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The slice filtration and mixed Tate motives

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Abstract

Using the 'slice filtration', defined by effectivity conditions on Voevodsky's triangulated motives, we define spectral sequences converging to their motivic cohomology and étale motivic cohomology. These spectral sequences are particularly interesting in the case of mixed Tate motives as their E_2-terms then have a simple description. In particular this yields spectral sequences converging to the motivic cohomology of a split connected reductive group. We also describe in detail the multiplicative structure of the motive of a split torus.

Introduction

In this paper we study the 'slice filtration' defined by effectivity conditions on Voevodsky's triangulated motives, and apply it to obtain spectral sequences converging to their motivic cohomology. These spectral sequences are particularly interesting in the case of *mixed Tate motives* as their E_2-terms then have a simple description. They generalise the spectral sequences introduced in [Kah99] for (geometrically) cellular varieties, in particular projective homogeneous varieties. The interest is that they can be computed for a much wider class of varieties, among which are connected reductive groups: in this case, a concrete computation of the E_2-terms is given in § 9.

Our approach is quite elementary. Given a perfect field F and an object M of the category $DM_-^{\mathrm{eff}}(F)$ of triangulated motivic complexes (see [Voe00b]), for any integer $n \geqslant 0$, the identity map $\underline{\mathrm{Hom}}(\mathbf{Z}(n), M) \to \underline{\mathrm{Hom}}(\mathbf{Z}(n), M)$ gives by adjunction a map

$$\underline{\mathrm{Hom}}(\mathbf{Z}(n), M)(n) \to M.$$

Here $\underline{\mathrm{Hom}}$ denotes the partially defined internal Hom of $DM_-^{\mathrm{eff}}(F)$. This turns out to define a very well-behaved 'filtration' on M: the *slice filtration*. The successive cones (chunks) of this filtration are unique up to unique isomorphism and functorial in M; taking morphisms to $\mathbf{Z}(n)$ for various n then gives the desired spectral sequences.

We can also take the image of this filtration in the category $DM_{-,\text{ét}}^{\mathrm{eff}}(F)$ of étale triangulated motivic complexes and get spectral sequences for étale motivic cohomology.

If M is mixed Tate, that is, belongs to the tensor localising subcategory of $DM_-^{\mathrm{eff}}(F)$ generated by $\mathbf{Z}(1)$, then the chunks of the slice filtration are just tensor products of Tate objects $\mathbf{Z}(n)$ by complexes of abelian groups. If M is 'geometrically Tate', the same is true in the étale situation, the complexes of abelian groups being replaced by complexes of sheaves over the small étale site of $\operatorname{Spec} F$. This applies for example when $M = M(G)$ where G is a connected reductive group.

The first to have considered mixed Tate motives and a weight filtration on them in a triangulated context is Levine [Lev93a]; our paper was inspired by it and of course by Voevodsky's work. It was started in 1998; some of its results were announced in [Kah00]. It has known a rather long evolution.

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Meanwhile related work has been done, by among others Voevodsky [Voe02a, Voe03a], Kahn and Sujatha [KS06], Huber [Hub03] and Biglari [Big04]. (The reader may also like to look at [Tot06].)

In particular, after hesitating between 'weight filtration' (inappropriate in general), 'niveau filtration' (same problem), 'level filtration'... for our object of study, we opted for 'slice filtration' as in [Voe02a], since the two notions are clearly similar. In fact, Voevodsky defines in [Voe03a, § 5] relative mixed Tate motives and a slice filtration on them in his triangulated category $DM_-^{\mathrm{eff}}(\mathcal{X})$ of effective motivic complexes over a simplicial scheme $\mathcal{X}$ in the same way as we do [Voe03a, proof of Lemma 5.9].

We now run briefly through the contents of this paper. The first section introduces the slice filtration and proves some of its basic properties. In the next one we relate it to the theory of birational motives of Kahn and Sujatha [KS06] and compute the slices in special cases, extending some of their results. The third section describes spectral sequences associated to the slice filtration, and we come back to this question in the sixth section for the étale topology; in particular, we solve there a question raised in [Kah99]. Sections 4 and 5 introduce and discuss mixed Tate motives.

The next three sections converge towards a computation of the motive of a split (connected) reductive group G: the sixth deals with split tori, the seventh with toric fibrations and the eighth with reductive groups. We should stress, however, that our computation of $M(G)$ is not completed here: we merely describe the E_1-terms of the spectral sequence obtained from the slice filtration and converging to its motivic cohomology. Biglari's thesis [Big04] contains the complete computation for split reductive groups and rational coefficients.

Finally, there are three appendices dealing with technical matters.

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The work on this project was to some extent done in parallel with [KS06]. Ideas from there have been used in § 2.

Convention. Throughout the paper, F is a perfect field. We use the notation of [Voe00b].

1. The slice filtration

Let $n \geqslant 0$. Let $DM_-^{\mathrm{eff}}(F)(n) \subset DM_-^{\mathrm{eff}}$ be the full subcategory of objects of the form $M(n)$ with $M \in DM_-^{\mathrm{eff}}(F)$. For $M \in DM_-^{\mathrm{eff}}(F)$, define

$$\nu^{\geqslant n} M = \underline{\mathrm{Hom}}(\mathbf{Z}(n), M)(n). \tag{1.1}$$

It is clear that $\nu^{\geqslant n}$ is a triangulated functor from $DM_-^{\mathrm{eff}}(F)$ to its subcategory $DM_-^{\mathrm{eff}}(F)(n)$. By adjunction, the identity map of $\underline{\mathrm{Hom}}(\mathbf{Z}(n), M)$ gives a canonical morphism (natural transformation)

$$a^n : \nu^{\geqslant n} M \to M. \tag{1.2}$$

More generally, for $n > 0$ define a natural transformation

$$f^n : \nu^{\geqslant n} M \to \nu^{\geqslant n-1} M \tag{1.3}$$

as follows: by adjunction, we get a morphism

$$\underline{\mathrm{Hom}}(\mathbf{Z}(n), M)(1) \to \underline{\mathrm{Hom}}(\mathbf{Z}(n-1), M)$$

whence (1.3) by tensoring with $\mathbf{Z}(n-1)$. It is immediate to check that $a^{n-1} \circ f^n = a^n$.

Proposition 1.1. *The functor $\nu^{\geqslant n}$ is right adjoint to the inclusion $DM_-^{\mathrm{eff}}(F)(n) \hookrightarrow DM_-^{\mathrm{eff}}(F)$.*

Proof. We have to show that for any $N, P \in DM_-^{\mathrm{eff}}(F)$

$$\mathrm{Hom}(P(n), \nu^{\geqslant n} N) \xrightarrow{\sim} \mathrm{Hom}(P(n), N)$$

where the map is induced by a^n. Now

$$\begin{aligned}\mathrm{Hom}(P(n), \nu^{\geqslant n} N) = \mathrm{Hom}(P(n), \underline{\mathrm{Hom}}(\mathbf{Z}(n), N)(n)) &\xleftarrow{\sim} \mathrm{Hom}(P, \underline{\mathrm{Hom}}(\mathbf{Z}(n), N))\\ &\simeq \mathrm{Hom}(P \otimes \mathbf{Z}(n), N) = \mathrm{Hom}(P(n), N)\end{aligned}$$

where the first isomorphism follows from the quasi-invertibility of the Tate object (Proposition A.1). It remains to check that the isomorphism described above is indeed the one induced by a^n: this is left to the reader. □

Remark 1.2. As a right adjoint, $\nu^{\geqslant n}$ commutes with existing inverse limits. It also commutes with direct sums because so does tensor product and because Rp_*p^* has this property, where $p : \mathbf{A}^n - \{0\} \to \operatorname{Spec} F$ is the structural morphism. (Recall that by [Voe00b, Proposition 3.2.8] the internal Hom is computed in terms of Rp_*p^*.)

DEFINITION 1.3. Let $\nu_{\leqslant n} DM_-^{\mathrm{eff}}(F)$ be the full subcategory of $DM_-^{\mathrm{eff}}(F)$ consisting of those objects on which $\nu^{\geqslant n+1}$ vanishes.

COROLLARY 1.4. *Let $M \in DM_-^{\mathrm{eff}}(F)$.*

(i) *Let $\nu_{<n} M = \nu_{\leqslant n-1} M$ be an object fitting in an exact triangle*

$$\nu^{\geqslant n} M \xrightarrow{a^n} M \to \nu_{<n} M \to \nu^{\geqslant n} M[1].$$

This object is uniquely defined up to unique isomorphism. For all $n \geqslant 0$, $\nu_{<n}$ defines a triangulated endofunctor of $DM_-^{\mathrm{eff}}(F)$. The natural transformations $a_n : Id \to \nu_{<n}$ factor canonically through natural transformations $f_n : \nu_{<n+1} \to \nu_{<n}$.

(ii) *One has $\nu_{\leqslant n}$ is left adjoint to the inclusion $\nu_{\leqslant n} DM_-^{\mathrm{eff}} \to DM_-^{\mathrm{eff}}$.*

(iii) *Let $\nu_n M$ be an object fitting in an exact triangle*

$$\nu_n M \to \nu_{<n+1} M \xrightarrow{f_n} \nu_{<n} M \to \nu_n M[1].$$

Again this object is uniquely defined up to unique isomorphism; ν_n defines another triangulated endofunctor of $DM_-^{\mathrm{eff}}(F)$. There is also a functorial exact triangle

$$\nu^{\geqslant n+1} M \xrightarrow{f^n} \nu^{\geqslant n} M \to \nu_n M.$$

(iv) *For any $M \in DM_-^{\mathrm{eff}}(F)$, one can write canonically*

$$\nu_n M = c_n(M)(n)[2n]$$

and the c_n also define triangulated endofunctors of $DM_-^{\mathrm{eff}}(F)$.

(v) *One has the identities*

$$\begin{aligned}\nu^{\geqslant n}(M(1)) &= (\nu^{\geqslant n-1} M)(1),\\ c_n(M(1)[2]) &= c_{n-1}(M).\end{aligned} \tag{1.4}$$

Proof. The first assertion (including the functoriality) is equivalent to the following: for all $M, N \in DM_-^{\mathrm{eff}}(F)$, $m \leqslant n$ and $r \in \mathbf{Z}$, we have

$$\mathrm{Hom}(\nu^{\geqslant n} M, \nu_{<m} N[r]) = 0$$

or

$$\mathrm{Hom}(\nu^{\geqslant n} M, \nu^{\geqslant m} N[r]) \xrightarrow{\sim} \mathrm{Hom}(\nu^{\geqslant n} M, N[r]),$$

which is a special case of Proposition 1.1. (See also [BBD82, § 1.4.4].) Let $M \in DM_-^{\mathrm{eff}}(F)$, and $N \in \nu_{\leqslant n} DM_-^{\mathrm{eff}}(F)$. The defining triangle for $\nu_{\leqslant n}N$ yields an exact sequence

$$\mathrm{Hom}(M, \nu^{\geqslant n+1}N) \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(M, \nu^{\leqslant n}N) \to \mathrm{Hom}(M, \nu^{\geqslant n+1}N[1]).$$

By assumption $\nu^{\geqslant n+1}N = 0$. This verifies the universal property of $\nu^{\leqslant n}$.

The existence of f_n and the functoriality of ν_n are proven in the same way. By the octahedral axiom, the commutative diagram of exact triangles

$$\begin{tikzcd}
\nu^{\geqslant n+1}M \arrow[r, "f^n"] \arrow[d, "a^{n+1}"'] & \nu^{\geqslant n}M \arrow[d, "a^n"'] \\
M \arrow[r, equal] \arrow[d, "a_{n+1}"'] & M \arrow[d, "a_n"'] \\
\nu_{<n+1}M \arrow[r, "f_n"] & \nu_{<n}M
\end{tikzcd}$$

implies the second exact triangle for $\nu_n M$. In particular, $\nu_n M \in DM_-^{\mathrm{eff}}(F)(n)$. In view of the quasi-invertibility of $\mathbf{Z}(n)$ this is enough to show the existence and uniqueness of c_n. The last assertion is obvious. □

Definition 1.5. The $c_n(M)$ are called the *fundamental invariants* of M. For a variety X we abbreviate $c_n(X) := c_n(M(X))$.

Corollary 1.6. *For all integers $n, n' \geqslant 0$, there exists a unique natural transformation of bifunctors $\nu^{\geqslant n} \otimes \nu^{\geqslant n'} \to \nu^{\geqslant n+n'}(\cdot \otimes \cdot)$ letting the diagram*

$$\begin{tikzcd}
\nu^{\geqslant n} \otimes \nu^{\geqslant n'} \arrow[r] \arrow[d, "a^n \otimes a^{n'}"'] & \nu^{\geqslant n+n'}(\cdot \otimes \cdot) \arrow[d, "a^{n+n'}"'] \\
\otimes \arrow[r, equal] & \otimes
\end{tikzcd}$$

commute. This natural transformation induces natural transformations

$$\begin{aligned} \nu_n \otimes \nu_{n'} &\to \nu_{n+n'}(\cdot \otimes \cdot), \\ c_n \otimes c_{n'} &\to c_{n+n'}(\cdot \otimes \cdot). \end{aligned} \tag{1.5}$$

Proposition 1.7. *Let $\operatorname{char} F = 0$. Let X be an F-variety of dimension $\leqslant d$. Then*

$$\nu^{\geqslant m}M^c(X) = \begin{cases} 0 & \text{if } m > d, \\ \underline{CH}_d(X)[0] \otimes \mathbf{Z}(d)[2d] & \text{if } m = d, \end{cases}$$

$$c_m M^c(X) = \begin{cases} 0 & \text{if } m > d, \\ \underline{CH}_d(X)[0] & \text{if } m = d, \end{cases}$$

where $\underline{CH}_d(X)$ denotes the homotopy invariant Nisnevich sheaf with transfers $U \mapsto CH_d(X \otimes_F F(U))$ where $F(U)$ is the total ring of fractions of U. Here we denote by $M^c(X)$ the object $\underline{C}_*^c(X) \in DM_-^{\mathrm{eff}}(F)$ of [Voe00b, p. 224], which belongs to $DM_{\mathrm{gm}}(F)$ by [Voe00b, Proposition 4.1.6] ('motive with compact supports' [Voe00b, p. 195]).

