ON THE TRANSCENDENTAL PART OF THE MOTIVE OF A SURFACE

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Introduction

Bloch's conjecture on surfaces [B1], which predicts the converse to Mumford's famous necessary condition for finite-dimensionality of the Chow group of 0-cycles [Mum1], has been a source of inspiration in the theory of algebraic cycles ever since its formulation in 1975. It is known for surfaces not of general type by [B-K-L] (see also [G-P1]), for certain generalised Godeaux surfaces [Voi] and in a few other scattered cases. Thirty years later, it remains open.

As was seen at least implicitely by Bloch himself early on, his conjecture is of motivic nature (see [B2, 1.11]). This was made explicit independently by Beilinson and the second author [Mu1]: we refer to Jannsen's article [J2] for an excellent overview. In particular, the

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Chow-Künneth decomposition for a surface S constructed in [Mu1] easily shows that the information necessary to study Bloch's conjecture is concentrated in the summand $h_2(S)$ of the Chow motive of S.[1]

The main purpose of this article is to introduce and study a finer invariant of S: the *transcendental part* $t_2(S)$ *of* $h_2(S)$. Let us immediately clarify to the reader that we will not give a proof of Bloch's conjecture! Instead, we study the endomorphism ring of $t_2(S)$ for a general S and prove the following two formulas (Theorems 7.4.3 and 7.4.8):

$$\text{End}_{\mathcal{M}_{\text{rat}}}(t_2(S)) \simeq \frac{A_2(S \times S)}{\mathcal{J}(S,S)} \simeq \frac{T(S_{k(S)})}{T(S_{k(S)}) \cap H_{\leq 1}}. \tag{7.1}$$

Here k is the base field, $\mathcal{M}_{\text{rat}}$ is the category of Chow motives over k with rational coefficients, A_* denotes Chow groups tensored with $\mathbb{Q}$, and

- $\mathcal{J}(S,S)$ is the subgroup of $A_2(S \times S)$ generated by those correspondences which are not dominant over S via either the first or the second projection;
- $T(S)$ is the Albanese kernel;
- $H_{\leq 1}$ is the subgroup of $A^2(S_{k(S)})$ generated by the images of the $A^2(S_L)$, where L runs through the subextensions of $k(S)/k$ of transcendence degree ≤ 1.

The first formula is a higher-dimensional analogue of a classical result of Weil concerning divisorial correspondences. The second formula provides a variant of Bloch's proof of Mumford's theorem in [B2, App. to Lecture 1] (with actually a slightly more precise result), cf. Corollary 7.4.9.

A conjecture generalising Bloch's conjecture:

$$\text{End}_{\mathcal{M}_{\text{rat}}}(t_2(S)) \overset{?}{\simeq} \text{End}_{\mathcal{M}_{\text{hom}}}(t_2^{\text{hom}}(S))$$

(here hom stands for homological equivalence) may therefore be reformulated as saying that *the cycles homologically equivalent to* 0 *should be contained in* $\mathcal{J}(S,S)$. This conjecture, in turn, appears in a wider generality in Beilinson's article [Bei]; a link with this point of view is outlined in the last section, via the theory of *birational motives* ([K-S], see §7.5 here; in the context of Bloch's conjecture this point of view goes right back to Bloch, Colliot-Thélène and Sansuc, see [B2, App. to Lect. 1]). In particular, it is proven that $t_2(S)$ does not depend on the choice of the refined Chow-Künneth decomposition of Propositions

[1] In this article we adopt a covariant convention for motives, hence write $h_i(S)$ rather than $h^i(S)$.

7.2.1 and 7.2.3, and is functorial in S for the action of correspondences (Corollary 7.8.10).

We now describe the contents of this paper in more detail. Section 7.1 fixes notation and reviews motives. Section 7.2 reviews the Chow-Künneth decomposition of a surface S (Proposition 7.2.1) and introduces the hero of our story, $t_2(S)$ (Proposition 7.2.3). Section 7.3 reviews the conjectures of [Mu2] on the Chow-Künneth decomposition as well as some of their consequences established by Jannsen in [J2], and proves a part of these consequences for the case of a product of two surfaces (Theorem 7.3.10). In Section 7.4 the isomorphisms (7.1) are established; they are reinterpreted in the next section in terms of birational motives. Section 7.6 studies the relationship of the previous results with Kimura's notion of finite-dimensional motives [Ki, G-P1, A-K]. The next section discusses some (conditional) higher-dimensional generalizations. Finally, Section 7.8 reproves and generalizes some of the previous results from a categorical viewpoint.

This article may be seen as a convergence point of the ways its 3 authors understand the theory of motives. The styles of the various sections largely reflect the styles of the various authors: we didn't attempt (too much) to homogenize them.

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7.1. Definitions and Notation

7.1.1. **Categories.** Throughout this paper we shall use interchangeably the notations $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $\mathcal{C}(X,Y)$ for Hom sets between objects of a category $\mathcal{C}$. In particular, the notation $\mathcal{C}(-,-)$ is more convenient when the symbol designating $\mathcal{C}$ is long, but the notation $\mathrm{End}_{\mathcal{C}}(X)$ may be more evocative than $\mathcal{C}(X,X)$.

7.1.2. **Pure motives.** Let k be a field and let $\mathcal{V} = \mathcal{V}_k$ be the category of smooth projective varieties over k. We shall sometimes write $X = X_d$ to say that $X \in \mathcal{V}$ is irreducible (or equidimensional) of dimension d.

We denote by $A^i(X) = A_{d-i}(X)$ the group $CH^i(X) \otimes \mathbb{Q}$ of cycles of codimension i (or dimension $d-i$) on X, modulo rational equivalence, with $\mathbb{Q}$ coefficients.

We shall assume that the reader is familiar with the definition of pure motives and will only give minimal recollections on it, except for one thing. In [Mu1, Mu2, J1] and [Sch], pure motives are defined in the Grothendieck tradition so that the natural functor sending a variety to its motive is contravariant. On the contrary, here we are going to consider *covariant* motives, in order to be compatible with Voevodsky's convention that his triangulated motives are covariant. Moreover, we change the sign of the weight. Since the translation thoroughly confused all three authors, we prefer to give a dictionary of how to pass from one convention to the other, for clarity and the benefit of the reader:

To start with, for $X, Y \in \mathcal{V}$ we introduce as in [Sch, p. 165] the groups of *contravariant Chow correspondences*

$$\operatorname{Corr}^i(X,Y) = \bigoplus_\alpha A^{d_\alpha+i}(X_\alpha \times Y)$$

if $X = \coprod_\alpha X_\alpha$ with X_α equidimensional of dimension d_α; composition of correspondences is given by the usual formula (ibid.):

$$g \circ_{contr} f = (p_{13})_*(p_{12}^* f \cdot p_{23}^* g).$$

Let us denote by $CH\mathcal{M}(k) = CH\mathcal{M}$ the category of Chow motives considered in [Mu1, Mu2, J1] or [Sch]. Thus an object of $CH\mathcal{M}$ is a triple $M = (X, p, m)$ where $X \in \mathcal{V}$, p is an idempotent in $\operatorname{Corr}^0(X,X)$ and $m \in \mathbb{Z}$, while morphisms are given by

$$CH\mathcal{M}((X,p,m),(Y,q,n)) = q \operatorname{Corr}^{n-m}(X,Y)p.$$

To $X \in \mathcal{V}$ we associate $ch(X) = (X, 1_X, 0) \in CH\mathcal{M}$ and to a morphism $f : X \to Y$ we associate $ch(f) = [\Gamma_f]^t$, where Γ_f is the graph of f and γ^t denotes the transpose of a correspondence γ[2]: this defines a contravariant functor $ch : \mathcal{V} \to CH\mathcal{M}$.

We could merely define $\mathcal{M}_{\mathrm{rat}}$ as the opposits category to $CH\mathcal{M}$. However it is much more comfortable to have an explicit description of it:

7.1.1. **Definition.** a) The groups of *covariant Chow correspondences* are defined as follows: for $X, Y \in \mathcal{V}$

$$\operatorname{Corr}_i(X,Y) = \operatorname{Corr}^{-i}(Y,X).$$

[2]On [Sch, p. 166], Scholl writes $ch(f) = [\Gamma_f]$, which is slightly misleading.

Composition of covariant correspondences is given by the formula

$$g \circ_{cov} f = (f^t \circ_{contr} g^t)^t.$$

(From now on, we drop the index ${}_{cov}$ for the composition sign.)
b) The category of *covariant Chow motives* $\mathcal{M}_{\mathrm{rat}}(k) = \mathcal{M}_{\mathrm{rat}}$ has objects triples $M = (X, p, m)$ as above, while morphisms are given by

$$\mathcal{M}_{\mathrm{rat}}((X, p, m), (Y, q, n)) = q \operatorname{Corr}_{m-n}(X, Y)p.$$

c) The "covariant motive" functor $h : \mathcal{V} \to \mathcal{M}_{\mathrm{rat}}$ is given by the formulas

$$h(X) = (X, 1_X, 0)$$
$$h(f) = [\Gamma_f].$$

Note that, by definition

$$\operatorname{Corr}_i(X, Y) = \bigoplus_{\alpha} A_{d_\alpha + i}(X_\alpha \times Y)$$

if $X = \coprod X_\alpha$ with $\dim X_\alpha = d_\alpha$. The reader will easily check the following

7.1.2. **Lemma.** *There is an anti-isomorphism of categories $F : CH\mathcal{M} \to \mathcal{M}_{\mathrm{rat}}$ defined by*

$$F(X, p, m) = (X, p^t, -m)$$
$$F(\gamma) = \gamma^t.$$

One has the formula $F \circ ch = h$. □

Except at the beginning of Section 7.2, we shall never mention the category $CH\mathcal{M}$ again and will work only with $\mathcal{M}_{\mathrm{rat}}$. Let us review a few features of this category:

7.1.2.1. *Effective motives.* Let $\mathcal{M}_{\mathrm{rat}}^{\mathrm{eff}}$ be the full subcategory of $\mathcal{M}_{\mathrm{rat}}$ consisting of the $(X, 1_X, 0)$ for $X \in \mathcal{V}$ (see [Sch]): this is the category of *effective Chow motives*.

7.1.2.2. *Tensor structure.* The product of varieties and of correspondences defines on $\mathcal{M}_{\mathrm{rat}}$ a tensor structure (= a symmetric monoidal structure which is distributive with respect to direct sums); $\mathcal{M}_{\mathrm{rat}}^{\mathrm{eff}}$ is stable under this tensor structure. Then $\mathcal{M}_{\mathrm{rat}}$ is an additive, $\mathbb{Q}$-linear, pseudoabelian tensor category. It is also rigid, in the sense that there exist internal Homs and dual objects $M^\vee$ satisfying suitable axioms. Namely one has

$$(X, p, m)^\vee = (X, p^t, -d - m)$$

if $\dim X = d$, and

$$\gamma^\vee = \gamma^t$$

if $\gamma \in \mathrm{Corr}_n(X, Y) = \mathcal{M}_{\mathrm{rat}}(X, 1_X, n), (Y, 1_Y, 0))$.

7.1.2.3. *The unit motive and the Lefschetz motive.* The *unit motive* is $\mathbf{1} = (\mathrm{Spec}(k), 1, 0)$: it is a unit for the tensor structure. The *Lefschetz motive* $\mathbb{L}$ is defined via the motive of the projective line over k:

$$h(\mathbb{P}^1_k) = \mathbf{1} \oplus \mathbb{L}.$$

We then have an isomorphism $\mathbb{L} \simeq (\mathrm{Spec}(k), 1, 1)$.

7.1.2.4. *Tate twists.* For every motive $M = (X, p, m)$ we define the Tate twist $M(r)$ to be the motive $(X, p, m + r)$. Note that, with our conventions, $M(r) \simeq M \otimes \mathbb{L}^{\otimes r}$ for $r \geq 0$.

7.1.2.5. *Inverse image morphisms.* For a morphism $f : X \to Y$ in $\mathcal{V}$, one often writes f_* instead of $h(f)$. One may also consider the map in $\mathcal{M}_{\mathrm{rat}}$:

$$f^* = [\Gamma_f^t] \in A_d(Y \times X) : h(Y) \to h(X)(e - d)$$

where $d = \dim X$, $e = \dim Y$.

7.1.2.6. *Action of correspondences on Chow groups.* Observe that, by definition

$$A^i(X) = \mathcal{M}_{\mathrm{rat}}(h(X), \mathbb{L}^i)$$
$$A_i(X) = \mathcal{M}_{\mathrm{rat}}(\mathbb{L}^i, h(X))$$

for any $X \in \mathcal{V}$. This gives us a way to let correspondences act on Chow groups:

- On the left: if $\alpha \in A_i(X)$ and $\gamma \in \mathrm{Corr}_n(X, Y)$, then $\gamma_*\alpha = \gamma \circ \alpha \in A_{i+n}(Y)$. (In terms of cycles: $\gamma_*(\alpha) = (p_2)_*(p_1^*(\alpha) \cdot \gamma)$.)
- On the right: if $\gamma \in \mathrm{Corr}_n(X, Y)$ and $\alpha \in A^i(Y)$, then $\gamma^*\alpha = \alpha \circ \gamma \in A^{i-n}(X)$.

7.1.3. *Remark.* Suppose that $\dim X = d$. If $\alpha \in A_i(X)$ is interpreted as a morphism from $\mathbb{L}^i$ to $h(X)$, then the dual morphism $\alpha^\vee : h(X)(-d) \to \mathbb{L}^{-i}$ is nothing else than α. The same applies to $\gamma_*\alpha$ and $\gamma^*\alpha$, with notation as above. If we view $\alpha = \alpha^\vee$ in $A^{d-i}(X)$, we thus get a formula comparing left and right actions:

$$\gamma^*\alpha = (\gamma^*\alpha)^\vee = (\alpha \circ \gamma)^\vee = \gamma^\vee \circ \alpha^\vee = \gamma^t \circ \alpha = (\gamma^t)_*\alpha.$$

In the same vein, note the formula

$$A^i(M) = A_{-i}(M^\vee).$$

7.1.2.7. *Chow groups of motives.* We now extend the functors $A^i : \mathcal{V} \to \mathrm{Vect}_\mathbb{Q}$ (contravariant) and $A_i : \mathcal{V} \to \mathrm{Vect}_\mathbb{Q}$ (covariant) to functors on $\mathcal{M}_{\mathrm{rat}}$:

$$A^i(X,p,m) = \mathcal{M}_{\mathrm{rat}}((X,p,m),\mathbb{L}^i) = p^* A^{i-m}(X)$$
$$A_i(X,p,m) = \mathcal{M}_{\mathrm{rat}}(\mathbb{L}^i,(X,p,m)) = p_* A_{i-m}(X).$$

7.1.3. **Weil cohomology theories.** If $\sim$ is any *adequate equivalence relation* on cycles (see [J3]), then a similar definition yields the category $\mathcal{M}_\sim$. In particular we will consider the cases where $\sim$ equals homological equivalence or numerical equivalence.

We give ourselves a Weil cohomology theory H^* on $\mathcal{V}$, as defined in [Kl] or [An, 3.3]; we shall denote its field of coefficients by K (by convention it is of characteristic 0). We also define

$$H_i(X) = \bigoplus_\alpha H^{2d_\alpha - i}(X_\alpha)$$

if $X = \coprod_\alpha X_\alpha$ with $\dim X_\alpha = d_\alpha$.

For an element $\alpha \in A^i(X)$ we denote by $\mathrm{cl}^i(\alpha)$ its image under the cycle map in $A^i(X) \to H^{2i}(X)$; we write

$$A^i(X)_{\mathrm{hom}} = \mathrm{Ker}\,\mathrm{cl}^i$$
$$A^i_{\mathrm{hom}}(X) = A^i(X)/A^i(X)_{\mathrm{hom}} = \mathrm{Coim}\,\mathrm{cl}^i$$
$$\bar{A}^i(X) = \mathrm{Im}\,\mathrm{cl}^i \simeq A^i_{\mathrm{hom}}(X).$$

Equivalently, we have "homological" cycle maps $\mathrm{cl}_i : A_i(X) \to H_{2i}(X)$ and vector spaces $A_i(X)^{\mathrm{hom}}, A_i^{\mathrm{hom}}(X)$ and $\bar{A}_i(X)$. One easily checks that the Künneth formula and Poincaré duality carry over to homology without any change.

We denote by $\mathcal{M}_{\mathrm{hom}}$ the (covariant) category of homological motives, which is is defined as above by considering correspondences modulo homological equivalence, and by h_{hom} the functor which associates to every $X \in \mathcal{V}_k$ its motive in $\mathcal{M}_{\mathrm{hom}}$.

7.1.3.1. *Action of correspondences on cohomology.* Let $X, Y \in \mathcal{V}$, equidimensional of dimensions d and e for simplicity. The cycle class map gives us a homomorphism

$$\mathrm{Corr}_i(X,Y) = A_{d+i}(X \times Y) = A^{e-i}(X \times Y) \xrightarrow{\mathrm{cl}^{e-i}} H^{2e-2i}(X \times Y).$$

Since we have

$$H^{2e-2i}(X\times Y)\simeq\bigoplus_j H^j(X)\otimes H^{2e-2i-j}(Y)\simeq\bigoplus_j H^j(X)\otimes H^{2i+j}(Y)^*$$
$$\simeq\prod_j \mathrm{Hom}(H^{2i+j}(Y),H^j(X))$$

by the Künneth formula and Poincaré duality, we get a contravariant action of correspondences

$$\gamma^*: H^*(Y)\to H^{*-2i}(X)$$

for $\gamma\in\mathrm{Corr}_i(X,Y)$, extending the action of morphisms. Similarly, using the homological cycle map, we get a covariant action

$$\gamma_*: H_*(X)\to H_{*+2i}(Y)$$

by means of the composition

$$\mathrm{Corr}_i(X,Y)=A_{d+i}(X\times Y)\xrightarrow{\mathrm{cl}_{d+i}} H_{2d+2i}(X\times Y)$$
$$\simeq\bigoplus_j H_{2d+2i-j}(X)\otimes H_j(Y)\simeq\bigoplus_j H_{j-2i}(X)^*\otimes H_j(Y)$$
$$\simeq\prod_j\mathrm{Hom}(H_{j-2i}(X),H_j(Y)).$$

7.1.3.2. *Homology and cohomology of motives.* As in 7.1.2.7, we may use 7.1.3.1 to extend H^i, H_i to tensor functors

$$H^*:\mathcal{M}_{\mathrm{hom}}\to\mathrm{Vect}_K^{\mathrm{gr}}\ \text{(contravariant)}$$
$$H_*:\mathcal{M}_{\mathrm{hom}}\to\mathrm{Vect}_K^{\mathrm{gr}}\ \text{(covariant)}$$

with values in graded vector spaces: the first functor corresponds to [An, 4.2.5.1]. Explicitly, $H^i(X,p,m)=p^*H^{i-2m}(X)$, $H_i(X,p,m)=p_*H_{i-2m}(X)$ and $H^*(\gamma)=\gamma^*$, $H_*(\gamma)=\gamma_*$. We also have

$$H_i(M)=H^{-i}(M^\vee).$$

7.1.4. **Definition** (cf. [An, 3.4])**.** A Weil cohomology theory H is *classical* if:

(1) $\mathrm{char} k=0$ and H is algebraic de Rham cohomology, l-adic cohomology for some prime number l or Betti cohomology relative to a complex embedding of k, or
(2) $\mathrm{char} k=p>0$ and H is crystalline cohomology (if k is perfect) or l-adic cohomology for some prime number $l\neq p$.

(Recall that, if the prime number l is different from char$k$, l-adic cohomology is defined by $H^i_l(X) = H^i_{et}(X \otimes_k k_s, \mathbb{Q}_l)$, where k_s is some separable closure of k.)

7.1.5. **Lemma.** *If* char$k = 0$*, homological equivalence does not depend on the choice of a classical Weil cohomology theory. Moreover, for any smooth projective* X, Y*, the Hom groups* $\mathcal{M}_{\rm hom}(h_{\rm hom}(X), h_{\rm hom}(Y))$ *are finite-dimensional* $\mathbb{Q}$*-vector spaces.*

Proof. Suppose first that k admits a complex embedding. Then the first statement follows from the comparison theorems between classical cohomology theories [An, 3.4.2]; the second one follows from taking Betti cohomology, which has rational coefficients.

In general, X and Y are defined over some finitely generated subfield k_0 of k, and k_0 has a complex embedding. If we pick two classical Weil cohomology theories over k, then the base change comparison theorems show that we may compare them with the corresponding ones over k_0; in turn we may compare the latter two with the help of the complex embedding. □

7.1.6. *Remark.* In arbitrary characteristic, the dimension of $H^i(M)$ for $M \in \mathcal{M}_{\rm rat}$ is independent of the choice of the (classical) Weil cohomology H [An, 4.2.5.2], and the Euler characteristic $\sum(-1)^i \dim H^i(M)$ is independent of the choice of the (arbitrary) Weil cohomology H because it equals the trace $tr(1_M)$ computed in the rigid category $\mathcal{M}_{\rm rat}$. For example, for a curve C of genus g one always has $\dim H^1(C) = 2g$.

Unless otherwise specified, all Weil cohomology theories considered in this paper will be classical.

7.1.4. **Chow-Künneth decompositions.** Let $X \in \mathcal{V}$, $X = X_d$. We say that X has a *Chow-Künneth decomposition in* $\mathcal{M}_{\rm rat}$ (C-K for short) if there exist orthogonal projectors $\pi_i = \pi_i(X) \in {\rm Corr}_0(X, X) = A^d(X \times X)$, for $0 \le i \le 2d$, such that ${\rm cl}^d(\pi_i)$ is the $(i, 2d-i)$-component of Δ_X in $H^{2d}(X \times X)$ and

$$[\Delta_X] = \sum_{0 \le i \le 2d} \pi_i.$$

This implies that in $\mathcal{M}_{\rm rat}$ the motive $h(X)$ decomposes as follows:

$$h(X) = \bigoplus_{0 \le i \le 2d} h_i(X) \tag{7.2}$$

where $h_i(X) = (X, \pi_i, 0)$. Moreover

$$H^*(h_i(X)) = H^i(X), \qquad H_*(h_i(X)) = H_i(X)$$

(see 7.1.3.2).

If we have $\pi_i = \pi^t_{2d-i}$ for all i, we say that the C-K decomposition is *self-dual*.

7.1.5. **Triangulated motives.** Let $DM^{\rm eff}_{\rm gm}(k)$ be the triangulated category of effective geometrical motives constructed by Voevodsky [Voev2]: there is a covariant functor $M : Sm/k \to DM^{\rm eff}_{\rm gm}(k)$ where Sm/k is the category of smooth schemes of finite type over k. We shall write $DM^{\rm eff}_{\rm gm}(k,\mathbb{Q})$ for the pseudo-abelian hull of the category obtained from $DM^{\rm eff}_{\rm gm}(k)$ by tensoring morphisms with $\mathbb{Q}$, and usually abbreviate it into $DM^{\rm eff}_{\rm gm}$. By abuse of notation, we shall denote by $\mathbb{Q}(1)$ the image of $\mathbb{Z}(1)$ in $DM^{\rm eff}_{\rm gm}(k,\mathbb{Q})$ under the natural functor $DM^{\rm eff}_{\rm gm}(k) \to DM^{\rm eff}_{\rm gm}(k,\mathbb{Q})$.

