ON UNIVERSAL MODULAR SYMBOLS

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Abstract. We clarify the relationship between works of Lee-Szczarba and Ash-Rudolph on the homology of the Steinberg module of a linear Tits building. This yields a simple proof of the Solomon-Tits theorem in this special case. We also give a (weak) relationship between this combinatorics and the one studied by van der Kallen, Suslin and Nesterenko to compute the homology of the general linear group with constant coefficients.

Introduction

In two related papers [5, 1], Lee-Szczarba and Ash-Rudolph study the homology of the Steinberg module of a Tits building by means of a canonical resolution [5, Th. 3.1] and an explicit set of generators called *universal modular symbols* [1, Prop. 2.3 and Th. 4.1]. A first purpose of this note is to clarify the relationship between the two approaches: we shall show in Theorem 2 that the generators provided by Lee and Szczarba coincide with the universal modular symbols of Ash and Rudolph: this answers a question asked in [1, end of introduction].

For this, we offer in Theorem 1 a shorter proof of Lee-Szczarba's Theorem 3.1, which has the advantage to generalise from principal ideal domains to any integral domain A. As a byproduct, we get in Corollary 1 a short proof of the Solomon-Tits theorem for GL_n. We use the categorical techniques of Quillen [7, §1].

Finally, we give in Proposition 3 a (rather disappointing) relationship between the Lee-Szczarba resolution and the complexes used by van der Kallen, Suslin and Nesterenko to study the homology of the general linear group of an infinite field.

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1. On the universal modular symbol for $n = 2$

Let us review the Ash-Rudolph construction of the universal modular symbol in [1, §2]. For coherence with the rest of this paper, we adopt a

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slightly different notation from theirs. For $r \geq 0$, let Δ_r be the standard (abstract) simplicial complex based on the set $[r] = \{0, \dots, r-1\}$: the simplices of Δ_r are the nonempty subsets of $[r]$. Let $\operatorname{sd}\Delta_r$ denote the first barycentric subdivision of Δ_r: the vertices of $\operatorname{sd}\Delta_r$ are the simplices of Δ_r and the simplices of $\operatorname{sd}\Delta_r$ are the nonempty sets of simplices of Δ_r which are totally ordered by inclusion (we shall call such a set a *flag* of simplices). Its boundary $\partial \operatorname{sd}\Delta_r$ is the full subcomplex whose vertices are the nonempty proper subsets of $[r]$.

Let now V be an n-dimensional vector space over a field K, with $n \geq 2$. The Tits building of V, denoted by $T(V)$, is the simplicial complex whose vertices are the (nonzero) proper subspaces of V and simplices are flags of proper subspaces. It has the homotopy type of a bouquet of $(n-2)$-spheres by the Solomon-Tits theorem ([9], [8, §2]; see also Corollary 1 below). Its $(n-2)$-th homology group is called the Steinberg module of V and denoted by $\operatorname{St}(V)$.

Let $Q = (v_0, \dots, v_{n-1})$ be a sequence of n nonzero vectors of V. It defines a simplicial map

$$\varphi_Q : \partial \operatorname{sd}\Delta_{n-1} \to T(V)$$

by sending each vertex $I \subsetneq [n-1]$ to $\langle v_i \rangle_{i \in I}$. For $n > 2$, the universal modular symbol $[v_0, \dots, v_{n-1}] \in \operatorname{St}(V)$ is defined as $(\varphi_Q)_*\zeta$, where $\zeta \in H_{n-2}(\partial \operatorname{sd}\Delta_{n-1})$ is the fundamental class corresponding to the canonical orientation of $\operatorname{sd}\Delta_{n-1}$. By [1, Prop. 2.2] the symbol $[Q] = [v_0, \dots, v_{n-1}]$ satisfies the following relations:

(a) It is anti-symmetric (transposition of two vectors changes the sign of the symbol).
(b) It is homogeneous of degree zero: $[av_0, \dots, v_{n-1}] = [v_0, \dots, v_{n-1}]$ for any nonzero $v_0, \dots, v_{n-1}$.
(c) $[Q] = 0$ if $\det Q = 0$.
(d) If $v_0, \dots, v_n$ are all non-zero, then

$$\sum_{i=0}^{n} (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_n] = 0.$$

(e) If $A \in GL(V)$, then $[AQ] = A \cdot [Q]$, the dot denoting the natural action of $GL(V)$ on $\operatorname{St}(V)$.

By [1, Prop. 2.3], the universal modular symbols generate $\operatorname{St}(V)$ (for $n > 2$). Relations (a) – (d) actually *present* $\operatorname{St}(V)$ (Corollary 2).