Proof. From [Voe00b, prop. 4.2.8], we have an isomorphism for any $m \geqslant 0$,

$$\nu^{\geqslant m}M^c(X) := \underline{\mathrm{Hom}}(\mathbf{Z}(m), M^c(X))(m) \cong \underline{C}_*(z_{\mathrm{equi}}(X, m))(m)[2m],$$

where $z_{\mathrm{equi}}(X,m)$ is a certain Nisnevich sheaf with transfers defined in the beginning of [Voe00b, § 4.2]. This definition shows immediately that $z_{\mathrm{equi}}(X,m)=0$ for $m>\dim X$ and that

$$z_{\mathrm{equi}}(X,d)(U)\simeq CH_d(X\otimes_F F(U))$$

as the free abelian group generated by the closed integral subschemes of $U\times X$ which are equidimensional of relative dimension d over U, that is, the (reduced) irreducible components of $U\times X$. □

In Appendix B we explain how to extend the definition of $M^c(X)$ to arbitrary characteristic under some conditions.

Proposition 1.8. *If X is smooth in Proposition 1.7, the assumption of characteristic 0 is not necessary in the following cases:*

(i) $d\leqslant 2$;

(ii) *any d provided we tensor everything by* $\mathbf{Q}$;

(iii) *any d provided X is smooth projective.*

Proof. Let $A=\mathbf{Z},\mathbf{Q}$ respectively. Assume X is connected of dimension d. Let $m\geqslant d$. By Lemma B.1

$$\nu^{\geqslant m}M^c(X)(-m)\cong\underline{\mathrm{Hom}}(M(X)(m-d)[-2d],A).$$

By Lemma A.2 this vanishes for $m>d$. For $m=d$ the cohomology sheaves are associated to

$$U\mapsto H^q(U\times X,A).$$

This cohomology vanishes for $q\neq 0$ because A is flasque. For $q=0$ it equals $CH_d(X\times U)\otimes A=CH_d(X\otimes_F F(U))\otimes A$. □

Corollary 1.9. *Suppose* $\mathrm{char}\,F=0$. *For any $M\in DM^{\mathrm{eff}}_{\mathrm{gm}}(F)$ we have $\nu^{\geqslant n}M=0$ for n large enough. Equivalently, $\nu_{\leqslant n}M=M$ for n large enough. The functors $(c_n)_{n\geqslant 0}$ are conservative on $DM^{\mathrm{eff}}_{\mathrm{gm}}(F)$: if $c_n(M)=0$ for all n, then $M=0$.*

If $\mathrm{char}\,F>0$ *all this remains true after tensorisation by* $\mathbf{Q}$.

Proof. Let $d_{\leqslant n}DM^{\mathrm{eff}}_{\mathrm{gm}}(F)$ be the thick subcategory of $DM^{\mathrm{eff}}_{\mathrm{gm}}(F)$ generated by the $M(X)$ for X smooth and $\dim X\leqslant n$, so that $DM^{\mathrm{eff}}_{\mathrm{gm}}(F)=\bigcup d_{\leqslant n}DM^{\mathrm{eff}}_{\mathrm{gm}}(F)$. By Lemma B.4, $d_{\leqslant n}DM^{\mathrm{eff}}_{\mathrm{gm}}(F)$ is also generated by the $M^c(X)$ for X smooth and $\dim X\leqslant n$. Now it follows from Proposition 1.7 that $\nu^{\geqslant n+1}M=0$ for all $M\in d_{\leqslant n}DM^{\mathrm{eff}}_{\mathrm{gm}}(F)$. The third statement is an immediate consequence. In characteristic p, the same argument works with Proposition 1.8 instead of Proposition 1.7. □

Remark 1.10. In the case of $DM^{\mathrm{eff}}_{-}(F)$, we have to be a little more careful. Note that the functors c_n are certainly not conservative on

$$DM^{\mathrm{eff}}_{-}(F)(\infty):=\bigcap_{n\geqslant 0}DM^{\mathrm{eff}}_{-}(F)(n).$$

Here is an example of an object in the right-hand side. Consider a sequence of units $\underline{t}=(t_0,\dots,t_n,\dots)$ in F^* and form the homotopy colimit (= mapping telescope)

$$\mathbf{Z}(\infty)_{\underline{t}}=\mathrm{hocolim}\;\mathbf{Z}(n)[n],$$

where the transition map $\mathbf{Z}(n)[n]\to\mathbf{Z}(n+1)[n+1]$ is given by cup-product by t_n. Note that there is a canonical map

$$\mathbf{Z}\to\mathbf{Z}(\infty)_{\underline{t}}$$

given by the sequence of elements $\{t_0,\dots,t_{n-1}\}\in K^M_n(F)=\mathrm{Hom}(\mathbf{Z},\mathbf{Z}(n)[n])$. To get an example where $\mathbf{Z}(\infty)_{\underline{t}}\neq 0$, it therefore suffices to find one where all these symbols are non-zero: take for instance $F=k(t_0,\dots,t_n,\dots)$, a rational function field in infinitely many variables.

2. Birational motives

By definition,

$$c_n M = \nu_{\leqslant 0}\underline{\mathrm{Hom}}(\mathbf{Z}(n)[2n], M), \tag{2.1}$$

i.e. the functors c_n actually take their values in the full triangulated subcategory $DM^{\mathrm{o}}_{-}(F) = \nu_{\leqslant 0} DM^{\mathrm{eff}}_{-}(F)$ of *birational motivic complexes* of [KS06]. In other words, for any dense open immersion $j : U \to X$ of smooth F-schemes, any $M \in DM^{\mathrm{eff}}_{-}(F)$ and any $n \geqslant 0$, the map

$$j^* : \mathrm{Hom}(M(X), c_n(M)) \to \mathrm{Hom}(M(U), c_n(M))$$

is an isomorphism. Equivalently by the Gysin exact triangles (compare [KS06]), for any $N \in DM^{\mathrm{eff}}_{\mathrm{gm}}(F)$ we have $\mathrm{Hom}(N(1), c_n M) = 0$: this follows immediately from Proposition 1.1.

The category $DM^{\mathrm{o}}_{-}(F)$ is not stable under the tensor product of $DM^{\mathrm{eff}}_{-}(F)$. However, by Proposition 1.7, $\nu_{\leqslant 0}$ is left adjoint–left inverse to the inclusion functor $i : DM^{\mathrm{o}}_{-}(F) \to DM^{\mathrm{eff}}_{-}(F)$ and one easily sees that the tensor product of $DM^{\mathrm{eff}}_{-}(F)$ descends via $\nu_{\leqslant 0}$ to a tensor product $\overline{\otimes}$ on $DM^{\mathrm{o}}_{-}(F)$ (compare [KS06]). Then a simple adjunction argument shows that *the pairing of functors* (1.5) *factors through* $\overline{\otimes}$.

We are now going to prove a number of vanishing results for the fundamental invariants, by relying on some results of [KS06]: these vanishing results will not be used in the rest of the paper. Let us start with the following lemma.

Lemma 2.1. *The functor $c_n[n]$ is right exact on motivic complexes, i.e. if M is concentrated in non-positive degrees, then the same if true for $c_n M[n]$.*

Proof. We have to consider the functor $\nu_{\leqslant 0}\, \underline{\mathrm{Hom}}(\mathbf{Z}(n)[n], \cdot)$. The functor $\nu_{\leqslant 0}$ is right exact by [KS06, Proposition 14.3.3]. We view $\mathbf{Z}(n)[n]$ as direct summand of $M(\mathbf{A}^n \smallsetminus \{0\})[1-n]$. Hence $H^i\, \underline{\mathrm{Hom}}(\mathbf{Z}(n)[n], M)$ is the sheafification of

$$U \mapsto H^{i+1-n}(U \times (\mathbf{A}^n \smallsetminus \{0\}), M)/H^i(U, M).$$

This is the functor considered in [Voe00a, Lemma 4.35]. As shown there, it is concentrated in degree 0 if M is concentrated in degree 0. □

In the case $M = M^c(X)$, much stronger vanishing statements are true. In [KS06], $c_0(M^c(X))$ is considered in the case where X is smooth projective and F is of characteristic 0. One finds that it is concentrated in non-positive degrees and that

$$H^0 c_0(M^c(X)) = \bar{h}_0(X)[0],$$

where $\bar{h}_0(X)$ denotes the sheaf $U \mapsto CH_0(X \otimes_F F(U))$ where $F(U)$ is the total ring of fractions of U (it is not *a priori* obvious that this is a presheaf!).

We are going to generalise this using the same line of arguments.

Theorem 2.2. *Let X be a variety, $n \geqslant 0$. We assume one of the following conditions:*

(i) *either* char $F = 0$*;*

(ii) *or* char $F > 0$ *and* $\dim X \leqslant 2$*;*

(iii) *or* char $F > 0$ *and we take* $\mathbf{Q}$ *coefficients;*

(iv) *or* char $F > 0$ *and X smooth projective.*

(Recall that any of these conditions permits one to define $M^c(X)$; cf. Definition B.3 in characteristic p.) Then the complex $c_n M^c(X)$ is concentrated in non-positive degrees and moreover

$$H^0(c_n(M^c(X))) = \underline{CH}_n(X),$$

where the values of the Nisnevich sheaf with transfers $\underline{CH}_n(X)$ are given by the formula

$$\underline{CH}_n(X)(U) = CH_n(X_{F(U)}).$$

Proof. We start with char $F = 0$ and X arbitrary. We consider the defining triangle for $c_n(M^c(X))$:

$$\underline{\mathrm{Hom}}(\mathbf{Z}(n+1)[2n+2], M^c(X))(1)[2] \to \underline{\mathrm{Hom}}(\mathbf{Z}(n)[2n], M^c(X)) \to c_n M^c(X).$$

By [Voe00b, Proposition 4.2.8], $\underline{\mathrm{Hom}}(\mathbf{Z}(n)[2n], M^c(X))$ is given by $\underline{C}_* z_{\mathrm{equi}}(X, n)$. In particular it is concentrated in non-positive degrees. Tensor product is right exact on $DM_-^{\mathrm{eff}}(F)$ (see Lemma 2.4 below). Hence the term

$$\underline{\mathrm{Hom}}(\mathbf{Z}(n+1)[2n+2], M^c(X))(1)[2] = \underline{\mathrm{Hom}}(\mathbf{Z}(n+1)[2n+2], M^c(X)) \otimes \mathbb{G}_m[1]$$

is concentrated in negative degrees. The long exact sequence of cohomology sheaves shows that $c_n(M^c(X))$ is concentrated in non-positive degrees and, moreover,

$$H^0(c_n(M^c(X))) \cong H^0\, \underline{\mathrm{Hom}}(\mathbf{Z}(n)[2n], M^c(X)).$$

As $c_n(M^c(X))$ is a birational motive, its cohomology sheaves are birational [KS06, Proposition 14.2.6(a)]: this is the case in particular for H^0.

Let $\mathcal{F}$ be a presheaf (with transfers) such that the associated sheaf $\widetilde{\mathcal{F}}$ for the Nisnevich topology is birational. Then for all connected smooth U,

$$\widetilde{\mathcal{F}}(U) = \widetilde{\mathcal{F}}_{F(U)} \cong \mathcal{F}_{F(U)} = \varinjlim_{V} \mathcal{F}(V),$$

where V runs through all dense open subschemes of U.

We apply this remark to the presheaf

$$\begin{aligned} U \mapsto\ & \mathrm{Hom}(M(U)(n)[2n], M^c(X)) \\ & \cong \mathrm{Hom}(\mathbf{Z}(n)[2n], M^c(U \times X)(-d_U)[-2d_U]) \\ & = CH_{n-d_U}(U \times X) \quad \text{[Voe00b, Proposition 4.2.9]}, \end{aligned}$$

where d_U and d_X are the dimensions of U and X. In the limit we obtain $CH_n(X_{F(U)})$.

Now let char F be arbitrary, X smooth connected of dimension d_X satisfying one of the assumptions, $A = \mathbf{Z}, \mathbf{Q}$ respectively. By Proposition 1.8 the assertion holds for $n \geqslant d_X$. Let $n < d_X$. By Lemma B.1

$$\nu^{\geqslant n} M^c(X)(-n) \cong \underline{\mathrm{Hom}}(M(X), A(d_X - n)[2d_X - 2n]).$$

Hence its qth cohomology sheaf is associated to the presheaf

$$\begin{aligned} U \mapsto\ & \cong H^{2d_X - 2n + q}(U \times X, A(d_X - n)) \\ & \cong CH^{d_X - n}(U \times X, -q) \otimes A \quad \text{[Voe02b]}. \end{aligned}$$

This presheaf vanishes for $q > 0$. For $q = 0$ this is the ordinary Chow group and the sheafification is computed as in the first case. ∎

Remark 2.3. As a by-product of the proof, we find that the Nisnevich sheafification of the presheaf $P(U) = CH_{n-d_U}(U \times X)$ is birational. Here is a direct proof. Observe that:

(a) $P(U) \to P(V)$ is surjective when $V \subseteq U$;

(b) if $x \in U$ and η is the generic point of U, then $P_x \xrightarrow{\sim} P_\eta$ (see [Voe00a, Corollary 4.18]).

It follows for the associated Zariski sheaf P_{Zar} that $P_{\mathrm{Zar}}(U) \xrightarrow{\sim} P_\eta$. Thus P_{Zar} is birational and *a fortiori* so is $P_{\mathrm{Nis}} = P_{\mathrm{Zar}}$.

Note that the tensor product on $DM_-^{\mathrm{eff}}(F)$ is Voevodsky's. It is characterised by the fact that $M(X)\otimes M(Y) = M(X\times Y)$ for all smooth varieties X, Y. At the referee's request, we give a proof of the following lemma.