By [Voev2, p. 197], M induces a covariant functor $\Phi : \mathcal{M}^{\rm eff}_{\rm rat} \to DM^{\rm eff}_{\rm gm} := DM^{\rm eff}_{\rm gm}(k,\mathbb{Q})$ which is a full embedding by [Voev3] and sends $\mathbb{L}$ to $\mathbb{Q}(1)[2]$ (this functor is already defined and fully faithful on the level of Chow motives with integral coefficients). As in [Voev2], we denote by $DM_{\rm gm} := DM_{\rm gm}(k,\mathbb{Q})$ the category obtained from $DM^{\rm eff}_{\rm gm}$ by inverting $\mathbb{Q}(1)$.

The category $DM^{\rm eff}_{\rm gm}$ admits a natural full embedding, as a tensor triangulated category, into the category $DM^{\rm eff}_{-} := DM^{\rm eff}_{-}(k)\otimes\mathbb{Q}$ of (bounded above) motivic complexes [Voev2, 3.2]. If the motive $h(X)$ of a smooth projective variety X has a Chow-Künneth decomposition in $\mathcal{M}_{\rm rat}$ as in (7.2), we write $M_i(X) = \Phi(h_i(X))$, so that in the category $DM^{\rm eff}_{\rm gm}$ we have the following decomposition

$$M(X) = \bigoplus_{0\le i\le 2d} M_i(X).$$

7.1.6. **Abelian varieties.** We denote by $\mathrm{Ab}(k)$ or Ab the category whose objects are abelian k-varieties and, for $A, B \in \mathrm{Ab}$, $\mathrm{Ab}(A,B) = \mathrm{Hom}(A,B)\otimes\mathbb{Q}$. Recall that these are finite-dimensional $\mathbb{Q}$-vector spaces (see [Mum2, p. 176]).

7.2. Chow-Künneth decomposition for surfaces

In this section we adapt in Proposition 7.2.1 the construction of a suitable Chow-Künneth decomposition (see [Mu1] and [Sch]) for a smooth projective surface S to our covariant setting for $\mathcal{M}_{\rm rat}$. Then we refine this decomposition in Proposition 7.2.3.

7.2.1. **The covariant Chow-Künneth decomposition for surfaces.** For a smooth projective variety $X = X_d$, we denote by Alb_X and by Pic^0_X the Albanese variety and the Picard variety of X; $T(X)$ denotes the Albanese kernel of X, i.e the kernel of the map $A^d(X)_0 \to \mathrm{Alb}_X(k)_{\mathbb{Q}}$, where $A^d(X)_0$ is the group of 0-cycles of degree 0.

From [Mu1], [Mu2] and [Sch, §4] it follows that, in $CH\mathcal{M}$, there exist projectors p_0, p_1, p_{2d-1}, p_{2d} in $\mathrm{End}_{CH\mathcal{M}}(ch(X)) = A^d(X \times X)$ with the following properties:

(i) $p_0 = (1/n)[P \times X]$ and $p_{2d} = (1/n)[X \times P]$, P a closed point on X of degree n with separable residue field.
(ii) p_{2d} operates as 0 on $A^i(X)$ for $i \neq d$; on $A^d(X)$ we have $F^1 A^d(X) = \mathrm{Ker}\, p_{2d} = A^d(X)_{\mathrm{num}}$.
(iii) p_1, the "Picard projector", operates as 0 on all $A^i(X)$ with $i \neq 1$; its image on $A^1(X)$ is $A^1(X)_{\mathrm{hom}}$, hence $A^1(ch^1(X)) = \mathrm{Pic}^0_X(k)_{\mathbb{Q}}$ where $ch^1(X) = (X, p_1, 0)$. Moreover, on $A^1(X)_{\mathrm{hom}}$ p_1 operates as the identity.
(iv) p_{2d-1}, the "Albanese projector", operates as 0 on all $A^i(X)$ with $i \neq d$; on $A^d(X)$ its image lies in $A^d(X)_{\mathrm{hom}}$ and on $A^d(X)_{\mathrm{hom}}$ its kernel is $T(X)$; hence $A^d(ch^{2d-1}(X)) = \mathrm{Alb}_X(k)_{\mathbb{Q}}$ where $ch^{2d-1}(X) = (X, p_{2d-1}, 0)$.
(v) $F^2 A^d(X) = \mathrm{Ker}(p_{2d-1}) \cap F^1 = T(X)$.
(vi) $p_0, p_1, p_{2d-1}, p_{2d}$ are mutually orthogonal.
(vii) $p_{2d-1} = p_1^t$.

Note that P exists by [EGA4, 17.15.10 (iii)]. Also, the motive $ch^0(X)$ is in general not isomorphic to **1**, but rather to $ch(\mathrm{Spec}\, k')$ where k' is the *field of constants* of X. If X is equidimensional but reducible, to get the "right" p_0 and p_{2d} we need to take the sum of the corresponding projectors for all irreducible components X_α of X; then we get

$$ch^0(X) = \bigoplus ch^0(X_\alpha) \simeq \bigoplus ch(\mathrm{Spec}\, k'_\alpha) = ch(\coprod \mathrm{Spec}\, k'_\alpha)$$

where k'_α is the field of constants of X_α: thus $ch^0(X)$ is an *Artin motive* (cf. [An, 4.1.6.1]). Note finally that the existence of $p_0, p_1, p_{2d-1}, p_{2d}$ with properties (i) – (vii) is part of a conjectural Chow-Künneth decomposition in $CH\mathcal{M}$ for any variety X (see [Mu2] and §7.3).

In the case of a smooth projective surface S a Chow-Künneth decomposition of the motive $ch(S)$ always exists by [Mu1] and [Sch, §4].

The following proposition is just a translation of these results in $\mathcal{M}_{\mathrm{rat}}$.

7.2.1. **Proposition.** *Let S be a smooth projective connected surface over k and let $P \in S$ be a separable closed point. There exists a Chow-Künneth decomposition of $h(S)$ in $\mathcal{M}_{\rm rat}$: $h(S) = \bigoplus_{0\le i\le 4} h_i(S)$, with $h_i(S) = (S, \pi_i, 0)$, $\pi_i = \pi_i(S) \in A^2(S\times S)$, with the following properties:*

(i) $\pi_i = \pi_{4-i}^t$ *for* $0 \le i \le 4$.

(ii) $\pi_0 = (1/\deg(P))[S \times P]$, $\pi_4 = (1/\deg(P))[P \times S]$.

(iii) *There exists a curve $C \subset S$ of the form $C_1 \cup C_2$, where $C_1 = S\cdot H$ is a general (smooth) hyperplane section of S, such that π_1 is supported on $S \times C$ (and hence π_3 is supported on $C \times S$).*

(iv) *Let $i_1 : C_1 \to S$ be the inclusion and $\xi = (i_1)_* i_1^* : h(S) \to h(S)(1)$. Then the compositions*

$$h_3(S) \to h(S) \xrightarrow{\xi} h(S)(1) \to h_1(S)(1)$$

$$h_4(S) \to h(S) \xrightarrow{\xi^2} h(S)(2) \to h_0(S)(2)$$

are isomorphisms.

We set $\pi_2 := \Delta(S) - \pi_0 - \pi_1 - \pi_3 - \pi_4$ and $h_2(S) = (S, \pi_2, 0)$. We have the following tables:

$M =$	$h_0(S)$	$h_1(S)$	$h_2(S)$	$h_3(S)$	$h_4(S)$
$A^0(M) =$	$A^0(S)$	0	0	0	0
$A^1(M) =$	0	$\mathrm{Pic}^0_S(k)_\mathbb{Q}$	$NS(S)_\mathbb{Q}$	0	0
$A^2(M) =$	0	0	$T(S)$	$\mathrm{Alb}_S(k)_\mathbb{Q}$	$A^2_{\rm num}(S)$

$M =$	$h_0(S)$	$h_1(S)$	$h_2(S)$	$h_3(S)$	$h_4(S)$
$A_0(M) =$	$A_0^{\rm num}(S)$	$\mathrm{Alb}_S(k)_\mathbb{Q}$	$T(S)$	0	0
$A_1(M) =$	0	0	$NS(S)_\mathbb{Q}$	$\mathrm{Pic}^0_S(k)_\mathbb{Q}$	0
$A_2(M) =$	0	0	0	0	$A_2(S)$

and

$$\mathrm{End}_{\mathcal{M}_{\rm rat}}(h_1(S)) = \mathrm{End}_{\rm Ab}(\mathrm{Alb}_S)$$
$$\mathrm{End}_{\mathcal{M}_{\rm rat}}(h_3(S)) = \mathrm{End}_{\rm Ab}(\mathrm{Pic}_S).$$

Proof. In conformity with Lemma 7.1.2, we take $\pi_i = p_i^t$, where p_i are the projectors defined in [Sch, §4]. (The first table is copied from [Sch, p.178]; by Remark 7.1.3, we have the general formula $A_i(h_j(X)) = A^{d-i}(h_{2d-j}(X))$ for a self-dual C-K decomposition.) □

For later use we add some precisions on the construction of the p_i, hence of the π_i, especially concerning rationality issues, making [Sch, 4.2] more specific. Contrary to the case of P, the curve C_1 may always be chosen as defined over k and geometrically connected: this is clear if k is infinite by Bertini's theorem [Ha, p. 179, Th. 8.18 and p. 245, Rem. 7.9.1], and in case k is finite this can also be achieved up to enlarging the projective embedding of S as in [Del-I, 5.7].

In Proposition 7.2.1 (ii) the curve C_2 enters the picture because, in the case of a surface, in order to get mutually orthogonal idempotents π_1 and π_3, one has to introduce a "correction term" (see [Mu1] and [Sch, p. 177]) as follows: in the contravariant setting one first takes projectors $p_1^?, p_3^?$ defined in [Mu1] verifying $p_3^? = (p_1^?)^t$ and $p_3^? p_1^? = 0$, and then one corrects them by $p_1 = p_1^? - \frac{1}{2}(p_1^? \circ p_3^?)$ and $p_3 = p_3^? - \frac{1}{2}(p_1^? \circ p_3^?)$.[3] The correction term $\frac{1}{2}(p_1^? \circ p_3^?)$ is supported on $C_2 \times C_2$ and is of the form $\sum A_\lambda \times A'_\lambda$, where each A_λ, A'_λ is a divisor on S, homologically equivalent to 0.

Supposing k perfect, the projector π_1 may be described more precisely as follows. Let $i : C \to S$ be the closed embedding. Replace C by its normalization $\tilde{C}$, which is smooth. There is a divisor class $D \in A^1(S \times \tilde{C})$ such that $\pi_1 = (1_S \times \tilde{\imath})(D)$, where $\tilde{\imath} : \tilde{C} \to S$ is the proper morphism induced by the projection $\tilde{C} \to C$ and $\tilde{\imath}_*$ is the correspondence given by the graph $\Gamma_{\tilde{\imath}}$ of. With this description we then have $\pi_3 = D^t \circ \tilde{\imath}^*$, where $\tilde{\imath}^* = \Gamma_{\tilde{\imath}}^t$. Note that $D^t(R)$ is a divisor homologically equivalent to 0 on S for every divisor R on C.

7.2.2. **The refined Chow-Künneth decomposition.** We now introduce the motive $t_2(S)$, whose construction had been outlined in [An, 11.1.3] in a special case. We start with a well-known lemma (cf. [Bo, §4, no 2, Formula (12) p. 77]):

7.2.2. **Lemma.** *Let $\mu : V \times W \to \mathbb{Q}$ be a perfect pairing between two finite dimensional $\mathbb{Q}$-vector spaces V, W. Let $(e_i)_{1 \le i \le n}$ be a basis of V and let (e_i^*) be the dual basis of W with respect to this pairing ($\mu(e_i, e_j^*) = \delta_{ij}$). Then, in $\mathrm{Hom}(V \otimes W, \mathbb{Q}) \simeq \mathrm{Hom}(V, W^*)$, we have the identity*

$$\mu^{-1} = \sum e_i \otimes e_i^*$$

where we have viewed μ as an isomorphism in $\mathrm{Hom}(W^*, V)$. In particular, the right hand side is independent of the choice of the basis (e_i). □

7.2.3. **Proposition.** *Let S be a surface provided with a C-K decomposition as in Proposition 7.2.1. Let k_s be a separable closure of k, $G_k = Gal(k_s/k)$ and*

$$\underline{NS}_S = NS(S \otimes_k k_s)_{\mathbb{Q}}$$

[3]This correction is the one from [Sch] which is different from the one in [Mu1]: its advantage is that $p_3 = p_1^t$.

be the ($\mathbb{Q}$-linear, geometric) Néron-Severi group of S viewed as a G_k-module. Then there is a unique splitting

$$\pi_2 = \pi_2^{\mathrm{alg}} + \pi_2^{tr}$$

which induces a decomposition

$$h_2(S) \simeq h_2^{\mathrm{alg}}(S) \oplus t_2(S)$$

where $h_2^{\mathrm{alg}}(S) = (S, \pi_2^{\mathrm{alg}}, 0) \simeq h(\underline{NS}_S)(1)$ *and* $t_2(S) = (S, \pi_2^{\mathrm{tr}}, 0)$. *Here* $h(\underline{NS}_S)$ *is the Artin motive associated to* $\underline{NS}_S$. *Moreover the tables of Proposition 7.3.6 refine as follows:*

$M =$	$h_2^{\mathrm{alg}}(S)$	$t_2(S)$
$A^0(M) =$	0	0
$A^1(M) =$	$NS(S)_{\mathbb{Q}}$	0
$A^2(M) =$	0	$T(S)$

$M =$	$h_2^{\mathrm{alg}}(S)$	$t_2(S)$
$A_0(M) =$	0	$T(S)$
$A_1(M) =$	$NS(S)_{\mathbb{Q}}$	0
$A_2(M) =$	0	0

Finally

$$\begin{aligned} H^2(S) = H^2_{\mathrm{alg}}(S) \oplus H^2_{\mathrm{tr}}(S) = \pi_2^{\mathrm{alg}} H^2(S) \oplus \pi_2^{\mathrm{tr}} H^2(S) \\ = (NS(S) \otimes K) \oplus H^2(t_2(S)) \end{aligned}$$

where $H^2_{\mathrm{tr}}(S)$ *is (by definition) the "transcendental cohomology".*

Proof. Choose a finite Galois extension E/k such that the action of G_k on $\underline{NS}_S$ factors through $G = Gal(E/k)$. Let $[D_i]$ be an orthogonal basis of $\underline{NS}_S = NS(S_E)_{\mathbb{Q}}$. It follows from Lemma 7.2.2 that

$$\sum_i \frac{1}{< [D_i], [D_i] >} [D_i] \otimes [D_i] \in NS(S_E)_{\mathbb{Q}} \otimes NS(S_E)_{\mathbb{Q}}$$

is G-invariant, where $< [D_i], [D_i] >$ are the intersection numbers. By Proposition 7.2.1, the k-rational projector π_2 defines a G-equivariant section σ of the projection $A^1(S_E) \to NS(S_E)_{\mathbb{Q}}$. The composition

$$\lambda : NS(S_E)_{\mathbb{Q}} \otimes NS(S_E)_{\mathbb{Q}} \xrightarrow{\sigma\otimes\sigma} A^1(S_E) \otimes A^1(S_E) \xrightarrow{\cap} A^2((S \times S)_E)$$

is also G-equivariant. It follows that

$$\pi_2^{\mathrm{alg}} = \lambda(\sum_i \frac{1}{< [D_i], [D_i] >} [D_i] \otimes [D_i]) \in A^2((S \times S)_E)$$

is a G-invariant cycle, hence descends uniquely to a correspondence $\pi_2^{\mathrm{alg}} \in A^2(S \times S)$.

Over E, the correspondences

$$\alpha_i = \frac{1}{<[D_i],[D_i]>}[D_i \times D_i] = \lambda(\frac{1}{<[D_i],[D_i]>}[D_i] \otimes [D_i])$$

are mutually orthogonal idempotents, are orthogonal to π_j for $j \neq 2$, and verify

$$\pi_2 \circ \alpha_i = \alpha_i \circ \pi_2 = \alpha_i.$$

It follows that their sum π_2^{alg} is an idempotent orthogonal to π_j for $j \neq 2$, and that $\pi_2 \circ \pi_2^{\mathrm{alg}} = \pi_2^{\mathrm{alg}} \circ \pi_2 = \pi_2^{\mathrm{alg}}$. We define

$$\pi_2^{\mathrm{tr}} := \pi_2 - \pi_2^{\mathrm{alg}}.$$

In order to prove the isomorphism $h_2^{\mathrm{alg}}(S) \simeq h(\underline{NS}_S)(1)$, it is enough to show that

$$M_i := (S_E, \alpha_i, 0) \simeq \mathbb{L} \text{ for } 1 \leq i \leq \rho$$

where ρ is the Picard number. Define $f_i : \mathbb{L} \to M_i$ by $f_i = \alpha_i \circ [D_i] \circ 1_{\operatorname{Spec} E} = [D_i]$. The transpose f_i^t is a morphism $M_i \to \mathbb{L}$ and, by taking $g_i = \frac{1}{<[D_i, D_i]>} f_i^t$, we get $g_i \circ f_i = 1_{\mathbb{L}}$ and $f_i \circ g_i = \alpha_i$, hence the required isomorphism.

From the construction above we also get $A^*(h_2^{\mathrm{alg}}(S)) = A^1(h_2(S)) = NS(S)_{\mathbb{Q}}$, $A^*(t_2(S)) = T(S)$. This shows that

$$\mathcal{M}_{\mathrm{rat}}(\mathbb{L}, t_2(S_E)) = \mathcal{M}_{\mathrm{rat}}(t_2(S_E), \mathbb{L}) = 0$$

for any extension E/k. Taking E as above, we get that

$$\mathcal{M}_{\mathrm{rat}}(h_2^{\mathrm{alg}}(S_E), t_2(S_E)) = \mathcal{M}_{\mathrm{rat}}(t_2(S_E), h_2^{\mathrm{alg}}(S_E)) = 0$$

hence by descent that

$$\mathcal{M}_{\mathrm{rat}}(h_2^{\mathrm{alg}}(S), t_2(S)) = \mathcal{M}_{\mathrm{rat}}(t_2(S), h_2^{\mathrm{alg}}(S)) = 0.$$

Therefore

$$\operatorname{End}_{\mathcal{M}_{\mathrm{rat}}}(h_2(S)) = \operatorname{End}_{\mathcal{M}_{\mathrm{rat}}}(h_2^{\mathrm{alg}}(S)) \times \operatorname{End}_{\mathcal{M}_{\mathrm{rat}}}(t_2(S))$$

which implies the uniqueness of the decomposition $\pi_2 = \pi_2^{\mathrm{alg}} + \pi_2^{\mathrm{tr}}$.

The assertions on cohomology immediately follow from the definition of π_2^{alg} and π_2^{tr}. □

7.2.4. **Corollary.** $t_2(S) = 0 \Rightarrow H^2_{\mathrm{tr}}(S) = 0$. *If* char$k = 0$, *this implies* $p_g = 0$.

Proof. The first assertion is obvious from Proposition 7.2.3. The second one is classical [B2]. □

7.2.5. **Definition.** For a surface S, we call the set of projectors

$$\{\pi_0, \pi_1, \pi_2^{\mathrm{alg}}, \pi_2^{\mathrm{tr}}, \pi_3, \pi_4\}$$

the *refined Chow-Künneth decomposition* associated to the C-K decomposition $\{\pi_0, \pi_1, \pi_2, \pi_3, \pi_4\}$.

7.3. On some conjectures

In this section, we show in Theorem 7.3.10 that part of the results proved by U. Jannsen in [J2, Prop 5.8] hold unconditionally for the Chow motives coming from surfaces. We first recall the conjectures about the existence of a C-K decomposition as formulated in [Mu2] (see also [J2]).

7.3.1. **Conjecture** (see Conj. A in [Mu2]). *Every smooth projective variety X has a Chow-Künneth decomposition.*

The conjecture is true in particular for curves, surfaces, abelian varieties, uniruled 3-folds and Calabi-Yau 3-folds.

If X and Y have a C-K decomposition, with projectors $\pi_i(X)$ and $\pi_j(Y)$ $(0 \le i \le 2d, d = \dim X$ and $0 \le j \le 2e, e = \dim Y)$ then $Z = X \times Y$ also has a C-K decomposition with projectors $\pi_m(Z)$ given by $\pi_m(Z) = \sum_{r+s=m} \pi_r(X) \times \pi_s(Y)$ whith $0 \le m \le 2(d+e)$.

If the motive $h(X)$ is finite-dimensional in the sense of Kimura [Ki] and the Künneth components of the diagonal are algebraic (i.e., are classes of algebraic cycles), then $h(X) = \bigoplus_{0\le i\le 2d} h_i(X)$ and the motives $h_i(X)$ are unique, up to isomorphism as follows from the results of [Ki] (see §7.6).

Now let X have a C-K decomposition and consider the action of the correspondence $\pi_i(X)$ on the Chow groups $A^j(X)$. Then Conjecture B in [Mu2] translates as follows in our covariant setting (identical statement):

7.3.2. **Conjecture** (Vanishing Conjecture). *The correspondences $\pi_i(X)$ act as 0 on $A^j(X)$ for $i < j$ and for $i > 2j$.*

Assuming that Conjectures 7.3.1 and 7.3.2 hold, one may define a decreasing filtration $F^\bullet$ on $A^j(X)$ as follows:

$$F^1A^j(X) = \operatorname{Ker}\pi_{2j}, F^2A^j(X) = \operatorname{Ker}\pi_{2j} \cap \operatorname{Ker}\pi_{2j-1}, \dots$$
$$F^\nu A^j(X) = \operatorname{Ker}\pi_{2j} \cap \operatorname{Ker}\pi_{2j-1}, \cap \cdots \cap \operatorname{Ker}\pi_{2j-\nu+1}.$$

Note that, with the above definitions, $F^{j+1}A^j(X) = 0$.

Also it easily follows from the definition of $F^\bullet$ (see [Mu2, 1.4.4]) that:

$$F^1A^j(X) \subset A^j(X)_{\mathrm{hom}}.$$

7.3.3. **Conjecture** (see Conj. D in [Mu2]). *$F^1A^j(X) = A^j(X)_{\mathrm{hom}}$, for all j.*

Finally we mention:

7.3.4. **Conjecture** (see Conj. C in [Mu2]). *The filtration $F^\bullet$ is independent of the choice of the $\pi_i(X)$.*

7.3.5. *Remark.* Jannsen [J2] has shown that if the Conjectures 7.3.1 ... 7.3.4 hold for every smooth projective variety over k, then the filtration $F^\bullet$ satisfies *Beilinson's Conjecture*. The converse also holds [J2, 5.2].

