A MOTIVIC FORMULA FOR THE L-FUNCTION OF AN ABELIAN VARIETY OVER A FUNCTION FIELD

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Abstract. Let A be an abelian variety over the function field of a smooth projective curve C over an algebraically closed field k. We compute the l-adic cohomology groups

$$H^i(C, j_*H^1(\bar{A}, \mathbf{Q}_l)), \quad j : \eta \hookrightarrow C$$

in terms of arithmetico-geometric invariants of A. We apply this, when k is the algebraic closure of a finite field, to a motivic computation of the L-function of A.

Contents

1. Introduction

1.1. **The L-function of an abelian variety.** Let K be a global field, and let A be an abelian variety over K. Its L-function is classically defined as

$$(1.1) \qquad L(A,s) = \prod_{v\in\Sigma_K^f} \det(1-\pi_v N(v)^{-s} \mid H^1_{l_v}(A)^{I_v})^{-1}$$

where Σ_K^f is the set of non-archimedean places of K and, given $v \in \Sigma_K^f$, $H^1_{l_v}(A) = H^1_{\text{ét}}(A \otimes_K K_s, \mathbf{Q}_{l_v})$ for a separable closure K_s of K and some

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prime l_v different from the residue characteristic, I_v is the absolute inertia group and π_v is the geometric Frobenius, well-defined modulo I_v as a conjugacy class. This function does not depend on the choice of the l_v's, as a consequence of Weil's proof of the Riemann hypothesis for curves and of the "weight-monodromy conjecture", which is known in this case by [SGA7, exp. IX, Th. 4.3 and Cor. 4.4].

In positive characteristic p, we may choose $l_v = l$ for a fixed prime $l \neq p$. We have the more precise theorem, due to Grothendieck ([GS, (13), p. 194], see also [Dc, §10]) and Deligne [W.II, Th. 3.2.3]:

Theorem A. *Suppose* $\operatorname{char} K > 0$*; let* $k = \mathbf{F}_q$ *be the field of constants of* K*. Then one has the formula*

$$L(A,s) = \frac{P_1(q^{-s})}{P_0(q^{-s})P_2(q^{-s})}$$

where $P_i \in \mathbf{Z}[t]$ *with* $P_i(0) = 1$*. The inverse roots of* P_i *are Weil* q*-numbers of weight* $i+1$*. Moreover,* $L(A,s)$ *has a functional equation of the form*

$$L(A, 2-s) = ab^s L(A,s)$$

for suitable integers a, b*.*

In this article, we give a formula for the polynomials P_i in terms of *pure motives over* k. To express the result, let us take some notation:

- $B = \operatorname{Tr}_{K/k} A$ is the K/k-trace of A (see [K1, App. A] or [C]); recall that this is (essentially) the largest abelian subvariety of A which is defined over k.
- $\operatorname{LN}(A, K\bar{k}/\bar{k}) = A(K\bar{k})/B(\bar{k})$ is the *geometric Lang-Néron group* of A, where $\bar{k}$ is an algebraic closure of k: it is finitely generated by the Lang-Néron theorem (e.g. [K1, App. B], [C] or [K2]). We view it as a Galois representation over k.
- $T_l(E) = \operatorname{Hom}(\mathbf{Q}_l/\mathbf{Z}_l, E)$; $V_l(E) = T_l(E) \otimes \mathbf{Q}$ for an abelian group E.

Theorem 1. *Let* $\mathcal{M}$ *be the category of pure motives over* k *with rational coefficients, modulo rational equivalence.*[1]
a) We have

$$P_0(t) = Z(h^1(B), t), \quad P_2(t) = Z(h^1(B), qt)$$

where $h^1(B) \in \mathcal{M}$ *is the degree* 1 *part of the Chow-Künneth decomposition of the motive of* B [DM]*.*

[1] Throughout this paper we use the contravariant convention for pure motives, as in [Kl] or [Sco].

b) We have

$$P_1(t) = Z(\ln(A, K/k), qt)^{-1} \cdot Z(\text{ш}(A, K/k), t)^{-1}$$

(a product of two polynomials), where $\ln(A, K/k)$ *is the Artin motive associated to* $\mathrm{LN}(A, K\bar{k}/\bar{k})$ *and* $\text{ш}(A, K/k) \in \mathcal{M}$ *is an effective Chow motive of weight* 2 *whose l-adic realization is* $V_l(\text{Ш}(A, K\bar{k}))(-1)$, *where* $\text{Ш}(A, K\bar{k})$ *is the geometric Tate-Šafarevič group of A.*

In Theorem 1, we used the *Z-function* of a motive $M \in \mathcal{M}$ [Kl, p. 81]. It is known to be a rational function of t, with a functional equation; more precisely, if M is homogeneous of weight w, then $Z(M, t)$ is a polynomial or the inverse of a polynomial according as w is odd or even. That its inverse roots are Weil q-numbers of weight w depends on [W.I] rather than [W.II], see Proposition 3.4 and Remark 3.5. Theorem 1 also provides a proof that $L(A, s)$ is independent of l avoiding [SGA7, Exp. IX]. Note that B already appears in [T1, (4.4)].

The motive $\text{ш}(A, K/k)$ is really the new character in this story. We construct it "by hand" in Proposition 3.2; however, we show in [K3, Cor. 8.4] that it is actually canonical and functorial in A (for homomorphisms of abelian varieties).

Theorem 1 "reduces" the Birch and Swinnerton-Dyer conjecture for A to the non-vanishing of $Z(\text{ш}(A, K/k), t)$ at $t = q^{-1}$. The existence of $\text{ш}(A, K/k)$ actually yields a simple proof of the following corollary by basically quoting the relevant literature [I, M2, IR]:

Corollary 2. *a) (cf. Schneider* [Scn, Lemma 2 (i)] *for $l \neq p$) One has the equality and inequality, for any prime number l ($l = p$ is allowed):*

$$\begin{aligned} (1.2)\quad \mathrm{ord}_{s=1} L(A, s) &= \mathrm{rk} A(K) + \dim V_l(\text{Ш}(A, K\bar{k}))^{(1)} \\ &\geq \mathrm{rk} A(K) + \dim V_l(\text{Ш}(A, K)) \end{aligned}$$

where $V_l(\text{Ш}(A, K\bar{k}))^{(1)}$ *is the generalised eigenspace for the eigenvalue* 1 *of the action of the arithmetic Frobenius on* $V_l(\text{Ш}(A, K\bar{k}))$. *(In particular,* $\dim V_l(\text{Ш}(A, K\bar{k}))^{(1)}$ *is independent of l.)*

b) (Kato-Trihan, [KT]*) The following conditions are equivalent:*

(i) $\mathrm{ord}_{s=1} L(A, s) = \mathrm{rk} A(K)$.
(ii) $\text{Ш}(A, K)\{l\}$ *is finite for some prime l.*
(iii) $\text{Ш}(A, K)\{l\}$ *is finite for all primes l.*
(iv) $\text{Ш}(A, K)$ *is finite.*

On the other hand, computing the special value at $s = 1$ factor by factor looks very uninspiring, see §6.3. It seems that a new idea is needed to nicely relate Theorem 1 to the formula in the Birch and Swinnerton-Dyer conjecture, for example by revisiting Schneider's height pairing [Scn, p. 507] as a determinant pairing on total derived complexes. I hope to come back to this in a further work.

To prove Theorem 1, we reduce to the case where A is the Jacobian J of a curve Γ. In this case, we also get a precise relationship between $L(J,s)$ and the zeta function of a smooth projective k-surface spreading Γ, which was my original motivation for this work. More precisely, let Γ be a smooth, projective, geometrically irreducible curve over K, C the smooth projective k-curve with function field K and S a smooth projective surface over k, fibred over C by a flat morphism f with generic fibre Γ (its existence is justified in footnote 2 below):

$$\begin{tikzcd}
\Gamma \arrow[r] \arrow[d, "f'"'] & S \arrow[d, "f"'] \\
\operatorname{Spec} K \arrow[r, "j"] & C \arrow[d, "p"'] \\
& \operatorname{Spec} k.
\end{tikzcd} \tag{1.3}$$

Define (cf. [S])

$$L(h^i(\Gamma),s) = \prod_{v\in\Sigma_K} \det(1-\pi_v N(v)^{-s} \mid H^i_l(\Gamma)^{I_v})^{(-1)^{i+1}}$$

$$L(h(\Gamma),s) = \prod_{i=0}^{2} L(h^i(\Gamma),s)$$

so that

$$L(h^0(\Gamma),s) = \zeta(C,s), \quad L(h^2(\Gamma),s) = \zeta(C,s-1),$$
$$L(h^1(\Gamma),s) = L(J,s)^{-1}$$

(beware the exponent change!).