Lemma 2.4. *In $DM_-^{\mathrm{eff}}(F)$, the tensor product $\otimes$ is right exact, i.e. if K, L are concentrated in non-positive degrees, then so is $K\otimes L$.*

Proof. By a spectral sequence argument, it is enough to show that, if $\mathcal{F}, \mathcal{G} \in DM_-^{\mathrm{eff}}(F)$ are concentrated in degree 0, then $H^q(\mathcal{F}[0]\otimes\mathcal{G}[0]) = 0$ for $q > 0$. For this we may use the resolutions of $\mathcal{F}$ and $\mathcal{G}$ considered in [Voe00b, p. 206] and apply the definition of $\otimes$ given therein on p. 210. $\square$

Proposition 2.5. *Let $p : X \to \operatorname{Spec} F$ be smooth of pure dimension $d \geqslant 1$. Let $p_0 : \pi_0(X) \to \operatorname{Spec} F$ be the scheme of constants of X: p_0 is étale and p factors through a morphism $X \to \pi_0(X)$ with geometrically connected fibres. Assume that one of the conditions of Theorem 2.2 is satisfied. Then*

$$H^q c_{d-1} M^c(X) = \begin{cases} p_*\mathbb{G}_m/(p_0)_*\mathbb{G}_m & q = -1,\\ R^1p_*\mathbb{G}_m = \underline{CH}_{d-1}(X) & q = 0,\\ 0 & \textit{otherwise}.\end{cases}$$

Proof. As in the proof of the last theorem we consider the defining triangle for $c_{d-1}M^c(X)$. By the case $n = d$ already calculated in Propositions 1.7 and 1.8, it reads

$$\underline{CH}_d(X)(1)[2] \to \underline{\mathrm{Hom}}(\mathbf{Z}(d-1)[2d-2], M^c(X)) \to c_{d-1}M^c(X). \tag{2.2}$$

We abbreviate $\pi_0(X) = \pi_0$. Note that this scheme is smooth of dimension zero. We have

$$\underline{CH}_d(X) \cong \underline{CH}^0(X) \cong L(\pi_0) \cong M(\pi_0)$$

and hence

$$\underline{CH}_d(X)(1)[2] \cong M(\pi_0)(1)[2] \cong \underline{\mathrm{Hom}}(\mathbf{Z}, M(\pi_0)(1)[2]) \cong \underline{\mathrm{Hom}}(M(\pi_0), \mathbb{G}_m[1]) \cong (p_0)_*\mathbb{G}_m[1].$$

In particular the left-hand term of (2.2) is concentrated in degree -1. By Lemma B.1 and [Voe00b, Proposition 3.2.8], the term in the middle of (2.2) is isomorphic to

$$\underline{\mathrm{Hom}}(M(X), \mathbf{Z}(1)[2]) \cong Rp_*p^*\mathbb{G}_m[1].$$

The higher direct images of $\mathbb{G}_m$ under p are known and vanish for degrees different from $0, 1$, hence the result. (For $q \geqslant 0$ we recover part of Theorem 2.2.) $\square$

Proposition 2.5 shows that $c_nM^c(X)$ is not concentrated in degree 0 in general. However we have the following corollary.

Corollary 2.6. *In Theorem 2.2, $c_n(M^c(X))$ is concentrated in degree 0 in the following cases:*

(i) *$n \geqslant \dim X$;*

(ii) *X smooth projective, $n = \dim X - 1$;*

(iii) *X cellular.*

Proof. Case (i) is Proposition 1.7 and Proposition 1.8.

For case (ii) we use Proposition 2.5: as X is proper, $p_*\mathbb{G}_m \cong (p_0)_*\mathbb{G}_m$.

Case (iii) follows from the stronger Proposition 4.11 below. $\square$

3. Spectral sequences

Consider the exact sequences

$$\cdots \to \mathrm{Hom}(\nu_{\leqslant q-1}M, \mathbf{Z}(n)[p+q]) \xrightarrow{f_q^*} \mathrm{Hom}(\nu_{\leqslant q}M, \mathbf{Z}(n)[p+q]) \\ \to \mathrm{Hom}(\nu_q M, \mathbf{Z}(n)[p+q]) \to \mathrm{Hom}(\nu_{\leqslant q-1}M, \mathbf{Z}(n)[p+q+1]) \to \cdots .$$

We get for any $M \in DM_-^{\mathrm{eff}}(F)$ and any $n \geqslant 0$ an exact couple, hence a spectral sequence

$$E_2^{p,q}(M,n) = H^{p+q}(\nu_q M, \mathbf{Z}(n)) \Rightarrow H^{p+q}(M, \mathbf{Z}(n))$$

or, by quasi-invertibility,

$$E_2^{p,q}(M,n) = H^{p-q}(c_q M, \mathbf{Z}(n-q)) \Rightarrow H^{p+q}(M, \mathbf{Z}(n)). \tag{3.1}$$

This will be called the *slice spectral sequence* associated to M in weight n. It is functorial in M and Corollary 1.6 provides pairings of spectral sequences

$$E(M,m) \times E(N,n) \to E(M \otimes N, m+n).$$

Similarly, we may pull back the situation to the étale topology. Let $\alpha^* : DM_-^{\mathrm{eff}}(F) \to DM_{-,\text{ét}}^{\mathrm{eff}}(F)$ be the natural functor: we may consider the functors $\alpha^*\nu^{\geqslant n}$, $\alpha^*\nu_{<n}$, $\alpha^*\nu_n$ and $\alpha^* c_n$. They are all triangulated functors; note that we have

$$\alpha^*\nu_n(M) = (\alpha^* c_n(M))(n)[2n]$$

for any $M \in DM_-^{\mathrm{eff}}(F)$, since α^* commutes with tensor product. We get a filtration on $\alpha^* M$ with 'associated graded' the $\alpha^*\nu_n(M)$.

Definition 3.1. Let (cf. [Kah99, Lemma 2.4])

$$\mathbf{Z}(r)_{\text{ét}} = \begin{cases} \alpha^*\mathbf{Z}(r) & \text{if } r \geqslant 0, \\ \bigoplus_{l \neq \mathrm{char}\, F} \mathbf{Q}_l/\mathbf{Z}_l(r)[-1] & \text{if } r < 0. \end{cases}$$

For all n we have similar long exact sequences

$$\cdots \to \mathrm{Hom}(\alpha^*\nu_{\leqslant q-1}M, \mathbf{Z}(n)_{\text{ét}}[p+q]) \xrightarrow{f_q^*} \mathrm{Hom}(\alpha^*\nu_{\leqslant q}M, \mathbf{Z}(n)_{\text{ét}}[p+q]) \\ \to \mathrm{Hom}(\alpha^*\nu_q M, \mathbf{Z}(n)_{\text{ét}}[p+q]) \to \mathrm{Hom}(\alpha^*\nu_{\leqslant q-1}M, \mathbf{Z}(n)_{\text{ét}}[p+q+1]) \cdots .$$

By 'quasi-invertibility' in $DM_{-,\text{ét}}^{\mathrm{eff}}(F)$ (see [Kah99, Lemma 2.4] and Propositions A.3 and A.4), we get a spectral sequence

$$E_2^{p,q}(M,n) = H_{\text{ét}}^{p-q}(c_q M, \mathbf{Z}(n-q)_{\text{ét}}) \Rightarrow H_{\text{ét}}^{p+q}(M, \mathbf{Z}(n)_{\text{ét}}), \tag{3.2}$$

where $H_{\text{ét}}^r(M, \mathbf{Z}(n)_{\text{ét}})$ is shorthand for $\mathrm{Hom}_{DM_{-,\text{ét}}^{\mathrm{eff}}(F)}(\alpha^* M, \mathbf{Z}(n)_{\text{ét}}[r])$.

4. Mixed Tate motives

Definition 4.1. Let $\mathbf{Z}(1) \in DM_{\mathrm{gm}}^{\mathrm{eff}}(F)$ be the Tate object. Denote by

$$TDM_{\mathrm{gm}}^{\mathrm{eff}}(F)$$

the thick tensor subcategory of $DM_{\mathrm{gm}}^{\mathrm{eff}}(F)$ generated by $\mathbf{Z}(0)$ and $\mathbf{Z}(1)$. An object of $TDM_{\mathrm{gm}}(F)$ is called a *mixed Tate motive*. We write $TDM_-^{\mathrm{eff}}(F)$ for the localising subcategory of $DM_-^{\mathrm{eff}}(F)$ generated by $TDM_{\mathrm{gm}}^{\mathrm{eff}}(F)$ (*effective mixed Tate motivic complexes*). (Recall that a triangulated subcategory of a triangulated category is *thick* if it is stable under direct summands, *localising* if it is stable under arbitrary direct sums.)

Remark 4.2. A less elegant version of these categories is also considered in [Hub00, § 1.2] in the case $F = \mathbf{Q}$ and with rational coefficients.

Lemma 4.3.

(i) *The partially defined bifunctor internal* Hom *on* DM_-^{eff} *restricts to an internal* Hom*-functor*

$$\underline{\mathrm{Hom}} : TDM_{\mathrm{gm}}^{\mathrm{eff}}(F) \times TDM_{\mathrm{gm}}^{\mathrm{eff}}(F) \to TDM_{\mathrm{gm}}^{\mathrm{eff}}(F).$$

(ii) *The functors* $\nu^{\geqslant n}$ *and* $\nu^{\leqslant n}$ *respect* $TDM_{\mathrm{gm}}^{\mathrm{eff}}$ *and* TDM_-^{eff}.

Proof. The objects $\mathbf{Z}(i)$ for $i \geqslant 0$ generate $TDM_{\mathrm{gm}}^{\mathrm{eff}}$ as thick triangulated subcategory of $DM_{\mathrm{gm}}^{\mathrm{eff}}$. It suffices to check the assertion for these generators. By quasi-invertibility we have

$$\underline{\mathrm{Hom}}(\mathbf{Z}(i), \mathbf{Z}(j)) = \begin{cases} \mathbf{Z}(j-i) & \text{for } j \geqslant i, \\ 0 & \text{otherwise.} \end{cases}$$

In particular these $\underline{\mathrm{Hom}}$ are always in $TDM_{\mathrm{gm}}^{\mathrm{eff}}$. By definition of the slice filtration this implies that all $\nu^{\geqslant n}$ (and hence also $\nu_{\leqslant n}$) respect $TDM_{\mathrm{gm}}^{\mathrm{eff}}$. The statement for TDM_-^{eff} follows because $\nu^{\geqslant n}$ commutes with direct sums. □

Remark 4.4. On the category of mixed Tate motives, the slice filtration in fact agrees with the weight filtration (whatever the latter is going to be).

Let Ab be the category of abelian groups, $D^b(Ab)$ the bounded derived category of Ab, $D^-(Ab)$ the bounded above derived category and $D^b_f(Ab)$ the full subcategory of $D^b(Ab)$ consisting of those objects whose cohomology groups are finitely generated. It is the category of perfect complexes, i.e. those isomorphic to bounded complexes of finitely generated free $\mathbf{Z}$-modules. It is equivalent to the bounded derived category of finitely generated abelian groups. The category $D^b_f(Ab)$ is rigid tensor triangulated.

Proposition 4.5. *There exists a unique triangulated functor*

$$i : D^b_f(Ab) \to DM_{\mathrm{gm}}^{\mathrm{eff}}(F)$$

sending $\mathbf{Z}$ *to* $\mathbf{Z}$. *It is fully faithful and respects the tensor structures. Its essential image is the thick tensor subcategory of* $DM_{\mathrm{gm}}^{\mathrm{eff}}$ *generated by* $\mathbf{Z}(0)$.

Similarly there is a fully faithful tensor functor

$$i : D^-(Ab) \to DM_-^{\mathrm{eff}}(F).$$

Its essential image is the localising subcategory of DM_-^{eff} *generated by* $\mathbf{Z}(0)$.

Proof. For $D^-(Ab)$ the existence of i is obvious; it is clear that i sends $D^b_f(Ab)$ in $DM_{\mathrm{gm}}^{\mathrm{eff}}(F)$. For the full faithfulness on $D^b_f(Ab)$, we reduce to the case $\mathrm{Hom}(\mathbf{Z}, \mathbf{Z}[j])$, which is obvious, and full faithfulness on $D^-(Ab)$ then follows by density. Similarly, uniqueness holds because $\mathbf{Z}$ generates $D^-(Ab)$. □

In particular, $D^b_f(Ab)$ can be viewed as a full subcategory of $TDM_{\mathrm{gm}}^{\mathrm{eff}}$ and $D^-(Ab)$ as a full subcategory of TDM_-^{eff}.

PROPOSITION 4.6. *Assume either* $\operatorname{char} F = 0$ *or coefficients in* $\mathbf{Q}$. *A motive* $M \in DM^{\mathrm{eff}}_{\mathrm{gm}}(F)$ *is in* $TDM^{\mathrm{eff}}_{\mathrm{gm}}(F)$ *if and only if* $c_n(M) \in D^b_f(Ab)$ *for all* n *and* $c_n(M) = 0$ *for* n *large enough. If* $M \in TDM^{\mathrm{eff}}_{-}(F)$, *then* $c_n(M) \in D^-(Ab)$.

Proof. Let $TDM^{\mathrm{eff}}_{\mathrm{gm}}(F)'$ be the full subcategory of $DM^{\mathrm{eff}}_{\mathrm{gm}}(F)$ formed of those motives M such that $c_n(M) \in D^b_f(Ab)$ for all n. It is triangulated, thick, stable under tensor product and contains $\mathbf{Z}(1)$, hence it contains $TDM^{\mathrm{eff}}_{\mathrm{gm}}(F)$. Conversely, let $M \in TDM^{\mathrm{eff}}_{\mathrm{gm}}(F)'$. By assumption $\nu_n(M) \in TDM^{\mathrm{eff}}_{\mathrm{gm}}(F)$ for all n. By induction on n, it follows that $\nu_{\leqslant n}M \in TDM^{\mathrm{eff}}_{\mathrm{gm}}(F)$ for all $n \geqslant 0$. As $M = \nu_{\leqslant n}M$ for some n by Corollary 1.9, this implies $M \in TDM^{\mathrm{eff}}_{\mathrm{gm}}(F)$. The statement for TDM^{eff}_{-} follows from the first case because c_n commutes with arbitrary direct sums. □

Remark 4.7. We do not expect that objects of DM^{eff}_{-} such that $c_n(M) \in D^-(Ab)$ are automatically in TDM^{eff}_{-}. Tensoring the object of the example in Remark 1.10 with $M(X)$ for some variety X gives a motivic complex M such that all $c_n(M)$ vanish, hence are in $D^-(Ab)$. However, we do not expect M to be contained in TDM^{eff}_{-}.

