_l = [D \otimes Q_l(-1)] \).

Proof. For simplicity, put \( B := \text{Tr}_{k/k} J \). Let \( K \) be the Grothendieck group of the category \( \text{Mot}_{\text{rat}}(k, Q) \) of Chow motives over \( k \), and let \( R_l : K \to K_l \) be the homomorphism given by \( l \)-adic realisation. (To avoid confusion, we adopt cohomological notation as in [Kl] but contrary to [KMP], also for Tate twists.) Then \( H_l = R_l(h) \), with \( h = h(S) \).

\[ \text{The homomorphism is constructed in [F, ex. 16.1.2 (c)], but its bijectivity is not mentioned...} \]
Similarly, there exists a canonical \( h' \in K \) such that \( H'_t = R_t(h') \). Indeed, Theorem 1.1 shows that

\[
[p_*j_*R^1f'_*Q_l] = R_t([h^1(B)]), \quad [R^2p_*j_*R^1f'_*Q_l] = R_t([h^1(B)(-1)]),
\]

\[
[R^1p_*j_*R^1f'_*Q_l] = R_t([t^2(S)] + [\ln(J, K/k)(-1)])
\]

where \( \ln(J, K/k) \) is the Artin motive associated to the Galois module \( LN(J, K/k) \) (see [KMP] for \( t^2(S) \)).

Using \( f'_*Q_l = Q_t \) and \( R^2f'_*Q_l = Q_t(-1) \), we similarly get

\[
[R^4p_*j_*f'_*Q_l] = R_t([h^q(C)]), \quad [R^4p_*j_*R^2f'_*Q_l] = R_t([h^q(C)(-1)]).
\]

Set

\[
h' = \sum_{q=0}^{2}(-1)^q[h^q(C)] + \sum_{q=0}^{2}(-1)^q[h^q(C)(-1)]
\]

\[
\quad - ([h^1(B)] + [h^1(B)(-1)] - ([t^2(S)] + [\ln(J, K/k)(-1)])) \in K.
\]

To prove Theorem 4.1, it therefore suffices to show:

\[
(4.1) \quad h - h' = [D(-1)].
\]

From Lemma 2.2, one gets identities in \( K \):

\[
[h^1(S)] = [h^1(C)] + [h^1(B), [h^3(S)] = [h^1(C)(-1)] + [h^1(B)(-1)],
\]

\[
[NS(S)] = [NS(C)] + [1 + [\ln(J, K/k)] + [D] = 2[1] + [\ln(J, K/k)] + [D]
\]

from which (4.1) easily follows. \( \square \)

**Remark 4.2.** The same proof gives a more precise identity in \( K \):

\[
[h^2(S)] - [R_{p_*j_*}h^1(J)] = [D(-1)] + 2[\mathbb{L}] - [h^1(B)] - [h^1(B)(-1)]
\]

where \( \mathbb{L} \) is the Lefschetz motive and \( [R_{p_*j_*}h^1(J)] \) stands for the canonical element of \( K \) whose \( l \)-adic realisation is \( \sum_{i=0}^{2}(-1)^i[R^i_{p_*j_*}f'_*Q_l] \).

Let us pass to the case of an abelian variety \( A \) over \( K \). We would like to interpret the terms in Corollary 1.2 as realisations of pure motives over \( k \). This is clearly possible, except perhaps for the term \( V_t(\mathcal{I}_2(A, Kk_s)) \).

**Proposition 4.3.** There exists an effective \( k \)-Chow motive \( \mathfrak{m}(A, K/k) \) such that \( R_t(\mathfrak{m}(A, K/k)) = V_t(\mathcal{I}_2(A, Kk_s))(-1) \).

**Proof.** Write \( A \) as a direct summand of a Jacobian \( J \), up to isogeny, where \( J \) comes from a situation (0.2). Via Proposition 3.1, the corresponding projector \( \pi \in \text{End}(A) \otimes \mathbb{Q} \) defines a projector \( r(\pi) \in CH^2_m(S \times_k S) \otimes \mathbb{Q} = \text{End}(t^2(S)) \) [KMP]. Define \( \mathfrak{m}(A, K/k) \) as the image of \( r(\pi) \). \( \square \)
Remark 4.4. We shall see in [K3] that the motive \( m(A, K/k) \) is independent of the choice of \( J \), and is \textit{functorial in} \( A \).