Now let X and Y be two smooth projective varieties: the following result, due to U. Jannsen, relates Conjectures 7.3.2 and 7.3.3 for $Z = X \times Y$, with the groups $\mathcal{M}_{\mathrm{rat}}(h_i(X), h_j(Y))$.

7.3.6. **Proposition** ([J2, Prop 5.8]). *Let X and Y be smooth projective varieties of dimensions respectively d and e, provided with C-K decompositions, and let $Z = X \times Y$ be provided with the product C-K decomposition.*
a) If Z satisfies the vanishing Conjecture 7.3.2, then:

$$\mathcal{M}_{\mathrm{rat}}(h_i(X), h_j(Y)) = 0 \ \text{if} \ j < i;\ 0 \le i \le 2d;\ 0 \le j \le 2e.$$

b) If Z satisfies Conjecture 7.3.3 then

$$\mathcal{M}_{\mathrm{rat}}(h_i(X), h_i(Y)) \simeq \mathcal{M}_{\mathrm{hom}}(h_i^{\mathrm{hom}}(X), h_i^{\mathrm{hom}}(Y)).$$

In particular, if $X \times X$ satifies Conjecture 7.3.3, then b) implies that the $\mathbb{Q}$-vector space $\mathrm{End}_{\mathcal{M}_{\mathrm{rat}}}(h_i(X))$ has finite dimension for $0 \le i \le 2d$, at least if $\mathrm{char} k = 0$ and the Weil cohomology is classical (cf. Lemma 7.1.5).

7.3.7. *Remark.* Note that, because of our covariant definition of the functor $h : \mathcal{V} \to \mathcal{M}_{\mathrm{rat}}$, in a) we have $j < i$, while in the contravariant setting (as in [J2, 5.8]) one has $i < j$.

7.3.8. **Corollary.** *Let S be a smooth projective surface and C a smooth projective curve. Let $\Delta_S = \sum_{0 \le i \le 4} \pi_i(S)$ and $\Delta_C = \sum_{0 \le i \le 2} \pi_j(C)$ be C-K decompositions respectively for S and for C. Then*

$$(1)\ \pi_j(C) \cdot \Gamma \cdot \pi_i(S) = 0 \ \text{if} \begin{cases} i > j \ \text{and}\ \Gamma \in A^1(S \times C) \\ i = j \ \text{and}\ \Gamma \in A^1(S \times C)_{\mathrm{hom}} \end{cases}$$

$$(2)\ \pi_j(S) \cdot \Gamma \cdot \pi_i(C) = 0 \ \text{if} \begin{cases} i > j \ \text{and}\ \Gamma \in A^2(C \times S) \\ i = j \ \text{and}\ \Gamma \in A^2(C \times S)_{\mathrm{hom}} \end{cases}$$

$$(3)\ \pi_r(S) \cdot \Gamma^t \cdot \pi_s(C) = 0 \ \text{if} \begin{cases} r < 2 + s \ \text{and}\ \Gamma \in A^1(S \times C) \\ r = 2 + s \ \text{and}\ \Gamma \in A^1(S \times C)_{\mathrm{hom}} \end{cases}$$

(4) $\pi_r(C)\cdot\Gamma^t\cdot\pi_s(S)=0$ *if* $\begin{cases} r+2<s \text{ and } \Gamma\in A^2(C\times S)\\ r+2=s \text{ and } \Gamma\in A^2(C\times S)_{\mathrm{hom}}.\end{cases}$

Proof. By the results in [Mu2, Prop. 4.1], Conjectures 7.3.1, 7.3.2 and 7.3.3 hold for the product $Z=S\times C$. Therefore Proposition 7.3.6 applies to $S\times C$ (and $C\times S$). Then (1) follows from the equality:

$$\pi_j(C)\cdot\Gamma\cdot\pi_i(S)\in A_2(S\times C)=\mathcal{M}_{\mathrm{rat}}(h_i(S),h_j(C))$$

and similarly for (2), (3) and (4). □

7.3.9. **Corollary.** *Let S and C be as in Corollary 7.3.8. Then:*
a) $\mathcal{M}_{\mathrm{rat}}(h_1(C)(1),h_2(S))=0$;
b) $\mathcal{M}_{\mathrm{rat}}(h_2(S)),h_1(C))=0$.

Proof. The first assertion follows from the equality :

$$\mathcal{M}_{\mathrm{rat}}(h_1(C)(1),h_2(S))=\pi_2(S)\circ A^1(C\times S)\circ\pi_1(C)$$

by applying (3) of Cor. 7.3.8 to Γ^t for any $\Gamma\in A^1(S\times C)$.

b) follows from

$$\mathcal{M}_{\mathrm{rat}}(h_2(S),h_1(C))=\pi_1(C)\circ A^1(S\times C)\circ\pi_2(S)$$

and from (1). □

The next result shows that, in the case of two surfaces S and S', part of Proposition 7.3.6 holds without assuming any conjecture for $S\times S'$.[4]

7.3.10. **Theorem.** *Let S and S' be smooth projective surfaces over the field k. Then for any C-K decompositions as in Proposition 7.2.1*

$$h(S)=\bigoplus_{0\le i\le 4}h_i(S);\qquad h(S')=\bigoplus_{0\le j\le 4}h_j(S')$$

where $h_i(S)=(S,\pi_i(S),0)$ and $h_j(S')=(S',\pi'_j(S'),0)$, we have

(i) $\mathcal{M}_{\mathrm{rat}}(h_i(S),h_j(S'))=0$ *for all* $j<i$ *and* $0\le i\le 4$

(ii) $\mathcal{M}_{\mathrm{rat}}(h_i(S),h_i(S'))\simeq\mathcal{M}_{\mathrm{hom}}(h_i^{\mathrm{hom}}(S),h_i^{\mathrm{hom}}(S'))$ *for* $i\ne 2$.

Proof. Let $\pi_i=\pi_i(S)$ and $\pi'_j=\pi_j(S')$. Then $S\times S'$ has a C-K decomposition defined by the projectors $\sum_{r+s=m}\pi_r\times\pi'_s$. For any correspondence $Z\in A^2(S\times S')$ let us define

$$\alpha_{ji}(Z)=\pi'_j\circ Z\circ\pi_i \text{ for } 0\le i,j\le 4.$$

Then, in order to prove part (i) it is enough to show that $\alpha_{ji}(Z)=0$ for $j<i$.

[4]Kenichiro Kimura recently informed the second author that he had also found a proof, for the case of the product of two surfaces, of conjecture 7.3.2 and of part of conjecture 7.3.3.

We will show that $\alpha_{12}(Z) = \alpha_{23}(Z) = 0$: the other cases are easier and follow from the same type of arguments.

Let $\alpha_{12} = \pi'_1 \circ Z \circ \pi_2$: from the construction of the projectors $\{\pi_i\}$ and $\{\pi'_j\}$ in Proposition 1, it follows that $\pi'_1 = j_* \circ D$ where $j : C' \to S'$ is the closed embedding of the curve C' in S' and $D \in A^1(S' \times C')$. By possibly taking a desingularization for each irreducible component of C' we get a morphism $Y' \to S'$ where Y' is a smooth projective curve. Also, by arguing componentwise, we may as well assume that Y' is irreducible and we may replace C' by such a Y'. Then $\alpha_{12} = j_* \circ D \circ Z \circ \pi_2 = j_* \circ D_1 \circ \pi_2$, with $D_1 = D \circ Z \in A^1(S \times Y')$ and $D \circ Z \circ \pi_2 \in \mathrm{Hom}_{\mathcal{M}_{\mathrm{rat}}}(h_2(S), h(Y'))$.

Let us take a C-K decomposition $h(Y') = \sum_{0 \le j \le 2} h_j(Y')$, where $h_j(Y') = (Y', \pi_j(Y'), 0)$. By applying Corollary 7.3.8 to $S \times Y'$, we get $\pi_j(Y') \circ D_1 \circ \pi_2(S) = 0$ for $j = 0, 1$ so that $D \circ Z \circ \pi_2(S) = \pi_2(Y') \circ D \circ Z \circ \pi_2(S)$. If $\pi_2(Y') = [R' \times Y']$, with R' a chosen rational point on Y', then $\pi_2(Y') \circ D_1 = D_1^t(R') \times Y'$ and $D_1^t(R') = Z^t(D^t(R'))$. From the chosen normalization in the construction of the projectors $\{\pi_i(S)\}$ and $\{\pi_j(S')\}$ (see the proof of Proposition 7.2.1) it follows that $D^t(R') \in A^1(S')_{\mathrm{hom}}$ and $D_1^t(R') \in A^1(S)_{\mathrm{hom}}$. Therefore we get:

$$\begin{aligned} D \circ Z \circ \pi_2(S) &= \pi_2(Y') \circ D \circ Z \circ \pi_2(S) = \pi_2(Y') \circ D_1 \circ \pi_2(S) \\ &= (D_1^t(R') \times Y') \circ \pi_2(S) = \pi_2(S)(D_1^t(R')) \times Y' = 0 \end{aligned}$$

since $\pi_2(S)(D_1^t(R')) = 0$ because $\pi_2(S)(A^1(S)_{\mathrm{hom}}) = 0$. Therefore $\alpha_{12}(Z) = 0$.

To show that $\alpha_{23}(Z) = \pi_2(S') \circ Z \circ \pi_3(S) = 0$ it is enough to look at the transpose correspondence α_{23}^t. Then $\alpha_{23}^t(Z) = \pi_1(S) \circ Z^t \circ \pi_2(S')$. By applying the previous case to $S' \times S$ we get $\alpha_{23}^t(Z) = 0$, hence $\alpha_{23}(Z) = 0$.

We now prove part (ii): for Z homologically equivalent to 0, $\alpha_{11}(Z) = \alpha_{33}(Z) = 0$ follows from the definition of $\{\pi_i(S)\}$ and $\{\pi_j(S')\}$ in Proposition 7.2.1 and from the following result in [Sch, 4.5] (by interchanging π_1 and π_3 because of our covariant set-up):

$$\begin{aligned} \mathcal{M}_{\mathrm{rat}}(h_1(S), h_1(S')) &= \mathrm{Ab}(\mathrm{Alb}_S, \mathrm{Alb}_{S'}) \\ \mathcal{M}_{\mathrm{rat}}(h_3(S), h_3(S')) &= \mathrm{Ab}(\mathrm{Pic}^0_S, \mathrm{Pic}^0_{S'}). \end{aligned}$$

Both equalities hold also with $\mathcal{M}_{\mathrm{rat}}$ replaced by $\mathcal{M}_{\mathrm{hom}}$ and therefore we get (ii).

The equalities $\alpha_{00}(Z) = \alpha_{44}(Z) = 0$ are trivial because

$$\mathcal{M}_{\mathrm{rat}}(h_j(S), h_j(S')) = \mathcal{M}_{\mathrm{hom}}(h_j^{\mathrm{hom}}(S), h_j^{\mathrm{hom}}(S')) \simeq \mathbb{Q}$$

for $j = 0, 4$. □

Summarizing what we have done so far with Proposition 7.3.6 in mind, let us display our information on the Hom groups $\mathcal{M}_{\mathrm{rat}}(h_i(S), h_j(S'))$ in matrix form ($r = h$ means "rational equivalence = homological equivalence"):

$$\begin{pmatrix} r=h & 0 & 0 & 0 & 0 \\ * & r=h & 0 & 0 & 0 \\ * & * & ? & 0 & 0 \\ * & * & * & r=h & 0 \\ * & * & * & * & r=h \end{pmatrix}$$

In the next section we study the remaining group on the diagonal: the one marked with a '?'.

7.4. The group $\mathcal{M}_{\mathrm{rat}}(h_2(S), h_2(S'))$

Let S, S' be smooth projective surfaces over the field k: from Proposition 7.2.1 and from Theorem 7.3.10 it follows that for any C-K decompositions $h(S) = \bigoplus_{0\le i\le 4} h_i(S)$ and $h(S') = \bigoplus_{0\le i\le 4} h_i(S')$ as in Proposition 7.2.1 in $\mathcal{M}_{\mathrm{rat}}$, the $\mathbb{Q}$-vector spaces $\mathcal{M}_{\mathrm{rat}}(h_i(S), h_i(S'))$ are finite dimensional for $i \neq 2$. In fact we have

$$\mathcal{M}_{\mathrm{rat}}(h_0(S), h_0(S')) \simeq \mathcal{M}_{\mathrm{rat}}(h_4(S), h_4(S')) \simeq \mathbb{Q}$$

(if S and S' are geometrically connected), and

$$\mathcal{M}_{\mathrm{rat}}(h_1(S), h_1(S')) \simeq \mathrm{Ab}(\mathrm{Alb}_S, \mathrm{Alb}_{S'})$$
$$\mathcal{M}_{\mathrm{rat}}(h_3(S), h_3(S')) \simeq \mathrm{Ab}(\mathrm{Pic}_S, \mathrm{Pic}_{S'}).$$

Moreover, from Proposition 7.3.6 (ii) it follows that, if $S\times S'$ satisfies Conjecture 7.3.3, then $\mathcal{M}_{\mathrm{rat}}(h_2(S), h_2(S'))$ is also a finite dimensional $\mathbb{Q}$-vector space, at least in characteristic 0 and for a classical Weil cohomology.

In the case $k = \mathbb{C}$, if the surface S has geometric genus 0 then the isomorphism $\mathcal{M}_{\mathrm{rat}}(h_2(S), h_2(S)) \simeq \mathcal{M}_{\mathrm{hom}}(h_2^{\mathrm{hom}}(S), h_2^{\mathrm{hom}}(S))$ in Proposition 7.3.6 (ii) holds if and only if Bloch's conjecture holds for S i.e. if and only if the Albanese kernel $T(S)$ vanishes (see §7.6).

It is therefore natural to ask how the group $\mathcal{M}_{\mathrm{rat}}(h_2(S), h_2(S'))$ may be computed. We have

7.4.1. **Lemma.** *There is a canonical isomorphism*

$$\mathcal{M}_{\mathrm{rat}}(h_2(S), h_2(S')) \simeq \mathcal{M}_{\mathrm{rat}}(h_2^{\mathrm{alg}}(S), h_2^{\mathrm{alg}}(S')) \oplus \mathcal{M}_{\mathrm{rat}}(t_2(S), t_2(S'))$$

where $t_2(S)$ and $t_2(S')$ are defined in Proposition 7.2.3.

Proof. It suffices to see that

$$\mathcal{M}_{\rm rat}(t_2(S), h_2^{\rm alg}(S')) = \mathcal{M}_{\rm rat}(h_2^{\rm alg}(S), t_2(S')) = 0$$

which follows immediately from Proposition 7.2.3 (see its proof). □

Since $\mathcal{M}_{\rm rat}(h_2^{\rm alg}(S), h_2^{\rm alg}(S')) \simeq \mathbb{Q}^{\rho\rho'}$, *this lemma reduces the study of* $\mathcal{M}_{\rm rat}(h_2(S), h_2(S'))$ *to that of* $\mathcal{M}_{\rm rat}(t_2(S), t_2(S'))$. In this section we give *two* descriptions of this group: one as a quotient of $A_2(S \times S')$ (Theorem 7.4.3) and the other in terms of Albanese kernels (theorem 7.4.8).

Then, in §7.5, we will relate these results with the *birational motives* of S and S' i.e. with the images of $h(S)$ and $h_2(S')$ in the category $\mathcal{M}^{\rm o}_{\rm rat}(k)$ of birational motives of [K-S].

7.4.1. **First description of $\mathcal{M}_{\rm rat}(t_2(S), t_2(S'))$.**

We start with

7.4.2. **Definition.** Let $X = X_d$ and $Y = Y_e$ be smooth projective varieties over k: we denote by $\mathcal{J}(X, Y)$ the subgroup of $A_d(X \times Y)$ generated by the classes supported on subvarieties of the form $X \times N$ or $M \times Y$, with M a closed subvariety of X of dimension $< d$ and N a closed subvariety of Y of dimension $< e$.

In other words: $\mathcal{J}(X, Y)$ is generated by the classes of correspondences which are not dominant over X and Y by either the first or the second projection.

Note that $\mathcal{J}(X, Y) = A_d(X \times Y)$ if $d < e$ (project to Y). In the case $X = Y$ $\mathcal{J}(X, X)$ is a two-sided ideal in the ring of correspondences $A_d(X \times X)$ (see [Fu, p. 309]).

Now let S and S' be smooth projective surfaces over k and let $\{\pi_i = \pi_i(S)\}$ and $\{\pi'_i = \pi_i(S')\}$, for $0 \le i \le 4$, be projectors giving C-K decompositions respectively for S and for S' as in Proposition 7.2.1. Then, as in Proposition 7.2.3, $\pi_2(S) = \pi_2^{\rm alg}(S) + \pi_2^{\rm tr}(S)$, $h_2(S) \simeq \rho\mathbb{L} \oplus t_2(S)$, where $t_2(S) = (S, \pi_2^{\rm tr}(S), 0)$ and ρ is the Picard number of S. Similarly $\pi_2(S') = \pi_2^{\rm alg}(S') + \pi_2^{\rm tr}(S')$, $h_2(S') \simeq \rho'\mathbb{L} \oplus t_2(S')$ where ρ' is the Picard number of S'. Let us define a homomorphism

$$\Phi : A_2(S \times S') \to \mathcal{M}_{\rm rat}(t_2(S), t_2(S'))$$

as follows: $\Phi(Z) = \pi_2^{\rm tr}(S') \circ Z \circ \pi_2^{\rm tr}(S)$. Then we have the following result.

7.4.3. **Theorem.** *The map* Φ *induces an isomorphism*

$$\bar{\Phi} : \frac{A_2(S \times S')}{\mathcal{J}(S, S')} \simeq \mathcal{M}_{\rm rat}(t_2(S), t_2(S')).$$

Proof. From the definition of the motives $t_2(S)$ and $t_2(S')$ it follows that

$$\mathcal{M}_{\mathrm{rat}}(t_2(S), t_2(S')) = \{\pi_2^{\mathrm{tr}}(S') \circ Z \circ \pi_2^{\mathrm{tr}}(S) \mid Z \in A_2(S \times S')\}.$$

We first show that $\mathcal{J}(S, S') \subset \operatorname{Ker}\Phi$. Let $Z \in \mathcal{J}(S, S')$: we may assume that Z is irreducible and supported either on $S \times Y'$ with $\dim Y' \leq 1$ or on $Y \times S'$ with $\dim Y \leq 1$.

Suppose Z is supported on $S \times Y'$. The case $\dim Y' = 0$ being easy, let us assume that Y' is a curve which, by possibly taking a desingularization (compare proof of Proposition 7.2.1), we may take to be smooth and irreducible. Let $j : Y' \to S'$; then $Z = j_* \circ D$, where j_* is the graph Γ_j and $D \in A^1(S \times Y')$. Using the identity $\pi_2(S) \circ \pi_2^{\mathrm{tr}}(S) = \pi_2^{\mathrm{tr}}(S) \circ \pi_2^{\mathrm{tr}}(S) = \pi_2^{\mathrm{tr}}(S)$ we get

$$\pi_2^{\mathrm{tr}}(S') \circ Z \circ \pi_2^{\mathrm{tr}}(S) = \pi_2^{\mathrm{tr}}(S') \circ j_* \circ D \circ \pi_2(S) \circ \pi_2^{\mathrm{tr}}(S).$$

Let $\Delta_{Y'} = \pi_0(Y') + \pi_1(Y') + \pi_2(Y')$ be a C-K decomposition. By Corollary 7.3.8 (a)

$$\pi_1(Y') \circ D \circ \pi_2(S) = \pi_0(Y') \circ D \circ \pi_2(S) = 0$$

hence $D \circ \pi_2(S) = \pi_2(Y') \circ D \circ \pi_2(S)$. Let R' be a rational point on Y' such that $\pi_2(Y') = [R' \times Y']$; then $\pi_2(Y') \circ D = [D(R')^t \times Y']$ and $D \circ \pi_2^{\mathrm{tr}}(S) = D \circ \pi_2(S) \circ \pi_2^{\mathrm{tr}}(S) = [D(R')^t \times Y'] \circ \pi_2^{\mathrm{tr}}(S) = [\pi_2^{\mathrm{tr}}(S)(D(R')^t) \times Y']$. From $A^2(t_2(S)) = T(S)$ it follows that $\pi_2^{\mathrm{tr}}(S)$ acts as 0 on divisors, hence $[\pi_2^{\mathrm{tr}}(S)(D(R')^t) \times Y'] = 0$ and

$$\pi_2^{\mathrm{tr}}(S') \circ Z \circ \pi_2^{\mathrm{tr}}(S) = 0.$$

This completes the proof in the case Z has support on $S \times Y'$.

Let us now consider the case when Z is supported on $Y \times S'$, Y a curve on S. In order to show that $\pi_2^{\mathrm{tr}}(S') \circ Z \circ \pi_2^{\mathrm{tr}}(S) = 0$ we can just take the transpose. Then we get $\pi_2^{\mathrm{tr}}(S) \circ Z^t \circ \pi_2^{\mathrm{tr}}(S')$ and this brings us back to the previous case.

Therefore Φ induces a map

$$\bar{\Phi} : A_2(S \times S')/\mathcal{J}(S, S') \to \mathcal{M}_{\mathrm{rat}}(t_2(S), t_2(S'))$$

which is clearly surjective, and we are left to show that $\bar{\Phi}$ is injective.

Let $Z \in A_2(S \times S')$ be such that $\pi_2^{\mathrm{tr}}(S') \circ Z \circ \pi_2^{\mathrm{tr}}(S) = 0$: we claim that $Z \in \mathcal{J}(S, S')$.