For convenience, we express the next theorem in terms of the *zeta function of a motive* $M\in\mathcal{M}$, rather than its Z-function:

$$\zeta(M,s) = Z(M,q^{-s});\quad \zeta(h(X),s)=\zeta(X,s) \text{ for } X \text{ smooth projective.}$$

Theorem 3. *a) We have*

$$\frac{\zeta(S,s)}{L(K,h(\Gamma),s)} = \zeta(a(D),s-1)$$

where $a(D)$ is the Artin motive associated to the "divisor of multiple fibres"

$$D = \bigoplus_{c\in C_{(0)}} \bar D_c,\quad \bar D_c = \operatorname{Coker}(\mathbf{Z} \xrightarrow{f^*} \bigoplus_{x\in Supp(f^{-1}(c))^{(0)}} \mathbf{Z}).$$

b) The function $\dfrac{L(K, h(\Gamma), s)}{\zeta(\ln(J, K/k), s-1)}$ *only depends on the function field* $K(\Gamma)$.

(Here as in the rest of this paper, we write $Z^{(p)}, Z_{(p)}$ for the set of points of codimension/dimension p of a scheme Z.)

Theorem 1, Corollary 2 and Theorem 3 are results on abelian varieties over a global field of positive characteristic. More intriguing to me is that Theorem 1 leads to a definition of the L-function of an abelian variety over a *finitely generated field of Kronecker dimension* 2, and to an analogue of Theorem 3 for the total L-function of a surface S over a number field k sitting in a fibration (1.3) (Theorem 7.5). This might be viewed a step towards answering the awkwardness of [T2, §4]: meanwhile, it raises more questions than it answers. See §7 for more details.

1.2. **The method.** To prove Theorem 1, we start from Grothendieck's formula for $P_i(t)$:

$$P_i(t) = \det(1 - \pi_k t \mid H^i(C \otimes_k \bar{k}, j_* H^1_l(A))) \tag{1.4}$$

where π_k is the geometric Frobenius of k, C is the smooth projective k-curve with function field K and $j : \operatorname{Spec} K \hookrightarrow C$ is the inclusion of the generic point. The issue is then to give an expression of the cohomology groups $H^i(C \otimes_k \bar{k}, j_* H^1_l(A)))$.

This follows from a cohomological computation valid in greater generality. Consider the situation of (1.3), where k is now any field.[2] For a prime number l invertible in k, we write

$$H^2_{\mathrm{tr}}(\bar{S}, \mathbf{Q}_l(1)) = \operatorname{Coker}\left(\mathrm{NS}(\bar{S}) \otimes \mathbf{Q}_l \to H^2(\bar{S}, \mathbf{Q}_l(1))\right)$$

where $\mathrm{NS}(\bar{S})$ is the Néron-Severi group of $\bar{S} := S \otimes_k \bar{k}$, $\bar{k}$ being an algebraic closure of k as before. Here are two other descriptions of this group:

$$H^2_{\mathrm{tr}}(\bar{S}, \mathbf{Q}_l(1)) \simeq V_l(\mathrm{Br}(\bar{S})) \simeq V_l(Ш(J, K\bar{k})) \tag{1.5}$$

where $\mathrm{Br}(\bar{S})$ is the Brauer group of $\bar{S}$, J is the Jacobian variety of Γ and $Ш(J, K\bar{k})$ denotes its geometric Tate-Šafarevič group: the first isomorphism follows from the Kummer exact sequence and [Br.II, Cor. 2.2], and the second one from [Br.III, pp. 120/121]. If k_s (resp. k_p) is the

[2] Given f' and assuming k perfect, to get such a diagram we can start from a projective model S_0 of Γ over $\operatorname{Spec} k$ obtained from some very ample divisor on Γ, then resolve the singularities of the closure in $S_0 \times_k C$ of the graph of the rational map $S_0 \dashrightarrow C$ [A]; the resulting f is automatically flat by [H, Ch. II, Prop. 9.7]. When k is imperfect, one may have to pass to a finite purely inseparable extension to get (1.3), but this is a harmless restriction, compare proof of Proposition 3.2.

separable (resp. perfect) closure of k in $\bar{k}$, then $G_k := Gal(k_s/k) \xrightarrow{\sim} Gal(\bar{k}/k_p)$ acts naturally on the groups above.

Theorem 4. *There are G_k-equivariant isomorphisms*

$$H^0(\bar{C}, j_*R^1f'_*\mathbf{Q}_l(1)) \simeq H^1(\mathrm{Tr}_{K/k} J \otimes_k \bar{k}, \mathbf{Q}_l(1))$$
$$H^2(\bar{C}, j_*R^1f'_*\mathbf{Q}_l(1)) \simeq H^1(\mathrm{Tr}_{K/k} J \otimes_k \bar{k}, \mathbf{Q}_l)$$

(where $\bar{C} = C \otimes_k \bar{k}$), and a G_k-equivariant exact sequence

$$0 \to \mathrm{LN}(J, K\bar{k}/\bar{k}) \otimes \mathbf{Q}_l \to H^1(\bar{C}, j_*R^1f'_*\mathbf{Q}_l(1)) \to H^2_{\mathrm{tr}}(\bar{S}, \mathbf{Q}_l(1)) \to 0.$$

Corollary 5. *Let $f' : A \to \operatorname{Spec} K$ be an abelian variety, where $K = k(C)$ as in (1.3). There are G_k-equivariant isomorphisms*

$$H^0(\bar{C}, j_*R^1f'_*\mathbf{Q}_l(1)) \simeq H^1(\mathrm{Tr}_{K/k} A \otimes_k \bar{k}, \mathbf{Q}_l(1))$$
$$H^2(\bar{C}, j_*R^1f'_*\mathbf{Q}_l(1)) \simeq H^1(\mathrm{Tr}_{K/k} A \otimes_k \bar{k}, \mathbf{Q}_l)$$

(where $\bar{C} := C \otimes_k \bar{k}$), and a G_k-equivariant exact sequence

$$0 \to \mathrm{LN}(A, K\bar{k}/\bar{k}) \otimes \mathbf{Q}_l \to H^1(\bar{C}, j_*R^1f'_*\mathbf{Q}_l(1)) \to V_l(\text{Ш}(A, K\bar{k})) \to 0.$$

Remarks 1.1. 1) It can be shown by a polarisation argument that the exact sequences of Theorem 4 and Corollary 5 are G_k-split.
2) If $k = \mathbf{C}$, it seems likely that similar results can be obtained for the analytic cohomology of $R^1f'_*\mathbf{Q}(1)$ with techniques similar to those used in the next section, replacing Kummer sequences by exponential sequences.

The main technical part of this work is to prove Theorem 4. The method is to "l-adify" Grothendieck's computations with $\mathbb{G}_m$ coefficients in [Br.III, §4][3]. In a forthcoming work with Amílcar Pacheco [KP], we shall extend these results to a general fibration of smooth projective k-varieties, with a different and (hopefully) less unpleasant proof.

Contents of this paper. Theorem 4 is proven in Section 2; Corollary 5 and Theorem 1 are proven in Section 3. In Section 4, we show how Theorem 4 yields an identity in K_0 of a category of l-adic representations or pure motives, see Theorem 4.1: this identity implies Theorem 3. In Section 5, we recall well-known facts on the crystalline realisation and present them in a convenient way. In Section 6, we examine what Theorem 1 teaches us on the functional equation and special values of $L(A, s)$; in particular, we prove Corollary 2 in §6.2. Finally, in Section

[3]These computations also appear with less generality in two other exposés of the volume *Dix exposés sur la cohomologie des schémas*: [Ra, §3] and [T1, Th. 3.1].

7, we examine what happens when we replace the finite field k by a global field.