LEMMA 4.8. *Let* $M \in DM^{\mathrm{eff}}_{-}(F)$ *and* $N \in TDM^{\mathrm{eff}}_{-}(F)$. *Then the morphisms (1.5) induce 'Künneth' isomorphisms*

$$\bigoplus_{p+q=n} c_p(M) \otimes c_q(N) \xrightarrow{\sim} c_n(M \otimes N).$$

(In this formula, the tensor product $\otimes$ of $DM^{\mathrm{eff}}_{-}(F)$ may be replaced by the tensor product $\overline{\otimes}$ of $DM^{\mathrm{o}}_{-}(F)$, cf. § 2.)

Proof. Reduce to the case where $N = \mathbf{Z}(n)$, when it follows from Corollary 1.4(v). □

DEFINITION 4.9. An object of DM_{gm} is called *pure Tate motive* if it is a (finite) direct sum of copies of $\mathbf{Z}(p)[2p]$ for $p \in \mathbf{Z}$.

Note that by Voevodsky's embedding theorem (see [Voe00b, Proposition 2.1.4, Corollary 4.2.6] and [Voe02b]), this category is equivalent to the full subcategory of Tate motives in the category of pure motives in Grothendieck's sense. Indeed, rational, homological and numerical equivalence agree on such motives.

PROPOSITION 4.10. *Let* X *be a smooth variety such that* $M(X)$ *is a pure Tate motive. Then there is a natural isomorphism*

$$M(X) \cong \bigoplus_p c_p(X)(p)[2p], \quad \text{with } c_p(X) = CH^p(X)^*[0],$$

where $\cdot^*$ *denotes the dual of a free abelian group.*

Proof. By [Voe02b] there is a natural isomorphism

$$CH^p(X) \cong \operatorname{Hom}(M(X), \mathbf{Z}(p)[2p]). \tag{4.1}$$

As $M(X)$ is mixed Tate, this group is free of finite type. Hence the isomorphism (4.1) yields a canonical morphism

$$M(X) \otimes CH^p(X) \to \mathbf{Z}(p)[2p].$$

Summing over all p we obtain dually a natural map

$$M(X) \to \bigoplus_p CH^p(X)^*(p)[2p].$$

We are going to check that it is an isomorphism. By a version of the Yoneda lemma on the category of pure Tate motives it suffices to prove that, for any q,

$$\mathrm{Hom}\Big(\bigoplus_p CH^p(X)^*(p)[2p],\, \mathbf{Z}(q)[2q]\Big) \xrightarrow{\sim} \mathrm{Hom}(M(X), \mathbf{Z}(q)[2q]).$$

This holds tautologically. □

Recall that a *cell* is a variety isomorphic to some $\mathbf{A}^n$. A variety is called *cellular* if it contains a cell as an open subvariety such that the closed complement is already cellular.

Proposition 4.11. *Let X be a cellular variety. If* $\operatorname{char} F = 0$, *then $M^c(X)$ is a pure Tate motive. Moreover,*

$$M^c(X) = \bigoplus_p c_p(M^c(X))(p)[2p], \quad \textit{with } c_p(M^c(X)) = CH_p(X)[0].$$

If $\operatorname{char} F$ *is arbitrary and X is smooth, then $M(X)$ is a pure Tate motive. Moreover,*

$$M(X) = \bigoplus_p c_p(X)(p)[2p], \quad \textit{with } c_p(X) = CH^p(X)^*[0].$$

Proof. The first statement was proved in [Kah99, Proposition 3.4]. The second statement was deduced there by duality in the case $\operatorname{char} F = 0$. Here is a direct proof in arbitrary characteristic.

By Proposition 4.10 it suffices to check that $M(X)$ is pure Tate. We do this by induction on the number of cells. Let d be the dimension of X, d' the dimension of the smallest cells occurring in the cellular decomposition of X. One easily sees that there is a closed cell C of dimension d' (remove an open cell of dimension d and argue by induction on the total number of cells). Note that, being a cell, C is smooth. Consider the Gysin triangle

$$M(U) \to M(X) \to M(C)(d-d')[2d-2d'] \to$$

with $U = X \setminus C$. By induction on the number of cells we can assume that $M(U)$ is a direct sum of the $\mathbf{Z}(q)[2q]$ with $q \geqslant d-d'$. By homotopy invariance, $M(C) = \mathbf{Z}$. The boundary map vanishes because $\mathrm{Hom}(\mathbf{Z}(d-d')[2d-2d'], \mathbf{Z}(q)[2q+1]) = 0$ for $q \geqslant d-d'$. Hence the triangle splits. □

5. Geometrically mixed Tate motives

Definition 5.1. An object $M \in DM_-^{\mathrm{eff}}(F)$ is a *geometrically Tate mixed motivic complex* if its restriction to $DM_-^{\mathrm{eff}}(F^{\mathrm{sep}})$ (where F^{sep} is the separable closure of F) is mixed Tate. The full subcategory of geometrically Tate mixed motivic complexes is denoted $T^gDM_-^{\mathrm{eff}}(F)$. Similarly, an object $M \in DM_{\mathrm{gm}}^{\mathrm{eff}}(F)$ is a *geometrically mixed Tate motive* if its restriction to $DM_{\mathrm{gm}}^{\mathrm{eff}}(F^{\mathrm{sep}})$ is mixed Tate. The full subcategory of geometrically Tate mixed motives is denoted $T^gDM_{\mathrm{gm}}^{\mathrm{eff}}(F)$.

Clearly, $T^gDM_-^{\mathrm{eff}}(F)$ is a localising tensor subcategory of $DM_-^{\mathrm{eff}}(F)$ and $T^gDM_{\mathrm{gm}}^{\mathrm{eff}}(F)$ is a thick tensor subcategory of $DM_{\mathrm{gm}}^{\mathrm{eff}}(F)$.

Example 5.2. The motive of a non-split torus is not mixed Tate but is geometrically mixed Tate.

As the functors c_n commute with extension of scalars, Lemma 4.3 implies that being geometrically mixed Tate can be tested on the $c_n(M)$.

Proposition 5.3. *An object $M \in DM_{\mathrm{gm}}^{\mathrm{eff}}(F)$ is geometrically mixed Tate if and only if there is a finite separable extension E of F such that the restriction of M to $DM_{\mathrm{gm}}^{\mathrm{eff}}(E)$ is mixed Tate.*

Proof. The key observation is the following. If M, M' are objects of $DM^{\mathrm{eff}}_{\mathrm{gm}}(E)$ and $f : M \times E^{\mathrm{sep}} \to M' \times E^{\mathrm{sep}}$ is a morphism, then there is a finite separable extension L/E over which f is defined. Moreover, if two such morphisms are equal over E^{sep}, they are equal over some finite extension L/E.

We introduce the notion of complexity of a mixed Tate motive. The object 0 is of complexity 0, and the objects $\mathbf{Z}(n)[j]$ are of complexity 1. If M and M' are of complexity at most c and at most c', and if $f : M \to M'$ is a morphism, then the third object N in the exact triangle $M \to M' \to N$ is of complexity at most $c + c'$. All mixed Tate motives are of finite complexity. The above observation shows by induction on the complexity that all mixed Tate motives over F^{sep} are defined over some finite separable extension E of F (where E depends on the object). If M is geometrically mixed Tate, it is isomorphic to such an object over F^{sep}, hence by the observation already over some finite extension of F. □

In contrast to the Tate case, there is no simple description of the fundamental invariants $c_n(M)$ for geometrically mixed Tate motives. However, such a description exists after passing to the étale site.

Lemma 5.4. *Let $M \in T^g DM^{\mathrm{eff}}_{-}(F)$. Then $\alpha^* c_n(M)$ is isomorphic to a complex of étale sheaves coming from the small étale site of* Spec F. *For $M \in T^g DM^{\mathrm{eff}}_{\mathrm{gm}}(F)$, the $\alpha^* c_n(M)$ are in addition bounded with finitely generated cohomology and vanish for n big enough.*

Proof. We have that α^* commutes with restriction to Spec F^{sep}, hence $\alpha^* c_n(M)$ is geometrically constant, i.e. isomorphic to a complex of constant étale sheaves over Spec F^{sep}. This implies that it is induced by a complex of sheaves on the small étale site. □

Remarks 5.5. (1) There is a competing notion of *étale mixed Tate motive*: $M \in DM^{\mathrm{eff}}_{-}(F)$ is étale mixed Tate if it satisfies the conclusion of this lemma. The two categories agree after tensoring with $\mathbf{Q}$ by [Voe00b, Proposition 3.3.2], but not integrally. Any torsion motive is étale mixed Tate by [Voe00b, Proposition 3.3.3], but not necessarily geometrically mixed Tate. For example, let A be an abelian variety over $F = F^{\mathrm{sep}}$. Then $U \mapsto A(U)/2$ is a homotopy invariant Nisnevich sheaf with transfers whose étale sheafification vanishes, so it is étale mixed Tate; on the other hand it is not geometrically mixed Tate. Indeed, it is well known that A is a birational sheaf. If our sheaf was étale mixed Tate, it would have to be constant.

(2) This example shows that the functor $\alpha^* : DM^{\mathrm{eff}}_{-}(F) \to DM^{\mathrm{eff}}_{-,\mathrm{\acute{e}t}}(F)$ is not conservative on étale mixed Tate motives. This is also true for the functor $DM^{\mathrm{eff}}_{-}(F) \to DM^{\mathrm{eff}}_{-}(F^{\mathrm{sep}})$ on geometrically mixed Tate motives: consider for example the sheaf $U \mapsto H^1_{\mathrm{\acute{e}t}}(\pi_0(U), \mathbf{Z}/2)$ where $\pi_0(U)$ is the scheme of constants of U.

Recall that the category of *mixed Artin motives* is the subcategory of $DM^{\mathrm{eff}}_{\mathrm{gm}}(F)$ generated by zero-dimensional varieties. The category of *mixed Artin–Tate motives* is the thick tensor subcategory of $DM^{\mathrm{eff}}_{\mathrm{gm}}(F)$ generated by mixed Artin motives and mixed Tate motives.

Proposition 5.6. *The category of mixed Artin–Tate motives is contained in $T^g DM^{\mathrm{eff}}_{\mathrm{gm}}(F)$. After extension of coefficients to $\mathbf{Q}$ it is even equal to the latter.*

Proof. Obviously Artin motives are geometrically Tate. This implies the first statement. For $M \in T^g DM^{\mathrm{eff}}_{\mathrm{gm}}(F) \otimes \mathbf{Q}$ it suffices to show that $c_n(M)$ is a mixed Artin motive. By [Voe00b, Proposition 3.3.2] we can work on the étale site. By Lemma 5.4, $c_n(M)$ is a bounded complex. It suffices to consider its cohomology sheaves which are finitely generated and constant over F^{sep}. Hence they correspond to finite-dimensional representations of $G(F^{\mathrm{sep}}/F)$, i.e. to Artin motives. □

6. Spectral sequences again

Recall that in [Kah99] were defined similar spectral sequences to (3.1) and (3.2) for (geometrically) cellular varieties. We are now going to compare these spectral sequences with those of § 3.

Recall that the spectral sequences of [Kah99] are constructed from a filtration

$$\mathbf{Z}(n,0,X) \to \mathbf{Z}(n,1,X) \to \cdots \to \mathbf{Z}(n,n,X),$$

where

$$\mathbf{Z}(n,i,X) = \underline{\mathrm{Hom}}(M(X),\mathbf{Z}(i))(n-i).$$

More generally, for any $M \in DM_-^{\mathrm{eff}}(F)$ and any integer $n \geqslant 0$, consider the filtration

$$\mathbf{Z}(n,0,M) \to \mathbf{Z}(n,1,M) \to \cdots \to \mathbf{Z}(n,n,M),$$

where

$$\mathbf{Z}(n,q,M) = \underline{\mathrm{Hom}}(M,\mathbf{Z}(q))(n-q).$$

Let $\mathbf{Z}(n,q/q-1,M)$ be the cone of the morphism $\mathbf{Z}(n,q-1,M) \to \mathbf{Z}(n,q,M)$ defined analogously to [Kah99, p. 153]. There is an associated spectral sequence

$$E_2^{p,q} = H^{p+q}(F,\mathbf{Z}(n,q/q-1,M)) \Rightarrow H^{p+q}(M,\mathbf{Z}(n)), \tag{6.1}$$

generalising [Kah99, Equation (4), p. 153].

We define a morphism

$$\varphi_q : \mathbf{Z}(n,q,M) \to \underline{\mathrm{Hom}}(\nu_{\leqslant q}M,\mathbf{Z}(n)) \tag{6.2}$$

as follows. Start from the evaluation morphism

$$\underline{\mathrm{Hom}}(M,\mathbf{Z}(q)) \otimes M \to \mathbf{Z}(q).$$

Note that $\underline{\mathrm{Hom}}(M,\mathbf{Z}(q)) \otimes \nu^{\geqslant q+1}M$ is in $DM_-^{\mathrm{eff}}(F)(q+1)$, hence the evaluation map to $\mathbf{Z}(q)$ vanishes on it. This implies that it factors, even canonically, through $\underline{\mathrm{Hom}}(M,\mathbf{Z}(q)) \otimes \nu_{\leqslant q}M$. We get the desired morphism by first tensoring both sides by $\mathbf{Z}(n-q)$ and then using adjunction. These morphisms are easily seen to be compatible with the transition maps.

Remark 6.1. The complexes $\mathbf{Z}(n,q,M)$ are unbounded in general. In order for the above calculations to be well defined we would have to work in the bigger category $DM^{\mathrm{eff}}(F)$ of unbounded motivic complexes; see e.g. [Voe03b] and [Wei04]. However, in the situation of [Kah99], where $M = M(X)$ for a cellular variety X, they are in fact bounded above and within the scope of [Voe00b].