Let ξ be the generic point of S. To prove our claim we are going to evaluate

$$(\pi_2^{\mathrm{tr}}(S') \circ Z \circ \pi_2^{\mathrm{tr}}(S))(\xi)$$

over $k(\xi)$. By using Chow's moving lemma on $S \times S'$ we may choose a cycle in the class of Z in $A_2(S \times S')$ (which we will still denote by Z) such that $\pi_2^{\mathrm{tr}}(S') \circ Z \circ \pi_2^{\mathrm{tr}}(S)$ is defined as a cycle and $\pi_2^{\mathrm{tr}}(S') \circ Z \circ \pi_2^{\mathrm{tr}}(S)(\xi)$

can be evaluated using the formula $(\alpha \circ \beta)(\xi) = \alpha(\beta(\xi))$, for $\alpha, \beta \in A_2(S \times S')$. From the definition of the projector $\pi_2^{\rm tr}(S)$ in Proposition 7.2.3, we have

$$\pi_2^{\rm tr}(S) = \Delta_S - \pi_0(S) - \pi_1(S) - \pi_2^{\rm alg}(S) - \pi_3(S) - \pi_4(S)$$

where $\pi_2^{\rm alg}(S), \pi_3(S)$ and $\pi_4(S)$ act as 0 on 0-cycles, while $\pi_0(S)(\xi) = P$ if $\pi_0(S) = [S \times P]$ and $\pi_1(S)(\xi) = D_\xi$, where D_ξ is a divisor (defined over $k(\xi)$) on the curve $C = C(S)$ used to construct $\pi_1(S)$. Therefore

$$\pi_2^{\rm tr}(S))(\xi) = \xi - P - D_\xi$$

and

$$(\pi_2^{\rm tr}(S') \circ Z \circ \pi_2^{\rm tr}(S))(\xi) = \pi_2^{\rm tr}(S')(Z(\xi) - Z(P) - Z(D_\xi)) = 0$$

By the same argument as before, applied to the projectors $\{\pi_i(S')\}$, we get

$$\begin{aligned} 0 &= (\pi_2^{\rm tr}(S')(Z(\xi) - Z(P) - Z(D_\xi)) \\ &= (Z(\xi) - Z(P) - Z(D_\xi)) - mP' - \pi_1(S')(Z(\xi) - Z(P) - Z(D_\xi)) \end{aligned}$$

where P' is a rational point defining $\pi_0(S')$ and m is the degree of the 0-cycle $Z(\xi) - Z(P) - Z(D_\xi)$. The cycle $\pi_1(S')(Z(P) + Z(D_\xi)) = D'_\xi$ is a divisor (defined over $k(\xi)$) on the curve $C' = C(S')$ appearing in the construction of $\pi_1(S')$. Therefore we get from $\pi_2^{\rm tr}(S') \circ Z \circ \pi_2^{\rm tr}(S) = 0$:

$$Z(\xi) = Z(P) + Z(D_\xi) + mP' + \pi_1(S')(Z(\xi)) - D'_\xi.$$

The cycle on the right hand side is supported on a curve $Y' \subset S'$, with Y' the union of $Z(C)$ and C'. Therefore, by taking the Zariski closure in $S \times S'$ of both sides of the above formula we get :

$$Z = Z_1 + Z_2$$

where $Z_1, Z_2 \in A_2(S \times S')$, Z_1 is supported on $S \times Y'$ with $\dim Y' \le 1$ and Z_2 is a cycle supported on $Y \times S$ with $Y \subset S$, $\dim Y \le 1$. Therefore $Z \in \mathcal{J}(S, S')$. □

7.4.4. *Remark.* Theorem 7.4.3 is an analogue in the case of surfaces of a well-known result for curves, namely the isomorphism:

$$\frac{A_1(C \times C')}{\mathcal{J}(C, C')} \simeq \mathcal{M}_{\rm rat}(h_1(C), h_1(C'))$$

which immediately follows from the definitions of $\mathcal{J}(C, C')$ and of the motives $h_1(C)$ and $h_1(C')$. In this case $\mathcal{J}(C, C')$ is the subgroup generated by the classes which are represented by "horizontal" and "vertical" divisors on $C \times C'$. The equivalence relation defined by $\mathcal{J}(C, C')$ is denoted in [Weil1, Chap. 6] as "three line equivalence". In the case of curves, since $h_1(C)$ and $h_1(C')$ have a "realization" as the Jacobians $J(C)$ and $J(C')$, the following result in [Weil1, Ch. 6, Thm 22] holds:

$$\frac{A_1(C \times C')}{\mathcal{J}(C, C')} \simeq \mathrm{Ab}(J(C), J(C'))$$

where $J(C)$, $J(C')$ are the Jacobians.

7.4.5. **Corollary.** *Keep the same notation and let $\Pi_r = \sum_{i+j=r} \pi_i(S) \times \pi_j(S')$ be the Chow-Künneth projectors on $S \times S'$ deduced from those of S and S'. Let $F^\bullet$ be the filtration on $A_j(S \times S')$ defined by the Π_r. Then*

$$\mathcal{J}(S, S') \cap A_2(S \times S')_{\mathrm{hom}} \simeq F^1 A_2(S \times S') = \operatorname{Ker} \Pi_4.$$

Therefore $S \times S'$ satisfies Conjecture 7.3.3 if and only if $A_2(S \times S')_{\mathrm{hom}} \subset \mathcal{J}(S, S')$.

Proof. For simplicity let us drop (S) and (S') from the notation for projectors and use π_i etc. for those of S and π'_i etc. for those of S'. Let $\Gamma \in A_2(S \times S')_{\mathrm{hom}}$: from Theorem 7.3.10 $\pi'_j \circ \Gamma \circ \pi_j = 0$ for $j \neq 2$. Therefore $\Gamma \in \operatorname{Ker} \Pi_4$ if and only if $\pi'_2 \circ \Gamma \circ \pi_2 = 0$. By Lemma 7.4.1, it suffices to consider separately the algebraic and transcendental parts.

Let $\Gamma \in \mathcal{J}(S, S') \cap A_2(S \times S')_{\mathrm{hom}}$: then $(\pi_2^{\mathrm{tr}})' \circ \Gamma \circ \pi_2^{\mathrm{tr}} = 0$, because $\Gamma \in \mathcal{J}(S, S')$. Since $\Gamma \in A_2(S \times S')_{\mathrm{hom}}$ we also have: $(\pi_2^{\mathrm{alg}})' \circ \Gamma \circ \pi_2^{\mathrm{alg}} = 0$. This follows from the isomorphism $h_2(S) = \rho\mathbb{L} \oplus t_2(S)$ where $\rho\mathbb{L} \simeq (S, \pi_2^{\mathrm{alg}}, 0)$, and the same for S'. In fact we have $\mathcal{M}_{\mathrm{rat}}(\rho\mathbb{L}, \rho'\mathbb{L}) \simeq \mathcal{M}_{\mathrm{hom}}(\rho\mathbb{L}, \rho'\mathbb{L})$, so that, if $\Gamma \in A_2(S \times S')_{\mathrm{hom}}$, then $(\pi_2^{\mathrm{alg}})' \circ \Gamma \circ \pi_2^{\mathrm{alg}}$ yields the 0 map in $\mathcal{M}_{\mathrm{hom}}(\rho\mathbb{L}, \rho'\mathbb{L})$, hence it is 0. Therefore $\pi'_2 \circ \Gamma \circ \pi_2 = 0$ which proves that $\Gamma \in \operatorname{Ker} \Pi_4$.

Conversely let $\Gamma \in F^1 A_2(S \times S') = \operatorname{Ker} \Pi_4$: then $\Gamma \in A_2(S \times S')_{\mathrm{hom}}$ by [Mu2, 1.4.4]. By Theorem 7.4.3, we also have $\Gamma \in \mathcal{J}(S, S')$ because

$$\pi'_2 \circ \Gamma \circ \pi_2 = (\pi_2^{\mathrm{alg}})' \circ \Gamma \circ \pi_2^{\mathrm{alg}} + (\pi_2^{\mathrm{tr}})' \circ \Gamma \circ \pi_2^{\mathrm{tr}} = 0$$

and $(\pi_2^{\mathrm{alg}})' \circ \Gamma \circ \pi_2^{\mathrm{alg}} = 0$, since $\Gamma \in A_2(S \times S')_{\mathrm{hom}}$. □

7.4.2. **Second description of $\mathcal{M}_{\mathrm{rat}}(t_2(S), t_2(S'))$.** Let us still keep the same notation.

7.4.6. **Definition.** We denote by $H_{\leq 1}$ be the subgroup of $A^2(S'_{k(S)})$ generated by the subgroups $A^2(S'_L)$, when L runs through all the subfields of $k(S)$ containing k and which are of transcendence degree ≤ 1 over k.

Theorem 7.4.8 below will give a description of $\mathcal{M}_{\mathrm{rat}}(t_2(S), t_2(S'))$ in terms of $T(S'_{k(S)})$ and $H_{\leq 1}$. We need a preparatory lemma:

7.4.7. **Lemma.** *Let S, S' and $H_{\leq 1}$ be as above. Let ξ be the generic point of S and let $Z \in A_2(S \times S')$. Then $Z \in \mathcal{J}(S, S')$ if and only if $Z(\xi) \in H_{\leq 1}$.*

Proof. Let us denote – by abuse – with the same letter Z both a cycle class and a suitable cycle in this class. Let $Z \in \mathcal{J}(S, S') \subset A_2(S \times S') = A^2(S \times S')$. If Z has support on $Y \times S$ with Y closed in S and of dimension ≤ 1, then $Z(\xi) = 0$. Therefore we may assume that Z has support on $S \times Y'$ and by linearity we may take Z to be represented by a k-irreducible subvariety of $S \times Y'$. Furthermore, by taking its desingularization if necessary, we may also assume that Y' is a smooth curve. Let $j : Y' \to S'$ be the corresponding morphism and $j_* : A^1(Y'_{k(\xi)}) \to A^2(S'_{k(\xi)})$ the induced homomorphim on Chow groups. Then $Z(\xi) = j_* D(\xi)$ where D is a k-irreducible divisor on $S \times Y'$. Then $D(\xi)$ has a smallest field of rationality L, in the sense of [Weil2, Cor 4 p. 269] with $k \subset L \subset k(\xi)$. $D(\xi)$ consists of a finite number of points $P_1, \dots, P_m$ on Y' each one conjugate to the others over L, and with the same multiplicity. Moreover L is contained in the algebraic closure of $k(P_1)$, where $P_1 \in Y'$. Therefore $tr \deg_k L \leq 1$. We have $D(\xi) \in A^1(Y'_L)$ and $Z(\xi) = \tilde{j}_* D(\xi)$ where $\tilde{j}_* : A^1(Y'_L) \to A^2(S'_L)$. Therefore $Z(\xi) \in H_{\leq 1}$.

Conversely suppose that $Z(\xi) \in H_{\leq 1}$; because of the definition of $H_{\leq 1}$ we may assume that $Z(\xi)$ is a cycle defined over a field L with $k \subset L \subset k(\xi)$ and $t = tr \deg_k L \leq 1$. If $t = 0$ then $Z(\xi)$ is defined over an algebraic extension extension of k, hence $Z \in \mathcal{J}(S, S')$. Assume $t = 1$ and let C be a smooth projective curve with function field L. Since $L \subset k(\xi)$ there is a dominant rational map f from S to C. Let $U \subset S$ be an open subset such that f is a morphism on U. Then $\eta = f(\xi)$ is the generic point of C. Moreover, $Z(\xi) \in A^2(S'_{k(\eta)}) \subset A^2(S'_{k(\xi)})$. Let Z' be the closure of $Z(\xi)$ in $C \times S'$ so that $Z'(\eta) = Z(\xi)$. Let $Y' \subset S'$ be the projection of Z': then $\dim Y' \leq 1$. Consider the morphism

$$(f_{|U} \times id_{S'})^* : A^2(C \times S') \to A^2(U \times S')$$

and let Z_1 be the cycle in $A^2(S \times S')$ obtained by taking the Zariski closure of $(f_{|U} \times id_{S'})^*(Z')$. Then Z_1 has support on $S \times Y'$ and $Z(\xi) = Z'(\eta) = Z_1(\xi)$. Therefore $Z = Z_1 + Z_2$ where Z_2 has support on $Y \times S'$, Y a curve on S, hence $Z \in \mathcal{J}(S \times S')$. □

7.4.8. **Theorem.** *Let S and S' be smooth projective surfaces over k and let $T(S'_{k(S)})$ and $H_{\leq 1}$ be as above. Let us define $H = T(S'_{k(S)}) \cap H_{\leq 1}$. Then there is an isomorphism*

$$\mathcal{M}_{\mathrm{rat}}(t_2(S), t_2(S')) \simeq \frac{T(S'_{k(S)})}{H}.$$

Proof. Let us define a homomorphism

$$A_2(S\times S')\xrightarrow{\beta} T(S'_{k(S)})$$

by $\beta(Z)=((\pi_2^{\mathrm{tr}})'\circ Z\circ\pi_2^{\mathrm{tr}})(\xi)$, with ξ the generic point of S. By Lemma 7.4.7, β induces a map

$$\bar\beta:\frac{A_2(S\times S')}{\mathcal{J}(S,S')}\to\frac{T(S'_{k(S)})}{H}$$

and, by Theorem 7.4.3

$$\frac{A_2(S\times S')}{\mathcal{J}(S,S')}\simeq\mathcal{M}_{\mathrm{rat}}(t_2(S),t_2(S')).$$

Therefore we are left to show that $\bar\beta$ is an isomorphism.

Let $[\sigma]\in\frac{T(S'_{k(S)})}{H}$, σ a representative in $T(S'_{k(S)})$ and Z the Zariski closure of σ in $S\times S'$: then $Z(\xi)=\sigma$. Let $Z_1=(\pi_2^{\mathrm{tr}})'\circ Z\circ\pi_2^{\mathrm{tr}}$ and $Z_2=Z-Z_1$: then $(\pi_2^{\mathrm{tr}})'\circ Z_2\circ\pi_2^{\mathrm{tr}}=0$. From Theorem 7.4.3 and Lemma 7.4.7 we get $Z_2\in\mathcal{J}(S,S')$ and $Z_2(\xi)\in H_{\le 1}$. On the other hand, both $Z(\xi)=\sigma\in T(S'_{k(S)})$ and $Z_1(\xi)\in T(S'_{k(S)})$, hence $Z_2(\xi)\in H=T(S'_{k(S)})\cap H_{\le 1}$. Therefore we get $\bar\beta(Z)=[Z_1(\xi)]=[Z(\xi)-Z_2(\xi)]=[\sigma-Z_2(\xi)]=[\sigma]$ and this shows that $\bar\beta$ is surjective.

Let $Z\in A_2(S\times S')$ be such that $\beta(Z)\in H$. Let $Z_1=(\pi_2^{\mathrm{tr}})'\circ Z\circ\pi_2^{\mathrm{tr}}$: then $Z_1\in H_{\le 1}$, and by Lemma 7.4.7 $Z_1\in\mathcal{J}(S,S')$. By taking $Z_2=Z-Z_1$ as before we have $Z_2\in\mathcal{J}(S,S')$, hence $Z=Z_1+Z_2\in\mathcal{J}(S,S')$. Therefore $\bar\beta$ is injective. □

7.4.9. **Corollary.** *a)* $t_2(S)=0\Leftrightarrow T(S_{k(S)})\subset H_{\le 1}\Leftrightarrow T(S_{k(S)})=0$.
b) Suppose that k is algebraically closed and has infinite transcendance degree over its prime subfield. Then $t_2(S)=0\Leftrightarrow T(S)=0$.
c) With the same assumption as in b), $T(S)=0$ *implies* $H^2_{\mathrm{tr}}(S)=0$, *and* $p_g=0$ *if* $\operatorname{char}k=0$.

Proof. From Proposition 7.2.3 we get the second implication in the following:

$$t_2(S)=0\Rightarrow t_2(S_{k(S)})=0\Rightarrow T(S_{k(S)})=0\Rightarrow T(S_{k(S)})\subset H_{\le 1}$$

the other two being obvious. The implication $T(S_{k(S)})\subset H_{\le 1}\Rightarrow t_2(S)=0$ follows from Theorem 7.4.8, hence a). To see b), in view of Proposition 7.2.3 we need only show that $T(S)=0\Rightarrow t_2(S)=0$. Note that there exists a finitely generated subfield $k_0\subset k$ and a smooth projective k_0-surface S_0 such that $S\simeq S_0\times_{k_0}k$. The assumption on k implies that the inclusion $k_0\subset k$ extends to an inclusion $k_0(S_0)\subset k$.

A standard transfer argument shows that $T((S_0)_{k_0(S_0)}) \to T(S)$ is injective. So $t_2(S_0) = 0$ by a) and therefore $t_2(S) = 0$. Finally, c) follows from b) and Corollary 7.2.4. □

7.5. The birational motive of a surface

In this section we first recall some definitions and results from [K-S] on the category of birational Chow motives (with rational coefficients) over k: this category is denoted there (see 6.1) by $\mathrm{Chow}^{\mathrm{o}}(k, \mathbb{Q})$ or $\mathrm{Mot}^{\mathrm{o}}_{\mathrm{rat}}(k, \mathbb{Q})$ while we shall denote it here by $\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}(k)$ or even $\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}$. Then we compute the group $\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}(\bar{h}(S), \bar{h}(S))$ of a surface S, where $\bar{h}(S)$ is the image of $h(S)$ in $\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}$.

7.5.1. **Lemma.** *For every smooth and projective varieties X and Y, with $\dim X = d$, let $\mathcal{I}$ be the subgroup of $\mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}}(h(X), h(Y)) = A_d(X \times Y)$ defined as follows:*

$$\mathcal{I}(X, Y) = \{f \in A_d(X \times Y) \mid f \text{ vanishes on } U \times Y, U \text{ open in } X\}.$$

Then $\mathcal{I}$ is a two-sided tensor ideal in $\mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}}$. In particular for any smooth projective variety X there is an exact sequence of rings:

$$0 \to \mathcal{I}(X, X) \to A^d(X \times X) \xrightarrow{\phi} A_0(X_{k(X)}) \to 0 \tag{7.3}$$

where $d = \dim X$ and $k(X)$ is the function field of X. If we denote by $\bullet$ the multiplication in $A_0(X_{k(X)})$ defined via (7.3) then, if P and Q are two rational points of X, we have:

$$[P] \bullet [Q] = [P]$$

in $A_0(X_{k(X)})$.

Proof. The fact that $\mathcal{I}$ is a tensor ideal in $\mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}}$ is proven in [K-S, 5.3]. We review the proof:

If X, Y, Z are smooth projective varieties and $U \subset X$ is open, then the usual formula defines a composition of correspondences:

$$A_{\dim X}(U \times Y) \times A_{\dim Y}(Y \times Z) \to A_{\dim X}(U \times Z)$$

and this composition is compatible with the restriction to any open subset $V \subset U$. Passing to the limit, since:

$$A_0(Y_{k(X)}) = A^{\dim Y}(Y_{k(X)}) = \lim_{U \subset X} A^{\dim Y}(U \times Y) = \lim_{U \subset X} A_{\dim X}(U \times Y)$$

we get a composition

$$A_0(Y_{k(X)}) \times A_{\dim Y}(Y \times Z) \to A_0(Z_{k(X)}).$$

If $\alpha \in A_0(Y_{k(X)})$ and $\beta \in \mathcal{I}(Y,Z)$, i.e. if β has support on a closed subset $M \times Z$ of $Y \times Z$ then $\beta \circ \alpha = 0 \in A_0(Z_{k(X)})$ as one sees by moving α away from M. Therefore we get a a pairing

$$A_0(Y_{k(X)}) \times A_0(Z_{k(Y)}) \to A_0(Z_{k(X)}) \tag{7.4}$$

which, in the case $X = Y = Z$ yields a multiplication $\bullet$ in $A_0(X_{k(X)})$ defined by

$$\bar{\beta} \bullet \bar{\alpha} = \overline{\beta \circ \alpha}$$

where for a correspondence Γ in $A^d(X \times X)$, $\bar{\Gamma}$ denotes its class in $A_0(X_{k(X)})$. Let η be the class of the generic point of X in $A_0(X_{k(X)})$, which is the image of the cycle $[\Delta_X]$ of $A^d(X \times X)$ under the map ϕ in (7.3): then η is the identity for $\bullet$. Let P and Q be closed points in X, and let $[P]$ and $[Q]$ be the corresponding elements in $A_0(X_{k(X)})$. By choosing representatives $[X \times P]$ and $[X \times Q]$ in $A^d(X \times X)$ we get $[X \times P] \circ [X \times Q] = [X \times P]$ in $A^d(X \times X)$. This shows that

$$[P] \bullet [Q] = [P]$$

in $A_0(X_{k(X)})$. □

7.5.2. **Definition.** We denote by $\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}$ the category of *birational Chow motives*, i.e the pseudo-abelian envelope of the factor category $\mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}}/\mathcal{I}$ and, if $M \in \mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}}$, by $\bar{M}$ its image in $\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}$. We also denote by $\bar{h}$ the (covariant) composite functor $\mathcal{V} \xrightarrow{h} \mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}} \to \mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}$.

Note that under the functor $\mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}} \to \mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}$ the Lefschetz motive $\mathbb{L}$ goes to 0. By Lemma 7.5.1, one has the following isomorphism in $\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}$:

$$\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}(\bar{h}(X), \bar{h}(Y)) \simeq A_0(Y_{k(X)}) \tag{7.5}$$

for $X, Y \in \mathcal{V}$. We also have:

7.5.3. **Proposition** ([K-S, 5.3 and 5.4])**.** *A morphism f in $\mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}}$ belongs to the ideal $\mathcal{I}$ if and only if it factors through an object of the form $M(1)$.*

7.5.4. *Remark.* The proof in [K-S, 5.4] is not correct because Chow's moving lemma is applied on a singular variety. However, N. Fakhruddin pointed out that it is sufficient to take the subvariety Z appearing in this proof minimal to repair it, and moreover Chow's moving lemma is then avoided. This correction will appear in the final version.

7.5.5. **Definition.** For all $n \geq 0$, we let

(i) $d_{\leq n}\mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}}$ denote the thick subcategory of $\mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}}$ generated by motives of varieties of dimension $\leq n$ (thick means full and stable under direct summands).

(ii) $d_{\leq n}\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}$ denote the thick image of $d_{\leq n}\mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}}$ in $\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}$.

(iii) $\mathcal{K}_{\leq n}$ denote the ideal of $\mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}}$ consisting of those morphisms that factor through an object of $d_{\leq n}\mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}}$.

(iv) $\mathcal{K}^{\mathrm{o}}_{\leq n}$ denote the thick image of $\mathcal{K}_{\leq n}$ in $\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}$.

For simplicity, we write $\mathcal{K}_{\leq n}(X,Y)$ and $\mathcal{K}^{\mathrm{o}}_{\leq n}(X,Y)$ for two varieties X,Y instead of $\mathcal{K}_{\leq n}(h(X),h(Y))$ and $\mathcal{K}^{\mathrm{o}}_{\leq n}(h(X),h(Y))$.