Let us look at the case $n = 2$. Then $T(V)$ is the discrete set of lines of V, hence $\operatorname{St}(V) = H_0(T(V))$ is the free $\mathbf{Z}$-module over this basis. The first problem is a definition of the "fundamental class" of the non connected discrete space $\partial \operatorname{sd}\Delta_1$. This space consists of the points $0, 1,$

which form a basis of $H_0(\partial\,\mathrm{sd}\,\Delta_1)$. If $\underline{v} = (v_0, v_1) \in (V - \{0\})^2$, $\varphi_{\underline{v}}(i)$ is the line generated by v_i. The proof of Relation (c) above given on top of *loc. cit.*, p. 244 is correct for $n > 2$, but breaks down for $n = 2$ since then $H_0(\partial\Delta_1) \neq 0$. If we want to save this relation, we must make the right choice of the fundamental class: namely, $\zeta = [1] - [0] \in H_0(\partial\Delta_1)$. But then $[v_0, v_1] = [v_1] - [v_0]$, which is in the kernel of the augmentation $\mathrm{St}(V) \to \mathbf{Z}$ sending each line to 1. hence the symbols $[v_0, v_1]$ do not generate $\mathrm{St}(V)$, but rather the "reduced Steinberg module" $\widetilde{\mathrm{St}}(V) = \mathrm{Ker}(\mathrm{St}(V) \to \mathbf{Z})$.

The above mistake is compounded by a parallel error a little further: in [1, Def. 3.1], the second isomorphism does not exist for $n = 2$ (the first author is indebted to Loïc Merel for pointing this out). The map goes the other way and yields an exact sequence

$$0 = H_1(\bar{X}) \to H_1(\bar{X}, \partial\bar{X}) \to H_0(\partial\bar{X}) \to H_0(\bar{X}) = \mathbf{Z}$$

which gives an isomorphism $H_0(\partial\bar{X}) \xrightarrow{\sim} \widetilde{\mathrm{St}}(V)$. This saves [1, Prop. 3.2] for $n = 2$. (In its proof, l. 4 one should read "surjective" instead of "injective".)

In the sequel, we shall write

$$\widetilde{\mathrm{St}}(V) = \begin{cases} \mathrm{St}(V) & \text{if } n > 2 \\ \mathrm{Ker}(\mathrm{St}(V) \to \mathbf{Z}) & \text{if } n = 2 \\ \mathbf{Z} & \text{if } n = 1 \\ \mathbf{Z} & \text{if } n = 0. \end{cases} \tag{1}$$

2. Categories and functors

We shall work with essentially 4 categories:

- **Set**, the category of (small) sets.
- **Ord**, the category of partially ordered sets. Recall that, as in Quillen [7], we may think of a poset as a category.
- **Spl**, the category of abstract simplicial complexes.
- **Top**, the category of topological spaces.

There are various functors between these categories: we write

- E : **Set** $\to$ **Spl** for the functor which sends a set X to the simplicial complex of nonempty finite subsets of X.
- B : **Ord** $\to$ **Spl** for the functor sending a poset to the simplicial complex of its totally ordered nonempty finite subsets.
- Simpl : **Spl** $\to$ **Ord** for the functor which associates to a simplicial complex the set of its simplices ordered by inclusion.
- $|\ |$: **Spl** $\to$ **Top** for the geometric realisation functor [10, 3.1].

For any set X, we have $\operatorname{Simpl} E(X) = \mathcal{P}_f(X)$, the poset of nonempty finite subsets of X. If $[n] \in \mathbf{Set}$ is the set $\{0, 1, \ldots, n-1\}$, then $E([n]) = \Delta_n$, the standard n-simplex.

If $\omega : \mathbf{Ord} \to \mathbf{Set}$ is the forgetful functor, there is an obvious natural transformation

$$\rho : B \Rightarrow E \circ \omega$$

and ρ_S is an *isomorphism* if S is totally ordered, for example if $S = [n]$.

Note also that $B \circ \operatorname{Simpl} = \operatorname{sd}$ is the functor "subdivision" on simplicial complexes (remark of Segal to Quillen, [7, p. 89]).