Proposition 6.2. *The morphism φ_n is an isomorphism. The collection of morphisms φ_q defines a morphism θ of spectral sequences from (3.1) to (6.1), with the same abutments. If $M \in TDM_-^{\mathrm{eff}}(F)$, the φ_q are isomorphisms, so θ is an isomorphism of spectral sequences. If $M \in T^gDM_-^{\mathrm{eff}}(F)$, then the $\alpha^*\varphi_q$ are isomorphisms.*

Proof. The only things to justify are the assertions on isomorphisms. By dévissage we reduce to the case where M is of the form $\mathbf{Z}(i)$ for some $i \geqslant 0$, and then it is obvious (the two sides of (6.2) are 0 for $i > q$ and isomorphic to $\mathbf{Z}(n-i)$ for $i \leqslant q$). If M is geometrically mixed Tate, we have an isomorphism over F^{sep}. This implies that it also becomes an isomorphism on the étale site. $\square$

Remark 6.3. The étale case is more complicated because α^* does not commute with the internal Homs: this accounts for the delicate description of the abutment of Equation (6), p. 159 in [Kah99]. Taking the pull-back of (6.2), we get a chain of morphisms,

$$\alpha^*\mathbf{Z}(n,q,M) \to \alpha^*\,\underline{\mathrm{Hom}}(\nu_{\leqslant q}M,\mathbf{Z}(n)) \to \underline{\mathrm{Hom}}_{\text{ét}}(\alpha^*\nu_{\leqslant q}M,\alpha^*\mathbf{Z}(n)), \tag{6.3}$$

where $\underline{\mathrm{Hom}}_{\text{ét}}$ is the partial internal Hom of $DM^{\text{eff}}_{-,\text{ét}}(F)$. The left objects yield a spectral sequence

$$E_2^{p,q} = H^{p+q}_{\text{ét}}(F, \alpha^*\mathbf{Z}(n, q/q-1, M)) \Rightarrow H^{p+q}_{\text{ét}}(F, \alpha^* \underline{\mathrm{Hom}}(M, \mathbf{Z}(n))), \tag{6.4}$$

generalising [Kah99, Equation (6), p. 159]. The middle objects yield an intermediate spectral sequence, with the same abutment as (6.4). By the last assertion of Proposition 6.2, this spectral sequence is isomorphic to (6.4) in the case of geometrically Tate motivic complexes. Finally, the right-most objects in (6.3) yield the 'right' spectral sequence (3.2): this solves the question in [Kah99, Remark 4.5].

7. The motive of a torus

Let T be a torus. In this section we compute the étale fundamental invariants of T explicitly. When T is split, this computation is valid in the Nisnevich topology. We then describe an almost functorial decomposition of $M(T)$ in the split case.

Recall that $M(\mathbb{G}_m) \cong \mathbf{Z} \oplus \mathbf{Z}(1)[1]$. Hence, *a priori*, $c_n(\mathbb{G}_m^r)$ is concentrated in degree n. By étale sheafification, the same is true for $\alpha^* c_n(T)$ for any torus.

Let $\chi : \mathbb{G}_m \to T$ be a cocharacter. It induces a morphism

$$c_1(\chi) : \mathbf{Z}[-1] = c_1(\mathbb{G}_m) \to c_1(T).$$

Collecting these gives a natural map

$$\Xi(F) \to \mathrm{Hom}(H^1 c_1(\mathbb{G}_m), H^1 c_1(T)) = H^1 c_1(T)(F), \tag{7.1}$$

where $\Xi(F)$ is the group of all F-rational cocharacters of T.

Lemma 7.1. *The map (7.1) is a group homomorphism.*

Proof. Let $\xi_1, \xi_2 \in \Xi(F)$. Then $c_1(\xi_1 \cdot \xi_2)$ is computed via

$$c_1(\mathbb{G}_m) \xrightarrow{c_1(\xi_1,\xi_2)} c_1(T \times T) \xrightarrow{c_1\mu} c_1(T).$$

The statement follows from the Künneth formula of Lemma 4.8, which implies that $c_1(\mu) = c_1(p_1) + c_1(p_2)$ (with p_i the projection to the factor i.) ☐

Proposition 7.2. *Let $\Xi = \mathrm{Hom}(\mathbb{G}_m, T)$ be the cocharacter group of T, viewed as a locally constant étale sheaf. Then there exists a natural isomorphism in $DM^{\text{eff}}_{-,\text{ét}}(F)$,*

$$\Lambda^n(\Xi)[-n] \xrightarrow{\sim} \alpha^* c_n M(T). \tag{7.2}$$

The isomorphism (7.2) is compatible with homomorphisms of tori and the Künneth formula. If T is split (hence Ξ is constant), these isomorphisms already hold in $DM^{\text{eff}}_{\text{gm}}(F)$.

Proof. We first construct the homomorphism of (7.2). As $M(T)$ is geometrically mixed Tate, the cohomology sheaves of its fundamental invariants come from the small étale site of F. We compute in the smaller category. By Lemma 7.1, the sheafification of (7.1),

$$\Xi \to H^1 \alpha^* c_1(T),$$

is a homomorphism of étale sheaves. We get a composite morphism for all $n > 0$,

$$(\Xi)^{\otimes n} \to (H^1 \alpha^* c_1(T))^{\otimes n} \to H^n \alpha^* c_n M(T), \tag{7.3}$$

where the second morphism is induced by multiplication. As usual, the symmetric group operates on this map via the signature. Therefore (7.3) factors into

$$\Lambda^n(\Xi) \to H^n \alpha^* c_n(T), \tag{7.4}$$

as promised. The map (7.3) (and hence (7.4)) is natural with respect to homomorphisms of tori. Clearly, if T is split, the sheaves are constant.

Let us now show that our morphism (7.2) is an isomorphism. It is sufficient to see that (7.4) is an isomorphism. The assertion is local for the étale topology, so we may and do assume that T is split.

We shall argue by induction on $d = \dim T$. If $d = 1$, this is trivial by construction. Let us assume that $d > 1$. Since T is split, we may write it as $T = T_1 \times T_2$ with T_1 and T_2 of smaller dimension. Let Ξ_1 and Ξ_2 be the corresponding cocharacter groups. For every $n > 0$ we have an isomorphism

$$\bigoplus_{i+j=n} \Lambda^i(\Xi_1) \otimes \Lambda^j(\Xi_2) \xrightarrow{\sim} \Lambda^n(\Xi)$$

given on the (i, j) factor by

$$v_1 \wedge \cdots \wedge v_i \otimes w_{j+1} \wedge \cdots \wedge w_n \mapsto v_1 \wedge \cdots \wedge v_i \wedge w_{j+1} \wedge \cdots \wedge w_n.$$

The collection of these isomorphisms is by construction compatible with the Künneth formula for $H^n\alpha^* c_n(T)$ (cf. Lemma 4.8) under the morphisms of (7.3) and hence (7.4). By induction, (7.2) is an isomorphism for T_1 and T_2, hence also for T. □

Example 7.3. The smallest-dimensional example of a non-split torus is

$$T = R^1_{E/F}\mathbb{G}_m =: \mathrm{Ker}(N_{E/F} : R_{E/F}\mathbb{G}_m \to \mathbb{G}_m),$$

where E/F is a quadratic extension and $R_{E/F}$ denotes Weil restriction of scalars. Here $\dim T = 1$. One easily sees that $\widetilde{M}(T) \simeq c(1)[1]$ where $c = \mathrm{Coker}(L(\mathrm{Spec}\, F) \to L(\mathrm{Spec}\, E))$, the map being transpose to the norm map. We leave the details to the reader.

Now assume that T is split. For any choice of splitting $\sigma : T \to \mathbb{G}_m^r$, there is an induced isomorphism

$$\Phi^\sigma : M(T) \to M(\mathbb{G}_m^r) \to \bigoplus \Lambda^n \mathbf{Z}^r(n)[n] \to \bigoplus \Lambda^n \Xi(n)[n]. \tag{7.5}$$

We are going to study to what extent this isomorphism is natural. Let $\alpha : T \to T'$ be a homomorphism of tori. It is equivalent to a homomorphism $\Xi \to \Xi'$, which we will also denote by α. Choices of trivialisations of T and T' induce matrix coefficients $\alpha_{ij} : \Lambda^j\Xi(j)[j] \to \Lambda^i\Xi'(i)[i]$. We have

$$\alpha_{ij} \in \mathrm{Hom}(\Lambda^j\Xi, \Lambda^i\Xi') \otimes K^M_{i-j}(F) \tag{7.6}$$

by [SV00, Theorem 3.4].

Lemma 7.4.

(i) *The matrix (α_{ij}) is lower triangular, i.e. $\alpha_{ij} = 0$ for $i < j$.*
(ii) *One has $\alpha_{i0} = 0$ for $i > 0$.*
(iii) *One has $\alpha_{ii} = \Lambda^i(\alpha)$ for all i.*

In particular, these terms are independent of the choices of splittings of T and T'.

Proof. Property (i) follows from weight reasons. The unit map is compatible with all morphisms of tori, hence (ii). Finally (iii) holds because $c_i(\Phi^\sigma)$ gives back the natural isomorphism (7.2). □

Let $m \neq 0$ and let $[m]$ denote multiplication by m on a (split) torus. The matrix of $[m]$ (in the above sense) is diagonal for $\mathbb{G}_m$ and hence in general, i.e. $[m]_{ij} = 0$ for $i > j$. By Lemma 7.4(iii), $[m]_{ii}$ is multiplication by m^i. Thus, by the same argument as using Adams operations (using the equality $M(\alpha)M([m]) = M([m])M(\alpha)$), we get that in general α_{ij} is torsion for $i > j$. This shows that (7.5) is independent of σ at least after tensoring with $\mathbf{Q}$. We are going to substantially refine this remark. We need a little preparation.

Lemma 7.5. *Let $t = (t_1, \dots, t_r)$ be a rational point of $\mathbb{G}_m^r$. Let Ξ be the cocharacter group of $\mathbb{G}_m^r$. Then the component $M(t)_p$ of*

$$M(t) \in \operatorname{Hom}(\mathbf{Z}, M(\mathbb{G}_m^r)) = \bigoplus \Lambda^p(\Xi) \otimes K_p^M(F)$$

on the summand $\Lambda^p(\Xi) \otimes K_p^M(F)$ is given by the formula

$$M(t)_p = \sum_{i_1 < \cdots < i_p} e_{i_1} \wedge \cdots \wedge e_{i_p} \otimes \{t_{i_1}, \dots, t_{i_p}\},$$

where $(e_1, \dots, e_r)$ denotes the standard basis of $\Xi = \mathbf{Z}^r$.

Proof. We immediately reduce to the case $r = 1$, where this follows from the definition of the isomorphism $\operatorname{Hom}(\mathbf{Z}, \mathbf{Z}(1)[1]) \simeq K_1^M(F)$ in [SV00, Theorem 3.4]. □

Notation 7.6. Let $E_r = \{1, \dots, r\}$ and, for $l \geqslant 0$, let $\mathcal{P}_l(E_r)$ be the set of all subsets of E_r with l elements. For $j = (j_1, \dots, j_p) \in (E_r)^p$, let $\underline{j}$ be its underlying set: we have $\underline{j} = \{k_1, \dots, k_l\} \in \mathcal{P}_l(E_r)$ with $1 \leqslant k_1 < \cdots < k_l \leqslant r$. We set $l = l(j)$ and, for $(t_1, \dots, t_r) \in F^{*r}$,

$$\{t_{\underline{j}}\} = \{t_{k_1}, \dots, t_{k_l}\} \in K_l^M(F).$$

Theorem 7.7. *Let T, T', Ξ, Ξ', α be as above, with $T = \mathbb{G}_m^r$ and $T' = \mathbb{G}_m^s$. As above let α_{pq} be the matrix coefficients of (7.6) with respect to the canonical trivialisations. Then one has the following.*

(a) *For $p > q$, the coefficient α_{pq} is of the form $\lambda_{pq}(\alpha) \otimes \{-1, \dots, -1\}$ for a unique*

$$\lambda_{pq}(\alpha) \in \operatorname{Hom}(\Lambda^q \Xi, \Lambda^p \Xi')/2.$$

(b) *Recall notation 7.6. Let $(e_1, \dots, e_r)$ and $(e'_1, \dots, e'_s)$ be the standard bases of Ξ and Ξ', and let $A = (a_{ij})_{i \in E_s, j \in E_r}$ be the matrix of α with respect to these bases. For $\underline{i} = \{i_1, \dots, i_p\} \in \mathcal{P}_p(E_s)$ with $i_1 < \cdots < i_p$ and $\underline{j} = \{j_1, \dots, j_q\} \in \mathcal{P}_q(E_r)$ with $j_1 < \cdots < j_q$, let $\delta_{\underline{i},\underline{j}} \in \operatorname{Hom}(\Lambda^q \Xi, \Lambda^p \Xi')$ be the map sending $e_{\underline{j}} := e_{j_1} \wedge \cdots \wedge e_{j_q}$ to $e_{\underline{i}}$ and $e_{\underline{j}'}$ to 0 for any $\underline{j}' \neq \underline{j}$. Then*

$$\lambda_{pq}(\alpha) = \sum_{\substack{\underline{i} \in \mathcal{P}_p(E_s) \\ j \in (E_r)^p,\, l(j) = q}} a_{i_1 j_1} \dots a_{i_p j_p} \delta_{\underline{i},\underline{j}}.$$

Proof. Let $t \in \mathbb{G}_m^r(F)$ and $u = \alpha(t)$, so that $M(u) = M(\alpha)M(t)$. Our strategy is to compute the matrices of both sides of this equation explicitly and compare coefficients.