7.5.6. **Lemma.** *a) The functor*

$$D^{(n)}:\mathcal{M}_{\mathrm{rat}}\to\mathcal{M}_{\mathrm{rat}}$$
$$M\mapsto\underline{\mathrm{Hom}}(M,\mathbb{L}^n),$$

where $\underline{\mathrm{Hom}}(M,\mathbb{L}^n)=M^\vee\otimes\mathbb{L}^n$ *is the internal Hom in* $\mathcal{M}_{\mathrm{rat}}$*, sends* $d_{\leq n}\mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}}$ *to itself and defines a self-duality of this category such that* $D^{(n)}(h(X))=h(X)$ *for any* n*-dimensional* X*. Moreover, for* X,Y *purely of dimension* n*,*
b) The map $D^{(n)}:A_n(X\times Y)\to A_n(Y\times X)$ *is the transposition of cycles and in particular*

$$D^{(n)}(\mathcal{J}(X,Y))=\mathcal{J}(Y,X)$$

where $\mathcal{J}(X,Y)$ *is the subgroup of Definition 7.4.2.*
c) $D^{(n)}(\mathcal{I}(X,Y))=\mathcal{K}_{\leq n-1}(Y,X)$ *and* $D^{(n)}(\mathcal{K}_{\leq n-1}(X,Y))=\mathcal{I}(Y,X)$*, where* $\mathcal{I}$ *is as in Lemma 7.5.1.*
d) For X,Y *purely of dimension* n *we have*

$$\mathcal{J}(X,Y)=\mathcal{I}(X,Y)+\mathcal{K}_{\leq n-1}(X,Y). \tag{7.6}$$

Proof. a) and b) are obvious. For c), the argument in [K-S, proof of 5.4] (see Remark 7.5.4) implies that $\mathcal{I}(X,Y)$ consists of those morphisms that factor through some $h(Z)(1)$, where $\dim Z=n-1$. A similar argument shows that $\mathcal{K}_{\leq n-1}(X,Y)$ consists of those morphisms that factor through some $h(Z)$ with $\dim Z=n-1$. The claim is now obvious. Finally, d) follows immediately from c) and the definition of $\mathcal{J}$. □

7.5.7. **Lemma.** *Let* S*,* S' *be smooth projective surfaces over a field* k*. For any C-K decompositions as in Proposition 7.2.1*

$$h(S)=\bigoplus_{0\leq i\leq 4}h_i(S),\qquad h(S')=\bigoplus_{0\leq i\leq 4}h_i(S'),$$

we have

$$\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}(\bar h_1(S),\bar h_2(S'))\oplus\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}(\bar h_2(S),\bar h_2(S'))\simeq T(S'_{k(S)})/T(S')$$

and

$$\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}(\bar h_2(S),\bar h_2(S'))=\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}(\overline{t_2(S)},\overline{t_2(S')}).$$

Proof. Let $\bar\pi_i = \bar\pi_i(S)$, $\bar\pi'_j = \bar\pi_j(S')$, $\bar h_i(S)$, $\bar h_j(S')$, be the images in $\mathcal{M}^o_{\mathrm{rat}}$ of the projectors π_i, π'_j and of the corresponding motives $h_i(S)$, $h_j(S')$) for the surfaces S and S' (as defined in Proposition 7.2.1).

It follows from Proposition 7.5.3 and Proposition 7.2.1 (iii) that $\bar\pi_3 = \bar\pi'_3 = \bar\pi_4 = \bar\pi'_4 = 0$.

From Proposition 7.2.3 we get isomorphisms: $h_2(S) \simeq \rho\mathbb{L} \oplus t_2(S)$ and $h_2(S') \simeq \rho'\mathbb{L} \oplus t_2(S')$. It follows that $\bar h_2(S) \simeq \overline{t_2(S)}$, in $\mathcal{M}^o_{\mathrm{rat}}$ and similarly for S': $\bar h_2(S') \simeq \overline{t_2(S')}$.

Therefore in $\mathcal{M}^o_{\mathrm{rat}}$ we have

$$\bar h(S) = 1 \oplus \bar h_1(S) \oplus \bar h_2(S) = 1 \oplus \bar h_1(S) \oplus \overline{t_2(S)}$$

and

$$\bar h(S') = 1 \oplus \bar h_1(S') \oplus \bar h_2(S') = 1 \oplus \bar h_1(S') \oplus \overline{t_2(S')}.$$

According to (7.5) we have

$$\mathcal{M}^o_{\mathrm{rat}}(\bar h(S), \bar h(S')) = A_0(S'_{k(S)})$$

and

$$\mathcal{M}^o_{\mathrm{rat}}(1, \bar h(S')) = \mathcal{M}^o_{\mathrm{rat}}(\bar h(\operatorname{Spec} k), \bar h(S')) \simeq A_0(S').$$

From Proposition 7.2.1 it follows:

$$\mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}}(h_1(S), 1) = A^0(S)\pi_1 = 0; \mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}}(h_2(S), 1) = A^0(S)\pi_2 = 0.$$

Therefore we get:

$$A_0(S'_{k(S)})/A_0(S') \simeq \mathcal{M}^o_{\mathrm{rat}}(\bar h_1(S) \oplus \bar h_2(S), \bar h_1(S') \oplus h_2(S')).$$

Theorem 7.3.10 (i) yields : $\mathcal{M}_{\mathrm{rat}}(h_2(S), h_1(S')) = 0$ while from [Sch, prop.4.5] it follows:

$$\mathcal{M}_{\mathrm{rat}}(h_1(S), h_1(S')) \simeq \mathrm{Ab}(\mathrm{Alb}_S, \mathrm{Alb}_{S'}).$$

Therefore we have:

$$A_0(S'_{k(S)})/A_0(S') \simeq \mathrm{Ab}(\mathrm{Alb}_S, \mathrm{Alb}_{S'}) \oplus \mathcal{M}^o_{\mathrm{rat}}(\bar h_1(S), \bar h_2(S')) \oplus \mathcal{M}^o_{\mathrm{rat}}(\bar h_2(S), \bar h_2(S')). \tag{7.7}$$

There is a canonical map $\alpha : A_0(S'_{k(S)}) \to \mathrm{Ab}(\mathrm{Alb}_S, \mathrm{Alb}_{S'})$ which is 0 on $A_0(S')$ (see [K-S, (9.5))]) as well as an isomorphism:

$$\mathrm{Ab}(\mathrm{Alb}_S, \mathrm{Alb}_{S'}) \simeq \frac{\mathrm{Alb}_{S'}(k(S))_{\mathbb{Q}}}{\mathrm{Alb}_{S'}(k)_{\mathbb{Q}}}.$$

Therefore we get the following exact sequence:

$$0 \to T(S'_{k(S)})/T(S') \to A_0(S'_{k(S)})/A_0(S') \to \frac{\mathrm{Alb}_{S'}(k(S))_{\mathbb{Q}}}{\mathrm{Alb}_{S'}(k)_{\mathbb{Q}}} \to 0.$$

Hence:

$$A_0(S'_{k(S)})/A_0(S') \simeq (\mathrm{Alb}_{S'}(k(S))/\mathrm{Alb}_{S'}(k))_{\mathbb{Q}} \oplus T(S'_{k(S)})/T(S'). \tag{7.8}$$

From (7.7) and (7.8) we get;

$$\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}(\bar h_1(S), \bar h_2(S')) \oplus \mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}(\bar h_2(S), \bar h_2(S')) \simeq T(S'_{k(S)})/T(S').$$

□

7.5.8. **Proposition.** *With the same notation as in Lemma 7.5.7, the projection map*

$$\Psi : \mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}}(t_2(S), t_2(S')) \to \mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}(\overline{t_2(S)}, \overline{t_2(S')})$$

is an isomorphism.

Proof. The map Ψ of the proposition is clearly surjective, and we have to show that it is injective.

Let $f \in \mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}}(t_2(S), t_2(S'))$ be such that $\Psi(f) = 0$. Then f, as a correspondence in $A^2(S \times S')$, belongs to the subgroup $\mathcal{I}(S, S')$: from the definition of $\mathcal{I}(S, S')$ and $\mathcal{J}(S, S')$ (see Definition 7.4.2) it follows that $\mathcal{I}(S, S') \subset \mathcal{J}(S, S')$. Thus $f \in \mathcal{J}(S, S')$ and from Theorem 7.4.3 we get that $(\pi_2^{\mathrm{tr}})' \circ f \circ \pi_2^{\mathrm{tr}} = 0$. Since $f \in \mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}}(t_2(S), t_2(S'))$, we also have $(\pi_2^{\mathrm{tr}})' \circ f \circ \pi_2^{\mathrm{tr}} = f$, hence $f = 0$. □

7.5.9. **Lemma.** *Let S be a smooth projective surface and C a smooth projective curve. Then for any C-K decompositions $h(S) = \bigoplus_{0 \le i \le 4} h_i(S)$ and $h(C) = \bigoplus_{0 \le j \le 2} h_j(C)$ as in Proposition 7.2.1, we have*

$$A_0(C_{k(S)})/A_0(C) \simeq \mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}(\bar h_1(S), \bar h_1(C))$$

where $\bar h_i(X)$ and $\bar h_j(C)$ are the images in $\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}$.

Proof. We have in $\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}$:

$$\bar h(S) = 1 \oplus \bar h_1(S) \oplus \bar h_2(S); \qquad \bar h(C) = 1 \oplus \bar h_1(C)$$

and, by Proposition 7.5.3,

$$A_0(C_{k(S)}) \simeq \mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}(\bar h(S), \bar h(C)); \qquad A_0(C) \simeq \mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}(1, \bar h(C))$$

with $A_0(C) \simeq \mathbb{Q} \oplus J_C(k)_{\mathbb{Q}}$, where J_C is the Jacobian of C. Therefore

$$A_0(C_{k(S)})/A_0(C) \simeq \mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}(\bar h_1(S), \bar h(C)) \oplus \mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}(\bar h_2(S), \bar h(C))$$

and

$$\begin{aligned}\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}(\bar h_i(S), \bar h(C)) &= \mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}(\bar h_i(S), 1) \oplus \mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}(\bar h_i(S), \bar h_1(C)) \\ &= \mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}(\bar h_i(S), \bar h_1(C))\end{aligned}$$

because $\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}(\bar h_i(S), 1) = 0$ for $i = 1, 2$.

From Corollary 7.3.9 (ii) we get $\mathcal{M}_{\rm rat}(h_2(S), h_1(C)) = 0$ hence

$$\mathcal{M}^{\rm o}_{\rm rat}(\bar h_2(S), \bar h(C)) = 0.$$

Therefore

$$A_0(C_{k(S)})/A_0(C) \simeq \mathcal{M}^{\rm o}_{\rm rat}(\bar h_1(S), \bar h_1(C)).$$

□

The following Theorem 7.5.10 is a reintepretation of Theorem 7.4.8 in terms of the birational motives $\bar h_2(S)$ and $\bar h_2(S')$.

Let $d_{\le 1}\mathcal{M}^{\rm o}_{\rm rat}$ be the thick subcategory of $\mathcal{M}^{\rm o}_{\rm rat}$ generated by motives of curves: by a result in [K-S, 9.5], $d_{\le 1}\mathcal{M}^{\rm o}_{\rm rat}$ is equivalent to the category $\mathrm{AbS}(k)$ of abelian k-schemes (extensions of a lattice by an abelian variety) with rational coefficients.

7.5.10. **Theorem.** *Let S, S' be smooth projective surfaces over k. Given any two refined C-K decompositions as in Propositions 7.2.1 and 7.2.3, there are two isomorphisms*

$$\mathcal{M}_{\rm rat}(t_2(S), t_2(S')) \simeq \mathcal{M}^{\rm o}_{\rm rat}(\overline{t_2(S)}, \overline{t_2(S')}) \simeq \frac{T(S'_{k(S)})}{\mathcal{K}^{\rm o}_{\le 1}(S,S') \cap T(S'_{k(S)})}$$

and $\mathcal{K}^{\rm o}_{\le 1}(S,S') = H_{\le 1}$. (See Definition 7.5.5 (iv) for the definition of $\mathcal{K}^{\rm o}_{\le 1}$ and Definition 7.4.6 for the definition of $H_{\le 1}$.)

Proof. Let $\{\pi_i\}$ and $\{\pi'_i\}$, for $0 \le i \le 4$, be the projectors giving a refined C-K decomposition respectively for S and S'. From Proposition 7.5.8 it follows that

$$\mathcal{M}_{\rm rat}(t_2(S), t_2(S')) \simeq \mathcal{M}^{\rm o}_{\rm rat}(\overline{t_2(S)}, \overline{t_2(S')}).$$

From Lemma 7.5.1 we get:

$$A_0(S'_{k(S)}) \simeq \frac{A_2(S \times S')}{\mathcal{I}(S,S')}$$

and from Lemma 7.5.6 (d):

$$\mathcal{J}(S,S') = \mathcal{I}(S,S') + \mathcal{K}_{\le 1}(S,S').$$

From Theorems 7.4.3 and 7.4.8, the map

$$\beta : A_2(S \times S') \to T(S'_{k(S)})$$

defined by

$$\beta(Z) = ((\pi_2^{tr})' \circ Z \circ \pi_2^{tr})(\xi),$$

where ξ is the generic point of S, induces isomorphisms:

$$\mathcal{M}_{\rm rat}(t_2(S), t_2(S')) \simeq \frac{A_2(S \times S')}{\mathcal{J}(S,S')} \simeq \frac{T(S'_{k(S)})}{H_{\le 1} \cap T(S'_{k(S)})}.$$

Moreover, it follows from Lemma 7.4.7 that, if $T \in A_2(S \times S')$ then $T \in \mathcal{J}(S, S')$ if and only if $T(\xi) \in H_{\leq 1}$. Hence

$$\beta(Z) \in H_{\leq 1} \cap T(S'_{k(S)}) \iff (\pi_2^{tr})' \circ Z \circ \pi_2^{tr} \in \mathcal{J}(S, S')$$
$$\iff (\pi_2^{tr})' \circ Z \circ \pi_2^{tr} = \Gamma_1 + \Gamma_2$$

where $\Gamma_1 \in \mathcal{I}(S.S')$ and $\Gamma_2 \in \mathcal{K}_{\leq 1}(S, S')$. Since $\Gamma_1(\xi) = 0$ we get, for any $Z \in A_2(S \times S')$

$$\beta(Z) \in H_{\leq 1} \cap T(S'_{k(S)}) \iff (\pi_2^{tr})' \circ Z \circ \pi_2^{tr}(\xi) = \Gamma_2(\xi)$$

with $\Gamma_2 \in \mathcal{K}_{\leq 1}(S, S')$. This proves that the image of $\mathcal{K}_{\leq 1}(S, S')$ under the map β is $\mathcal{K}^o_{\leq 1}(S, S') \cap T(S'_{k(S)})$ and coincides with $H_{\leq 1} \cap T(S'_{k(S)})$.

Therefore we get:

$$\mathcal{M}_{\rm rat}(t_2(S), t_2(S')) \simeq \frac{T(S'_{k(S)})}{\mathcal{K}^o_{\leq 1}(S, S') \cap T(S'_{k(S)})}$$

and $\mathcal{K}^o_{\leq 1}(S, S') \cap T(S'_{k(S)}) = H_{\leq 1} \cap T(S'_{k(S)})$.

So we are left to show that

$$\mathcal{K}^o_{\leq 1}(S, S') = H_{\leq 1}.$$

From the definitions of $\mathcal{K}^o_{\leq 1}(S, S')$ and $H_{\leq 1}$ it follows that $H_{\leq 1} \subset \mathcal{K}^o_{\leq 1}(S, S')$.

We have $\mathcal{M}^o_{\rm rat}(\bar{h}(S), \bar{h}(S')) \simeq A_0(S'_{k(S)})$, $\mathcal{M}^o_{\rm rat}(1, \bar{h}(S')) \simeq A_0(S')$ and

$$\mathcal{M}^o_{\rm rat}(\bar{h}_1(S), \bar{h}_1(S')) = \{\bar{\pi}'_1 \circ \bar{\Gamma} \circ \bar{\pi}_1 | \Gamma \in A_0(S'_{k(S)})\}.$$

From the construction of the projector $\pi_1(S')$ as in Proposition 7.2.3, it follows that there exists a curve $C' \subset S'$ such that $\pi_1(S')$, as a map in $\mathcal{M}_{\rm rat}(h(S'), h(S'))$, factors through the motive $h_1(C')$. Therefore, every map α in $\mathcal{M}^o_{\rm rat}(\bar{h}_1(S), \bar{h}_1(S'))$ factors trough the birational motive of a curve C', i.e. it is in the image $\mathcal{K}^o_{\leq 1}(S, S')$ of $\mathcal{K}_{\leq 1}(S, S')$. Moreover, the same argument, as in the proof of lemma 7.4.7 shows that $\alpha \in H_{\leq 1}$.

From Corollary 7.3.9 (ii) it follows that the only map in the group $\mathcal{M}^o_{\rm rat}(\bar{h}_2(S), \bar{h}_2(S'))$ that factors through $\bar{h}(C)$ for some curve C is 0.Therefore we get:

$$\mathcal{K}^o_{\leq 1}(S, S') = \mathcal{M}^o_{\rm rat}(1, \bar{h}(S')) + \mathcal{M}^o_{rat}(\bar{h}_1(S), \bar{h}_1(S')) + \mathcal{M}^o_{\rm rat}(\bar{h}_1(S), \bar{h}_2(S'))$$

because $\mathcal{M}^o_{\rm rat}(\bar{h}_2(S), \bar{h}_1(S')) = 0$. Furthermore

$$\mathcal{M}^o_{\rm rat}(1, \bar{h}(S')) + \mathcal{M}^o_{\rm rat}(\bar{h}_1(S), \bar{h}_1(S') \subset H_{\leq 1}.$$

From Lemma 7.5.7

$$\mathcal{M}^o_{\rm rat}(\bar{h}_1(S), \bar{h}_2(S')) \oplus \mathcal{M}^o_{\rm rat}(\bar{h}_2(S), \bar{h}_2(S')) \simeq T(S'_{k(S)})/T(S')$$

hence:

$$\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}(\bar h_1(S),\bar h_2(S'))=\frac{\mathcal{K}^{\mathrm{o}}_{\le 1}(S,S')\cap T(S'_{k(S)})}{T(S')}=\frac{H_{\le 1}\cap T(S'_{k(S)})}{T(S')}.$$

This proves that $\mathcal{K}^{\mathrm{o}}_{\le 1}(S,S')\subset H_{\le 1}$. □

7.5.11. *Remarks.* 1) According to Proposition 7.3.6 (ii), if S and S' are surfaces such that $S\times S'$ satisfies Conjecture 7.3.3 then the group $\mathcal{M}_{\mathrm{rat}}(t_2(S),t_2(S'))$ has finite rank. From Theorem 7.4.3 and Theorem 7.5.10 it follows that this group is isomorphic to a quotient of the group $T(S'_{k(S)})/T(S')$. The following example, suggested to us by Schoen and Srinivas, shows that, if S is a surface, the group $T(S_{k(S)})/T(S)$ may have infinite rank.

Let $E\subset\mathbb{P}^2_{\bar{\mathbb{Q}}}$ denote the elliptic curve defined by $X^3+Y^3+Z^3=0$, $L=\bar{\mathbb{Q}}(E)$ and $S=E\times E$. Then from the results in [Schoen] it follows that the group $A^2(S_L)_{\deg 0}/A^2(S)_{\deg 0}$ has infinite rank. Now, applying the exact sequence (7.8) with $X=E, Y=S=E\times E$ and $k=\bar{\mathbb{Q}}$, we get an exact sequence

$$0\to T(S_L)/T(S)\to A_0(S_L)/A_0(S)\to \mathrm{Ab}(E,E\times E)\to 0.$$

Since $\mathrm{Ab}(E,E\times E)$ has finite rank, $T(S_L)/T(S)$ has infinite rank; since $L\subset k(S)$, so does $T(S_{k(S)})/T(S)$.

2) In [B2, 1.8] (see also [J2, 1.12]) Bloch conjectured that, if S is a smooth projective surface and $\Gamma\in A_2(S\times S)_{\mathrm{hom}}$, then Γ acts trivially on $T(S_\Omega)$, where Ω is a universal domain containing k. This conjecture implies that, if $H^2_{\mathrm{tr}}(S)=0$, then the Albanese kernel $T(S)$ vanishes. We claim that, from the results in §§7.4 and 7.5, it follows that the above conjecture also implies $A_2(S\times S)_{\mathrm{hom}}\subset\mathcal{J}(S,S)$, hence that $\mathrm{End}_{\mathcal{M}_{\mathrm{rat}}}(t_2(S))\simeq A_2(S\times S)/\mathcal{J}(S,S)$ (Theorem 7.4.3) is finite-dimensional as a quotient of $A^2_{\mathrm{hom}}(S\times S)$ (at least in characteristic 0 for a "classical" Weil cohomology in Bloch's conjecture).

To show the claim, observe that if $\alpha\in A_0(S_{k(S)})$, then $\alpha(\beta)=\beta\circ\alpha$ for every $\alpha\in A_0(S_{k(S)})$ (see (7.4)). Therefore, if $\Gamma\in A_2(S\times S)_{\mathrm{hom}}$, then $\bar\Gamma(\pi_2^{\mathrm{tr}})=0$ because $\pi_2^{\mathrm{tr}}(\xi)\in T(S_{k(S)})$ and $k(S)\subset\Omega$. This implies that $\bar\pi_2^{\mathrm{tr}}\circ\bar\Gamma\circ\pi_2^{\mathrm{tr}}=0$ in $\mathrm{End}_{\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}}(\bar t_2(S))$. From Theorem 7.4.3 and Proposition 7.5.8 it follows that $\Gamma\in\mathcal{J}(S,S)$.

7.6. Finite-dimensional motives

In this section we first recall from [Ki] and [G-P2] some definitions and results on finite dimensional motives. Then we relate the finite dimensionality of the motive of a surface S with Bloch's Conjecture on the vanishing of the Albanese kernel and with the results in §§7.4 and 7.5.

Let $\mathcal{C}$ be a pseudoabelian, $\mathbb{Q}$-linear, rigid tensor category and let X be an object in $\mathcal{C}$. Let Σ_n be the symmetric group of order n: any $\sigma \in \Sigma_n$ defines a map $\sigma : (x_1, \dots, x_n) \to (x_{\sigma(1)}, \dots, x_{\sigma(n)})$ on the n-fold tensor product $X^{\otimes n}$ of X by itself. There is a one-to-one correspondence between all irreducible representations of the group Σ_n (over $\mathbb{Q}$) and all partitions of the integer n. Let V_λ be the irreducible representation corresponding to a partition λ of n and let χ_λ be the character of the representation V_λ. Let

$$d_\lambda = \frac{\dim(V_\lambda)}{n!} \sum_{\sigma \in \Sigma_n} \chi_\lambda(\sigma) \cdot \Gamma_\sigma$$

where Γ_σ is the correspondence associated to σ. Then $\{d_\lambda\}$ is a set of pairwise orthogonal idempotents in $\mathrm{Hom}_{\mathcal{C}}(X^{\otimes n}, X^{\otimes n}))$ such that $\sum d_\lambda = \Delta_{X^{\otimes n}}$. The category $\mathcal{C}$ being pseudoabelian, they give a decomposition of $X^{\otimes n}$. The n-th symmetric product $S^n X$ of X is then defined to be the image $\mathrm{Im}(d_\lambda)$ when λ corresponds to the partition (n), and the n-th exterior power $\wedge^n X$ is $\mathrm{Im}(d_\lambda)$ when λ corresponds to the partition $(1, \dots, 1)$.