Finally, we note the natural transfrmation

$$\theta : \operatorname{Simpl} \circ B \Rightarrow Id_{\mathbf{Ord}}$$

such that for $S \in \mathbf{Ord}$, θ_S maps $\sigma \in \operatorname{Simpl} B(S)$ to $\sup(\sigma) \in S$. Applying B on the left, we get a natural transformation

$$B * \theta : \operatorname{sd} B \Rightarrow B.$$

Applying this to $S = [n]$, we get a canonical map

$$\operatorname{sd} \Delta_n \to \Delta_n \tag{2}$$

which is natural for morphisms in $\mathbf{Ord}$ and induces a homotopy equivalence of (contractible) spaces after geometric realisation. From the definition of the latter, it extends to a homotopy equivalence

$$\varepsilon_\Gamma : |\operatorname{sd} \Gamma| \xrightarrow{\sim} |\Gamma| \tag{3}$$

which is natural in $\Gamma \in \mathbf{Spl}$.

For any $S \in \mathbf{Ord}$, $|B(S)|$ is naturally homeomorphic to $|N(S)|$, where $N(S)$ is the nerve of the category S; conversely, if $\Gamma \in \mathbf{Spl}$, the relation $B \circ \operatorname{Simpl} = \operatorname{sd}$ and (3) yield a natural homotopy equivalence $|N(\operatorname{Simpl}(\Gamma))| \xrightarrow{\sim} |\Gamma|$ (compare [7, p. 89]). Thus we can work equivalently with simplicial complexes or posets, and use Quillen's techniques from [7] when dealing with the latter. Following the practice in [7] and [8], we shall say that a poset, a simplicial complex, or a morphism in $\mathbf{Ord}$ or $\mathbf{Spl}$ have a certain homotopical property if their topological realisations have.

Remark 1 (J. Riou). The morphism ε_Γ of (3) is not a homeomorphism in general, as the example $\Gamma = \Delta_1$ shows. On the other hand, the homeomorphism $|\operatorname{sd} \Gamma| \approx |\Gamma|$ constructed in [10, 3.3] is not natural in Γ, as seen by considering the morphism $\Delta_2 \to \Delta_1$ identifying the vertices $1, 2$.

The naturality of (3) is critical for the proof of Theorem 2 below.

3. Some well-known lemmas

Lemma 1. *$E(X)$ is contractible if X is nonempty.*

Proof. Here is one "à la Quillen" (it is a version of the proof for simplicial sets):

Let $x \in X$ and let $\mathcal{P}_f(X)_x$ be the subset of $\mathcal{P}_f(X)$ consisting of those finite subsets that contain x. This poset has a smallest element $\{x\}$, hence is contractible. But the inclusion $\mathcal{P}_f(X)_x \subset \mathcal{P}_f(X)$ (viewed as a functor) has the left adjoint $Y \mapsto Y \cup \{x\}$. □

If $K \in \mathbf{Spl}$ and $r \geq 0$, we denote by $\operatorname{Sk}^r K$ its r-th skeleton: it has the same vertices as K and its simplices are the simplices of K of dimension $\leq r$.

Lemma 2. *Let $\Gamma \in \mathbf{Spl}$, and let v be a vertex of Γ. Then, for any $r \geq 0$, the map $\pi_i(\operatorname{Sk}^r \Gamma, v) \to \pi_i(\Gamma, v)$ is bijective for $i < r$ and surjective for $i = r$.*

Proof. An equivalent statement is: $\pi_i(\Gamma, \operatorname{Sk}^r \Gamma) = 0$ for $i \leq r$. But the pair $(|\operatorname{Sk}^{r+1} \Gamma|, |\operatorname{Sk}^r \Gamma|)$ is r-connected by [10, Ch. 7, §6, Lemma 15]. By induction on s this gives $\pi_i(\operatorname{Sk}^{r+s} \Gamma, \operatorname{Sk}^r \Gamma) = 0$ for $i \leq r$ and any $s \geq 1$, hence the conclusion in the limit. □

Lemma 3. *Let X be a r-dimensional CW-complex which is $(r-1)$-connected. Then X has the homotopy type of a bouquet of r-spheres.*

Since we could not find a reference for this classical fact, here is a proof: si $r \leq 1$, the statement is easy. If $r \geq 2$, the homology exact sequence

$$0 = H_r(\operatorname{Sk}^{r-1} X) \to H_r(X) \to H_r(X, \operatorname{Sk}^{r-1} X)$$

injects $H_r(X)$ in the homology of a bouquet of r-spheres (see previous proof), showing that this group is *free*[1]. Let $(e_i)_{i \in I}$ be a basis of $\pi_r(X, x) \xrightarrow{\sim} H_r(X)$, where x is some base-point, hence a map

$$f : \bigvee_{i \in I} S^r \to X$$

which is an isomorphism on H_r, hence a homology equivalence, hence a homotopy equivalence (Whitehead's theorem, [10, Ch. 7, §5, Th. 9]). □

Lemma 4. *Let $\Gamma \in \mathbf{Spl}$. If Γ is contractible, then $\operatorname{Sk}^r \Gamma$ has the homotopy type of a bouquet of r-spheres for any $r \geq 0$. Moreover,*

[1] The first author thanks G. Masbaum for showing him this argument.