We have $u_n = \prod_m t_m^{a_{nm}}$, hence

$$\{u_{i_1}, \dots, u_{i_p}\} = \sum_{(j_1, \dots, j_p) \in (E_r)^p} a_{i_1 j_1} \dots a_{i_p j_p} \{t_{j_1}, \dots t_{j_p}\}.$$

If $l = p$, we have

$$\{t_{j_1}, \dots, t_{j_p}\} = \varepsilon(j)\{t_{\underline{j}}\},$$

where $\varepsilon(j)$ is the signature of the permutation necessary to put the j_i in increasing order. If $l < p$, we have

$$\{t_{j_1}, \dots, t_{j_p}\} = \{-1, \dots, -1\}\{t_{\underline{j}}\}$$

thanks to the well-known identity in Milnor K-theory

$$\{a, a\} = \{-1, a\}.$$

The sign is unimportant since $\{-1, \dots, -1\}$ is killed by 2. Thus

$$\{u_{i_1}, \dots, u_{i_p}\} = \sum_{j \in (E_r)^p,\, l(j) = p} a_{i_1 j_1} \dots a_{i_p j_p} \varepsilon(j)\{t_{\underline{j}}\} + \sum_{j \in (E_r)^p, l(j) < p} a_{i_1 j_1} \dots a_{i_p j_p} \{-1\}^{p - l(j)}\{t_{\underline{j}}\}$$

and hence finally by Lemma 7.5

$$M(u)_p = \sum_{\underline{i}=(i_1<\cdots<i_p)} e_{\underline{i}} \otimes \Bigg(\sum_{j\in(E_r)^p,\, l(j)=p} a_{i_1j_1}\dots a_{i_pj_p}\varepsilon(j)\{t_{\underline{j}}\} + \sum_{j\in(E_r)^p,\, l(j)<p} a_{i_1j_1}\dots a_{i_pj_p}\{-1\}^{p-l(j)}\{t_{\underline{j}}\}\Bigg). \tag{7.7}$$

On the other hand, we may write

$$\alpha_{pq} = \sum_{\underline{i},\underline{k}} \delta_{\underline{i},\underline{k}} \otimes \alpha_{pq}^{\underline{i},\underline{k}}$$

with $\alpha_{pq}^{\underline{i},\underline{k}} \in K^M_{p-q}(F)$. Via Lemma 7.5,

$$(M(\alpha)M(t))_p = \sum_q \alpha_{pq}M(t)_q = \sum_{\underline{i}} e_{\underline{i}} \otimes \sum_{q,\underline{k}} \alpha_{pq}^{\underline{i},\underline{k}}\cdot\{t_{\underline{k}}\}. \tag{7.8}$$

The terms (7.7) and (7.8) are equal. The coefficient of $\underline{i}$ gives

$$\sum_{\substack{q\leqslant p\\ \underline{k}=(k_1<\cdots<k_q)}} \alpha_{pq}^{\underline{i},\underline{k}}\cdot\{t_{\underline{k}}\} = \sum_{j\in(E_r)^p,\, l(j)=p} a_{i_1j_1}\dots a_{i_pj_p}\varepsilon(j)\{t_{\underline{j}}\} + \sum_{j\in(E_r)^p, l(j)<p} a_{i_1j_1}\dots a_{i_pj_p}\{-1\}^{p-l(j)}\{t_{\underline{j}}\}. \tag{7.9}$$

As this computation is in $DM_{-}^{\mathrm{eff}}(F)$, we need not assume that F is perfect. The formulae are natural in F, hence we may extend scalars and apply (7.9) to the generic point of $\mathbb{G}_m^r$. Then the t_i are independent indeterminates. We take iterated residues with respect to all indeterminates. This shows that

$$\alpha_{pq}^{\underline{i},\underline{k}} = \begin{cases} \sum_{\underline{j}=\underline{k}} a_{i_1j_1}\dots a_{i_pj_p}\varepsilon(j) & \text{if } q=p,\\ \sum_{l(j)=q,\,\underline{j}=\underline{k}} a_{i_1j_1}\dots a_{i_pj_p}\{-1\}^{p-l(j)} & \text{if } q<p. \end{cases}$$

For $q = p$ we recognise an entry of $\Lambda^p(\alpha)$ (cf. Lemma 7.4(iii)). For $q < p$ we get the announced result. ☐

Example 7.8. Consider $\lambda_{21}(\alpha)(e_i) = \sum_{j<k} \lambda_{21}(\alpha)_i^{jk} e_j \wedge e_k$. Then we have

$$\lambda_{21}(\alpha)_i^{jk} = a_{ji}a_{ki}$$

where a_{ij} are the entries of α.

Corollary 7.9.

(a) *Let $\mu : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ be the multiplication map. Then $M(\mu)$ is diagonal.*

(b) *Let $\Delta : \mathbb{G}_m \to \mathbb{G}_m^2$ be the diagonal map. Then $\Delta_{21} = \{-1\}$.*

(c) *Let $\tau : \mathbb{G}_m^2 \to \mathbb{G}_m^2$ be the transposition of the two factors. Then $\tau_{21} = 0$, i.e. $M(\tau)$ is diagonal.*

(d) *For any permutation σ of $\{1,\dots,r\}$, let $[\sigma]$ be the corresponding action of σ on $\mathbb{G}_m^r$ by permutation of the coordinates. Then $M([\sigma])$ is diagonal.*

Proof. (a) This is obvious from Lemma 7.4. Parts (b) and (c) are obvious from Theorem 7.7. One could also get (d) from the explicit computation in the proof of this theorem, but an alternative argument is that any permutation is a composition of permutations of type $(i, i+1)$, which reduces to (c). ☐

Corollary 7.10.

(a) *The isomorphism (7.5) is invariant by permutations of the basis and commutes with the algebra structures given by the multiplication of T and the addition of Ξ.*

(b) *If* $\operatorname{char} F = 2$ *or after inverting* 2, *(7.5) is independent of the choice of σ and is a natural isomorphism of Hopf objects, where the Hopf object structure on $\bigoplus_{n\geqslant 0} \Lambda^n(\Xi)(n)[n]$ is induced by the Hopf algebra structure on the exterior algebra. The isomorphism is compatible with morphisms of tori and with the Künneth formula.*

Proof. (a) Invariance is clear from Corollary 7.9(d); for the second statement, we reduce to the case $T = \mathbb{G}_m^r$ and note that its multiplication may then be factored as

$$\mathbb{G}_m^r \times \mathbb{G}_m^r \xrightarrow{\sigma} (\mathbb{G}_m \times \mathbb{G}_m)^r \xrightarrow{\mu_0^r} \mathbb{G}_m^r,$$

where σ is a shuffle permutation and μ_0 is the multiplication of $\mathbb{G}_m$; the conclusion then follows from Corollary 7.9(a) and (d).

(b) Independence of the choice of trivialisation is clear from Theorem 7.7; compatibility with the Künneth formula follows by reduction to the trivial case. Naturality also follows from this independence. Note that the comultiplication of T is a homomorphism of tori. By naturality it is compatible with the decomposition. $\square$

Remarks 7.11. (1) If $\operatorname{char} F \neq 2$, then the matrix of $M(\Delta)$ is not diagonal by Corollary 7.9(b). This implies that there cannot be a natural transformation as in Corollary 7.10 compatible with the Künneth formula with *integral* coefficients.

(2) Corollary 7.9 shows that the matrix of $M(\alpha)$ is diagonal for quite a few α. Here is an algorithm to compute this matrix differently from the proof of Theorem 7.7. The homomorphism α can be factored into a product of standard operations. First embed $\mathbb{G}_m^r \to (\mathbb{G}_m^s)^r$ diagonally, then permute the factors to $(\mathbb{G}_m^r)^s$, and finally project each factor $\mathbb{G}_m^r$ to $\mathbb{G}_m$ by means of a row of α. In this factorisation of α, all terms except the initial one have a diagonal motivic matrix.

(3) Any scheme-theoretic morphism of split tori may be factored as a translation by a rational point followed by a homomorphism of tori. A translation by $t \in T(F)$ may be further factored as

$$T = T \times \operatorname{Spec} F \xrightarrow{1_T \times t} T \times T \xrightarrow{\mu} T,$$

where μ is the multiplication of T. Hence the above computations give an expression of the matrix of $M(f)$ for any morphism f of split tori. We leave the details to the interested reader.

(4) Suppose that $\operatorname{char} F \neq 2$. Then $\{-1,-1\} = 0$ as soon as F is 'non-exceptional' in the sense of Harris and Segal, i.e. that the image of $Gal(F(\mu_{2^\infty})/F)$ in $\mathbf{Z}_2^*$ does not contain -1 [Kah02, proof of Lemma B.3b]. This is true in particular if -1 is a square in F or if $\operatorname{char} F > 0$. In this case, only the coefficients of the matrix of $M(\alpha)$ on the principal and on the first lower diagonal may be non-zero. On the other hand, if F is an ordered field, then $\{-1\}^n \neq 0$ for all n.

(5) It would be interesting to understand the situation for non-split tori and the étale topology.

8. Relative slice filtration for toric bundles

Let T be a split torus of dimension r and X a principal T-bundle over a smooth variety Y over a field F (of arbitrary characteristic). In this section we want to study $M(X)$.

Definition 8.1. Let $E \to Y$ be a vector bundle, i_0 its zero section. The *motivic Euler class* is the composition of the Gysin morphism for i_0 with the isomorphism induced by homotopy invariance,

$$e(E) : M(Y) \cong M(E) \xrightarrow{i_0^*} M(Y)(r)[2r]. \tag{8.1}$$

Lemma 8.2.

(a) *One has* $e(E \oplus E') = e(E)e(E')$*; in particular,* $e(E)e(E') = e(E')e(E)$.

(b) *If* E *is trivial, then* $e(E) = 0$.

Lemma 8.3 (see Proposition C.1). *If* $L \to Y$ *is a line bundle, then* $e(L)$ *is multiplication by the Chern class* $c_1(L) \in H^2(Y, \mathbf{Z}(1))$.

Corollary 8.4. *If* L *and* L' *are line bundles, then* $e(L \otimes L') = e(L) + e(L')$.

This implies that the Euler class induces an operation of the group $\mathrm{Pic}(Y)$ on motivic cohomology. This operation is in fact the cup-product.

Definition 8.5. Let $\hat{T} = \mathrm{Hom}(T, \mathbb{G}_m)$ be the character group of T and $\Xi = \mathrm{Hom}(\mathbb{G}_m, T)$ its $\mathbf{Z}$-dual as in the previous section. For $\chi \in \hat{T}$ let L_χ be the line bundle associated to the $\mathbb{G}_m$-bundle obtained by push-out of X by χ.

Cup-product with $c_1(L_\chi)$ is a map $M(Y) \to M(Y)(1)[2]$. This induces a canonical map

$$\hat{T} \otimes M(Y) \to M(Y)(1)[2]. \tag{8.2}$$

Remark 8.6. The choice of a splitting $T \cong \mathbb{G}_m^r$ induces an isomorphism $\hat{T} \cong \mathbf{Z}^r$. Note, however, that the map (8.2) is independent of such a choice of basis.

Definition 8.7. Let $d^0 : M(Y) \to M(Y)(1)[2] \otimes \Xi$ be the dual of the map (8.2). Let

$$d^p : M(Y)(p)[2p] \otimes \Lambda^p(\Xi) \to M(Y)(p+1)[2p+2] \otimes \Lambda^{p+1}(\Xi)$$

be its extension to the exterior powers (induced by the algebra structure of $\Lambda^*(\Xi)$).

Theorem 8.8. *There is a filtration in* $DM_{\mathrm{gm}}^{\mathrm{eff}}(F)$,

$$\nu_Y^{\geqslant p+1} M(X) \to \nu_Y^{\geqslant p} M(X) \to \cdots \to M(X),$$

with $M(X) \cong \nu_Y^{\geqslant 0} M(X)$, $0 = \nu_Y^{\geqslant r+1} M(X)$, *together with distinguished triangles*

$$\nu_Y^{\geqslant p+1} M(X) \to \nu_Y^{\geqslant p} M(X) \to M(Y)(p)[p] \otimes \Lambda^p(\Xi)$$

for $0 \leqslant p \leqslant r$. *The induced map*

$$M(Y)(p)[p] \otimes \Lambda^p(\Xi) \to \nu_Y^{\geqslant p+1} M(X)[1] \to M(Y)(p+1)[p+2] \otimes \Lambda^{p+1}(\Xi)$$

equals $d^p[-p]$, *where* d^p *is as in Definition 8.7. We call* $\nu_Y^{\geqslant p} M(X)$ *the* relative slice filtration of X over Y.

Remark 8.9. The construction of the $\nu_Y^{\geqslant p} M(X)$ will depend on the choice of a splitting of T. Note that graded pieces of the filtration and the d^p are independent of this choice. If $X \cong T \times Y$, the proof will show that

$$\nu_Y^{\geqslant p} M(X) \cong M(Y) \otimes \nu^{\geqslant p} M(T),$$

i.e. the relative slice filtration is indeed induced by the slice filtration of $M(T)$. In particular it is independent of the choice of the splitting. In general, the relative slice filtration could be described as the image of the slice filtration of the motive of X in $DM_-^{\mathrm{eff}}(Y)$ under the restriction of scalars functor, where $DM_-^{\mathrm{eff}}(Y)$ is Voevodsky's category of motivic sheaves over Y (see [Voe03a]). However, here we shall use a more elementary approach based on the method of Levine in [Lev93b], which unfortunately is not sufficient to prove independence.

Before proving Theorem 8.8, we are going to point out the consequences.

Proposition 8.10. *Let*

$$H : DM^{\mathrm{eff}}_{\mathrm{gm}}(F) \to \mathcal{A}$$

be a covariant homological functor with values in some abelian category. Then there is a spectral sequence

$$E_1^{pq} = H^{q+2p}(M(Y)(p)) \otimes \Lambda^p(\Xi) \Rightarrow H^{p+q}(M(X))$$

with first differential on E_1^{pq} induced by d^p.

Proof. We apply H to the filtration of Theorem 8.8. □

Remark 8.11. If $M(Y)$ is a (geometrically) mixed Tate motive, then the homological functor H needs only to be defined on (geometrically) mixed Tate motives. By Theorem 8.8 all $\nu_Y^{\geqslant p}M(X)$ (in particular $M(X)$) stay in the subcategory.

Corollary 8.12. *There is a spectral sequence*

$$E_1^{pq} = H^{-q-2p}(Y, \mathbf{Z}(-p+i)) \otimes \Lambda^{-p}(\Xi) \Rightarrow H^{-p-q}(X, \mathbf{Z}(i)).$$

Proof. This is Proposition 8.10 with the cohomological functor $\mathrm{Hom}(\cdot, \mathbf{Z}(i))$. Note that this is a contravariant functor, so the signs of the indices have to be inverted. □

Remark 8.13. The same constructions can also be carried out for $M^c(X)$. In this case we need the assumption $\mathrm{char}\, F = 0$ and Y arbitrary, or $\mathrm{char}\, F > 0$ and Y smooth, coefficients in $\mathbf{Q}$; see Definition B.3. We leave it to the reader to work out the indices of the spectral sequences in this case. The two versions are dual to each other where both apply.

The rest of this section is devoted to the construction of the $\nu_Y^{\geqslant p}M(X)$ and the proof of the theorem. We follow Levine's method in [Lev93b, § 1].