If $\mathcal{C} = \mathcal{M}_{\mathrm{rat}}$ and $M = h(X) \in \mathcal{M}_{\mathrm{rat}}$ for a smooth projective variety X, then $\wedge^n M$ is the image of $M(X^n)$ under the projector $(1/n!)(\sum_{\sigma \in \Sigma_n} sgn(\sigma)\Gamma_\sigma)$, while $S^n M$ is its image under the projector $(1/n!)(\sum_{\sigma \in \Sigma_n} \Gamma_\sigma)$.

7.6.1. **Definition** (see [Ki] and [G-P1])**.** The object X in $\mathcal{C}$ is said to be *evenly (oddly) finite-dimensional* if $\wedge^n X = 0$ ($S^n X = 0$) for some n. An object X is finite-dimensional if it can be decomposed into a direct sum $X_+ \oplus X_-$ where X_+ is evenly finite-dimensional and X_- is oddly finite-dimensional.

Kimura's nilpotence theorem [Ki, 7.2] says that if M is finite-dimensional, any numerically trivial endomorphism of M is nilpotent. We shall need the following more precise version in the proof of Theorem 7.6.9:

7.6.2. **Theorem.** *Let $M \in \mathcal{M}_{\mathrm{rat}}$ be a finite-dimensional motive. Then the ideal of numerically trivial correspondences in $\mathrm{End}_{\mathcal{M}_{\mathrm{rat}}}(M, M)$ is nilpotent.*

Recall [A-K, 9.1.4] that the proof is simply this: Kimura's argument shows that the nilpotence level is uniformly bounded. On the other hand, a theorem of Nagata and Higman says that if I in a non unital and not necessarily commutative ring such that there exists $n > 0$ for which $f^n = 0$ for all $f \in I$, then I is nilpotent.

7.6.3. *Examples.* 1) If two motives are finite-dimensional so is their direct sum and their tensor product.
2) A direct summand of an evenly (oddly) finite-dimensional motive is evenly (oddly) finite-dimensional. If a motive M is evenly and oddly finite-dimensional then $M = 0$ [Ki, 6.2]. A direct summand of a finite-dimensional motive is finite-dimensional [Ki, 6.9].
3) The dual motive M^* is finite-dimensional if and only if M is finite-dimensional.
4) The motive of a smooth projective curve is finite-dimensional: hence the motive of an abelian variety X is finite-dimensional. Also if X is the quotient of a product $C_1 \times \cdots \times C_n$ of curves under the action of a finite group G acting freely on $C_1 \times \cdots \times C_n$ then $h(X)$ is finite-dimensional.

More generally:

7.6.4. **Proposition** (Kimura's lemma, [Ki, 6.6 and 6.8])**.** *If $f : X \to Y$ is a surjective morphism of smooth projective varieties, then $h(Y)$ is a direct summand of $h(X)$. Hence, by Example 7.6.3 2), if $h(X)$ is finite-dimensional then $h(Y)$ is also finite-dimensional.*

Here is a simple proof, in the spirit of Kimura's: let $g = 1_Y \times f : Y \times X \to Y \times Y$ and $T = g^{-1}(\Delta)$, where Δ is the diagonal of $Y \times Y$. Pick a closed point p of the generic fibre of $g_{|T} : T \to \Delta$: the closure of p in T defines a closed subvariety Z in $Y \times X$ which is finite surjective over Δ. Then Z defines a correspondence $[Z]$ from Y to X, and one checks immediately that the composition

$$h(Y) \xrightarrow{[Z]} h(X) \xrightarrow{f_*} h(Y)$$

is multiplication by the generic degree of Z over Δ. □

We also have Kimura's conjecture:

7.6.5. **Conjecture** ([Ki])**.** *Any motive in $\mathcal{M}_{\mathrm{rat}}$ is finite-dimensional.*

7.6.6. **Lemma.** *For any smooth projective surface S, the motives $h_0(S)$, $h_1(S)$, $h_2^{\mathrm{alg}}(S)$, $h_3(S)$ and $h_4(S)$ appearing in Propositions 7.2.1 and 7.2.3 are finite-dimensional. Hence all direct summands of $h(S)$ appearing in these propositions are finite-dimensional, except perhaps $t_2(S)$.*

Proof. The lemma is clear for $h_0(S), h_2^{\mathrm{alg}}(S)$ and $h_4(S)$ since these motives are tensor products of Artin motives and Tate motives. Since $h_3(S) \simeq h_1(S)(1)$, it remains to deal with $h_1(S)$. But the construction of the projector defining $h_1(S)$ in [Mu1, Sch] shows that it is a direct summand of the motive of a curve; the claim therefore follows from Examples 7.6.3 2) and 4). □

7.6.7. **Lemma.** *Let U be a group acting transitively on a set E. Suppose that the following condition is verified:*

(*) If $e \in E$, $u \in U$ and $n \geq 1$ are such that $u^n e = e$, then $ue = e$.

On the other hand, let G be a group "acting on this action": there is an action of G on U and an action of G on E such that

$$^g(ue) = {}^g u {}^g e$$

for any $(g, u, e) \in G \times U \times E$. Suppose moreover that G is finite, U has a finite G-invariant composition series $\{1\} = Z_r \subset \cdots \subset Z_1 = U$ such that, for all i,

(i) *$Z_i \lhd U$;*
(ii) *Z_i/Z_{i+1} is central in U/Z_{i+1} and uniquely divisible.*

Then G has a fixed point e on E. If f is another fixed point, there exists $u \in U$, invariant by G, such that $f = ue$.

Proof. We argue by induction on r, the case $r = 1$ being trivial. Suppose $r > 1$ and let $Z = Z_{r-1}$. Since G preserves Z, it acts on U/Z and E/Z, preserving the induced action. Moreover, the fact that Z is central in U and divisible implies that Condition (*) is preserved.

By induction, there is $e \in E$ such that

$$^g e = z_g e \quad \forall g \in G$$

with $z_g \in Z$.

Let U_e be the stabilizer of e in U. Since the z_g are central, U_e is stable under the action of G; in particular, G acts on $Z/Z \cap U_e$. An easy computation shows that, for all $g, h \in G$:

$$z_{gh}^{-1} {}^g z_h z_g \in Z \cap U_e.$$

Now $Z/Z \cap U_e$ is divisible as a quotient of Z, and moreover Condition (*) implies that it is torsion-free. Therefore, $H^1(G, Z/Z \cap U_e) = 0$ and there is some $z \in Z$ such that $z_g \equiv {}^g z^{-1} z \pmod{Z \cap U_e}$ for all $g \in G$. Then ze is G-invariant.

For uniqueness, we argue in the same way. By induction, there exists $u_0 \in U$ such that $f = u_0 e$ and ${}^g u_0 = z_g u_0$ for all $g \in G$, with $z_g \in Z$. Applying $g \in G$ to the equation $f = u_0 e$ shows that $z_g \in U_f$. Thus, $g \mapsto z_g$ defines a 1-cocycle with values in $Z \cap U_f$. Since Z and $Z/Z \cap U_f$ are uniquely divisible, so is $Z \cap U_f$, hence this 1-cocycle is a 1-coboundary and we are done. □

7.6.8. **Lemma.** *Let A be a $\mathbb{Q}$-algebra, π a subset of A and ν a nilpotent element of A. Suppose that there exists a polynomial $P = \sum a_i t^i$, with $a_1 \neq 0$, such that $P(\nu)$ commutes with all the elements of π. Then ν commutes with all the elements of π.*

Proof. Let us denote by C the centralizer of π and let r be such that $\nu^r = 0$. We prove that $\nu^i \in C$ for all i by descending induction on i. The case $i \geq r$ is clear. Note that we may (and do) assume that $P(0) = 0$. Then $P(\nu)$ is nilpotent. Let $i < r$: then

$$C \ni P(\nu)^i = a_1^i \nu^i + \dots$$

where the next terms are higher powers of ν. By induction, $a_1^i \nu^i \in C$, hence $\nu^i \in C$. □

The following is a slight improvement of [G-P2, Th. 3]:

7.6.9. **Theorem.** *Let X be a smooth projective variety over k of dimension d, such that the the Künneth components of the diagonal are algebraic. Assume that the motive $h(X) \in \mathcal{M}_{\mathrm{rat}}$ is finite-dimensional. Then*
a) $h(X)$ has a Chow-Künneth decomposition

$$h(X) = \bigoplus_{0 \leq i \leq 2d} h_i(X)$$

with $h_i(X) = (X, \pi_i, 0)$.
If $\{\tilde{\pi}_i\}$ is another set of such orthogonal idempotents, then there exists a nilpotent correspondence n on X such that

$$\tilde{\pi}_i = (1+n)\pi_i(1+n)^{-1} \tag{7.9}$$

for all i. In particular,

$$h_i(X) \simeq \tilde{h}_i(X)$$

in $\mathcal{M}_{\mathrm{rat}}$, where $\tilde{h}_i(X) = (X, \tilde{\pi}_i, 0)$.
b) Moreover, the π_i may be chosen so that $\pi_i^t = \pi_{2d-i}$. If $\{\tilde{\pi}_i\}$ is another such choice, there exists a nilpotent correspondence n on X such that (7.9) *holds and moreover, $(1+n)^t = (1+n)^{-1}$.*

Proof. a) The existence and "uniqueness" of the π_i follow immediately from Kimura's nilpotence theorem (Theorem 7.6.2) and from [J2, 5.4]. For b), let $\mathcal{N} = A_d(X \times X)_{\mathrm{hom}}$: this is a nilpotent ideal of

$\mathrm{End}_{\mathcal{M}_{\mathrm{rat}}}(h(X))$ by Theorem 7.6.2. We apply Lemma 7.6.7 with

$$U = 1 + \mathcal{N}$$
$$Z_i = 1 + \mathcal{N}^i$$
$$E = \{\{\pi_i\} \mid \pi_i \mapsto \pi_i^{\mathrm{hom}}\}$$
$$G \simeq \mathbb{Z}/2.$$

We let U act on E by conjugation: this action is transitive by a). We let G act on this action as follows: if g is the nontrivial element of G, then

$$^g u = (u^{-1})^t; \qquad {}^g\{\pi_i\} = \{{\pi_i}^t\}.$$

Note that the action on E exists because the π_i^{hom} are stable under transposition (Poincaré duality). We now check the hypotheses of Lemma 7.6.7: clearly the Z_i are normal in U, G-invariant and verify the centrality assumption. Moreover, $Z_i/Z_{i+1} \simeq \mathcal{N}^i/\mathcal{N}^{i+1}$ is uniquely divisible. It remains to verify Condition (*): but this follows from Lemma 7.6.8 applied to $P(\nu) = (1+\nu)^n$. The proof is complete. □

7.6.10. **Theorem** ([G-P2, Th. 7]). *Let S be a smooth projective surface over an algebraically closed field k of characteristic 0 with $p_g(S) = 0$, and suppose that k has infinite transcendence degree over $\mathbb{Q}$: then the motive $h(S)$ is finite-dimensional if and only if the Albanese kernel $T(S)$ vanishes.*

Proof. "If" follows from Corollary 7.4.9 b) (see proof of (1) ⇔ (2) in Theorem 7.6.12 below). For "only if", note that the hypothesis $p_g = 0$ implies $H^2_{\mathrm{tr}}(S) = 0$ and therefore $(\pi_2^{\mathrm{tr}}(S))^{\mathrm{hom}} = 0$. By Kimura's nilpotence theorem (Theorem 7.6.2), the finite-dimensionality hypothesis now implies that $\pi_2^{\mathrm{tr}}(S) = 0$, and we conclude by Proposition 7.2.3. □

The following corollary may be viewed as a "birational" version of a result by S. Bloch in [B2, Lect. 1, Prop. 2].

7.6.11. **Corollary.** *Let S be a smooth projective surface over an algebraically closed field k of characteristic 0 and infinite transcendence degree over $\mathbb{Q}$. Then the following conditions are equivalent:*

(i) *$p_g(S) = 0$ and the motive $h(S)$ is finite-dimensional;*
(ii) *the Albanese Kernel $T(S)$ vanishes;*
(iii) *$t_2(S) = 0$;*
(iv) *$\bar{t}_2(S) = 0$ in $\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}$;*
(v) *the motive $\bar{h}(S)$ in $\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}$ is a direct summand of the birational motive of a curve.*

Proof. By Theorem 7.6.10, (i) $\Rightarrow$ (ii). The equivalence of (ii) and (iii) has been seen in Corollary 7.4.9 b) and the equivalence of (iii) and (iv) follows from Proposition 7.5.8. If $t_2(S) = 0$, then $p_g = 0$ by Corollary 7.4.9 c) and $h(S)$ is finite-dimensional by Lemma 7.6.6. Thus we have (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

Note that in general, $\bar{h}_2^{\mathrm{alg}}(S) = \bar{h}_3(S) = \bar{h}_4(S) = 0$ by Proposition 7.2.1 (iv). Therefore (iv) $\Rightarrow$ (v) (see proof of Lemma 7.6.6). Conversely, if $\bar{h}(S)$ is a direct summand of the birational motive of a (not necessarily connected) curve D, then so is $\bar{t}_2(S)$. But Corollary 7.3.9 b) implies that $\mathcal{M}_{\mathrm{rat}}(t_2(S), h(D))) = 0$, hence $\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}(\bar{t}_2(S), \bar{h}(D)) = 0$, which implies that $\mathrm{End}_{\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}}(\bar{t}_2(S)) = 0$ and therefore that $\bar{t}_2(S) = 0$. So (v) $\Rightarrow$ (iv) and the proof is complete. □

7.6.12. **Theorem.** *Let S be a smooth projective surface and let $h(S) = \bigoplus_{0\le i\le 4} h_i(S) = \bigoplus_{0\le i\le 4}(S, \pi_i, 0)$ be a refined Chow-Künneth decomposition as in Prop. 7.2.1 and 7.2.3. Let us consider the following conditions:*

1. *the motive $h(S)$ is finite-dimensional;*
2. *the motive $t_2(S)$ is evenly finite-dimensional;*
3. *every endomorphism $f \in \mathrm{End}_{\mathcal{M}_{\mathrm{rat}}}(h(S))$ which is homologically trival is nilpotent;*
4. *for every correspondence $\Gamma \in A_2(S\times S)_{\mathrm{hom}}$, $\alpha_{i,i} = \pi_i\circ\Gamma\circ\pi_i = 0$, for $0 \le i \le 4$;*
5. *for all i, the map $\mathrm{End}_{\mathcal{M}_{\mathrm{rat}}}(h_i(S)) \to \mathrm{End}_{\mathcal{M}_{\mathrm{hom}}}(h_i^{\mathrm{hom}}(S))$ is an isomorphism (hence $\mathrm{End}_{\mathcal{M}_{\mathrm{rat}}}(h_i(S))$ has finite rank in characteristic 0);*
6. *the map $\mathrm{End}_{\mathcal{M}_{\mathrm{rat}}}(t_2(S)) \to \mathrm{End}_{\mathcal{M}_{\mathrm{hom}}}(t_2^{\mathrm{hom}}(S))$ is an isomorphism;*
7. *let $\mathcal{J}(S)$ be the 2-sided ideal of $A_2(S \times S)$ defined in Definition 7.4.2: then $A_2(S \times S)_{\mathrm{hom}} \subset \mathcal{J}(S)$.*

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Leftarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7).

Proof. (1) $\Leftrightarrow$ (2) by Lemma 7.6.6. ($t_2(S)$ is evenly finite dimensional because it is a direct summand of $h_2(S)$.)

(1) $\Rightarrow$ (3) follows from [Ki, 7.2] (see Theorem 7.6.2).

(4) $\Rightarrow$ (3): (4) implies that $h_i(S)$, for $0 \le i \le 4$, satisfy (i) and (ii) in Theorem 7.3.10. Therefore, by [G-P2, Cor. 3], every endomorphism $f \in \mathrm{End}_{\mathcal{M}_{\mathrm{rat}}}(h(S))$ which is homologically trivial is nilpotent.

(4) $\Rightarrow$ (5). We have

$$\mathrm{End}_{\mathcal{M}_{\mathrm{rat}}}(h_1(S)) \simeq \mathrm{End}_{\mathrm{Ab}}(\mathrm{Alb}_S) \simeq \mathrm{End}_{\mathcal{M}_{\mathrm{hom}}}(h_1^{\mathrm{hom}}(S)).$$

By duality the same result holds for $\mathrm{End}_{\mathcal{M}_{\mathrm{rat}}}(h_3(S))$. From (4) it follows that also the map

$$\mathrm{End}_{\mathcal{M}_{\mathrm{rat}}}(h_2(S)) \to \mathrm{End}_{\mathcal{M}_{\mathrm{hom}}}(h_2^{\mathrm{hom}}(S))$$

is an isomorphism.

(5) $\Rightarrow$ (6) is obvious.

(6) $\Rightarrow$ (7). If $\Gamma \in A_2(S \times S)_{\mathrm{hom}}$ then $\pi_2^{\mathrm{tr}} \circ \Gamma \circ \pi_2^{\mathrm{tr}}$ yields the 0 map in $\mathrm{End}_{\mathcal{M}_{\mathrm{hom}}}(t_2^{\mathrm{hom}}(S))$, therefore it is 0. Since $\Gamma \in A_2(S \times S)_{\mathrm{hom}}$ we also have $\pi_2^{\mathrm{alg}} \circ \Gamma \circ \pi_2^{\mathrm{alg}} = 0$, hence $\Gamma \in \mathcal{J}(S)$ (see Lemma 7.4.1).

(7) $\Rightarrow$ (4). Let $\Gamma \in A_2(S \times S)_{\mathrm{hom}}$: then $\Gamma \in \mathcal{J}(S)$ which proves that $\pi_2^{\mathrm{tr}} \circ \Gamma \circ \pi_2^{\mathrm{tr}} = 0$. □

7.6.13. *Remark* (Abelian varieties and Kummer surfaces). Let A be an abelian variety of dimension d. Then $h(A)$ has a Chow-Künneth decomposition (see [Sch]) $h(A) = \bigoplus_{0 \le i \le 2d} h_i(A)$ where $h_i(A) = (A, \pi_i^A, 0)$ and $n^* \circ \pi_i^A = n^i \pi_i^A$, for every $n \in \mathbb{Z}$. Here $n^* = (id \times n)^*$ is the correspondence induced by multiplication by n on A. The motive $h(A)$ is finite-dimensional, hence the above decomposition is unique (up to isomorphism).

Now suppose that $d = 2$, and let S be the Kummer surface associated to the involution $a \to -a$ on A (with singularities resolved). The rational map $f : A \to S$ induces an isomorphism between the Albanese kernels : $T(A) \simeq T(S)$ (see [B-K-L, A.11]). Let $h(S) = \bigoplus_{0 \le i \le 4} h_i(S)$, with $h_i(S) = (S, \pi_i^S, 0)$. Reasoning as in [A-J, Th 3.2], we get that the formula

$$\pi_i^S = (1/2)(f \times f)_* \pi_i^A$$

defines a C-K decomposition on S. From the exact sequence in (7.3) and from Proposition 7.5.3 it follows that the map f induces homomorphisms $f^* : A_0(S_{k(S)}) \to A_0(A_{k(A)})$ and $f_* : A_0(A_{k(A)}) \to A_0(S_{k(S)})$. Then

$$f^*(f_*(\alpha)) = \alpha + [-1] \cdot \alpha$$

for all $\alpha \in A_0(A_{k(A)})$. From the equality $n^* \circ \pi_2^A = n^2 \pi_2^A$ it follows that $\bar{\pi}_2^A \in A_0(S_{k(S)})$, where $\bar{\pi}_2^A$ is the image of π_2^A under the map in (7.3). From the isomorphism

$$A_0(S_{k(S)}) \simeq \mathrm{End}_{\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}}(\bar{h}(S))$$

we get $\bar{h}_2(S) \simeq \bar{h}_2(A)$ in $\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}$. The Kummer surface S has $q = \dim H^1(S, \mathcal{O}_S) = 0$, hence $h_1(S) = h_3(S) = 0$. Therefore we get:

$$\bar{h}(S) \simeq 1 \oplus \bar{h}_2(A)$$

and:

$$\bar{h}_2(A) \simeq 1 \oplus \bar{h}_1(A) \oplus \bar{h}_2(A).$$

In particular, f induces an isomorphism $f_* : t_2(A) \xrightarrow{\sim} t_2(S)$.

7.7. Higher-dimensional refinements

The next results extend, under the assumption of certain conjectures, some of the properties proven in Propositions 7.2.1 and 7.2.3 for the refined Chow-Künneth decomposition of the motive of a surface to varieties of higher dimension. In particular these results apply to abelian varieties. To avoid questions of rationality we shall assume that k is separably closed; the reader will have no difficulties to extend these results to the general case along the lines of the proof of Proposition 7.2.3.

In the following we will denote by

$$\bar{A}^i(X) \subset H^{2i}(X)$$

the image of the cycle class map $\mathrm{cl}^i : A^i(X) \to H^{2i}(X)$ for a smooth projective variety X (see §7.1.3).

7.7.1. **Definition.** We say that the Hard Lefschetz theorem holds for the Weil cohomology H if, for any smooth projective variety X of dimension d, any smooth hyperplane section $W \subset X$ and any $i \le d$, the Lefschetz operator

$$L^{d-i} : H^i(X) \to H^{2d-i}(X)$$

given by cup product by $\mathrm{cl}(W)^{d-i}$, is an isomorphism.

It is known that every classical Weil cohomology satisfies the Hard Lefschetz theorem.

Let us choose a classical Weil cohomology theory H. Following [Kl], let $B(X)$ and $Hdg(X)$ denote respectively the Lefschetz standard conjecture and the Hodge standard conjecture for a smooth projective variety X. The conjecture $B(X)$ is equivalent to the following for any L as in Definition 7.7.1 (see [Kl, 4.1]):

> $\theta(X)$ For each $i \le d$, there exists an algebraic correspondence θ^i inducing the isomorphism $H^{2d-i}(X) \to H^i(X)$ inverse to L^{d-i}.

Recall also the conjectures:

> $A(X, L)$ The restriction $L^{d-2i} : \bar{A}^i(X) \to \bar{A}^{d-i}(X)$ is an isomorphism for all i.
> $C(X)$ The Künneth projectors are algebraic.
> $D(X)$ Numerical equivalence equals homological equivalence.

Under $D(X)$, $A^*_{\mathrm{hom}}(X)$ is a finite-dimensional $\mathbb{Q}$-vector space. By [Kl, 4.1 and 5.1], we have the following implications (for any L):

$$\begin{gathered} A(X \times X, L \otimes 1 + 1 \otimes L) \Rightarrow B(X) \Rightarrow A(X, L) \\ B(X) \Rightarrow C(X) \\ A(X, L) + Hdg(X) \Rightarrow D(X) \Rightarrow A(X, L). \end{gathered} \tag{7.10}$$

Finally, $B(X)$ is satisfied by curves, surfaces[5], abelian varieties and it is stable under products and hyperplane sections [Kl, 4.1 and 4.3]. Also $Hdg(X)$ is true in characteristic 0 and holds in arbitrary characteristic if X is a surface [Kl, §5].