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Theorems 1 (for the Jacobian of a curve), 3, 4 and 4.1 were obtained in the fall 2008 at the Tata Institute of Fundamental Research of Mumbai during its p-adic semester; I thank this institution for its hospitality and R. Sujatha for having invited me, and would like to add a thought for the Mumbai attacks which took place in this period. These results were initially part of a preliminary version of [K3], from which I extracted them. The other results were obtained more recently.

2. Proof of Theorem 4

In this section, we assume $k = \bar{k}$ to avoid carrying notation like $C \otimes_k \bar{k}$, etc. This is harmless because, in general, $\operatorname{Tr}_{K\bar{k}/\bar{k}}(J \otimes_k \bar{k}) = (\operatorname{Tr}_{K/k} J) \otimes_k \bar{k}$ ([K1, Prop. 6] or [C, Th. 6.8]). It is immediate to check that all exact sequences and isomorphisms appearing below are Galois-equivariant. We follow the notation of [Br.III, §4] as much as possible; in particular, B denotes here $j_* \operatorname{Pic}_{\Gamma/K}$ rather than $\operatorname{Tr}_{K/k} J$. We shall use:

Lemma 2.1 ([EGA3, Cor. 4.3.12])**.** *The fibres of $f : S \to C$ are connected and $f_*\mathbb{G}_m = \mathbb{G}_m$.*

2.1. Reduction to the cohomology of the Néron model.

Lemma 2.2. *Let $\mathcal{J} = j_*J$ be the Néron model of J over C. There are short exact sequences*

$$0 \to (\varprojlim H^{p-1}(C, \mathcal{J})/l^\nu) \otimes \mathbf{Q} \to H^p(C, j_*R^1 f'_*\mathbf{Q}_l(1)) \\ \to V_l(H^p(C, \mathcal{J})) \to 0.$$

Proof. Given $c \in C_{(0)}$, write $i_c : c \hookrightarrow C$ for the corresponding closed immersion and let Φ_c be the group of connected components of the special fibre of $\mathcal{J}$ at c. Then Φ_c is finite for any c and is 0 except for a finite number of c's. Write $\mathcal{J}^0 = \operatorname{Ker}(\mathcal{J} \to \bigoplus_{c \in C_{(0)}} (i_c)_* \Phi_c)$: it is divisible. Since k is algebraically closed, we have isomorphisms $H^p(C, \mathcal{J}^0) \xrightarrow{\sim} H^p(C, \mathcal{J})$ for $p > 0$ and an injection with finite cokernel

$H^0(C,\mathcal{J}^0)\hookrightarrow H^0(C,\mathcal{J})$. So,

$$(\varprojlim H^*(C,\mathcal{J}^0)/l^\nu)\otimes\mathbf{Q}\xrightarrow{\sim}(\varprojlim H^*(C,\mathcal{J})/l^\nu)\otimes\mathbf{Q},$$
$$V_l(H^*(C,\mathcal{J}_0))\xrightarrow{\sim}V_l(H^*(C,\mathcal{J})).$$

To handle the cohomology of $\mathcal{J}^0$, we use the Kummer exact sequences

$$0\to{}_{l^\nu}\mathcal{J}^0\to\mathcal{J}^0\xrightarrow{l^n}\mathcal{J}^0\to 0$$

which yield exact sequences with finite central terms

$$0\to H^{p-1}(C,\mathcal{J}^0)/l^\nu\to H^p(C,{}_{l^\nu}\mathcal{J}^0)\to{}_{l^\nu}H^p(C,\mathcal{J}^0)\to 0$$

hence other exact sequences

$$0\to\varprojlim H^{p-1}(C,\mathcal{J}^0)/l^\nu)\otimes\mathbf{Q}\to H^p(C,V_l(\mathcal{J}^0))\to V_l(H^p(C,\mathcal{J}^0))\to 0.$$

But $V_l(\mathcal{J}^0)\xrightarrow{\sim}V_l(\mathcal{J})$; as $R^1f'_*\mu_{l^\nu}\xrightarrow{\sim}{}_{l^\nu}J$ and $j_{*l^\nu}J={}_{l^\nu}\mathcal{J}$, the lemma follows. □

2.2. **Cohomology of $B:=j_*\operatorname{Pic}_{\Gamma/K}$.** Define

$$D=\operatorname{Coker}(\operatorname{Div}(C)\xrightarrow{f^*}\operatorname{Div}(S-\Gamma))$$

(the right hand side may be thought of as "vertical divisors of S relative to f"). We may describe D as follows: for $c\in C_{(0)}$, write $f^*(c)$ as an effective divisor $\sum_{i\in I}n_iC_i$ on S, with the C_i's irreducible and $n_i>0$. Let $D_c=\bigoplus_{i\in I}\mathbf{Z}$ and $\bar D_c=\operatorname{Coker}(\mathbf{Z}\xrightarrow{f_c}D_c)$, where $f_c(1)=(n_i)_{i\in I}$: thus $\bar D_c=0$ whenever f is smooth over c (Lemma 2.1). Then:

$$D=\bigoplus_{c\in C_{(0)}}\bar D_c \tag{2.1}$$

(a finite sum).

Lemma 2.3. *There is an isogeny*

$$\operatorname{Pic}^0_{S/k}/\operatorname{Pic}^0_{C/k}\to\operatorname{Tr}_{K/k}J$$

and a complex

$$0\to\operatorname{NS}(C)\to\operatorname{NS}(S)\to\operatorname{Pic}(\Gamma)/\operatorname{Tr}_{K/k}J(k)\to 0$$

which, modulo finite groups, is acyclic except at $\operatorname{NS}(S)$, *where its homology is* D.

Proof. This follows from [HP, Prop. 3.3 and 3.8] or [K2, 3.2 a)]. □

Lemma 2.4. *a) There is an exact sequence*

$$0\to D\to\operatorname{Pic}(S/C)\to H^0(C,B)\to 0 \tag{2.2}$$

where $\operatorname{Pic}(S/C)=H^0(C,\operatorname{Pic}_{S/C})$ *and* $B=j_*\operatorname{Pic}_{\Gamma/K}$.
b) There is an exact sequence

$$0\to\operatorname{Pic}(C)\to\operatorname{Pic}(S)\to\operatorname{Pic}(S/C)\to 0 \tag{2.3}$$

and isomorphisms

$$H^n(S,\mathbb{G}_m) \xrightarrow{\sim} H^{n-1}(C,B) \text{ for } n>1. \tag{2.4}$$

In particular,

$$\mathrm{Br}(S) \xrightarrow{\sim} H^1(C,B) \tag{2.5}$$

and $H^n(C,B)=0$ *for* $n>3$.

Proof. This follows from the computations in [Br.III, §4]. For simplicity, write $P=\mathrm{Pic}_{S/C}$. The homomorphism $\varphi: P\to B$ is epi and its kernel is a skyscraper sheaf whose global sections are D: indeed, epi follows from the fact that all residue fields $k(c)$ for $c\in C_{(0)}$ are perfect (they are all equal to k) [Br.III, p. 114], and $H^0(C,\operatorname{Ker}\varphi)$ is identified to D by the computation of [Br.III, p. 116, esp. (4.21)]. This yields a), as well as isomorphisms

$$H^n(C,P) \xrightarrow{\sim} H^n(C,B), \quad n>0. \tag{2.6}$$

To prove b), we use the long cohomology exact sequence from [Br.III, (4.1)] (a consequence of Lemma 2.1 and *op. cit.*, (3.2)):

$$\dots \to H^n(C,\mathbb{G}_m)\to H^n(S,\mathbb{G}_m)\to H^{n-1}(C,P)\to \dots \tag{2.7}$$

For any smooth k-variety Z of dimension d, one has $H^n(Z,\mathbb{G}_m)=0$ for $n>2d$ by [SGA4, Exp. X, Cor. 4.3 and 5.2], the Kummer exact sequences

$$0\to\mu_m\to\mathbb{G}_m\xrightarrow{m}\mathbb{G}_m\to 0, \quad m \text{ invertible in } k$$

(resp. $0\to\mathbb{G}_m\xrightarrow{p}\mathbb{G}_m\to\mathbb{G}_m/p\to 0$ if $p=\operatorname{char} k$) and the fact that $H^n(Z,\mathbb{G}_m)$ is torsion for $n>1$ [Br.II, Prop. 1.4]. Thus $H^n(S,\mathbb{G}_m)=0$ for $n>4$. Moreover, $H^2(C,\mathbb{G}_m)=0$ by [Br.III, Cor. 1.2] and [Br.II, Cor. 2.2]. b) follows from these facts and (2.6), (2.7). □

2.3. l-adic conversion.