$H_r(\mathrm{Sk}^r\,\Gamma)$ *is the r-th homology group of the (naïvely) truncated complex* $\sigma^{\leq r}\mathrm{Or}_*(\Gamma)$*, where* $\mathrm{Or}_*(\Gamma)$ *is the oriented chain complex of* Γ [10, pp. 158–159].

Proof. The first statement follows from Lemmas 2 and 3. For the second one, we have $\mathrm{Or}_*(\mathrm{Sk}^r\,\Gamma) = \sigma^{\leq r}\mathrm{Or}_*(\Gamma)$ tautologically. □

4. A homotopy equivalence

Let A be an Noetherian domain with quotient field K, and let M be a torsion-free finitely generated A-module. Write $V = K \otimes_R M$, so that M is a lattice in V: we assume $\dim V = n \geq 2$.

A submodule N of M is *pure* if M/N is torsion-free. Let $G^*(M)$ be the poset of proper pure submodules of M (those different from 0 and M). For $A = K$ we have $BG^*(V) = T(V)$ by definition, and by [4, Prop. 4.2.4], the map $N \mapsto K \otimes_R N$ yields a bijection

$$G^*(M) \xrightarrow{\sim} G^*(V).$$

If $N \subset M$ is a submodule, the *saturation* of N is the smallest pure submodule N_{sat} of M which contains N: it can be constructed as the kernel of the composition

$$M \to M/N \to (M/N)/\mathrm{torsion}.$$

The following lemma is tautological:

Lemma 5. *Let $N \subseteq M$ be a pure submodule, and let P be a submodule of N. Then $P_{\mathrm{sat}} \subseteq N$.* □

The *rank* of a subset X of M is the dimension of the subvector space of V generated by X. We write $E^*(M)$ for the set of nonempty finite subsets of rank $< n$ in $M - \{0\}$, viewed as a sub-simplicial complex of $E(M - \{0\})$. We then have a non-decreasing map:

$$\begin{aligned} \mathrm{AR} : \mathrm{Simpl}\, E^*(M) &\to G^*(M) \\ Y &\mapsto \langle Y \rangle_{\mathrm{sat}}. \end{aligned} \tag{4}$$

We take Quillen's viewpoint in [7] and consider AR as a functor between the corresponding categories.

Theorem 1. AR *is a homotopy equivalence.*

Proof. (Compare [5, proof of Prop. 3.2].) For $N \in G^*(M)$, we have by Lemma 5

$$\mathrm{AR}/N = \mathcal{P}_f(N - \{0\})$$

which is contractible (Lemma 1). Apply [7, Th. A]. □

Corollary 1 (Solomon-Tits)**.** *$T(V)$ has the homotopy type of a bouquet of $(n-2)$-spheres.*

Proof. We choose $A = K$ in Theorem 1. On the one hand, the p-chains of $E^*(V)$ and $E(V - \{0\})$ coincide for $p \leq n-2$, hence $T(V)$ is $(n-3)$-connected by Lemmas 1 and 2. On the other hand, $\dim T(V) \leq n-2$. We conclude with Lemma 3. □

5. The case of a principal ideal domain

Keep the notation of the previous section. An element $v \in M$ is *unimodular* if there exists a linear form $\theta : M \to A$ such that $\theta(v) = 1$. We write $U(M)$ for the set of unimodular vectors of M.

Lemma 6. *If A is principal, $U(M) \cap N$ is nonempty for any nonzero pure submodule $N \subseteq M$.*

Proof. It suffices to prove this when N has rank 1. Then N is free, with generator v. Since M/N is torsion-free, it is free, hence N is a direct summand in M. This readily implies that v is unimodular. □

If A is principal, let $U^*(M)$ be the set of nonempty finite subsets of rank $< n$ in $U(M)$: this is a sub-simplicial complex of $E^*(M)$.