We shall also need the following easy lemma:

7.7.2. **Lemma.** *Let H be a classical Weil cohomology theory. Let $M = (X_d, p, m) \in \mathcal{M}^{\mathrm{eff}}_{\mathrm{hom}}$. Then*
a) $m \geq -d$.
b) If $p^*H^i(X) \neq 0$ then we have the sharper inequality $m \geq -[i/2]$.

Proof. a) Let $\alpha : M \to h(Y)$ and $\beta : h(Y) \to M$ be two morphisms such that $\beta \circ \alpha = 1_M$. In particular, $0 \neq \alpha \in \mathrm{Corr}_m(X, Y) = A_{d+m}(X \times Y)$, hence $d + m \geq 0$.

b) We have $H^{i+2m}(M) = p^*H^i(X) \neq 0$. On the other hand, the correspondence α of a) realises $H^{i+2m}(M)$ as a direct summand of $H^{i+2m}(Y)$. The inequality follows. □

7.7.3. **Theorem.** *Let X be a smooth projective variety of dimension d such that Conjecture $B(X)$ holds and that the ideal* $\mathrm{Ker}(\mathrm{End}_{\mathcal{M}_{\mathrm{rat}}}(h(X)) \to \mathrm{End}_{\mathcal{M}_{\mathrm{hom}}}(h_{\mathrm{hom}}(X)))$ *is nilpotent (by Theorem 7.6.2, this is true if the motive $h(X)$ is finite-dimensional). Let $X \hookrightarrow \mathbb{P}^N$ be a fixed projective embedding. Then*

(i) *There exists a self-dual C-K decomposition $h(X) = \bigoplus h_i(X)$ ($\pi_i^t = \pi_{2d-i}$).*
(ii) *Let $i : W \hookrightarrow X$ be a smooth hyperplane section of X and $L = i_*i^* : h(X) \to h(X)(1)$ be the corresponding "Lefschetz operator". Then, for each $i \in [0, d]$, the composition*

$$\ell_i : h_{2d-i}(X) \to h(X) \xrightarrow{L^{d-i}} h(X)(d-i) \to h_i(X)(d-i) \tag{7.11}$$

is an isomorphism.
If moreover Conjecture $D(X \times X)$ holds, then:

[5] In [Kl, 4.3], Kleiman requires that $\dim H^1(X) = 2 \dim \mathrm{Pic}^0_X$, but this assumption is verified by all classical Weil cohomologies.

(iii) *For each* $i \in [0, 2d]$ *there exists a further decomposition*

$$h_i(X) \simeq \bigoplus_{j=0}^{[i/2]} h_{i,j}(X)(j) \tag{7.12}$$

such that, for each j, $h^{\hom}_{i,j}(X)$ *is effective but* $h^{\hom}_{i,j}(X)(-1)$ *is not effective. Moreover, the isomorphism from* (ii) *induces isomorphisms*

$$h_{2d-i,d-i+j}(X) \xrightarrow{\sim} h_{i,j}(X).$$

(iv) *Let* $(\pi_{i,j})$ *be the orthogonal set of projectors defining this decomposition. If* $(\pi'_{i,j})$ *is another such set of projectors, then there exists a correspondence* n*, homologically equivalent to* 0*, such that*

$$\pi'_{i,j} = (1+n)\pi_{i,j}(1+n)^{-1} \text{ for all } (i,j).$$

In particular, the $h_{i,j}(X)$ *are unique up to isomorphism.*

Proof. We first prove (i), (ii), (iii) and (iv) modulo homological equivalence. (i) is immediate since $B(X) \Rightarrow C(X)$ (see (7.10)). The homological version of (ii) follows immediately from the form $\theta(X)$ of Conjecture $B(X)$.

We now come to the homological versions of (iii) and (iv). By $D(X \times X)$ and Jannsen's theorem [J1], the algebra $\operatorname{End}_{\mathcal{M}_{\hom}}(h_{\hom}(X))$ is semi-simple. Given $i \in [0, 2d]$, write $M = h^{\hom}_i(X)$ as the direct sum of its isotypical components M_α: for each α, we have $M_\alpha \simeq S_\alpha^{n_\alpha}$ where S_α is a simple motive and $n_\alpha > 0$. By Lemma 7.7.2 b), the largest integer j_α such that $S_\alpha(-j_\alpha)$ is effective exists and verifies $j_\alpha \le i/2$. We set

$$h^{\hom}_{i,j}(X) = \bigoplus_{j_\alpha = j} M_\alpha(-j).$$

This proves the first claim of (iii). Moreover, this construction shows that the homological version of (7.12) is unique; in particular, the corresponding projectors $\pi^{\hom}_{i,j}$ are central in $\operatorname{End}_{\mathcal{M}_{\hom}}(h^{\hom}_i(X))$. This proves [the homological version of] (iv).

To see the second claim in (iii) (still in its homological version), let S_α be a simple summand of $h^{\hom}_{2d-i}(X)$; then clearly $\ell_i(S_\alpha)(-j_\alpha)$ is effective but $\ell_i(S_\alpha)(-j_\alpha - 1)$ is not effective. This proves that $\ell_i(h^{\hom}_{2d-i,d-i+j}(S)(d-i+j)) = h^{\hom}_{i,j}(S)(j)$, hence an isomorphism

$$h^{\hom}_{2d-i,d-i+j}(S) \xrightarrow{\sim} h^{\hom}_{i,j}(S).$$

Lifting these results from homological equivalence to rational equivalence follows from the nilpotency hypothesis, as in the proof of Theorem

7.6.9 a). The fact that ℓ_i still induces an isomorphism $h_{2d-i,d-i+j}(S) \xrightarrow{\sim} h_{i,j}(S)$ is also a standard consequence of nilpotence (cf. [An, 5.1.3.3]): we first note that numerically trivial endomorphisms of $h_{2d-i,d-i+j}(S)$ and $h_{i,j}(S)$ are nilpotent as both motives are direct summands of $h(X)$, up to Tate twists. Let θ be a cycle giving the inverse isomorphism to ℓ_i in $\mathcal{M}_{\mathrm{hom}}$. Then, in $\mathcal{M}_{\mathrm{rat}}$, $m = \ell_i \circ \theta - 1$ and $n = \theta \circ \ell_i - 1$ are numerically equivalent to 0, hence nilpotent. But then, $1 + m$ and $1 + n$ are isomorphisms. It follows that ℓ_i is left and right invertible, hence is an isomorphism. □

Moreover, we have:

7.7.4. **Theorem.** *Let X verify the hypotheses of Theorem 7.7.3. Then, for a C-K decomposition $h(X) = \bigoplus_{0 \le i \le 2d} h_i(X)$ as in this theorem, for every $i < d$, the projector π_i factors through $h(Y_i)$, where $Y_i = X \cdot H_1 \cdot H_2 \cdot \cdots \cdot H_{d-i}$ is a smooth hyperplane section of X of dimension i (with H_i hyperplanes). Hence $h_i(X)$ is a direct summand of $h(Y_i)$ for all $i < d$. Similarly, h_{2d-i} is a direct summand of $h(Y_i)$.*

Proof. By $B(X)$, for each $i \le d$ there exists an algebraic correspondence θ^i inducing the isomorphism $H^{2d-i}(X) \to H^i(X)$ inverse to the isomorphism $L^{d-i} : H^i(X) \to H^{2d-i}(X)$. Let $j : Y_i \to X$ be the closed embedding and let $\Gamma_i = j^* \circ \theta^i \in A^{d-i}(X \times Y_i)$ and $q_i = j_* \circ \Gamma_i \in A^d(X \times X)$.Then Γ_i and hence also q_i factor trough Y_i: furthermore q_i^{hom} operates as the identity on $H^{2d-i}(X)$ because $j_* \cdot j* = L^{d-i}$. Let

$$f_i = \pi_i \circ q_i \circ \pi_i \in \mathcal{M}_{rat}(h_i(X), h_i(X)).$$

Then f_i^{hom} is a projector on $H^*(X)$ and in fact is the $(i, 2d-i)$-Künneth projector. Therefore the map $a_i = \pi_i - f_i$ is homologically trivial, hence nilpotent by hypothesis, i.e. $a_i^n = 0$ for some $n > 0$. Let $b_i = (1+a_i+a_i^2+\cdots+a_i^{n-1}) = (1-a_i)^{-1}$. We have: $a_i \circ \pi_i = \pi_i \circ a_i = a_i$. Therefore $(1 - a_i) \circ \pi_i = \pi_i - a_i = f_i$ and we get

$$\pi_i = (1 - a_i)^{-1} \circ f_i = (1 + a_i + a_i^2 + \cdots + a_i^{n-1}) \circ f_i = b_i \circ f_i;$$

$$\pi_i = f_i \circ (1 - a_i)^{-1} = f_i \circ b_i.$$

Since q_i and therefore also f_i factor trough $h(Y_i)$ it follows that π_i factors trough $h(Y_i)$. Let $g_i = \Gamma_i \circ \pi_i : h(X) \to h(Y_i)$and $g_i' = b_i \circ \pi_i \circ j_* : h(Y_i) \to h_i(X)$: then we have $g_i' \circ g_i = \pi_i$, hence g_i has a left inverse. Therefore $h_i(X)$ is a direct summand of $h(Y_i)$ for all $i < d$.

The case of π_{2d-i} follows from the above since the C-K decomposition of Theorem 7.7.3 is self-dual. □

7.7.5. *Remark.* Theorem 7.7.3 notably applies to abelian varieties in characteristic 0. Also note that Theorem 7.7.4 answers – in a slightly weaker form – a question raised in [Mu2, p. 187].

We shall complement Theorem 7.7.3 with a somewhat more explicit result. For this we need lemma 7.7.6 which is just a reformulation of a result in [Ki, Prop. 2.11]:

7.7.6. **Lemma.** *Let X be a smooth projective variety of dimension d and let $a \in A^i(X)$, $b \in A^{d-i}(X)$, with $i \leq d$, be such that $< b, a >= \deg(a \cdot b) = 1$. Let $\alpha = p_1^* a \cdot p_2^* b \in A^d(X \times X)$, where $p_i : X \times X \to X$ are the projections. Then α is a projector and the motive $M = (X, \alpha, 0)$ is isomorphic to $\mathbb{L}^i$.*

Proof. We have $\alpha \circ \alpha =< a, b > \alpha = \alpha$, hence α is a projector. On the other hand

$$\mathcal{M}_{\mathrm{rat}}(M, \mathbb{L}^i) = A_{d-i}(X) \circ \alpha = A^i(X) \circ \alpha$$

and

$$\mathcal{M}_{\mathrm{rat}}(\mathbb{L}^i, M) = \alpha \circ A_i(X) = \alpha \circ A^{d-i}(X).$$

Therefore $\alpha \circ b \in \mathcal{M}_{\mathrm{rat}}(\mathbb{L}^i, M)$ and $a \circ \alpha \in \mathcal{M}_{\mathrm{rat}}(M, \mathbb{L}^i)$.

Considering a as an element of $A_{d-i}(X \times \operatorname{Spec} k)$ and b as an element of $A_i(\operatorname{Spec} k \times X)$, we have $a \circ \alpha =< a, b > a = a$ and $\alpha \circ b =< a, b > b = b$. Moreover $a \circ b = 1_{\operatorname{Spec} k}$ and $b \circ a = p_1^* a \cdot p_2^* b = \alpha = 1_M$. Hence a and b yield an isomorphism between M and $\mathbb{L}^i$. □

7.7.7. **Theorem.** *Keep the notation and hypotheses of Theorem 7.7.3. For all $i \in [0, d]$, the motive $h_{2i,i}(X)$ contains $h(\underline{\bar{A}}_{i,X})$ as a direct summand, where $h(\underline{\bar{A}}_{i,X})$is the Artin motive associated to the finite-dimensional vector space $\underline{\bar{A}}_{i,X}$.*

Proof. We prove this for $i \leq d/2$; the result then follows for $i \geq d/2$ by Poincaré duality.

We proceed as in the proof of Proposition 7.2.3. By $B(X)$, the homomorphism

$$L^{d-2i} : H^{2i}(X) \to H^{2d-2i}(X)$$

restricts to an isomorphism

$$L^{d-2i} : \bar{A}^i(X) \xrightarrow{\sim} \bar{A}^{d-i}(X)$$

where we use the same notation for the restriction of L^{d-2i}.

From $D(X \times X)$ it follows that the restriction of the Poincaré duality pairing on $H^{2i}(X) \times H^{2d-2i}(X)$ is still nondegenerate on $\bar{A}^i(X) \times$

$\bar{A}^{d-i}(X)$. Therefore there exist elements $a_{i,\alpha} \in A^i(X)$ and $b_{i,\alpha} \in A^{d-i}(X)$ ($\alpha = 0, \dots, \rho_i$) such that

1) $\tilde{e}_{i,\alpha} = \mathrm{cl}^i(a_{i,\alpha})$ and $\tilde{e}_{i,\alpha} = \mathrm{cl}^i(b_{i,\alpha})$ form dual bases for Poincaré duality.

Now we claim that we can choose the $a_{i,\alpha}$ and $b_{i,\alpha}$ so that they also satisfy

2) $\pi_k^t(a_{i,\alpha}) = \pi_k(b_{i,\alpha}) = 0$ for $k \neq 2i$; $\pi_{2i,j}^t(a_{i,\alpha}) = \pi_{2i,j}(b_{i,\alpha}) = 0$ for $j < i$.

To prove the claim, let $\pi' = \sum_{k\neq 2i} \pi_k$ and $\pi'' = \sum_{j<i} \pi_{2i,j}$: then π'_{hom} acts as 0 on $H^{2d-2i}(X)$ and π''_{hom} acts as 0 on $\bar{A}^{d-i}(X)$, so replacing $b_{i,\alpha}$ by $b_{i,\alpha} - (\pi' + \pi'')(b_{i,\alpha})$ we do not change the $\tilde{e}_{i,\alpha}$. Similarly for the $a_{i,\alpha}$.

Let us take $a_{i,\alpha}$ and $b_{i,\alpha}$ satisfying 1) and 2) and define

$$p_{i,\alpha} = a_{i,\alpha} \times b_{i,\alpha} \in A^d(X \times X).$$

We have $< a_{i,\alpha}, b_{i\alpha} >= 1$ and, by Lemma 7.7.6, the $p_{i,\alpha}$ are projectors and each motive $M_{i,\alpha} = (X, p_{i,\alpha}, 0)$ is isomorphic to $\mathbb{L}^i$. Moreover the $p_{i,\alpha}$ are pairwise orthogonal. Let $\pi_{2i}^{\mathrm{alg}} = \sum_{1\leq\alpha\leq\rho_i} p_{i,\alpha}$: by Condition 2) we have

$$\pi_k \circ \pi_{2i}^{\mathrm{alg}} = \pi_{2i}^{\mathrm{alg}} \circ \pi_k = 0 \text{ for } k \neq 2i;$$
$$\pi_{2i,j} \circ \pi_{2i}^{\mathrm{alg}} = \pi_{2i}^{\mathrm{alg}} \circ \pi_{2i,j} = 0 \text{ for } j < i.$$

Therefore

$$\pi_{2i,i} \circ \pi_{2i}^{\mathrm{alg}} = \pi_{2i}^{\mathrm{alg}} \circ \pi_{2i,i} = \pi_{2i}^{\mathrm{alg}}$$

and we can split $\pi_{2i,i}$ as a sum $\pi_{2i}^{\mathrm{alg}} + p$ of two orthogonal projectors. The theorem is proven. □

7.8. Return to birational motives

Throughout this section, we assume that k is perfect. We recover and strengthen some of the previous results in two steps:

(1) In §7.8.2 we show that, for a surface S provided with a refined C-K decomposition as in Propositions 7.2.1 and 7.2.3, the image of $t_2(S)$ under the full embedding of [Voev2, p. 197 and 3.2]

$$\Phi : \mathcal{M}_{\mathrm{rat}}^{\mathrm{eff}} \to DM_{\mathrm{gm}}^{\mathrm{eff}} \to DM_{-}^{\mathrm{eff}} \tag{7.13}$$

is a birational motive. See Theorem 7.8.4. This gives back some of the previous computations.

(2) In §7.8.3, we interpret the computation of the endomorphism ring of $t_2(S)$ as the existence of adjoints between certain categories of motives: see Theorem 7.8.8.

Finally, in §7.8.4 we show that "nothing more happens" for surfaces when we pass from the category of pure motives to Voevodsky's triangulated category of motives: see Corollary 7.8.13.

7.8.1. **Categorical trivialities.** Let $\mathcal{A}$ be a pseudo-abelian additive category and $\mathcal{B}$ be a thick subcategory of $\mathcal{A}$ (thick means full and closed under direct summands). To $\mathcal{B}$ one may associate the following ideal $\mathcal{I}$ of $\mathcal{A}$ (cf. [A-K, 1.3.1]):

$$\mathcal{I}(A, A') = \{f : A \to A' \mid f \text{ factors through an object of } \mathcal{B}\}.$$

Let $\mathcal{C} = (\mathcal{A}/\mathcal{I})^\natural$ be the pseudo-abelian envelope of the corresponding factor category, and $P : \mathcal{A} \to \mathcal{C}$ the corresponding projection functor. Recall that, for two objects $A, A' \in \mathcal{A}$,

$$\mathcal{C}(PA, PA') = \mathcal{A}(A, A')/\mathcal{I}(A, A').$$

Let us now define

$$\begin{aligned} {}^\perp\mathcal{I} &= \{A \in \mathcal{A} \mid \mathcal{I}(\mathcal{A}, A) = 0\} \\ &= \{A \in \mathcal{A} \mid \mathcal{A}(\mathcal{B}, A) = 0\} \\ &= \{A \in \mathcal{A} \mid \forall A' \in \mathcal{A}, P : \mathcal{A}(A', A) \xrightarrow{\sim} \mathcal{C}(PA', PA)\}. \end{aligned} \tag{7.14}$$

Note that ${}^\perp\mathcal{I}$ is stable under direct sums and direct summands: we view it as a thick subcategory of $\mathcal{A}$.

As usual, we say that "the" right adjoint of P is *defined* at an object $C \in \mathcal{C}$ if the functor

$$\mathcal{A} \ni A \mapsto \mathcal{C}(PA, C) \tag{7.15}$$

is representable. Let $\mathcal{C}'$ be the full subcategory of $\mathcal{C}$ consisting of such objects: it is a thick subcategory of $\mathcal{C}$.

The following is an abstraction of the arguments in [K-S, proof of 9.5]:

7.8.1. **Proposition.** *a) If $P^\#$ is "the" partial right adjoint of P (defined on $\mathcal{C}'$), then $P^\#(\mathcal{C}') \subseteq {}^\perp\mathcal{I}$.*
b) For any $C \in \mathcal{C}'$, the counit map of the adjunction

$$\varepsilon : PP^\# C \to C$$

is an isomorphism.
c) $\mathcal{C}'$ coincides with the essential image of $P' = P_{|{}^\perp\mathcal{I}}$.
d) For any $B \in {}^\perp\mathcal{I}$, the unit morphism

$$\eta : B \to P^\# PB$$

is an isomorphism. In particular, $P^{\#}(\mathcal{C}') = {}^{\perp}\mathcal{I}$ *and the functors*

$$P' : {}^{\perp}\mathcal{I} \to \mathcal{C}'$$
$$P^{\#} : \mathcal{C}' \to {}^{\perp}\mathcal{I}$$

form a pair of quasi-inverse equivalences of categories.

Proof. a) is obvious.

b) By definition of adjunction, for any $A \in \mathcal{A}$ the composition

$$\mathcal{A}(A, P^{\#}C) \xrightarrow{P} \mathcal{C}(PA, PP^{\#}C) \xrightarrow{\varepsilon_*} \mathcal{C}(PA, C)$$

is an isomorphism. Since $P^{\#}C \in {}^{\perp}\mathcal{I}$ by a), the left map is an isomorphism and hence so is the right one. It follows that, for any $C' \in \mathcal{C}$ (which may be written as a direct summand of PA for some A), the map

$$\mathcal{C}(C', PP^{\#}C) \xrightarrow{\varepsilon_*} \mathcal{C}(C', C)$$

is an isomorphism. By Yoneda's lemma, this implies that ε is an isomorphism.

c) Let for a moment $\mathcal{C}''$ denote the essential image of P'. If $C = PB \in \mathcal{C}''$, with $B \in {}^{\perp}\mathcal{I}$, then clearly the functor (7.15) is represented by B, so $\mathcal{C}'' \subseteq \mathcal{C}'$. Conversely, if $C \in \mathcal{C}'$, then $C \in \mathcal{C}''$ by a) and b).

d) Note that, by c), $P^{\#}PB$ is defined. Let $A \in \mathcal{A}$. As in any adjunction the composition

$$\mathcal{A}(A, B) \xrightarrow{\eta_*} \mathcal{A}(A, P^{\#}PB) \xrightarrow{\sim} \mathcal{C}(PA, PB)$$

is equal to P. Since it is an isomorphism, so is η_* and hence η is an isomorphism by Yoneda. The other conclusions follow immediately. □

7.8.2. **Corollary.** *P has an everywhere defined right adjoint if and only if $\mathcal{A} = \mathcal{B} \oplus {}^{\perp}\mathcal{I}$.* □

For future reference, we state the dual results (same proofs):

7.8.3. **Proposition.** *Let $\mathcal{I}^{\perp} = \{A \in \mathcal{A} \mid \mathcal{I}(A, \mathcal{A}) = 0\}$, viewed as a thick subcategory of $\mathcal{A}$. Then the thick subcategory $\mathcal{C}'$ of $\mathcal{C}$ of those objects where a left adjoint ${}^{\#}P$ of P is defined coincides with the essential image of $P' = P_{|\mathcal{I}^{\perp}}$; P' is an equivalence of categories, ${}^{\#}P(\mathcal{C}') = \mathcal{I}^{\perp}$ and ${}^{\#}P$ is a quasi-inverse of P'. Finally, ${}^{\#}P$ is everywhere defined if and only if $\mathcal{A} = \mathcal{B} \oplus \mathcal{I}^{\perp}$.* □

7.8.2. **$t_2(S)$ as a birational motive.** Recall from [Voev2] that the category $DM_-^{\text{eff}}(k)$ admits a partially defined internal Hom

$$\underline{\mathrm{Hom}} : DM_{\text{gm}}^{\text{eff}}(k) \times DM_-^{\text{eff}}(k) \to DM_-^{\text{eff}}(k)$$

which extends by $\mathbb{Q}$-linearity to an internal Hom

$$\underline{\mathrm{Hom}} : DM^{\mathrm{eff}}_{\mathrm{gm}}(k,\mathbb{Q}) \times DM^{\mathrm{eff}}_{-}(k)_{\mathbb{Q}} \to DM^{\mathrm{eff}}_{-}(k)_{\mathbb{Q}}.$$

This gives a meaning to the following

7.8.4. **Theorem.** *a) For any smooth projective variety X, one has*

$$\underline{\mathrm{Hom}}(\mathbb{Q}(1),\Phi(h_i(X))) = 0 \text{ for } i = 0,1$$

in $DM^{\mathrm{eff}}_{-} = DM^{\mathrm{eff}}_{-}(k,\mathbb{Q})$, where Φ is the full embedding of (7.13).
b) Let S be a surface provided with a refined C-K decomposition as in Propositions 7.2.1 and 7.2.3. Then

$$\underline{\mathrm{Hom}}(\mathbb{Q}(1),\Phi(t_2(S))) = 0.$$

Therefore, $\Phi(h_0(X)),\Phi(h_1(X))$ and $\Phi(t_2(S))$ belong to the image of the inclusion functor $i : DM^{\mathrm{o}}_{-} \to DM^{\mathrm{eff}}_{-}$, where $DM^{\mathrm{o}}_{-} := DM^{\mathrm{o}}_{-}(k)_{\mathbb{Q}}$ is the category of [K-S, 6.1].