Lemma 2.5. *We have natural isomorphisms*

$$T_l(\mathrm{Tr}_{K/k} J) \xrightarrow{\sim} T_l(H^0(C,B)) \tag{2.8}$$

$$(\varprojlim H^0(C,B)/l^\nu) \xrightarrow{\sim} (\mathrm{Pic}(\Gamma)/(\mathrm{Tr}_{K/k} J)(k))\otimes \mathbf{Z}_l. \tag{2.9}$$

Proof. The isomorphism $H^0(C,B)=\mathrm{Pic}(\Gamma)$ gives a tautological exact sequence:

$$0\to \mathrm{Tr}_{K/k} J(k)\to H^0(C,B)\to \mathrm{Pic}(\Gamma)/\mathrm{Tr}_{K/k} J(k)\to 0. \tag{2.10}$$

The first term is divisible and, by Lemma 2.3, the third term is finitely generated. Hence in the exact sequences derived from (2.10):

$$(2.11)\quad 0 \to {}_{l^\nu} \operatorname{Tr}_{K/k} J(k) \to {}_{l^\nu} H^0(C,B) \to {}_{l^\nu} \operatorname{Pic}(\Gamma)/\operatorname{Tr}_{K/k} J(k) \to \operatorname{Tr}_{K/k} J(k)/l^\nu \to H^0(C,B)/l^\nu \to (\operatorname{Pic}(\Gamma)/\operatorname{Tr}_{K/k} J(k))/l^\nu \to 0$$

we have $\operatorname{Tr}_{K/k} J(k)/l^\nu = 0$ and $({}_{l^\nu} \operatorname{Pic}(\Gamma)/\operatorname{Tr}_{K/k} J(k))$, viewed as an inverse system, is Mittag-Leffler null. Whence the lemma by passing to the inverse limit. □

2.4. **From B to $\mathcal{J}$.** To pass from B to $\mathcal{J}$, we work in the abelian category $\mathcal{C}$ of abelian groups modulo the Serre subcategory of finite groups, which does not affect the functor V_l.

Lemma 2.6. *In $\mathcal{C}$, we have*

(1) *A split exact sequence* $0 \to H^0(C,\mathcal{J}) \to H^0(C,B) \to \mathbf{Z} \to 0$.
(2) *An isomorphism* $Ш(J,K/k) = H^1(C,\mathcal{J}) \simeq \operatorname{Br}(S)$.
(3) *An isomorphism* $H^2(C,\mathcal{J})\{l\} \simeq \operatorname{Im}_{K/k} J\{l\}(-1)$; $H^2(C,\mathcal{J})$ *is torsion.*
(4) $H^p(C,\mathcal{J})\{l\} = 0$ *for* $p \geq 3$.

Proof. (1) and (2) are contained in [Br.III, §4]; we include their proof for completeness. (For the first isomorphism of (2), see [Br.III, (4.44) and (4.45)].)

In the exact sequence $0 \to J \to \operatorname{Pic}_{\Gamma/K} \xrightarrow{\deg} \mathbf{Z}$, the map deg is split up to an integer by the choice of a closed point of Γ. Applying j_*, this yields an exact sequence, split in $\mathcal{C}$

$$(2.12)\qquad 0 \to \mathcal{J} \to B \to \mathbf{Z} \to 0$$

which gives (still in $\mathcal{C}$) split exact sequences

$$0 \to H^p(C,\mathcal{J}) \to H^p(C,B) \to H^p(C,\mathbf{Z}) \to 0.$$

For $p=0$, we get (1). For $p=1$, we get (2) in view of $H^1(C,\mathbf{Z})=0$ and (2.5). The group $H^p(C,\mathcal{J})$ is torsion for $p>0$ (compare [SGA4, IX.4.2]), and $cd_l(C)=2$ (loc. cit., 4.6); the structure of $\mathcal{J}$ recalled in §2.1 then yields (4).

It remains to prove (3). Using (2) and the fact that $\operatorname{Br}(S)\{l\}$ is a group of cofinite type [Br.II, 3.2, 3.3], we get surjective maps $H^2(C, {}_{l^\nu}\mathcal{J}) \twoheadrightarrow {}_{l^\nu} H^2(C,\mathcal{J})$ with bounded kernels. The Weil pairings

$${}_{l^\nu} J \times {}_{l^\nu} J \to \mu_{l^\nu}$$

and Poincaré duality then show that $H^2(C,\mathcal{J})\{l\}$ is Pontrjagin dual in $\mathcal{C}$ to $H^0(C,T_l(\mathcal{J})) \simeq T_l(H^0(C,\mathcal{J}))$, and we conclude by (1) and (2.8). □

2.5. **Conclusion.** From Lemmas 2.5 and 2.6, we derive

$$V_l(\operatorname{Tr}_{K/k} J) \xrightarrow{\sim} V_l(H^0(C,\mathcal{J})) \text{ (Lemma 2.6 (1) and (2.8))}$$
$$V_l(H^1(C,\mathcal{J})) \simeq V_l(\operatorname{Br}(S)) \simeq H^2_{\mathrm{tr}}(S,\mathbf{Q}_l(1)) \text{ (Lemma 2.6 (2) and (1.5))}$$
$$\operatorname{LN}(J,K/k)\otimes\mathbf{Q}_l \xrightarrow{\sim} (\varprojlim H^0(C,\mathcal{J})/l^\nu)\otimes\mathbf{Q} \text{ (Lemma 2.6 (1) and (2.9))}$$
$$(\varprojlim H^1(C,\mathcal{J})/l^\nu)\otimes\mathbf{Q} = 0 \text{ (Lemma 2.6 (2))}$$
$$V_l(H^2(C,\mathcal{J})) \simeq V_l(\operatorname{Im}_{K/k} J)(-1) \simeq V_l(\operatorname{Tr}_{K/k} J)(-1) \text{ (Lemma 2.6 (3))}$$
$$(\varprojlim H^2(C,\mathcal{J})/l^\nu)\otimes\mathbf{Q} = 0 \text{ (Lemma 2.6 (3))}$$
$$V_l(H^p(C,\mathcal{J})) = (\varprojlim H^p(C,\mathcal{J})/l^\nu)\otimes\mathbf{Q} = 0 \text{ for } p>2 \text{ (Lemma 2.6 (4))}$$

whence finally the isomorphisms and the exact sequence of Theorem 4, using Lemma 2.2 and the isomorphisms

$$V_l(A)(-1) \simeq V_l(A)^* \simeq H^1(A,\mathbf{Q}_l)$$

valid for any abelian variety A over an algebraically closed field. □

3. Proofs of Corollary 5 and Theorem 1

3.1. **The refined Chow-Künneth decomposition for a surface.** Let k be an arbitrary field, and S a smooth projective k-surface. Recall that, by Murre and Scholl [Sco, §4], there exists a decomposition in $CH^2(S\times S)\otimes\mathbf{Q}$

$$\Delta_S = \sum_{i=0}^{4} \pi^i \tag{3.1}$$

where the π^i's are orthogonal idempotents lifting the Künneth projectors relative to any Weil cohomology theory H such that $H^1(\operatorname{Alb}(S)) \xrightarrow{f^*} H^1(S)$ is an isomorphism for an Albanese map $S \xrightarrow{f} \operatorname{Alb}(S)$. Recall that any "classical" Weil cohomology theory in the sense of [KMP, Def. 7.1.4] has this property: this follows for l-adic cohomology from [M1, Prop. 4.11], for crystalline cohomology from [I, Rem. II.3.11.2] and for the other classical cohomologies in characteristic 0 from the comparison theorems with l-adic cohomology. In [KMP, Prop. 7.2.3], (3.1) was refined to

$$\pi^2 = \pi^2_{\mathrm{alg}} + \pi^2_{\mathrm{tr}}$$

where π^2_{alg} and π^2_{tr} are orthogonal projectors, which cut off the "algebraic" and "transcendental" part of $H^2(S)$, e.g. $\operatorname{NS}(\bar S)\otimes\mathbf{Q}_l(-1)$ and $H^2_{\mathrm{tr}}(\bar S,\mathbf{Q}_l)$ from $H^2_{\text{ét}}(\bar S,\mathbf{Q}_l)$.