Proof of Theorem 8.8. We fix an isomorphism $T \cong \mathbb{G}_m^r$. This induces $\hat{T} \cong \mathbf{Z}^r$. Let $\{e_1, \ldots, e_r\}$ be the standard basis and $L_i = L_{e_i}$. Then

$$E \cong \prod_{Y, i=1}^{r} L_i \to Y$$

is a partial compactification of X by a vector bundle. Fibrewise it is induced by the partial compactification of $\mathbb{G}_m^r$ by $\mathbf{A}^r$.

For $I = \{i_1, \ldots, i_t\} \subset \{1, \ldots, r\}$ we put

$$E_I = 0_1 \times_Y \cdots \times_Y L_{i_1} \times_Y \cdots \times_Y L_{i_t} \times_Y \cdots \times_Y 0_r.$$

Note that

$$E_{\{1,\ldots,r\}} \cong E, \quad E_{\{i\}} \cong L_i, \quad X = E \smallsetminus \bigcup_{|I|<r} E_I.$$

For $I \subset J$, we have a closed immersion $E_I \subset E_J$. These closed immersions give rise to Gysin exact triangles. The Gysin maps in these triangles are nothing but the Euler classes $e(E_{J \smallsetminus I})$. We follow Levine's method in [Lev93b, § 1] in order to organise these triangles.

We repeat Levine's construction in the model category of complexes of Nisnevich sheaves with transfer rather than in the category of pointed topological spaces. The E_I for all I form a cube of subvarieties of $E = \prod L_i$. Let

$$\widetilde{M}_{E_I}(E) = \underline{C}_*([L(E \smallsetminus E_I) \to L(E)])$$

as complex of Nisnevich sheaves with homotopy invariant cohomology. The motive of E with support in E_I (denoted $M_{E_I}(E)$) is the same object viewed in $DM_{\mathrm{gm}}(F)$. By definition the triangle

$$M(E \smallsetminus E_I) \to M(E) \to M_{E_I}(E)$$

is distinguished. By localisation and homotopy invariance

$$M_{E_I}(E) \cong M(E_I)(r-k)[2r-2k] \cong M(Y)(r-k)[2r-2k] \quad (k = |I|).$$

The $\widetilde{M}_{E_I}(E)$ form a cube in the category of complexes of sheaves. Let C be the total complex of this cube. In detail, put

$$C^p = \bigoplus_{|I|=r-p} \widetilde{M}_{E_I}(E).$$

Then C is nothing but the total complex of the double complex

$$C^0 \to C^1 \to \cdots \to C^r.$$

As in [Lev93b, p. 419]

$$C^p \cong \bigoplus_{|I|=r-p} M(Y)(p)[2p] \cong \Lambda^p(\Xi) \otimes M(Y)(p)[2p] \quad \text{in } DM_{\mathrm{gm}}.$$

As in [Lev93b, p. 416]

$$C \cong M(X) \quad \text{in } DM_{\mathrm{gm}}.$$

Let $\nu_Y^{\geqslant p} M(X)$ be the subobject of C with respect to the stupid truncation, i.e. the total complex of $C^p \to \cdots \to C^r$. The short exact sequence

$$0 \to \nu_Y^{\geqslant p+1} M(X) \to \nu_Y^{\geqslant p} M(X) \to C^p[-p] \to 0$$

gives rise to the distinguished triangle of the theorem.

The map $M(Y)(p)[p] \to M(Y)(p+1)[p+2]$ of the theorem is nothing but the boundary map $C^p \to C^{p+1}$. By construction it is induced by a linear combination of Euler classes $e(E_{J \smallsetminus I})$ with $|J| = r-p$, $|I| = r-p-1$. By Lemma 8.3 this Euler class is multiplication by the Chern class of the line bundle corresponding to the additional index. This finishes the proof of Theorem 8.8. ☐

In the special case $X = Y \times T$, we have $E = Y \times V$ where V is an affine space which partially compactifies T. All constructions in the proof are concerned with V and closed subsets of V. Hence it suffices to consider the case $Y = \operatorname{Spec} F$. By the universal property of the slice filtration, and the computation of the graded pieces of the relative slice filtration, they agree. This proves the claim of Remark 8.9.

9. Motives of reductive groups

Let G be a split reductive group over F. Hence G has a split maximal torus T defined over F (see [DG70, Exposé XIV, Théorème 1.1]). Let B be a Borel subgroup of G containing T.

Lemma 9.1. *One has that $M(G)$ and $M(G/T) \xrightarrow{\sim} M(G/B)$ are mixed Tate motives. The fundamental invariants of G/B are*

$$c_p(G/B) = CH^p(G/B)^*[0],$$

where $\cdot^*$ denotes the dual of a free abelian group.

Proof. The statements for G/B are a special case of Proposition 4.11 since it is known to be a cellular variety. Since G/T is an affine bundle over G/B, the map $M(G/T) \to M(G/B)$ is an isomorphism by homotopy invariance. Then, the fact that $M(G)$ is mixed Tate follows from Theorem 8.8. ☐

Remark 9.2. Köck computed the Chow motive of G/B explicitly in [Köc91, § 2] in terms of the root system. His result can be viewed in $DM_{\mathrm{gm}}^{\mathrm{eff}}(F)$, yielding an explicit description of $M(G/B)$.

PROPOSITION 9.3. *The complex $c_N(G) \in D^b_f(Ab)$ is isomorphic to the complex $K(G,N)^*$:*

$$\begin{aligned} CH^N(G/B)^* &\to \Lambda^1(\Xi) \otimes CH^{N-1}(G/B)^* \to \cdots \\ &\to \Lambda^s(\Xi) \otimes CH^{N-s}(G/B)^* \xrightarrow{\gamma_s} \\ &\Lambda^{s-1}(\Xi) \otimes CH^{N-s-1}(G/B)^* \xrightarrow{\gamma_{s-1}} \cdots \longrightarrow CH^{N-r}(G/B)^*, \end{aligned} \tag{9.1}$$

with $CH^N(G/B)^*$ in degree 0 and γ_s the map (8.2).

Proof. We consider the homological functor

$$H^0 c_N : TDM^{\mathrm{eff}}_{\mathrm{gm}} \to Ab.$$

Note that $c_N(M(p)) = c_{N-p}(M)[-2p]$. By Proposition 8.10 applied to $G \to G/T$ there is a spectral sequence

$$E_1^{pq} = H^q(c_{N-p}(G/B)) \otimes \Lambda^p(\mathbf{Z}^r) \Rightarrow H^{p+q}(c_N(G)).$$

By Lemma 9.1 the spectral sequence is concentrated in the $q = 0$ row. It degenerates at E_2. Its E_1-term is the complex given in the assertion. □

COROLLARY 9.4. *For all $n \geqslant 0$ there is a spectral sequence*

$$E_2^{p,q}(G,n) = H^{p-q}(F, K(G,q) \otimes \mathbf{Z}(n-q)) \Rightarrow H^{p+q}(G, \mathbf{Z}(n)),$$

where $K(G,q)$ is the dual of the complex of Proposition 9.3.

Proof. This is the slice spectral sequence for $M(G)$ together with Proposition 9.3. □

Remark 9.5. The complex $K(G,q)$ is the same as the one considered in [EKLV98, § 3.14].

Remark 9.6. If G is reductive but not split reductive, then $M(G)$ is geometrically mixed Tate. We expect the same isomorphism as in Proposition 9.3, hence the same spectral sequence as in Corollary 9.4, in the étale topology. We do not know how to deduce this from our construction, but it should be possible along the lines of Remark 8.9.

Example 9.7. Let $G = GL_2(F)$, T the diagonal torus, and B the subgroup of upper triangular matrices. Hence $G/B = \mathbf{P}^1$ and

$$CH^p(G/B)^* = \begin{cases} \mathbf{Z} & p = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, $\Xi = \mathbf{Z}^2$. The complexes $K(G,N)$ vanish for $N < 0$ and $N > 3$ and have at most two terms in the remaining cases. In detail,

$$\begin{aligned} c_0 &= \mathbf{Z}[0], \\ c_1 &= [\mathbf{Z} \to \mathbf{Z}^2 \otimes \mathbf{Z}][0], \\ c_2 &= [\mathbf{Z}^2 \otimes \mathbf{Z} \to \Lambda^2\mathbf{Z}^2 \otimes \mathbf{Z}][-1], \\ c_3 &= [\Lambda^2\mathbf{Z}^2 \otimes \mathbf{Z}][-2], \end{aligned}$$

where the left-hand term of the complex is always put in degree 0. The differential is induced by cup-product with the first Chern class of a line bundle, in particular by an isomorphism $CH^1(G/B)^* \to CH^0(G/B)^*$. This allows one to compute the cohomology of the complexes:

$$H^i c_N(GL_2(F)) = \begin{cases} \mathbf{Z} & (N,i) = (0,0),(1,1),(2,1),(3,2), \\ 0 & \text{otherwise.} \end{cases}$$

As expected

$$H^* c_*(GL_2(F)) \cong \Lambda^* \mathbf{Z}^2$$

with generators in bidegrees $(1,1),(2,1)$.

Remark 9.8. Analogous results for $H^*c_*(G)$ and even $M(G)$ for general split reductive groups G are obtained by Biglari in [Big04].

Appendix A. Quasi-invertibility

Let F be a perfect field.

Proposition A.1. *The functor*

$$DM_-^{\mathrm{eff}}(F) \to DM_-^{\mathrm{eff}}(F)$$
$$A \mapsto A(1)$$

is fully faithful.

Proof. Let $(A,B) \in DM_-^{\mathrm{eff}}(F) \times DM_-^{\mathrm{eff}}(F)$. We have to show that the natural homomorphism

$$\mathrm{Hom}_{DM_-^{\mathrm{eff}}(F)}(A,B) \to \mathrm{Hom}_{DM_-^{\mathrm{eff}}(F)}(A(1),B(1))$$

is an isomorphism. By adjunction, this amounts to showing that the natural map

$$B \to \underline{\mathrm{Hom}}_{\mathrm{eff}}(\mathbf{Z}(1),B(1))$$

is an isomorphism.

Consider the full subcategory $\mathcal{T}$ of $DM_-^{\mathrm{eff}}(F)$ consisting of those B for which this map is an isomorphism. It is clearly triangulated. It is even localising (i.e. stable under infinite direct sums) by using [Voe00b, Proposition 3.2.8] for $X = \mathbf{P}^1$ and because Nisnevich cohomology commutes with infinite direct sums. By [Voe02c] it contains the image of $DM_{\mathrm{gm}}^{\mathrm{eff}}(F)$, which is dense in $DM_-^{\mathrm{eff}}(F)$ by [Voe00b, Theorem 3.2.6]. Therefore, $\mathcal{T} = DM_-^{\mathrm{eff}}(F)$. □

Lemma A.2. *For any $A \in DM_-^{\mathrm{eff}}(F)$, one has* $\mathrm{Hom}(A(1),\mathbf{Z}) = \mathrm{Hom}(A(1),\mathbf{Q}) = 0$.

Proof. Again, consider the full subcategory of those A verifying either of the conclusions of the lemma. It is triangulated and localising, because Hom transforms direct sums into products. By [Kah99, Lemma 2.1(a)], it contains $DM_{\mathrm{gm}}^{\mathrm{eff}}(F)$, so it is equal to $DM_-^{\mathrm{eff}}(F)$. □

Proposition A.3. *The functor*

$$DM_{-,\text{ét}}^{\mathrm{eff}}(F) \to DM_{-,\text{ét}}^{\mathrm{eff}}(F)$$
$$A \mapsto A(1)$$

is fully faithful.

Proof. We reduce again to proving that the natural map

$$B \to \underline{\mathrm{Hom}}_{\mathrm{eff}}^{\text{ét}}(\mathbf{Z}(1),B(1))$$

is an isomorphism, where the right-hand side is now the partially defined internal Hom of $DM^{\text{eff}}_{-,\text{ét}}(F)$, which exists by the same argument as [Voe00b, Proposition 3.2.8]. For notational simplicity, let us denote the right-hand side by $f(B)$. We have the exact triangle

$$B \longrightarrow B \otimes \mathbf{Q} \longrightarrow B \otimes \mathbf{Q}/\mathbf{Z} \xrightarrow{+1} \tag{A.1}$$

which yields a commutative diagram of exact triangles

$$\begin{array}{ccccccc} B & \longrightarrow & B \otimes \mathbf{Q} & \longrightarrow & B \otimes \mathbf{Q}/\mathbf{Z} & \xrightarrow{+1} & \\ \downarrow & & \downarrow & & \downarrow & & \\ f(B) & \longrightarrow & f(B \otimes \mathbf{Q}) & \longrightarrow & f(B \otimes \mathbf{Q}/\mathbf{Z}) & \xrightarrow{+1} & \end{array}$$

and it suffices to show that the middle and right vertical maps are isomorphisms. For the middle one, this follows from the previous proposition and [Voe00b, Proposition 3.3.2]. For the right one, we use [Voe00b, Proposition 3.3.3] which implies that $B \otimes \mathbf{Q}/\mathbf{Z}$ is quasi-isomorphic to a complex of ind-locally constant étale sheaves of order prime to the characteristic; then the result follows from the known étale cohomology of $\mathbf{P}^1$ with such coefficients and the fact that in $DM^{\text{eff}}_{-,\text{ét}}(F)$ the object $\mathbf{Z}/m(1)$ equals μ_m for $(m, \operatorname{char} F) = 1$. $\square$

Recall from Definition 3.1 the objects $\mathbf{Z}(n)_{\text{ét}} \in DM^{\text{eff}}_{-,\text{ét}}(F)$ for $n \in \mathbf{Z}$. For $n > 0$, the isomorphisms

$$\mathbf{Z}(n)_{\text{ét}} \otimes \mathbf{Z}/m \xrightarrow{\sim} \mu_m^{\otimes n} \qquad ((m, \operatorname{char} F) = 1)$$

together with the connecting morphism of the triangle

$$\mathbf{Z} \to \mathbf{Z}_{(p)} \longrightarrow \bigoplus_{l \neq p} \mathbf{Q}_l/\mathbf{Z}_l \longrightarrow \tag{A.2}$$

yield a natural composite map

$$\mathbf{Z}(-n)_{\text{ét}}(n) = \mathbf{Z}(n)_{\text{ét}} \otimes \mathbf{Z}(-n)_{\text{ét}} \xrightarrow{\sim} \bigoplus_{l \neq \operatorname{char} F} \mathbf{Q}_l/\mathbf{Z}_l[-1] \to \mathbf{Z}. \tag{A.3}$$

Proposition A.4. *For any $M \in DM^{\text{eff}}_{-,\text{ét}}(F)$ and $n > 0$, the map (A.3) induces an isomorphism*

$$\operatorname{Hom}(M, \mathbf{Z}(-n)_{\text{ét}}) \xrightarrow{\sim} \operatorname{Hom}(M(n), \mathbf{Z}(-n)_{\text{ét}}(n)) \xrightarrow{\sim} \operatorname{Hom}(M(n), \mathbf{Z})$$

where the first isomorphism is that of Proposition A.3.