Proposition 1. *The restriction AR^u of the functor AR of (4) to $\mathrm{Simpl}\, U^*(M)$ is a homotopy equivalence.*

Proof. Same as for Theorem 1, using Lemma 6: here, $\mathrm{AR}^u/N = \mathcal{P}_f(U(M) \cap N)$. □

6. Comparison of the Ash-Rudolph and Lee-Szczarba constructions

From (3) we get a zig-zag of isomorphisms

$$H_{n-2}(E^*(M)) \xleftarrow{\sim} H_{n-2}(\mathrm{sd}\, E^*(M)) \xrightarrow{\sim} \mathrm{St}(V) \tag{5}$$

induced by $B(\mathrm{AR})$ and $\varepsilon_{E^*(M)}$.

The singular chain complex of $E(M - \{0\})$ is given by

$$C_p(E(M - \{0\})) = \mathbf{Z}[(v_0, \dots, v_p) \mid v_i \in M - \{0\}].$$

That of $E^*(M)$ is given by

$$C_p(E^*(M)) = \mathbf{Z}[(v_0, \dots, v_p) \mid \mathrm{rk}\langle v_0, \dots, v_p\rangle < n].$$

Write $\bar{C}_* = C_*(E(M - \{0\}), E^*(M))$ for the quotient complex. As $E(M - \{0\})$ is contractible, we have by Theorem 1:

$$H_i(\bar{C}_*) \xrightarrow{\sim} \begin{cases} \widetilde{\mathrm{St}}(V) & \text{if } i = n-1 \\ 0 & \text{else} \end{cases}$$

(see (1) for $\widetilde{\mathrm{St}}(V)$).

Now $\bar{C}_p$ is isomorphic to the free $\mathbf{Z}$-module with basis the $(v_0,\dots,v_p)$ with $\dim\langle v_0,\dots v_p\rangle = n$. In particuliar, $\bar{C}_p = 0$ for $p < n-1$. Hence a resolution à la Lee-Szczarba [5, th. 3.1]:

$$\dots \xrightarrow{d_{n+1}} \bar{C}_n \xrightarrow{d_n} \bar{C}_{n-1} \xrightarrow{\mathrm{ar}} \widetilde{\mathrm{St}}(V) \to 0. \tag{6}$$

To get back [5, th. 3.1] in the case where A is principal (replacing $C_*(E^*(M))$ by $C_*(U^*(M))$), we use Proposition 1.

Theorem 2. *Modulo the isomorphisms of* (5)*, the map* ar *of* (6) *sends a generator $Q = (v_0,\dots,v_{n-1})$ to the universal modular symbol $[v_0,\dots,v_{n-1}]$ of Ash-Rudolph.*

Proof. The point is to get rid of subdivisions "without calculation". For simplicity, write $\varphi := \varphi_Q$. Observe first that φ factors as

$$\partial\,\mathrm{sd}\,\Delta_{n-1} \xrightarrow{\tilde\varphi} \mathrm{sd}\,E^*(M) = B\,\mathrm{Simpl}\,E^*(M) \xrightarrow{B(\mathrm{AR})} T(V)$$

where $\tilde\varphi$ is the simplicial map sending a vertex s of $\partial\,\mathrm{sd}\,\Delta_{n-1}$ to $\{v_i \mid i \in s\}$.

There is an isomorphism of simplicial complexes (induced by the inclusion $\partial\Delta_{n-1} \subset \Delta_{n-1}$)

$$\lambda : \mathrm{sd}\,\partial\Delta_{n-1} \xrightarrow{\sim} \partial\,\mathrm{sd}\,\Delta_{n-1}.$$

The composition $\tilde\varphi \circ \lambda$ is just $\mathrm{sd}\,\psi$, where $\psi : \partial\Delta_{n-1} \to E^*(V)$ is the restriction of $E(\Psi)$ with

$$\Psi : [n-1] \to M - \{0\}, \quad i \mapsto v_i.$$

By the naturality of ε (*cf.* (3)), we therefore have a commutative diagram

$$\begin{tikzcd}
{|\mathrm{sd}\,\partial\Delta_{n-1}|} \arrow[r, "|\lambda|", "\sim"'] \arrow[d, "\varepsilon_{\partial\Delta_{n-1}}"', "\wr"] & {|\partial\,\mathrm{sd}\,\Delta_{n-1}|} \arrow[r, "|\tilde\varphi|"] & {|\mathrm{sd}\,E^*(V)|} \arrow[r, "|B(\mathrm{AR})|"] \arrow[d, "\varepsilon_{E^*(V)}"', "\wr"] & {|T(V)|} \\
{|\partial\Delta_{n-1}|} \arrow[rr, "|\psi|"] & & {|E^*(V)|.} &
\end{tikzcd}$$