Proof. We do the proof for $t_2(S)$: the other cases are similar and easier (reduce to X a curve).

By definition, $\underline{\mathrm{Hom}}(\mathbb{Q}(1),\Phi(t_2(S)))$ is a complex of sheaves on the category of smooth k-schemes provided with the Nisnevich topology. The fact that it is 0 may be checked locally; moreover, using [Voev1, Prop. 4.20], it suffices to check that for any function field extension K/k we have

$$\mathbb{H}^*(K,\underline{\mathrm{Hom}}(\mathbb{Q}(1),\Phi(t_2(S)))) = 0.$$

($\mathbb{H}^*$ denotes Nisnevich hypercohomology.)

Since $t_2(S)$ is a direct summand of $h(S)$, $\underline{\mathrm{Hom}}(\mathbb{Q}(1),\Phi(t_2(S)))$ is a direct summand of $\underline{\mathrm{Hom}}(\mathbb{Q}(1),M(S))$. By [H-K, Lemma B.1], we have an isomorphism

$$\underline{\mathrm{Hom}}(\mathbb{Q}(1),M(S)) \simeq \underline{\mathrm{Hom}}(M(S),\mathbb{Q}(1)[4]).$$

This isomorphism is induced by the duality isomorphism $M(S) \simeq M(S)^*(2)[4]$. The latter is the image under Φ of the duality isomorphism $\theta : h(S) \simeq h(S)^\vee(2)$ in $\mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}}$; since θ carries $t_2(S)$ to $t_2(S)^\vee(2)$, $\Phi(\theta)$ carries $\Phi(t_2(S))$ to $\Phi(t_2(S))^*(2)[4]$ and thus $\underline{\mathrm{Hom}}(\mathbb{Q}(1),\Phi(t_2(S)))$ is isomorphic to the direct summand $\underline{\mathrm{Hom}}(t_2(S),\mathbb{Q}(1)[4])$ of the complex $\underline{\mathrm{Hom}}(M(S),\mathbb{Q}(1)[4])$.

Let U be a smooth k-scheme. We have

$$\begin{aligned}\mathbb{H}^q_{Nis}(U,\underline{\mathrm{Hom}}(M(S),\mathbb{Q}(1)[4])) &\simeq H^{q+4}_{Nis}(U\times S,\mathbb{Q}(1))\\ &= H^{q+4}_{Zar}(U\times S,\mathbb{Q}(1)) = H^{q+3}_{Zar}(U\times S,\mathbb{G}_m)_{\mathbb{Q}}.\end{aligned}$$

Passing to the function field K of U we get that, for $q\in\mathbb{Z}$, the group $\mathbb{H}^q(K,\underline{\mathrm{Hom}}(\mathbb{Q}(1),t_2(S)))$ is a direct summand of $H^{q+3}_{Zar}(S_K,\mathbb{G}_m)_{\mathbb{Q}}$.

It is therefore 0 except perhaps for $q = -3, -2$. But for $q = -3$, $H^0(S_K, \mathbb{G}_m) = K^*$ is "caught" by the direct summand $\mathbf{1}$ of $h(S)$. For $q = -2$, $H^1(S_K, \mathbb{G}_m)_\mathbb{Q} = \mathrm{Pic}(S_K)_\mathbb{Q}$ decomposes into $\mathrm{Pic}^0(S_K)_\mathbb{Q} \oplus NS(S_K)_\mathbb{Q}$. The first summand is obtained from $h_1(S)$ and the second from $h(\underline{NS}_S)(1)$ as a direct summand of $h_2(S)$. Hence the vanishing. The last claim now follows from [K-S, 6.2]. □

7.8.5. **Corollary.** *Keep the notation of Theorem 7.8.4. Then*

(1) $h_0(X), h_1(X), t_2(S) \in {}^\perp\mathcal{I}$, *where $\mathcal{I}$ is the ideal of Lemma 7.5.1 and ${}^\perp\mathcal{I}$ is defined in* (7.14). *(Here, $\mathcal{A} = \mathcal{M}_{\mathrm{rat}}^{\mathrm{eff}}$, $\mathcal{B} = \mathcal{A} \otimes \mathbb{L}$.)*

(2) $h_i(X) \in \mathcal{K}_{\leq i-1}^\perp$ *($i = 0, 1$) and $t_2(S) \in \mathcal{K}_{\leq 1}^\perp$, where $\mathcal{K}_{\leq n}$ is the ideal of Definition 7.5.5 (iii) and $\mathcal{K}_{\leq n}^\perp$ is defined in Proposition 7.8.3.*

Moreover, for any smooth projective variety Y of dimension d, one has

$$\mathcal{M}_{\mathrm{rat}}^{\mathrm{eff}}(h(Y), h_0(X)) \simeq A_0^{\mathrm{num}}(X_{k(Y)})$$
$$\mathcal{M}_{\mathrm{rat}}^{\mathrm{eff}}(h(Y), h_1(X)) \simeq \mathrm{Alb}_X(k(Y))$$
$$\mathcal{M}_{\mathrm{rat}}^{\mathrm{eff}}(h(Y), t_2(S)) \simeq T(S_{k(Y)}).$$

Proof. (1) is just a special case of Theorem 7.8.4 by the full faithfulness of (7.13), via Proposition 7.5.3. (2) follows from (1) by duality. For the isomorphisms, let us treat the case of $t_2(S)$. We first observe that

$$\mathcal{M}_{\mathrm{rat}}^{\mathrm{eff}}(\mathbf{1}, t_2(S)) = T(S) \tag{7.16}$$

(see Proposition 7.2.3). Then the isomorphism follows from (1), Proposition 7.5.3 and (7.16) applied over the function field of Y. The cases of $h_0(X)$ and $h_1(X)$ are similar. □

7.8.6. **Corollary.** *a) Let $P : \mathcal{M}_{\mathrm{rat}}^{\mathrm{eff}} \to \mathcal{M}_{\mathrm{rat}}^{\mathrm{o}}$ denote the projection functor. Then the right adjoint $P^\#$ of P is defined on $d_{\leq 2}\mathcal{M}_{\mathrm{rat}}^{\mathrm{o}}$.*
b) Let $S : \mathcal{M}_{\mathrm{rat}}^{\mathrm{o}} \to (\mathcal{M}_{\mathrm{rat}}^{\mathrm{o}}/\mathcal{K}_{\leq 1}^{\mathrm{o}})^\natural$ be the projection functor. The the left adjoint ${}^\# S$ of S is defined on the thick image of $d_{\leq 2}\mathcal{M}_{\mathrm{rat}}^{\mathrm{o}}$ by S.

Proof. a) Consider the thick subcategory $d_{\leq 2}^{\mathrm{o}}\mathcal{M}_{\mathrm{rat}}^{\mathrm{eff}}$ of $\mathcal{M}_{\mathrm{rat}}^{\mathrm{eff}}$ generated by those motives of the form $h_0(X), h_1(X)$ and $t_2(S)$ as in Theorem 7.8.4. Clearly $P(d_{\leq 2}^{\mathrm{o}}\mathcal{M}_{\mathrm{rat}}^{\mathrm{eff}}) = d_{\leq 2}\mathcal{M}_{\mathrm{rat}}^{\mathrm{o}}$, and Corollary 7.8.5 (1) gives the inclusion $d_{\leq 2}^{\mathrm{o}}\mathcal{M}_{\mathrm{rat}}^{\mathrm{eff}} \subset {}^\perp\mathcal{I}$. The conclusion now follows from Proposition 7.8.1.

b) The proof is the same, using this time Corollary 7.8.5 (2) (or rather its projection into $\mathcal{M}_{\mathrm{rat}}^{\mathrm{o}}$) and Proposition 7.8.3. □

7.8.7. *Remark.* It is natural to ask what is the largest full subcategory of $\mathcal{M}_{\mathrm{rat}}^{\mathrm{o}}$ on which $P^\#$ is defined. We don't know the answer to this question but at least, $P^\#$ is *not defined on* $\bar{h}(X)$ for any 3-fold X

such that the group $A^2_{\mathrm{alg}}(X)$ of codimension 2 cycles modulo algebraic equivalence is not finitely generated (cf. Griffiths' examples). This will be proven in the final version of [K-S], the core of the argument being due to Joseph Ayoub.

On the other hand we expect that the functor S_n of (7.18) below always has a left adjoint: this will be the object of a further work.

7.8.3. **Birational motives and motives at the generic point.** Let

$$d_n\mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}} = \left(\frac{d_{\le n}\mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}}}{\mathcal{K}_{\le n-1}\cap d_{\le n}\mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}}}\right)^\natural \tag{7.17}$$

$$d_n\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}} = \left(\frac{d_{\le n}\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}}{\mathcal{K}^{\mathrm{o}}_{\le n-1}\cap d_{\le n}\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}}\right)^\natural$$

where $^\natural$ means taking the pseudo-abelian envelope (cf. Definition 7.5.5 for the definitions of the objects appearing in (7.17).) We thus have a diagram of categories and functors

$$\begin{array}{ccc} d_{\le n}\mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}} & \xrightarrow{P_n} & d_{\le n}\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}} \\ {\scriptstyle R_n}\downarrow & & {\scriptstyle S_n}\downarrow \\ d_n\mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}} & \xrightarrow{Q_n} & d_n\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}. \end{array} \tag{7.18}$$

(With a previous notation, $P_n(M) = \bar{M}$ for $M \in d_{\le n}\mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}}$.)

Note that the duality $D^{(n)}$ of Lemma 7.5.6 acts on (7.18) by exchanging the categories $d_{\le n}\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}$ and $d_n\mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}}$ which are therefore anti-equivalent, and also induces a duality on $d_n\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}$.

If $\dim X \le n$, then we write $h_{\mathrm{gen}}(X)$ for $S_n(\bar{h}(X)) = S_nP_n(h(X))$: this is the *motive of X at the generic point* (relative to dimension n). We have $h_{\mathrm{gen}}(X) = 0$ if $\dim X < n$. Lemma 7.5.6 c) shows that, for two n-dimensional smooth projective k-varieties X, Y,

$$d_n\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}(h_{\mathrm{gen}}(X), h_{\mathrm{gen}}(Y)) = A_n(X\times Y)/\mathcal{J}(X,Y) \tag{7.19}$$

where $\mathcal{J}$ is the ideal of Definition 7.4.2.

The name "motive at the generic point" is in reference to Beilinson's paper [Bei], where he calls the right hand side of (7.19) *correspondences at the generic point.* His prediction (conjecture) (*) on p. 35[6], deduced from some "standard" conjectures on mixed motives, implies that, for $X = Y$, *this $\mathbb{Q}$-algebra is semi-simple finite-dimensional and that $A_n(X\times X)_{\mathrm{hom}} \subset \mathcal{J}(X,X)$.* (Compare with Theorem 7.6.12 (7) in the case of a surface.)

[6]Also due to Rovinski and Bloch as he points out.

7.8.8. **Theorem.** *a) Suppose that $n \leq 2$. In (7.18), P_n and Q_n have a right adjoint while R_n and S_n have a left adjoint. All these adjoints are right inverse to the corresponding functors. In particular, the functor*

$$S_n P_n = Q_n R_n : d_{\leq n}\mathcal{M}_{\mathrm{rat}}^{\mathrm{eff}} \to d_n \mathcal{M}_{\mathrm{rat}}^{\mathrm{o}}$$

has a canonical section

$$\Sigma_n = P_n^{\#} \circ {}^{\#}S_n = {}^{\#}R_n \circ Q_n^{\#}.$$

b) Suppose $n = 1$: if C is a smooth projective curve, then for any C-K decomposition of C we have

$$\begin{aligned} P_1^{\#}\bar{h}(C) &\simeq h_0(C) \oplus h_1(C) \\ Q_1^{\#}h_{\mathrm{gen}}(C) &\simeq R_2(h_1(C)) \\ {}^{\#}R_1 R_1(h(C)) &\simeq h_1(C) \oplus h_2(C) \\ {}^{\#}S_1 h_{\mathrm{gen}}(C) &\simeq \bar{h}_1(C) \\ \Sigma_1(h_{\mathrm{gen}}(C)) &\simeq h_1(C). \end{aligned}$$

c) Suppose $n = 2$: if S is a smooth projective surface, then for any refined C-K decomposition of S as in Propositions 7.2.1 and 7.2.3, we have

$$\begin{aligned} P_2^{\#}\bar{h}(S) &\simeq h_0(S) \oplus h_1(S) \oplus t_2(S) \\ Q_2^{\#}h_{\mathrm{gen}}(S) &\simeq R_2(t_2(S)) \\ {}^{\#}R_2 R_2(h(S)) &\simeq t_2(S) \oplus h_3(S) \oplus h_4(S) \\ {}^{\#}S_2 h_{\mathrm{gen}}(S) &\simeq \bar{t}_2(S) \\ \Sigma_2(h_{\mathrm{gen}}(S)) &\simeq t_2(S). \end{aligned}$$

Note that, as a composite of a left and a right adjoint, Σ_n has no special adjunction property.

Proof. a) For P_n and S_n this follows immediately from Corollary 7.8.6; the cases of Q_n and R_n follow by using the duality $D^{(n)}$.

b) and c) Let us prove the first formula in c): the other cases are similar. Writing $h(S) = \bigoplus_{i=0}^4 h_i(S)$, we have $\bar{h}(S) \simeq \bar{h}_0(S) \oplus \bar{h}_1(S) \oplus \bar{t}_2(S)$ (see proof of Lemma 7.5.7). By Corollary 7.8.5 (1), $h_0(S) \oplus h_1(S) \oplus t_2(S) \in {}^{\perp}\mathcal{I}$; the conclusion then follows from Proposition 7.8.1 d). □

7.8.9. **Corollary.** *The functors P_n, Q_n, R_n, S_n are essentially surjective for $n \leq 2$.* □

(This fact is not obvious a priori since we added projectors when defining the quotient categories: it amounts to saying that this operation was not necessary.)

From Theorem 7.8.8 a), we have for $n \leq 2$ a commutative diagram of natural transformations in $d_{\leq n}\mathcal{M}_{\text{rat}}^{\text{eff}}$:

$$\begin{array}{ccc} Id & \longrightarrow & P_n^\# P_n \\ \uparrow & & \uparrow \\ R_n^\# R_n & \longrightarrow & R_n^\# Q_n^\# Q_n R_n = P_n^\# S_n^\# S_n P_n \end{array}$$

given by the units and counits of the respective adjunctions. Applying this diagram to $h(C)$ (resp. $h(S)$) for C a curve (resp. S a surface) and using Theorem 7.8.8 b) (resp. c)), we get the following corollary, which gives a partial positive answer to Conjecture 7.3.4:

7.8.10. **Corollary.** *c) Given a curve C, in the diagram*

$$\begin{array}{ccc} h(C) & \longrightarrow & h_0(C) \oplus h_1(C) \\ \uparrow & & \uparrow \\ h_1(C) \oplus h_2(C) & \longrightarrow & h_1(C) \end{array}$$

all maps and objects are independent of the choice of a C-K decomposition, and are natural in C for the action of correspondences.
b) Given a surface S, in the diagram

$$\begin{array}{ccc} h(S) & \longrightarrow & h_0(S) \oplus h_1(S) \oplus t_2(S) \\ \uparrow & & \uparrow \\ t_2(S) \oplus h_3(S) \oplus h_4(S) & \longrightarrow & t_2(S) \end{array}$$

all maps and objects are independent of the choice of a refined C-K decomposition as in Propositions 7.2.1 and 7.2.3, and are natural in S for the action of correspondences. □

In the case of a surface S, we may think of $h_{bir}(S) := P_2^\# \bar{h}(S)$ as the *largest birational quotient* of $h(S)$ and think of $\Sigma_2(h_{\text{gen}}(S))$ as the *largest submotive at the generic point* of $h_{bir}(S)$. Similarly, $R_2^\# R_2(h(S))$ may be thought of as the largest subobject of $h(S)$ "purely of dimension 2". Note that both maps $h(S) \to t_2(S)$ and $t_2(S) \to h(S)$ given by a projector π_2^{tr} from a refined C-K decomposition do depend on the choice of this decomposition. Nevertheless, it is unambiguous to write $t_2(S)$ for $\Sigma_2(h_{\text{gen}}(S))$, viewed as a functor in S.

7.8.11. **Corollary.** *Let $d_{\leq 2}Sm$ be the category of smooth (open) varieties of dimension ≤ 2 over k. Assume that k is of characteristic 0. There are functors*

$$h_{bir}, t_2 : d_{\leq 2}Sm \to \mathcal{M}_{\text{rat}}^{\text{eff}}$$

extending the above functors. These functors are (stably) birationally invariant. There are similar contravariant functors starting from the category $d_{\leq 2}$place of function fields of transcendence degree ≤ 2 over k, with morphisms the k-places.

Proof. By [K-S, 5.6] there are canonical functors

$$d_{\leq 2}T^{-1}\text{place}^{op} \to d_{\leq 2}S_r^{-1}Sm \to d_{\leq 2}\mathcal{M}_{\text{rat}}^{\text{o}}$$

the latter extending the natural functor from smooth projective varieties. Here T^{-1}place denotes the category of finitely generated extensions of k with morphisms the k-places, localized by inverting morphisms of the form $K \hookrightarrow K(t)$, $S_r^{-1}Sm$ denotes the category of smooth k-varieties localized by inverting the dominant morphisms which induce a purely transcendental extension of function fields, and $d_{\leq 2}$ denotes the full subcategories respectively of function fields of transcendence degree ≤ 2 and of varieties of dimension ≤ 2. □

7.8.4. **The triangulated birational motive of a surface.** Suppose k perfect. Recall from [K-S] that the projection functor $P : \mathcal{M}_{\text{rat}}^{\text{eff}} \to \mathcal{M}_{\text{rat}}^{\text{o}}$ inserts into a naturally commutative diagram of categories and functors

$$\begin{array}{ccc} \mathcal{M}_{\text{rat}}^{\text{eff}} & \xrightarrow{\Phi} & DM_{-}^{\text{eff}} \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle \nu_{\leq 0}} \\ \mathcal{M}_{\text{rat}}^{\text{o}} & \xrightarrow{\bar{\Phi}} & DM_{-}^{\text{o}} \end{array}$$

where DM_{-}^{o} is a birational analogue of DM_{-}^{eff}, and that $\nu_{\leq 0}$ has an everywhere-defined right adjoint/right inverse i (in fact DM_{-}^{o} is a priori defined as the full subcategory of DM_{-}^{eff} consisting of those objects C such that $\underline{\text{Hom}}(\mathbb{Q}(1), C) = 0$, and it is then proven that the inclusion functor i has a left adjoint $\nu_{\leq 0}$ which inserts itself in the above commutative diagram). If X is a smooth projective variety, we set

$$\bar{M}(X) = \bar{\Phi}\bar{h}(X) = \nu_{\leq 0}M(X) \in DM_{-}^{\text{o}}.$$

Suppose that $P^{\#}$ is defined at some Chow birational motive $\bar{M} \in \mathcal{M}_{\text{rat}}^{\text{o}}$. Starting from the natural isomorphism $\nu_{\leq 0}\Phi = \bar{\Phi}P$, the two adjunctions give a "base change" morphism

$$\Phi P^{\#}\bar{M} \to i\nu_{\leq 0}\Phi P^{\#}\bar{M} = i\bar{\Phi}PP^{\#}\bar{M} \to i\bar{\Phi}\bar{M}. \tag{7.20}$$

7.8.12. **Proposition.** *Let $\bar{M} \in \mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}$ be such that $P^{\#}\bar{M}$ is defined. Then the following conditions are equivalent:*

(i) (7.20) *is an isomorphism.*
(ii) $\Phi P^{\#}\bar{M}$ *is in the essential image of* i *(i.e.* $\underline{\mathrm{Hom}}(\mathbb{Q}(1), \Phi P^{\#}\bar{M}) = 0$*).*

Proof. (i) ⇒ (ii) is obvious. Conversely, suppose that $\Phi P^{\#}\bar{M} \simeq iN$ for some $N \in DM^{\mathrm{eff}}_{-}$. Since i is right inverse to $\nu_{\le 0}$, we find first

$$N \simeq \nu_{\le 0} iN \simeq \nu_{\le 0}\Phi P^{\#}\bar{M} \simeq \bar{\Phi} PP^{\#}\bar{M}.$$

Since i is fully faihtful, the morphism $iN \to i\bar{\Phi}\bar{M}$ comes from a unique morphism $\bar{\Phi} PP^{\#}\bar{M} \simeq N \to \bar{\Phi}\bar{M}$, which clearly is the morphism induced by the counit $PP^{\#}\bar{M} \to \bar{M}$. This counit is an isomorphism by Proposition 7.8.1 b), hence (7.20) is an isomorphism. □

By Theorem 7.8.4 and Corollary 7.8.6 a), $P^{\#}\bar{M}$ is defined and Condition (ii) of Proposition 7.8.12 is verified for all $\bar{M} \in d_{\le 2}\mathcal{M}^{\mathrm{o}}_{\mathrm{rat}}$. Hence (i) holds for them. Taking $\bar{M} = \bar{h}(C)$ ($\bar{h}(S)$) for a curve C (a surface S), we get:

7.8.13. **Corollary.** *a) For any curve C, $i\bar{M}(C) \simeq \Phi(P^{\#}\bar{h}(C))$. Given a C-K decomposition, this motive is isomorphic to $M_0(C) \oplus M_1(C)$.*
b) For any surface S, $i\bar{M}(S) \simeq \Phi(P^{\#}\bar{h}(S))$. Given a C-K decomposition as in Propositions 7.2.1 and 7.2.3, this motive is isomorphic to $M_0(S) \oplus M_1(S) \oplus \Phi(t_2(S))$. □

In particular, $i\bar{M}(S)$ is a direct summand of $M(S)$ for any surface S, which answers positively a question of Ayoub.

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