Write $\mathcal{M}(k)=\mathcal{M}$ for the category of Chow motives over k with rational coefficients [Sco]. The above yields a *refined Chow-Künneth decomposition* of the motive $h(S)\in\mathcal{M}$:

$$h(S) = h^0(S)\oplus h^1(S)\oplus h^2_{\mathrm{alg}}(S)\oplus t^2(S)\oplus h^3(S)\oplus h^4(S). \tag{3.2}$$

Supposing S geometrically connected, we have $h^0(S) \simeq \mathbf{1}$ (unit motive), $h^3(S) \simeq h^1(S) \otimes \mathbb{L}$ ($\mathbb{L}$ the Lefschetz motive), $h^4(S) \simeq \mathbb{L}^2$ and $h^2_{\mathrm{alg}}(S) \simeq \mathrm{NS}_S \otimes \mathbb{L}$, where NS_S is the Artin motive corresponding to $\mathrm{NS}(\bar{S})$ viewed as a G_k-module.

Recall also that any abelian variety A has a Chow-Künneth decomposition by [DM]; in particular one has the motive $h^1(A) \in \mathcal{M}$, a direct summand of $h(A)$.

3.2. **Correspondences at the generic point.** If X is a smooth projective variety of dimension d over a field F, we write $CH^d_{\equiv}(X \times_F X)$ for the quotient of the ring of Chow correspondences on X by the ideal generated by those $Z \subset X \times X$ such that $p_1(Z) \neq X$ or $p_2(Z) \neq X$, where p_1, p_2 are the two projections $X \times X \to X$ (*cf.* [F, ex. 16.1.2 (b)].)

Proposition 3.1. *a) In the situation of* (1.3)*, there is a ring isomorphism* $CH^1_{\equiv}(\Gamma \times_K \Gamma) \xrightarrow{\sim} \mathrm{End}_K(J)$*, and a ring homomorphism*

$$r : CH^1_{\equiv}(\Gamma \times_K \Gamma) \to CH^2_{\equiv}(S \times_k S).$$

b) The rings $\mathrm{End}_K(J) \otimes \mathbf{Q}$ *and* $CH^2_{\equiv}(S \times_k S) \otimes \mathbf{Q}$ *act compatibly on the isomorphisms and the exact sequence of Theorem 4, as well as on* (1.5)*.*

Proof. (See [K3, Th. 9.3 b)] for a more conceptual proof.) a) The first isomorphism is due to Weil [W, ch. 6, th. 22][4]. We have a homomorphism

$$R : Z^1(C \times_K C) \to Z^2(S \times_k S)$$

defined as follows: let $Z \subset \Gamma \times_K \Gamma$ be an irreducible cycle of codimension 1. Write $\mathcal{Z}$ for its closure in $S \times_C S$. We set $R(Z)$ = image of $\mathcal{Z}$ in $Z^2(S \times_k S)$. One checks that R passes to rational equivalence and to the equivalences $\equiv$, and that the induced map r is compatible with composition of correspondances.

b) This is a long but eventless verification. $\square$

Proposition 3.2. *Let A be an abelian variety over K. There exists an effective Chow motive* $\text{ш}(A, K/k) \in \mathcal{M}(k)$ *such that* $R_l(\text{ш}(A, K/k)) = V_l(\text{Ш}(A, K\bar{k}))(-1)$*, for any prime* $l \neq \mathrm{char}\, k$*;* $\text{ш}(A, K/k)$ *is a direct summand of* $t^2(S)$ *for some smooth projective k-surface S. Here* $R_l : \mathcal{M} \to \mathrm{Rep}^*_{\mathbf{Q}_l}(G_k)$ *denotes the l-adic realisation, with values in finite dimensional graded* $\mathbf{Q}_l$*-vector spaces with continuous* G_k*-action.*

[4] The homomorphism is constructed in [F, ex. 16.1.2 (c)], but its bijectivity is not mentioned...

Proof. If k is perfect, write A as a direct summand of a Jacobian J, up to isogeny, where J comes from a situation (1.3). *Via* Proposition 3.1, the corresponding projector $\pi \in \operatorname{End}(J) \otimes \mathbf{Q}$ defines a projector $r(\pi) \in CH^2_{\equiv}(S \times_k S) \otimes \mathbf{Q} = \operatorname{End}(t^2(S))$ [KMP, Th. 7.4.3]. Define $Ш(A, K/k)$ as the image of $r(\pi)$. If k is imperfect, we get a situation (1.3) over some finite purely inseparable extension of k, and we conclude by using a Frobenius trick as in [KS, Prop. 1.7.2]. □

Remark 3.3. We show in [K3, Cor. 8.4] that the motive $Ш(A, K/k)$ is independent of the choice of J, and is *functorial in* A.

3.3. **Proofs of Corollary 5 and Theorem 1.** To prove Corollary 5, write A as a direct summand up to isogeny of the Jacobian J of a curve Γ as in Theorem 4: A corresponds to a projector $\pi \in \operatorname{End}_K(J) \otimes \mathbf{Q}$. Proposition 3.1 shows that π acts on the isomorphisms and the exact sequence of Theorem 4. Thus, Corollary 5 is obtained as a "direct summand" of Theorem 4.

For completeness, recall:

Proposition 3.4. *Let $M \in \mathcal{M} = \mathcal{M}(\mathbf{F}_q)$. Suppose that there is a smooth projective $\mathbf{F}_q$-variety X whose Chow motive $h(X)$ admits a Chow-Künneth decomposition relative to l-adic cohomology for some $l \nmid q$, and such that M is a direct summand of $h^i(X)$ for some $i \geq 0$. Then $Z(M, t) = P_M(t)^{(-1)^{i+1}}$, where P_M is the inverse characteristic polynomial of the action of the Frobenius endomorphism π_M on $H^i_l(M)$. Moreover, $P_M \in \mathbf{Z}[t]$ and its inverse roots are Weil q-numbers of weight i.*

Proof. The definition from [Kl, p. 81] amounts to $Z(M, t) = \exp(\sum_{i \geq 1} \operatorname{tr}(\pi_M^n)\frac{t^n}{n})$, where $\operatorname{tr}(\pi^n) \in \mathbf{Q}$ is computed in the rigid $\otimes$-category $\mathcal{M}$. Then $\operatorname{tr}(\pi_M^n) = \operatorname{tr}(R_l(\pi_M)^n)$. Hence the first statement, since $H^j(M) = 0$ for $j \neq i$ by hypothesis. The second one follows from [W.I], since P_M obviously divides $\det(1 - \pi_X t \mid H^i_l(X))$. □

Remark 3.5. Using [KM], one can show that the conclusion of Proposition 3.4 holds if one only assumes that M is effective and that $H^j_l(M) \neq 0$ for $j \neq i$ (one may even use crystalline cohomology instead of l-adic cohomology). But [KM] relies on [W.II], hence this result is less elementary than the one in Proposition 3.4.

In view of Theorem A, Theorem 1 immediately follows from (1.4), Corollary 5, Proposition 3.2 and Proposition 3.4.

4. Comparing classes in K_0; proof of Theorem 3

We come back again to the situation of (1.3), k being arbitrary. One might want to compare

$$R(pf)_*\mathbf{Q}_l$$

and
$$Rp_*j_*Rf'_*\mathbf{Q}_l$$
in the derived category of $\mathrm{Rep}^*_{\mathbf{Q}_l}(G_k)$ (cf. Proposition 3.2). Unfortunately this has no meaning, because j_* has no meaning in this derived category.

On the other hand, let $\mathbf{K}_l$ be the Grothendieck group of the abelian category $\mathrm{Rep}^*_{\mathbf{Q}_l}(G_k)$. We may consider in $\mathbf{K}_l$:

(1) $H_l = [H^*_l(\bar{S})]$, the alternating sum of the l-adic cohomology groups of S;
(2) $H'_l = [R^*p_*j_*R^*f'_*\mathbf{Q}_l]$ (9 terms).

We may also consider D as a discrete topological G_k-module.