Theorem 3 immediately follows from Theorem 4.1.

5. \textbf{The crystalline realisation}

In this section, \( k \) is any perfect field of characteristic \( p > 0 \).

5.1. \textbf{Isocrystals}. We rely here on the crystal-clear exposition of Saavedra [Saa, Ch. VI, §3].

Let \( W(k) \) be the ring of Witt vectors on \( k \) and \( K(k) \) be the field of fractions of \( W(k) \). The Frobenius automorphism \( x \mapsto x^p \) of \( k \) lifts to an endomorphism on \( W(k) \) and an automorphism of \( K(k) \), written \( \sigma \): we have \( K(k) \sigma = \mathbb{Q}_p \). A \( k \)-isocrystal is a finite-dimensional \( K(k) \)-vector space \( M \) provided with a \( \sigma \)-linear automorphism \( F_M \). \( k \)-isocrystals form a \( \mathbb{Q}_p \)-linear tannakian category \( \text{Fcriso}(k) \), provided with a canonical \( K(k) \)-valued fibre functor (forgetting \( F_M \)) [Saa, VI.3.2.1]. We have

\[
\text{Fcriso}(k)(1, M) = M^{F_M} = \{ m \in M \mid F_M m = m \}
\]

for \( M \in \text{Fcriso}(k) \), where \( 1 = (K(k), \sigma) \) is the unit object. For \( n \in \mathbb{Z} \), we write more generally

\[
(5.1) \quad M(n) = M^{F_M = p^n} = \text{Fcriso}(k)(\mathbb{L}_\text{crys}^n, M) = \text{Fcriso}(k)(1, M(n))
\]

where \( M(n) = M \otimes \mathbb{L}_\text{crys}^{-n} \) with \( \mathbb{L}_\text{crys} := (K(k), ps) \).

5.2. \textbf{The realisation}. By [Saa, VI.4.1.4.3], the formal properties of crystalline cohomology yield a \( \otimes \)-functor

\[
R_p : \text{Mot}_{\text{rat}}(k, \mathbb{Q}) \rightarrow \text{Fcriso}(k).
\]

This functor sends the motive of a smooth projective variety \( X \) to \( H^\ast_{\text{crys}}(X/W(k)) \otimes_{W(k)} K(k) \) and the Lefschetz motive \( \mathbb{L} \) to \( \mathbb{L}_\text{crys} \).

5.3. \textbf{The case of a finite field}. Suppose that \( k = \mathbb{F}_q \), with \( q = p^m \). Then any object \( M \in \text{Mot}_{\text{rat}}(k, \mathbb{Q}) \) has its \textit{Frobenius endomorphism} \( \pi_M \): if \( M = h(X) \) for a smooth projective variety \( X \), \( \pi_M = \pi_X \) is the graph of the Frobenius endomorphism \( F^m \) on \( X \). This implies:

Lemma 5.1. The action of \( \pi_M \) on \( R_p(M) \) equals that of \( F^m \).

Let \( \bar{k} \) be an algebraic closure of \( k \). There is an obvious functor

\[
(5.2) \quad \text{Fcriso}(k) \rightarrow \text{Fcriso}(\bar{k}), \quad M \mapsto \bar{M} := M \otimes_{K(k)} K(\bar{k})
\]

which is compatible with the extension of scalars \( \text{Mot}_{\text{rat}}(k, \mathbb{Q}) \rightarrow \text{Mot}_{\text{rat}}(\bar{k}, \mathbb{Q}) \) via the realisation functors \( R_p \) for \( k \) and \( \bar{k} \). Moreover \( F^m \) is \( K(k) \)-linear, therefore one can talk of its eigenvalues. We have the following result of Milne [M, Lemma 5.1]:

\[
\text{Fcriso}(k)(1, M) = M^{F_M} = \{ m \in M \mid F_M m = m \}
\]
Lemma 5.2. One has an equality
\[
\det(1 - \gamma t \mid \bar{M}^{(n)}) = \prod_{v(a) = v(q^n)} (1 - (q^n/at))
\]
where \(\gamma\) is the arithmetic Frobenius and \(a\) runs through the eigenvalues of \(F^n\) having same valuation as \(q^n\).