For $n > 2$, if ζ' denotes the fundamental class of $H_{n-1}(\mathrm{sd}\,\partial\Delta_{n-1})$ and ζ'' denotes that of $H_{n-1}(\partial\Delta_{n-1})$, we have

$$\begin{aligned}
\zeta &= \lambda_*\zeta' \\
\zeta'' &= (\varepsilon_{\partial\Delta_{n-1}})_*\zeta' \\
[v_0,\dots,v_{n-1}] &= B(\mathrm{AR})_* \circ \tilde\varphi_*(\zeta) = B(\mathrm{AR})_* \circ \tilde\varphi_* \circ \lambda_*(\zeta').
\end{aligned}$$

For $n = 2$, define (*cf.* §1) the fundamental class ζ'' of $H_{n-2}(\partial\Delta_{n-1})$ as the image of the "positive" generator of $H_{n-1}(\Delta_{n-1},\partial\Delta_{n-1})$, namely

$[1]-[0]$, and ζ, ζ' as the corresponding classes: the same identities hold. It now suffices to show that $\psi_*(\zeta'') = (v_0, \dots, v_{n-1}) \in H_{n-1}(E^*(V))$.

For this, consider the commutative diagram of exact sequences of complexes

$$\begin{tikzcd}
0 \arrow[r] & C_*(\partial\Delta_{n-1}) \arrow[r] \arrow[d] & C_*(\Delta_{n-1}) \arrow[r] \arrow[d] & C_*(\Delta_{n-1}, \partial\Delta_{n-1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_*(E^*(V)) \arrow[r] & C_*(E(V-\{0\}) \arrow[r] & \bar{C}_* \arrow[r] & 0
\end{tikzcd}$$

hence a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H_{n-1}(\Delta_{n-1}, \partial\Delta_{n-1}) \arrow[r] \arrow[d] & H_{n-2}(\partial\Delta_{n-1}) \arrow[r] \arrow[d] & H_{n-2}(\Delta_{n-1}) \arrow[d] \\
0 \arrow[r] & H_{n-1}(\bar{C}_*) \arrow[r] & H_{n-2}(E^*(V)) \arrow[r] & H_{n-2}(E(V-\{0\})
\end{tikzcd}$$

For any $n \geq 2$, ζ'' is the image of the element in $H_{n-1}(\Delta_{n-1}, \partial\Delta_{n-1})$ represented by the cycle $z \in C_{n-1}(\Delta_{n-1}, \partial\Delta_{n-1})$, image of the class of the identity $\Delta_{n-1} \to \Delta_{n-1}$ in $C_{n-1}(\Delta_{n-1})$. By functoriality, the image of z in $H_{n-1}(\bar{C}_*)$ is the image of $(v_0, \dots, v_{n-1}) \in C_{n-1}(E(V-\{0\})$. □

Corollary 2. *The group $\widetilde{\mathrm{St}}(V)$ is presented by the Ash-Rudolph relations (a)–(d) of §1.*

Proof. Indeed, we may take $A = K$ in Theorem 2; one should view $\bar{C}_{n-1} = C_{n-1}(E(V-\{0\})/C_{n-1}(E^*(V)))$ as the quotient of the free $\mathbf{Z}$-module with basis the $(v_0, \dots, v_{n-1})$ by the relations $(v_0, \dots, v_{n-1}) \equiv 0$ if $\dim\langle v_0, \dots v_{n-1}\rangle < n$. This gives Relation (c), and Relation (d) comes from d_n. On the other hand, one easily checks that Relations (a) and (b) formally follow from (c) and (d). Namely, by (c) and (d) we have the identity:

$$\begin{aligned}
&\partial[g_0, \dots, g_{i+1}, g_i, g_{i+1}, \dots, g_{n-1}] \\
&= (-1)^i[g_0, \dots, g_{i+1}, g_i, \dots, g_{n-1}] + (-1)^{i+2}[g_0, \dots, g_i, g_{i+1}, \dots, g_{n-1}] = 0
\end{aligned}$$

which implies (a). For (b), we have

$$\partial[g_0, ag_0, g_1, \dots, g_{n-1}] = [ag_0, g_1, \dots, g_{n-1}] - [g_0, g_1, \dots, g_{n-1}] = 0.$$

□

Remark 2. Together with Proposition, 1, Theorem 2 also shows that Theorem 3.1 of [5] implies Theorem 4.1 of [1], *cf.* [1, end of introduction].