Theorem 4.1. *$H_l - H'_l = [D \otimes \mathbf{Q}_l(-1)]$.*

Proof. For simplicity, put $B := \mathrm{Tr}_{K/k} J$. Let $\mathbf{K}$ be the Grothendieck group of the additive category $\mathcal{M}$, and let $R_l : \mathbf{K} \to \mathbf{K}_l$ be the homomorphism given by l-adic realisation. (To avoid any confusion, we adopt cohomological notation as in [Kl, Sco], but contrary to [KMP].) Then $H_l = R_l(h)$, with $h = h(S)$. Similarly, there exists a canonical $h' \in \mathbf{K}$ such that $H'_l = R_l(h')$. Indeed, Theorem 4 shows that
$$[p_*j_*R^1f'_*\mathbf{Q}_l] = R_l([h^1(B)]), \quad [R^2p_*j_*R^1f'_*\mathbf{Q}_l] = R_l([h^1(B)\otimes\mathbb{L}]),$$
$$[R^1p_*j_*R^1f'_*\mathbf{Q}_l] = R_l([t^2(S)] + [\mathrm{ln}(J,K/k)\otimes\mathbb{L}])$$
where $\mathrm{ln}(J,K/k)$ is the Artin motive associated to the Galois module $\mathrm{LN}(J,K\bar{k}/\bar{k})$. (Recall that $R_l(\mathbb{L}) = \mathbf{Q}_l(-1)$.)

Using $f'_*\mathbf{Q}_l = \mathbf{Q}_l$ and $R^2f'_*\mathbf{Q}_l = \mathbf{Q}_l(-1)$, we similarly get
$$[R^qp_*j_*f'_*\mathbf{Q}_l] = R_l([h^q(C)]), \quad [R^qp_*j_*R^2f'_*\mathbf{Q}_l] = R_l([h^q(C)\otimes\mathbb{L}]).$$

We may then take
$$h' = \sum_{q=0}^{2}(-1)^q[h^q(C)] + \sum_{q=0}^{2}(-1)^q[h^q(C)\otimes\mathbb{L}]$$
$$- \left([h^1(B)] + [h^1(B)\otimes\mathbb{L}] - ([t^2(S)] + [\mathrm{ln}(J,K/k)\otimes\mathbb{L}])\right).$$

To prove Theorem 4.1, it therefore suffices to show:
$$h - h' = [D\otimes\mathbb{L}]. \tag{4.1}$$

From Lemma 2.3, one gets identities in $\mathbf{K}$:
$$[h^1(S)] = [h^1(C)] + [h^1(B)], \quad [h^3(S)] = [h^1(C)\otimes\mathbb{L}] + [h^1(B)\otimes\mathbb{L}],$$
$$[\mathrm{NS}_S] = [\mathrm{NS}_C] + [\mathbf{1}] + [\mathrm{ln}(J,K/k)] + [D] = 2[\mathbf{1}] + [\mathrm{ln}(J,K/k)] + [D] \tag{4.2}$$
from which (4.1) easily follows. □

Theorem 3 a) immediately follows from Theorem 4.1. For b), we note that all terms appearing in (3.2) are birational invariants, except for $h^2_{\mathrm{alg}}(S) = \mathrm{NS}_S \otimes \mathbb{L}$. Using (4.2) again, we get from (4.1) the identity

$$h - [\mathrm{NS}_S \otimes \mathbb{L}] = h' - 2[\mathbb{L}] - [\ln(J, K/k) \otimes \mathbb{L}]$$

so that the right hand side only depends on $k(S) = K(\Gamma)$.

5. The crystalline realisation

This section prepares the proof of Corollary 2, which will be given in the next section. Here k is any perfect field of characteristic $p > 0$.

5.1. **Isocrystals.** We rely here on the crystal-clear exposition of Saavedra [Saa, Ch. VI, §3].

Let $W(k)$ be the ring of Witt vectors on k and $K(k)$ be the field of fractions of $W(k)$. The Frobenius automorphism $x \mapsto x^p$ of k lifts to an endomorphism on $W(k)$ and an automorphism of $K(k)$, written σ: we have $K(k)^\sigma = \mathbf{Q}_p$. A *$k$-isocrystal* is a finite-dimensional $K(k)$-vector space V provided with a σ-linear automorphism F_V. k-isocrystals form a $\mathbf{Q}_p$-linear tannakian category $\mathbf{Fcriso}(k)$, provided with a canonical $K(k)$-valued fibre functor (forgetting F_M) [Saa, VI.3.2.1]. We have

$$\mathbf{Fcriso}(k)(\mathbf{1}, V) = V^{F_V} = \{v \in V \mid F_V v = v\}$$

for $V \in \mathbf{Fcriso}(k)$, where $\mathbf{1} = (K(k), \sigma)$ is the unit object. For $n \in \mathbf{Z}$, we write more generally

$$V^{(n)} = V^{F_V = p^n} = \mathbf{Fcriso}(k)(\mathbb{L}^n_{\mathrm{crys}}, V) = \mathbf{Fcriso}(k)(\mathbf{1}, V(n)) \tag{5.1}$$

where $V(n) = V \otimes \mathbb{L}^{-n}_{\mathrm{crys}}$ with $\mathbb{L}_{\mathrm{crys}} := (K(k), p\sigma)$.

5.2. **The realisation.** By [Saa, VI.4.1.4.3], the formal properties of crystalline cohomology yield a $\otimes$-functor

$$R_p : \mathcal{M}(k) \to \mathbf{Fcriso}(k).$$

This functor sends the motive of a smooth projective variety X to $H^*_{\mathrm{crys}}(X/W(k)) \otimes_{W(k)} K(k)$ and the Lefschetz motive $\mathbb{L}$ to $\mathbb{L}_{\mathrm{crys}}$.

5.3. **The case of a finite field.** Suppose that $k = \mathbf{F}_q$, with $q = p^m$. Then any object $M \in \mathcal{M}(k)$ has its *Frobenius endomorphism* π_M: if $M = h(X)$ for a smooth projective variety X, $\pi_M = \pi_X$ is the graph of the Frobenius endomorphism F^m on X. This implies:

Lemma 5.1. *The action of π_M on $R_p(M)$ equals that of $F^m_{R_p(M)}$.* □

Let $\bar{k}$ be an algebraic closure of k. There is an obvious functor

$$\mathbf{Fcriso}(k) \to \mathbf{Fcriso}(\bar{k}), \quad V \mapsto \bar{V} := V \otimes_{K(k)} K(\bar{k}) \tag{5.2}$$

which is compatible with the extension of scalars $\mathcal{M}(k) \to \mathcal{M}(\bar{k})$ via the realisation functors R_p for k and $\bar{k}$. Moreover F^m_M is $K(k)$-linear, therefore one can talk of its eigenvalues. We have the following result of Milne [M2, Lemma 5.1]:

Lemma 5.2. *One has an equality*

$$\det(1-\gamma t \mid \bar{M}^{(n)}) = \prod_{v(a)=v(q^n)} (1-(q^n/at))$$

where γ is the arithmetic Frobenius and a runs through the eigenvalues of F_M^m having same valuation as q^n.

5.4. **Logarithmic Hodge-Witt cohomology.**

Proposition 5.3. *Let X/k be smooth projective. Then, for any $i,n \in$ $\mathbf{Z}$, there is a canonical isomorphism*

$$H^i(X,\mathbf{Q}_p(n)) \xrightarrow{\sim} (H^i_{\text{crys}}(X/W(k)) \otimes_{W(k)} K(k))^{(n)}$$

where the left hand side is logarithmic Hodge-Witt cohomology as in Milne [M2, p. 309].

Proof. This is [M2, Prop. 1.15], but unfortunately its proof is garbled (the last line of loc. cit., p. 310 is wrong). Let us recapitulate it. For simplicity, let $W=W(k)$ and $K=K(k)$.

1) The slope spectral sequence

$$E_1^{i,j} = H^j(X, W\Omega^i) \Rightarrow H^{i+j}(X,W\Omega^{\cdot}) \simeq H^{i+j}_{\text{crys}}(X/W)$$

degenerates up to torsion, yielding canonical isomorphisms of k-isocrystals

$$H^{i-n}(X,W\Omega^n)\otimes_W K \xrightarrow{\sim} (H^i(X/W)\otimes_W K)_{[n,n+1[}$$

where the index $[n,n+1[$ means the sum of summands of slope λ for $n\le \lambda < n+1$ [I, Th. 3.2 p. 615 and (3.5.4) p. 616].