5.4. Logarithmic de Rham-Witt cohomology.

Proposition 5.3. Let \(X/k\) be smooth projective. Then, for any \(i, n \in \mathbb{Z}\), there is a canonical isomorphism
\[
H^i(X, \mathbb{Q}_p(n)) \overset{\sim}{\longrightarrow} (H^i_{\text{crys}}(X/W(k)) \otimes_{W(k)} K(k))^{(n)}
\]
where the left hand side is logarithmic Hodge-Witt cohomology as in Milne [M, p. 309].

Proof. This is [M, Prop. 1.15], but unfortunately its proof is garbled (the last line of loc. cit., p. 310 is wrong). Let us recapitulate it. For simplicity, let \(W = W(k)\) and \(K = K(k)\).

1) The slope spectral sequence
\[
E^{i,j}_1 = H^i(X, W\Omega^j) \Rightarrow H^{i+j}(X, W\Omega) \simeq H^{i+j}_{\text{crys}}(X/W)
\]
degenerates up to torsion, yielding canonical isomorphisms of \(k\)-isocrystals
\[
H^{i-n}(X, W\Omega^n) \otimes_W K \overset{\sim}{\longrightarrow} (H^i(X/W) \otimes_W K)_{[n,n+1]}
\]
where the index \([n, n+1]\) means the sum of summands of slope \(\lambda\) for \(n \leq \lambda < n+1\) [I, Th. 3.2 p. 615 and (3.5.4) p. 616].

2) If \(k\) is algebraically closed, the homomorphism
\[
H^i(X, \mathbb{Z}_p(n)) := H^{i-n}(X, W\Omega^n_{\text{log}}) \to H^{i-n}(X, W\Omega^n)^F
\]
is bijective [IR, Cor. 3.5 p. 194].

3) In general, descend from \(\bar{k}\) to \(k\) by taking Galois invariants. \(\square\)

By [M, §2] and [G], Chow correspondences act on logarithmic Hodge-Witt cohomology by respecting the isomorphisms of Proposition 5.3: this yields functors
\[
H^i(-, \mathbb{Q}_p(n)) : \text{Mot}_{\text{rat}}(k, \mathbb{Q}) \to \text{Vec}_{\mathbb{Q}_p}^\ast
\]
and natural isomorphisms
\[
(5.3) \quad H^i(M, \mathbb{Q}_p(n)) \overset{\sim}{\longrightarrow} \text{Fcriso}(k)\left[\mathbb{L}^n_{\text{crys}}, R_p(M)\right], \quad M \in \text{Mot}_{\text{rat}}(k, \mathbb{Q}).
\]
5.5. The Brauer group and the Tate-Šafarevič group. We have Proposition 5.4 ([I, (5.8.5) p. 629]). Let \( k \) be algebraically closed and \( X/k \) be smooth projective. Then there is an exact sequence

\[
0 \to \text{NS}(X) \otimes \mathbb{Z}_p \to H^2(X, \mathbb{Z}_p(1)) \to T_p(\text{Br}(X)) \to 0.
\]

As before, Chow correspondences act on this exact sequence. Therefore if \( X = S \) is a surface, applying the projector \( \pi_2^{tr} \) defining \( t^2(S) \), we get an isomorphism

\[
H^2(t^2(X), \mathbb{Q}_p(1)) \simeq V_p(\text{Br}(X))
\]

hence, taking (5.3) into account:

\[
\text{Fcriso}(k)(\mathbb{L}_{\text{crys}}, R_p(t^2(X))) \simeq V_p(\text{Br}(X)).
\]

If now \( K/k \) is a function field in one variable and \( A \) is an abelian variety over \( K \), using the projector \( r(\pi) \) from the proof of Proposition 4.3, we get an isomorphism

\[
(5.4) \quad \text{Fcriso}(k)(\mathbb{L}_{\text{crys}}, R_p(\mathfrak{m}(A, K/k))) \simeq V_p(\text{III}(A, K)).
\]