7. The case of a Dedekind domain

If A is a Dedekind domain but is not principal, Lemma 6 is false even for $M = A^2$. Indeed, let $I \subset A$ be a nonprincipal ideal: it is generated by 2 elements [2, §1, Ex. 11 a) or §2, Ex. 1 a)], hence a surjection $A^2 \twoheadrightarrow I$ and an injection

$$I^* \hookrightarrow A^2$$

where $I^* = \operatorname{Hom}(I, A)$. By construction I^* is pure in A^2; if it contained a unimodular vector, there would be a linear form $\theta : A^2 \to A$ such that $\theta_{|I^*}$ is surjective, hence bijective. But then I^*, hence I, would be free, a contradiction. In fact:

Lemma 7. *If A is Dedekind, $U(M) \cap N \neq \emptyset$ for any pure submodule $N \subseteq M$ such that* $\operatorname{rk} N > 1$*. If* $\operatorname{rk} N = 1$*, $U(M) \cap N \neq \emptyset$ if and only if N is free.*

Proof. The case of rank 1 is clear. In the other, recall that all torsion-free finitely generated A-modules are projective; by Steinitz's structure theorem for projective modules [2, §4, no 10, Prop. 24], N contains A as a direct summand. Since N is itself a direct summand of M, it thus contains unimodular vectors. □

As in Proposition 1, we thus get an equivalence

$$\operatorname{AR}^u : \operatorname{Simpl} U^*(M) \xrightarrow{\sim} G^{**}(M) := G^*(M) - G^1(M) \tag{7}$$

where $G^1(M) = \{L \in G^*(M) \mid \operatorname{rk} L = 1 \text{ and } L \not\simeq A\}$. To compute further, we observe that the inclusion functor $T : G^1(M) \hookrightarrow G^*(M)$ is *cellular* in the sense of [4, Def. 2.3.2].[2] By [4, Prop. 2.3.5], we thus get a homotopy cocartesian square

$$\begin{array}{ccc} G^{**}(M) \int \mathbf{F}_T & \xrightarrow{p} & G^1(M) \\ {\scriptstyle\varepsilon}\big\downarrow & & {\scriptstyle T}\big\downarrow \\ G^{**}(M) & \xrightarrow{\iota} & G^*(M) \end{array}$$

where the category $G^{**}(M) \int \mathbf{F}_T$ has objects the inclusions $L \hookrightarrow N$ for $L \in G^1(M)$, $N \in G^{**}(M)$, and morphisms the commutative squares. This category splits as a coproduct

$$G^{**}(M) \int \mathbf{F}_T = \coprod_{L \in G^1(M)} L \downarrow G^{**}(M).$$

[2] Recall that, by definition, this means that T is fully faithful and $\operatorname{Hom}(d, c) = \emptyset$ for $d \in G^{**}(M)$ and $c \in G^1(M)$.

The map $N \mapsto N/L$ induces an isomorphism of posets

$$L \downarrow G^{**}(M) \xrightarrow{\sim} G^*(M/L).$$

Thus the square above becomes

$$\begin{array}{ccc} \coprod_{L \in G^1(M)} G^*(M/L) & \xrightarrow{p} & G^1(M) \\ {\scriptstyle\varepsilon}\big\downarrow & & {\scriptstyle T}\big\downarrow \\ G^{**}(M) & \xrightarrow{\iota} & G^*(M) \end{array}$$

where p projects $G^*(M/L)$ onto $\{L\}$ and ε is the inverse image. Note that $H_{n-2}(T(M/L)) = 0$ for all such L, and that $H_{n-3}(G^{**}(M)) = 0$ by (7) and by considering the chains of $C_*(E(M - \{0\}), E^*(M))$ as in the previous section. Hence an exact sequence

$$0 \to H_{n-2}(G^{**}(M)) \to \tilde{\mathrm{St}}(M) \to \bigoplus_{L \in G^1(M)} \tilde{\mathrm{St}}(M/L) \to 0$$

which gives a recursive computation of $\tilde{\mathrm{St}}(M)$ in terms of Ash-Rudolph symbols. In particular, taking homology, we find a long exact sequence

$$(8) \quad \cdots \to H_p(\mathrm{Aut}(M), H_{n-2}(G^{**}(M)) \to H_p(\mathrm{Aut}(M), \tilde{\mathrm{St}}(M))$$
$$\to H_p(\mathrm{Aut}(M), \bigoplus_{L \in G^1(M)} \tilde{\mathrm{St}}(M/L)) \to H_{p-1}(\mathrm{Aut}(M), H_{n-2}(G^{**}(M)) \to \dots$$