2) If k is algebraically closed, the homomorphism

$$H^i(X,\mathbf{Z}_p(n)) := H^{i-n}(X,W\Omega^n_{\log}) \to H^{i-n}(X,W\Omega^n)^F$$

is bijective [IR, Cor. 3.5 p. 194].

3) In general, descend from $\bar{k}$ to k by taking Galois invariants. □

By [M2, §2] and [Gr], Chow correspondences act on logarithmic Hodge-Witt cohomology by respecting the isomorphisms of Proposition 5.3: this yields a functor

$$H^*(-,\mathbf{Q}_p(n)) : \mathcal{M}(k) \to \mathbf{Vec}^*_{\mathbf{Q}_p}$$

and a natural isomorphism

$$H^*(M,\mathbf{Q}_p(n)) \xrightarrow{\sim} \mathbf{Fcriso}(k)(\mathbb{L}^n_{\text{crys}}, R_p(M)),\ M\in\mathcal{M}(k). \tag{5.3}$$

5.5. **The Brauer group and the Tate-Šafarevič group.** We have

Proposition 5.4 ([I, (5.8.5) p. 629])**.** *Let k be algebraically closed and X/k be smooth projective. Then there is an exact sequence*

$$0 \to \mathrm{NS}(X) \otimes \mathbf{Z}_p \to H^2(X, \mathbf{Z}_p(1)) \to T_p(\mathrm{Br}(X)) \to 0.$$

As before, Chow correspondences act on this exact sequence. Therefore if $X = S$ is a surface, applying the projector π^2_{tr} defining $t^2(S)$, we get an isomorphism

$$H^2(t^2(S), \mathbf{Q}_p(1)) \simeq V_p(\mathrm{Br}(S))$$

hence, taking (5.3) into account:

$$\mathbf{Fcriso}(k)(\mathbb{L}_{\mathrm{crys}}, R_p(t^2(S))) \simeq V_p(\mathrm{Br}(S)).$$

If now K/k is a function field in one variable and A is an abelian variety over K, using the projector $r(\pi)$ from the proof of Proposition 3.2, we get an isomorphism

$$\mathbf{Fcriso}(k)(\mathbb{L}_{\mathrm{crys}}, R_p(ш(A, K/k))) \simeq V_p(Ш(A, K)) \tag{5.4}$$

using Lemma 2.6 (2).

6. Functional equation, order of zero and special value; proof of Corollary 2

6.1. **Functional equation.** Recall the functional equation of the zeta function of a pure motive M of weight w over $k = \mathbf{F}_q$:

$$\zeta(M^*, -s) = \det(M)(-q^{-s})^{\chi(M)}\zeta(M, s)$$

where M^* is the dual of M, $\chi(M)$ is the Euler characteristic of M (computed for example with the help of its l-adic realisation) and

$$\det(M) = \pm q^{w\chi(M)/2}$$

is the determinant of the Frobenius endomorphism of M. Applying this to Theorem 1, we get the following functional equation for $L(K, A, s)$:

$$L(K, A, 2 - s) = a(-q^{-s})^{\beta} L(K, A, s)$$

with

$$\begin{aligned}
\beta &= -2\chi(h^1(B)) - \chi(ш(A, K/k)) - \chi(\ln(A, K/k))\\
&= 4\dim B - \dim V_l(Ш(A, K\bar{k})) - \mathrm{rk} A(K\bar{k}) \quad \text{for any prime } l\\
a &= \left(\det h^1(B) \det h^1(B)(-1) \det ш(A, K/k) \det \ln(A, K/k)(-1)\right)^{-1}\\
&= \pm q^{\beta}.
\end{aligned}$$

Note that [GS, (C') p. 193] yields a different expression for β:

$$\beta = \chi(j_* H^1(\bar{A}, \mathbf{Q}_l)) = 2 - 2g - \deg(\mathfrak{f})$$

where g is the genus of C and $\mathfrak{f}$ is the conductor of A (relative to K/k), and the second equality follows from [Ra, Th. 1]. Is there a direct proof that these two expressions coincide?

6.2. **Order of zero: proof of Corollary 2.** a) It suffices to show that $\operatorname{ord}_{s=1} Z(ш(A, K/k)) = \dim V_l(Ш(A, K\bar{k}))^{(1)}$ for any prime l. This order can be computed through the action of the Frobenius endomorphism $\pi_ш$ of $ш(A, K/k)(1)$ on $R(ш(A, K/k)(1)) = R_l(ш(A, K/k))(-1)$ for any realisation functor R on $\mathcal{M}(k)$. If we use the l-adic realisation R_l for $l \neq p$, the claim is clear by Proposition 3.2 since $R_l(\pi_ш)$ acts as the inverse of γ. If we now take R_p, we find from (5.1), (5.2) and (5.4):

$$R_p(ш(A, K\bar{k}/\bar{k})(1))^{(0)} \simeq \overline{R_p(ш(A, K/k)(1))}^{(0)} \simeq V_p(Ш(A, K\bar{k})).$$

By Lemma 5.2, we have

$$\det(1 - \gamma t \mid \overline{R_p(ш(A, K/k)(1))}^{(0)}) = \prod_{v(a)=0} (1 - 1/at)$$

where a runs through the eigenvalues of $F_ш^m$ acting on $R_p(ш(A, K/k)(1))$, with valuation 0. By lemma 5.1, $F_ш^m = R_p(\pi_ш)$, so we are done.

Note that this argument does *not* show that $\dim V_l(Ш(A, K))$ is independent of l.

b) For any l, $V_l(Ш(A, K\bar{k}))^{(1)} = 0 \iff V_l(Ш(A, K\bar{k}))^{G_k} = V_l(Ш(A, K)) = 0 \iff Ш(A, K)\{l\}$ is finite. In view of a), this shows (i) $\iff$ (ii) $\iff$ (iii). Of course (iii) $\Rightarrow$ (iv) is classical, but let us give a proof using the motive $ш(A, K/k)$. Let P be the characteristic polynomial of $\pi_ш$ acting on the realisations of $ш(A, K/k)$: by Propositions 3.2 and 3.4, it is independent of l and (ii) implies that $P(q) \neq 0$. Hence γ acts invertibly on $Ш(A, K\bar{k})\{l\}$ for $l \nmid P(q)$ and $Ш(A, K\bar{k})\{l\}^\gamma = 0$ for such l. It remains to study $\operatorname{Ker}(Ш(A, K) \to Ш(A, K\bar{k})^\gamma)$: by the Hochschild-Serre spectral sequence for the Galois cohomology of A, this kernel is contained in the group $H^1(G_k, A(K\bar{k}))$. The exact sequence

$$0 \to B(\bar{k}) \to A(K\bar{k}) \to \mathrm{LN}(A, K\bar{k}/\bar{k}) \to 0,$$

Lang's theorem and the cohomological dimension of G_k yield an isomorphism

$$H^1(G_k, A(K\bar{k})) \xrightarrow{\sim} H^1(G_k, \mathrm{LN}(A, K\bar{k}/\bar{k})).$$

But the right hand side is finite since $\mathrm{LN}(A, K\bar{k}/\bar{k})$ is finitely generated. Hence $Ш(A, K)\{l\} = 0$ for l large enough, concluding the proof of (iv).

6.3. **Special value.** It is less obvious to relate Theorem 1 to the value of the principal part in the Birch and Swinnerton-Dyer conjecture ([Scn, Theorem p. 509], [KT]):

$$\lim_{s\to 1}\frac{L(A,s)}{(s-1)^{-\rho}}\sim \pm q^{\rho}\frac{|Ш(A,K)||\det\langle,\rangle_{A(K)}|}{|A(K)_{\mathrm{tors}}||A'(K)_{\mathrm{tors}}|}\prod_{c\in C}|\Phi_c(k(c))|, \tag{6.1}$$

where $\rho = \mathrm{rk}A(K)$, $\langle,\rangle_{A(K)}$ is the height pairing constructed in [Scn, p. 502], A' is the dual abelian variety to A and the Φ_c are the groups of connected components of the Néron model of A over C, as in §2.1.