6. Functional equation, order of zero and special value

6.1. Functional equation. Recall the functional equation of the zeta function of a pure motive \( M \) of weight \( w \) over a finite field \( k \) with \( q \) elements:

\[
\zeta(M^*, -s) = \det(M)(-q^{-s})\chi(M)\zeta(M, s)
\]

where \( M^* \) is the dual of \( M \), \( \chi(M) \) is the Euler characteristic of \( M \) (computed for example with the help of its \( l \)-adic realisation) and

\[
\det(M) = \pm q^{w\chi(M)/2}
\]

is the determinant of the Frobenius endomorphism of \( M \). Applying this to Theorem 2, we get the following functional equation for \( L(K, A, s) \):

\[
L(K, A, 2 - s) = a(-q^{-s})^\beta L(K, A, s)
\]

with

\[
\beta = -2\chi(h^1(B)) - \chi(\mathfrak{m}(A, K/k)) - \chi(\ln(A, K/k))
= 4 \dim B - \text{corkIII}(A, K\bar k) - \text{rk}A(K\bar k)
\]

\[
a = (\det h^1(B)\det h^1(B)(-1)\det \mathfrak{m}(A, K/k)\det \ln(A, K/k)(-1))^{-1}
= \pm q^\beta.
\]

The exponent \( \beta \) compares mysteriously with the one appearing in the functional equation of Grothendieck:

\[
\beta = \chi(j_*H^1(\bar A, \mathbb{Q}_l)) = 2 - 2g - \deg(f)
\]
where \( g \) is the genus of \( C \) and \( \mathfrak{f} \) is the conductor of \( A \) (relative to \( K/k \)), 
and the second equality follows from [Ra, Th. 1].

6.2. **Order of zero.** From Theorem 2, one immediately gets the well-known equality and inequality

\[
(6.1) \quad \text{ord}_{s=1} L(A, s) = \text{rk} A(K) + \text{cork}_1 \mathbb{I}(A, K) \\
\geq \text{rk} A(K) + \text{cork}_1 \mathbb{I}(A, K)
\]

where \( \text{cork}_1 \mathbb{I}(A, K) \) is the corank of the generalised eigensubgroup for the eigenvalue 1 of the action of the arithmetic Frobenius \( \gamma \) on an arbitrary \( l \)-primary component of \( \mathbb{I}(A, K) \) (cf. [Sch, Lemma 2 (i)] for \( l \neq p \)).

Indeed, let us show that \( \text{ord}_{s=1} Z(\mathfrak{m}, A(K/k)) = \text{cork}_1 \mathbb{I}(A, K) \) for any prime \( l \). This order can be computed through the action of the Frobenius endomorphism \( \pi_\mathfrak{m} \) of \( \mathfrak{m}, A(K/k)(1) \) on \( R(\mathfrak{m}, A(K/k)(1)) = R_\mathfrak{m}(A, K/k)(-1) \) for any realisation functor \( R \) on \( \text{Mot}_{\text{rat}}(k, \mathbb{Q}) \).

If we use the \( l \)-adic realisation \( R_\mathfrak{m} \) for \( l \neq p \), the claim is clear by Proposition 4.3 since \( R_\mathfrak{m}(\pi_\mathfrak{m}) \) acts like the inverse of \( \gamma \). If we now take \( R_\mathfrak{p} \), we find from (5.1), (5.2) and (5.4):

\[
R_\mathfrak{p}(\mathfrak{m}, A(K/k)(1))^{(0)} \simeq R_\mathfrak{p}(\mathfrak{m}, A(K/k)(1))^{(0)} \simeq V_\mathfrak{p}(\mathbb{I}(A, K)).
\]

By Lemma 5.2, we have

\[
\det(1 - \gamma t \mid R_\mathfrak{p}(\mathfrak{m}, A(K/k)(1)))^{(0)} = \prod_{v(a)=0} (1 - 1/at)
\]

where \( a \) runs through the eigenvalues of \( F^m \) acting on \( R_\mathfrak{p}(\mathfrak{m}, A(K/k)(1)) \), with valuation 0. By lemma 5.1, \( F^m = R_\mathfrak{p}(\pi_\mathfrak{m}) \), so we are done.