The group $\mathrm{Aut}(M)$ permutes the L's, and permutes transitively those in a given isomorphism class (because L is a direct summand of M). Hence in (8), we have by Shapiro's lemma

$$H_p(\mathrm{Aut}(M), \bigoplus_{L \in G^1(M)} \tilde{\mathrm{St}}(M/L)) \simeq \bigoplus_{\bar{L} \in \mathrm{Pic}(A) - \{0\}} H_p(\mathrm{Stab}_M(L), \tilde{\mathrm{St}}(M/L))$$

where $\mathrm{Stab}_M(L)$ denotes the stabiliser of some $L \in \bar{L}$ in M (note that its action on $\tilde{\mathrm{St}}(M/L)$ factors through the projection $\mathrm{Stab}_M(L) \to \mathrm{Aut}(M/L)$). For $p = 0$, this boils down to $\bigoplus_{\bar{L} \in \mathrm{Pic}(A) - \{0\}} \tilde{\mathrm{St}}(M/L)_{\mathrm{Aut}(M/L)}$. This gives a recursive method to compute $\tilde{\mathrm{St}}(M)_{\mathrm{Aut}(M)}$ in terms of unimodular symbols.

8. Relatinship with the van der Kallen-Suslin-Nesterenko complexes

Let us now assume that K is infinite. We say that a (finite, nonempty) subset of $Y \subset M$ is a *frame* if the elements of Y are linearly independent over K. We say that Y is *in general position* if any subset of Y

with at most n elements is a frame. This defines two subcomplexes of $E(M-\{0\})$:

$$\operatorname{Fr}(M)=\operatorname{Sk}^{n-1}\operatorname{GP}(M)\subset\operatorname{GP}(M)\subset E(M-\{0\}).$$

Proposition 2. $\operatorname{GP}(M)$ *is contractible.*

Proof. We adapt the proof of Lemma 1 in the style of [6, Proof of Lemma 3.5]: for $v\in M$, let $\operatorname{GP}(M)_v=\{Y\in\operatorname{GP}(M)\mid v\in Y\}$ and $\operatorname{GP}(M)^v=\{Y\in\operatorname{GP}(M)\mid Y\cup\{v\}\in\operatorname{GP}(M)\}$. Since $\operatorname{Simpl}\operatorname{GP}(M)_v$ has a minimal element, it is contractible, hence so is $\operatorname{GP}(M)^v$ by the argument in the proof of Lemma 1. Using that K is infinite, for any $Y_1,\dots,Y_r\in\operatorname{GP}(M)$ there exists $v\in M-Y$ such that $Y_i\cup\{v\}\in\operatorname{GP}(M)$ for $i=1,\dots,r$. Hence any finite subcomplex C of $\operatorname{GP}(M)$ is contained in some $\operatorname{GP}(M)^v$; thus the inclusion $C\to\operatorname{GP}(M)$ is nullhomotopic, hence the lemma. □

Corollary 3. $\operatorname{Fr}(M)$ *has the homotopy type of a bouquet of* $(n-1)$-*spheres.*

Proof. Apply Lemma 4. □

There is an obvious inclusion $\operatorname{Sk}^{n-2}\operatorname{GP}(M)\subset E^*(M)$, hence a map

$$H_{n-2}(\operatorname{Sk}^{n-2}\operatorname{GP}(M))\to H_{n-2}(E^*(M))\xrightarrow{\sim}\operatorname{St}(V). \tag{9}$$

Proposition 3. *The map* (9) *is surjective.*

Proof. Equivalently, we show that the map

$$H_{n-1}(\operatorname{GP}(M),\operatorname{Sk}^{n-2}\operatorname{GP}(M))\to H_{n-1}(E(M-\{0\}),E^*(M))$$

is surjective. Using Lemma 4, these groups are obtained as the homology of the morphism of complexes

$$\operatorname{Or}_*(\operatorname{GP}(M))/\sigma^{\le n-2}\operatorname{Or}_*(\operatorname{GP}(M))\to\operatorname{Or}_*(E(M-\{0\})/\operatorname{Or}_*(E^*(M)) \tag{10}$$

(oriented chains). Both complexes are 0 in degree $<n-1$, and (10) is an isomorphism in degree $n-1$. □

Unfortunately, (9) is far from being an isomorphism: for $n=2$ for example, its left hand side is free on the nonzero elements of M while its right hand side is free on the lines of M (or V). In particular, unlike its right hand side, the left hand side of (9) heavily depends on the choice of A inside its field of fractions K. For a general n, the left hand side of (9) is presented by Relation (d) of p. 2.

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