It seems that the explicit expression of $L(A,s)$ could actually be used to provide an expression of the left hand side of (6.1) independently of the Birch and Swinnerton-Dyer conjecture, in the spirit of (1.2). This can presumably be done by the method of [Scn]: I hope to come back to it in the future. Let me only note that in Theorem 1, the factors $Z(h^1(B),q^{-s})$ and $Z(h^1(B),q^{1-s})$ respectively contribute by $q^{-g(B)}|B(k)|$ and $|B(k)|$ (as usual, $B := \mathrm{Tr}_{K/k}A$), while $Z(\mathrm{ln}(A,K/k),q^{1-s})$ contributes by

$$\pm q^{-\mathrm{rk}A(K)}\frac{\det\langle,\rangle_{A(K\bar k)}}{\det\langle,\rangle_{A(K)}}\frac{|A(K)_{\mathrm{tors}}/B(k)|}{|(\mathrm{LN}(A,K\bar k/\bar k)_{\gamma})_{\mathrm{tors}}|}$$

where $\langle,\rangle_{A(K)}$ and $\langle,\rangle_{A(K\bar k)}$ are as above. This folllows from the elementary lemma, in the spirit of [T1, Lemma z.4]:

Lemma 6.1. *Let $\langle,\rangle : M\times M'\to \mathbf{Q}$ be a $\mathbf{Q}$-non-degenerate pairing between finitely generated abelian groups. Suppose M and M' are provided with operators γ,γ' which are adjoint for the pairing, and $\mathbf{Q}$-semi-simple (e.g., γ is of finite order). Let $P=\det(1-\gamma T)$ be the inverse characteristic polynomial of γ acting on $M_{\mathbf{Q}}$. Then $\rho := \mathrm{ord}_{T=1}P=\mathrm{rk}M^{\gamma}$ and, if $P'=P/(1-T)^{\rho}$,*

$$|P'(1)|=\frac{\det\langle,\rangle}{\det\langle,\rangle^{\gamma}}\frac{|(M^{\gamma})_{\mathrm{tors}}|}{|(M_{\gamma})_{\mathrm{tors}}|}$$

where M^{γ} (resp. M_{γ}) denotes the γ-invariants (resp. coinvariants) of γ and $\langle,\rangle^{\gamma}$ is the ($\mathbf{Q}$-non-degenerate) pairing induced by $\langle,\rangle$ on $M^{\gamma}\times M'^{\gamma'}$.

7. Surfaces over a global field

Let K be a finitely generated field [over its prime subfield]. In [T2, §4], Tate associates to a smooth, projective, geometrically irreducible K-variety V the collection of zeta functions $\zeta(X,s)$ for smooth projective models $X\to Y$of V, where X,Y are of finite type over $\mathbf{Z}$, Y is a regular model of K and X is irreducible. For $y\in Y_{(0)}$, the zeta function of the fibre X_y may be factored as an alternating product of polynomials $P_i(X_y)$ corresponding to the l-adic cohomology groups of X_y, and the $P_i(X_y)$ are now known to be independent of l by [W.I]. This provides a factorisation

$$\zeta(X,s)=\prod_{i=0}^{2d}\Phi_i(X/Y,s)^{(-1)^i},\quad d=\dim V$$

and, for each i, $\Phi_i(X/Y,s)$ converges absolutely for $\Re(s)>\delta+i/2$ where $\delta=\dim Y$ is the *Kronecker dimension of* K, again by [W.I].

Tate then observes that, assuming one knows that $\Phi_i(X/Y,s)$ has an analytic continuation for $\Re(s)>\delta+i/2-1$, the order its of zeroes and poles in the strip $\delta+i/2-1<\Re(s)\leq\delta+i/2$ depends only on V and not on the choice of $X\to Y$, which allows him to formulate his conjectures on these orders (for $s\in\mathbf{N}$).

The question arises whether one can do better and associate canonical analytic functions to V rather that functions $\Phi_i(X/Y,s)$ depending on a model, for example to make sense of special values. If K is a global field, this is the subject of Serre's paper [S]. To the best of my knowledge, this question is open when $\delta>1$; the purpose of this section is to offer a partial answer when $\delta=2$, in the cases $i=0,1$.

We may write K as a function field in one variable over some global field k; without loss of generality, we may assume k algebraically closed in K. There is a finite purely inseparable extension k'/k such that $Kk'=k'(C)$ for some smooth projective k'-curve C; of course $k'=k$ if $\operatorname{char}K=0$, and k is then unique as the algebraic closure of $\mathbf{Q}$ in K. In view of Theorem 3 b), we set:

Definition 7.1. a) If $\operatorname{char}K=0$, $L(K,h^0(V),s)=L(k,h(C),s)$.
b) If $\operatorname{char}K>0$, $L(K,h^0(V),s)=\dfrac{L(k',h(C),s)}{\zeta(\operatorname{ln}(J(C),k'/\mathbf{F}_q),s-1)}$. Here $J(C)$ is the Jacobian of C and $\mathbf{F}_q$ is the field of constants of k'.

Note that if Y is a regular model of K over $\mathbf{Z}$, then $L(K,h^0(V),s)$ and $\zeta(Y,s)$ only differ by a finite number of Euler factors.

We now pass to the case $i=1$. In view of Theorem 1, we set:

Definition 7.2. a) If A is an abelian variety over K,

$$\begin{aligned}L(K,h^1(A),s)=L(k,h^1(\operatorname{Tr}_{K/k}A),s)L(k,h^1(\operatorname{Tr}_{K/k}A),s-1)\\ L(k,Ш(A,K/k),s)L(k,\operatorname{ln}(A,K/k),s-1)\end{aligned}$$

where the right hand side is defined in terms of l-adic realisations.
b) $L(K,h^1(V),s)=L(K,h^1(A),s)$, where A is the Albanese variety of V.

Definition 7.2 is independent of the choice of l (invertible in k) because this is so for each individual factor in a): for Ш $(A, K/k)$ it follows from [RZ, Satz 2.13] and [Sa, Cor. 0.6], using Proposition 3.2.

Question 7.3. If char $K > 0$, is Definition 7.2 independent of the choice of k?

One may have to do a normalization similar to the one in Definition 7.1 b) to get a positive answer.

Question 7.4. Can one interpret $L(K, h^1(A), s)$, *via* a trace formula, as an "Euler" product of the form

$$L(C, j_* H^1_l(A), s) = \prod_{x \in C_{(0)}} L(k(x), i_x^* H^1_l(\mathcal{A}), s)$$

where $\mathcal{A}$ is the Néron model of A over C?

(It is not even clear that the right hand side converges!)

Let us now place ourselves again in the situation of (1.3). It is natural to set $L(K, h^2(\Gamma), s) = L(k, h^0(\Gamma), s-1)$, and $L(K, h(\Gamma), s) = \prod_{i=0}^2 L(k, h^i(\Gamma), s)$. Theorem 4.1 then gives the following analogue to Theorem 3 a):

Theorem 7.5. *If* char $K = 0$*, one has*

$$\frac{L(k, h(S), s)}{L(K, h(\Gamma), s)} = L(k, a(D), s-1). \quad \square$$

If char $K > 0$, one has to take the normalization of Definition 7.1 b) into account; since Definition 7.2 may not be optimal, we skip this.

Question 7.6. The height pairing defined by Schneider in [Scn, p. 507]:

$$\mathcal{A}^0(\bar{C}) \times A'(K\bar{k}) \to \operatorname{Pic}(\bar{C})$$

induces a Galois-equivariant pairing

$$\mathcal{A}^0(\bar{C})/B(\bar{k}) \times \operatorname{LN}(A', K\bar{k}/\bar{k}) \to \mathbf{Z} \tag{7.1}$$

because $B(\bar{k})$ and $B'(\bar{k})$ are divisible; moreover, it presumably restricts to a pairing

$$B(\bar{k}) \times \operatorname{LN}(A', K\bar{k}/\bar{k}) \to \operatorname{Pic}^0(\bar{C}). \tag{7.2}$$

One way to justify (7.2) would be to show that the functor $S \mapsto \Gamma(C \times_k S, \mathcal{A}' \times_k S)$ on k-schemes of finite type is representable by a k-group scheme of finite type with connected component B, and that Schneider's pairing emanates from a pairing of k-group schemes. Then (7.2) would induce a Galois-equivariant homomorphism

$$\operatorname{LN}(A', K\bar{k}/\bar{k}) \to \operatorname{Hom}_{\bar{k}}(B, J). \tag{7.3}$$

Can one use (7.1) and (7.3) to describe the special values of $L(K, h^1(A), s)$?

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