This argument does not show that \( \text{cork}_1 \mathbb{I}(A, K) = \text{cork}_1 \mathbb{I}(A, K) \gamma \) is independent of \( l \). However, it does yield:

**Proof of Corollary 1.** For any \( l \), \( \text{cork}_1 \mathbb{I}(A, K) = 0 \iff \mathbb{I}(A, K) \{l\} \) is finite. In view of (6.1), this shows (i) \( \iff \) (ii) \( \iff \) (iii). To get (iii) \( \Rightarrow \) (iv), consider a surface \( S/k \) used to construct the motive \( \mathfrak{m}, A(K/k) \).

The projector \( r(\mathfrak{m}) \) in the proof of Proposition 4.3 is represented by an algebraic correspondence with \( \mathbb{Q} \) coefficients, which have a common denominator \( D \). For \( l \) prime to \( D \), \( \mathbb{I}(A, K) \{l\} \) is then a direct summand of \( \text{Br}(S) \{l\} \), a group of cofinite type whose finite quotient is 0 for almost all \( l \). \( \square \)
6.3. Special value. It is less obvious to relate Theorem 2 to the value of the principal part in the Birch and Swinnerton-Dyer conjecture ([Sch, Theorem p. 509], [KT]):

\[ \lim_{s \to 1} L(A, s) \frac{L(A, s)}{(s-1)\rho} \sim \pm q^\rho |\mathcal{H}(A, K)| |\det\langle , \rangle_{A(K)}| |A(K)_{\text{tors}}| |A'(K)_{\text{tors}}| \prod_{c \in \mathcal{C}} |\Phi_c(k(c))|, \]

where $\rho = \text{rk} A(K)$, $\langle , \rangle_{A(K)}$ is the height pairing constructed in [Sch, p. 502] and the $\Phi_c$ are the groups of connected components of the Néron model of $A$ over $C$, as in §2.1.

It seems that the explicit expression of $L(A, s)$ could actually be used to provide an expression of the left hand side of (6.2) independently of the Birch and Swinnerton-Dyer conjecture, in the spirit of (6.1). This can presumably be done by the method of [Sch]: I did not succeed and leave it to better experts. Let me only note that in Theorem 2, the factors $Z(h^1(B), q^{-s})$ and $Z(h^1(B), q^{1-s})$ respectively contribute by $|B'(k)|$ and $|B(k)|$ (as usual, $B := \text{Tr}_{K/k} A$), while $Z(\ln(A, K/k), q^{1-s})$ contributes by

\[ \pm q^{-\text{rk} A(K)} \frac{\det\langle , \rangle_{A(K\bar{k})}}{\det\langle , \rangle_{A(K)}} \frac{|A(K)_{\text{tors}}/B(k)|}{|(\ln(A, K/k)_{\bar{k}})_{\text{tors}}|}. \]

Where $\langle , \rangle_{A(K)}$ and $\langle , \rangle_{A(K\bar{k})}$ are the height pairings constructed in [Sch, p. 502]. This follows from the elementary lemma, in the spirit of [T1, Lemma z.4]:

**Lemma 6.1.** Let $\langle , \rangle : M \times M' \to \mathbb{Q}$ be a $\mathbb{Q}$-non-degenerate pairing between finitely generated abelian groups. Suppose $M$ and $M'$ are provided with operators $F, F'$ which are adjoint for the pairing, and $\mathbb{Q}$-semi-simple (e.g., $F$ is of finite order). Let $P = \det(1 - FT)$ be the inverse characteristic polynomial of $F$ acting on $M_{\mathbb{Q}}$. Then $\rho := \text{ord}_{T=1} P = \text{rk} M^F$ and, if $P' = P/(1 - T)\rho$,

\[ |P'(1)| = \frac{\det\langle , \rangle^F |(M^F)_{\text{tors}}|}{\det\langle , \rangle |(M^F)_{\text{tors}}|} \]

where $M^F$ (resp. $M_F$) denotes the $F$-invariants (resp. coinvariants) of $F$ and $\langle , \rangle^F$ is the ($\mathbb{Q}$-non-degenerate) pairing induced by $\langle , \rangle$ on $M^F \times M'^F$.

7. Surfaces over a global field

In §4, suppose $k$ global: $K$ is a function field in one variable over $k$. 
**Definition 7.1.** If $A$ is an abelian variety over $K$, we set

$$L(K, h^1(A), s) = L(k, h^1(\text{Tr}_{K/k} A), s)L(k, h^1(\text{Tr}_{K/k} A), s - 1)$$

$$L(k, \mathfrak{m}(A, K/k), s)L(k, \ln(A, K/k), s - 1)$$

where the right hand side is defined in terms of $l$-adic realisations.

In the right hand side, the motive $\ln(A, K/k)$ is of weight 0 (it is an Artin motive), $h^1(\text{Tr}_{K/k} A)$ is of weight 1 and $\mathfrak{m}(A, K/k)$ is of weight 2, a direct summand of $h^2$ of a suitable surface. Definition 7.1 is independent of the choice of $l$ (invertible in $k$) because this is so for each individual factor (for $\mathfrak{m}(A, K/k)$, it follows from [RZ, Satz 2.13] and [Sa, cor. 0.6]).

If $k$ is a number field, it may always be chosen as the algebraic closure of $\mathbb{Q}$ in $K$, and this choice is unique. On the other hand, I don’t know the answer to:

**Question 7.2.** If char $k > 0$, is Definition 7.1 independent of the choice of $k$?

(Said differently: does Definition 7.1 only depend on $K$, a function field in 2 variables over a finite field, and on $A$?)

**Question 7.3.** Can one interpret $L(K, h^1(A), s)$, via a trace formula, as an “Euler” product of the form

$$L(C, j_* H^1_i(A), s) = \prod_{x \in C_{(o)}} L(k(x), i^*_x H^1_i(A), s)$$

where $A$ is the Néron model of $A$ over $C$?

(It is not even clear that the right hand side converges!)

Let us now place ourselves in the situation of (0.2). Set $L(K, h^1(\Gamma), s) = L(K, h^1(J), s)$, and define similarly:

$$L(K, h^0(\Gamma), s) = L(k, h(C), s)$$

$$L(K, h^2(\Gamma), s) = L(k, h(C), s - 1)$$

$$L(K, h(\Gamma), s) = \prod_{i=0}^{2} L(k, h^i(\Gamma), s).$$

Theorem 4.1 then gives the following analogue to Theorem 3:

**Theorem 7.4.** One has

$$\frac{L(k, h(S), s)}{L(K, h(\Gamma), s)} = L(k, D(-1), s) = L(k, D, s - 1).$$
Question 7.5. The height pairing defined by Schneider in [Sch, p. 507]:
\[ A^0(C_s) \times A'(Kk_s) \to \text{Pic}(C_s) \]
induces a pairing
\[ A^0(C_s)/B(k_s) \times \text{LN}(A', Kk_s/k_s) \to \mathbb{Z} \]
because \( B(k_s) \) and \( B'(k_s) \) are divisible; moreover it presumably restricts to a pairing
\[ B(k_s) \times \text{LN}(A', Kk_s/k_s) \to \text{Pic}^0(C_s). \tag{7.1} \]

One way to justify (7.1) would be to show that the functor \( S \mapsto \Gamma(S, A \times_k S) \) on \( k \)-schemes of finite type is representable by a \( k \)-group scheme of finite type with connected component \( B \), and that Schneider’s pairing emanates from a pairing of \( k \)-group schemes. Then (7.1) would induce a Galois-equivariant homomorphism
\[ \text{LN}(A', Kk_s/k_s) \to \text{Hom}_{k_s}(B, J). \]

Can one use these pairings to describe the special values of \( L(K, A, s) \)?

References


B. Kahn Motifs et adjoints, in preparation.


B. Kahn, A. Pacheco Fibrations et valeurs spéciales de fonctions $L$, in preparation.


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