Proof. Let $p = \operatorname{char} F$. By the above exact triangle (A.2) it suffices to prove that $\operatorname{Hom}(M(n), \mathbf{Z}_{(p)}) = 0$. If $p > 0$, then $\operatorname{Hom}(M(n), \mathbf{Q}_p/\mathbf{Z}_p) = 0$ by [Voe00b, Proposition 3.3.3], hence we are reduced to checking that $\operatorname{Hom}(M(n), \mathbf{Q}) = 0$. As the second argument is rational, we may replace $DM^{\text{eff}}_{-,\text{ét}}(F)$ by $DM^{\text{eff}}_{-}(F)$ in this computation (see [Voe00a, Proposition 5.28]). The statement now follows from Lemma A.2. $\square$

Appendix B. Duality

By [Voe00b, § 4.3], $DM_{\text{gm}}(F)$ enjoys a perfect duality when F is of characteristic 0. In [Lev98, Part I, ch. IV, §§ 1.4 and 1.5], Levine gives a simple argument which may replace the one of Voevodsky and easily extends to yield a perfect duality on $d^{\otimes}_{\leqslant 2} DM_{\text{gm}}(F)$ (the thick tensor triangulated subcategory of $DM_{\text{gm}}(F)$ generated by motives of smooth varieties of dimension at most 2) or $DM_{\text{gm}}(F, \mathbf{Q})$ when F is of characteristic greater than 0. Let us briefly review this argument:

(i) There is a tensor functor $Chow^{\text{eff}}(F) \to DM^{\text{eff}}_{\text{gm}}(F)$ [Voe00b, Proposition 2.1.4], which induces a tensor functor $Chow(F) \to DM_{\text{gm}}(F)$.

(ii) The category $Chow(F)$ is rigid. In particular, any pure motive $M \in Chow(F)$ has a dual in the sense of [DP80]. It follows that the image of M in $DM_{\mathrm{gm}}(F)$ has a dual in the same sense.

(iii) The full subcategory of $DM_{\mathrm{gm}}(F)$ consisting of objects that have a dual is tensor-triangulated and thick.

(iv) In characteristic 0, by Hironaka [Hir64] the image of $Chow(F)$ is dense in $DM_{\mathrm{gm}}(F)$, so every object has a dual by (2) and (3). In characteristic p, we get the same conclusion either in dimension at most 2 by Abhyankar [Abh69] or Lipman [Lip78], or in all dimensions but with rational coefficients by de Jong [dJo96]. To apply de Jong, note that, if $f : U \to V$ is an étale finite morphism of degree d between two smooth varieties, then $M(f) \circ M({}^{\mathrm{t}}f)$ is multiplication by d, where ${}^{\mathrm{t}}f$ is the transpose of the graph of f viewed as a finite correspondence, hence $M(V)$ is a direct summand of $M(U)$ in $DM_{\mathrm{gm}}(F, \mathbf{Q})$.

By construction $M^*(n)$ is effective for $M \in d_{\leqslant n}DM_{\mathrm{gm}}$ because this is true for smooth projective varieties.

Lemma B.1. *Let X be a variety of dimension at most d such that $M(X)$ has a dual in $DM_{\mathrm{gm}}(F, A)$ with $A = \mathbf{Z}, \mathbf{Q}$. Let $m \geqslant 0$. Then*

$$\underline{\mathrm{Hom}}(A(m), M(X)^*(d)[2d]) \cong \begin{cases} \underline{\mathrm{Hom}}(M(X), A(d-m)[2d]) & \text{if } m \leqslant d, \\ 0 & \text{if } m > d. \end{cases}$$

Proof. Let U be smooth, $q \in \mathbf{Z}$, $m \leqslant d$. Using the universal properties of $\underline{\mathrm{Hom}}$ and duality together with quasi-invertibility, we have

$$\begin{aligned} \mathrm{Hom}(M(U),\ & \underline{\mathrm{Hom}}(A(m), M(X)^*(d)[2d])[q]) \\ & \cong \mathrm{Hom}(M(U)(m), M(X)^*(d)[2d+q]) \\ & \cong \mathrm{Hom}(M(U \times X)[-2d], A(d-m)[2d+q]) \\ & \cong \mathrm{Hom}(M(U), \underline{\mathrm{Hom}}(M(X), A(d-m))[q]). \end{aligned}$$

This proves the formula in the first case. The second case is proved in the same way, using Lemma A.2. Note that the case distinction is necessary in order to ensure that the arguments of $\underline{\mathrm{Hom}}$ are always effective. □

This allows one to deduce the same formula for the dual that is used in Voevodsky's approach [Voe00b, Corollary 4.3.6].

Corollary B.2. *Let X be a variety of dimension at most d which has a dual in $DM_{\mathrm{gm}}(F, A)$ for $A = \mathbf{Z}$ or $\mathbf{Q}$ respectively. Then*

$$M(X)^* = \underline{\mathrm{Hom}}(M(X), A(d))(-d).$$

In particular, $\underline{\mathrm{Hom}}(M(X), A(d))$ is in $DM^{\mathrm{eff}}_{\mathrm{gm}}$.

Proof. Apply the $m = 0$ case of Lemma B.1. As remarked before, $M(X)^*(d)$ is geometric and effective. □

Duality allows us to define the motive with compact supports of a smooth variety X even in characteristic p.

Definition B.3. Let X be a smooth variety of pure dimension d. Assume one of the following cases:

(i) char $F = 0$;

(ii) char $F > 0$, F is perfect and $d \leqslant 2$;

(iii) $\operatorname{char} F > 0$ and we take $\mathbf{Q}$ coefficients;

(iv) $\operatorname{char} F > 0$ and X is smooth and projective.

In either of these cases, we put

$$M^c(X) = M(X)^*(d)[2d].$$

In the case $\operatorname{char} F = 0$ this is isomorphic to the motive with compact support as defined in [Voe00b]. Dualising the Gysin exact triangles gives localisation exact triangles when all terms are smooth.

LEMMA B.4. *Let $n \geqslant 0$. Then $d_{\leqslant n} DM_{\mathrm{gm}}(F)$ is generated by the $M^c(X)$ for X smooth of dimension at most n if either:*

(i) $\operatorname{char} F = 0$*;*

(ii) $\operatorname{char} F > 0$*, F is perfect and $n \leqslant 2$;*

(iii) $\operatorname{char} F > 0$ *and we take $\mathbf{Q}$ coefficients.*

Proof. Let $d^c_{\leqslant n} DM_{\mathrm{gm}}(F)$ be the thick subcategory generated by the said motives: we want to show that $d^c_{\leqslant n} DM_{\mathrm{gm}}(F) = d_{\leqslant n} DM_{\mathrm{gm}}(F)$. It suffices to show that, for any X of dimension at most n, $M^c(X) \in d_{\leqslant n} DM_{\mathrm{gm}}(F)$ and $M(X) \in d^c_{\leqslant n} DM_{\mathrm{gm}}(F)$. Using the form of resolution of singularities suited to the context, we reduce by Gysin or localisation to the case where X is smooth projective, and then we have $M(X) = M^c(X)$. $\square$

Appendix C. Motivic Euler class of a line bundle

The purpose of this appendix is to prove the following proposition.

PROPOSITION C.1. *Let L be a line bundle over a smooth base scheme S. Then the motivic Euler class of L*

$$e(L) : M(S) \to M(S)(1)[2]$$

(see Definition 8.1) is given by cup-product with $c_1(L)$.

Remark C.2. Depending on the normalisation a sign might have to be introduced in the proposition.

We fix some notation. Local parameters of L are denoted by l. We write $i_0 : S \to L$ for the zero section. Recall that the Euler class is induced by the Gysin sequence for the smooth pair $(L, L \setminus i_0(S))$:

$$M(S) \cong M(L) \to M(S)(1)[2]. \qquad (*)$$

We are going to prove the proposition by going through the definition of the Gysin map in [Voe00b, § 3.5].

DEFINITION C.3. The projective closure of L is the projective bundle

$$\bar{L} = \mathbf{P}(L \times_S \mathbf{A}^1).$$

There is a natural inclusion

$$\begin{aligned} j : L &\to \bar{L}; \\ l &\mapsto [l : 1]. \end{aligned}$$

The section at infinity $i_\infty : S \to \bar{L}$ is given by $s \mapsto (s, [1:0])$. Consider the diagram in the proof of [Voe00b, Proposition 3.5.3]. We have $X = L$, $Z = i_0(S)$. The blow-up $X_Z = X = L$ because Z only has codimension 1. In particular $p^{-1}(Z) = Z = i_0(S)$. The diagram also contains the blow-up of $L \times \mathbf{A}^1$ in $i_0(S) \times 0$. We denote it by $\widetilde{L \times \mathbf{A}^1}$.

Lemma C.4. *The exceptional fibre E of $\widetilde{L \times \mathbf{A}^1} \to L \times \mathbf{A}^1$ is isomorphic to $\bar{L}$. $\widetilde{L \times \mathbf{A}^1}$ is an $\mathbf{A}^1$-bundle over E. In particular $M(\widetilde{L \times \mathbf{A}^1}) \cong M(\bar{L})$.*

Proof. A point on L has coordinates (s, l) with $s \in S$ and l a local parameter of the line bundle. A point on $\widetilde{L \times \mathbf{A}^1}$ has local coordinates $(s, l, t) \times [l' : t']$ with equation $lt' = tl'$. The exceptional fibre has equation $l = t = 0$. There is a natural map

$$\begin{aligned} \mathbf{P}(L \times \mathbf{A}^1) &\to E \\ (s, [l:t]) &\mapsto (s, 0, 0) \times [l:t]. \end{aligned}$$

It is clearly an isomorphism. The bundle structure of $\widetilde{L \times \mathbf{A}^1}$ is given via

$$\begin{aligned} \widetilde{L \times \mathbf{A}^1} &\to E \\ (s, l, t) \times [l' : t'] &\mapsto (s, 0, 0) \times [l' : t']. \end{aligned}$$

□

Using homotopy invariance, our version of the diagram in [Voe00b, Proposition 3.5.3] now reads as follows:

$$\begin{array}{ccc} M(i_0(S)) & \xrightarrow{1} & M(\bar{L}) \\ {\scriptstyle 3}\downarrow & & \downarrow{\scriptstyle 4} \\ M(S) \oplus M(L) & \xrightarrow{2} & M(S) \oplus M(\bar{L}) \\ {\scriptstyle 5}\downarrow & & \downarrow{\scriptstyle 6} \\ M(S) & \xrightarrow{id} & M(S) \end{array}$$

The map 1 is the inclusion via the infinity section. The map 3 is the diagonal and the map 5 is the difference between the entries. The map 2 is the composition

$$L \xrightarrow{id \times 0} \widetilde{L \times \mathbf{A}^1} \to E = \bar{L}.$$

It maps $(s, l) \mapsto (s, l, 0) \times [1:0] \mapsto s \times [1:0]$, i.e. it agrees with the infinity section of $\bar{L}$. The map 6 is the identity on the first summand and minus the natural projection on the second. We also identify the other morphisms in the proof of [Voe00b, Proposition 3.5.3]. The map $id \times 1 : X \to X \times \mathbf{A}^1$ lifts to $(X \times \mathbf{A}^1)_{Z \times 0}$. In our case it is given by $(s, l) \mapsto (s, l, 1) \times [l : 1]$. Composed with the projection to the exceptional fibre this yields the standard inclusion $L \to \bar{L}$. We sum this up in the following lemma.

Lemma C.5. *The map $M(L) \to M(\bar{L})$ lifting the f constructed in [Voe00b, p. 221] is given by the difference $M(j) - M(i_\infty)$.*

Proof. One has that f is the difference between two morphisms. The first is the natural section of 5 (the map $(0, -id)$ in our case) composed with 2. This yields the section at infinity. The second map is induced by the section $id \times 1$. It yields the natural inclusion. Their difference is lifted via the splitting of 4. □

The Gysin map in [Voe00b, § 3.5] is given by the composition of this f with the natural projection $M(\bar{L}) \to M(S)(1)[2]$. We make its definition explicit following [Voe00b, Proposition 3.5.1]. Let $\mathcal{O}(1)$

be the standard line bundle on $\bar{L} = \mathbf{P}(L \times \mathbf{A}^1)$. Let $c = c_1(\mathcal{O}(1)) \in H^2(\bar{L}, \mathbf{Z}(1))$. Consider the composition

$$\sigma_1 : M(\bar{L}) \xrightarrow{\Delta} M(S) \otimes M(\bar{L}) \xrightarrow{id \otimes c} M(S)(1)[2].$$

This is nothing but cup-product with the Chern class of the standard line bundle.

LEMMA C.6. *We have $\sigma_1 \circ M(j) = 0$ and $\sigma_1 \circ M(i_\infty) = \cup c_1(L)$.*

Proof. Let $\iota : L \to \bar{L}$ be a morphism. Then the diagram

$$\begin{array}{ccccc} M(\bar{L}) & \longrightarrow & M(S) \otimes M(\bar{L}) & \xrightarrow{id \otimes c_1(\mathcal{O}(1))} & M(S)(1)[2] \\ {\scriptstyle M(\iota)}\uparrow & & \uparrow & & \uparrow \\ M(L) & \longrightarrow & M(S) \otimes M(L) & \xrightarrow{id \otimes c_1(\iota^* \mathcal{O}(1))} & M(S)(1)[2] \end{array}$$

commutes by the functoriality of c_1. We apply this to $\iota = j, i_\infty$. The line bundle $\mathcal{O}(1) \to \bar{L}$ has local coordinates $(s, l, t) \mapsto s \times [l : t]$. Pull-back via j is restriction to $t = 1$. It yields the line bundle with coordinates $(s, l, t) \mapsto (s, l)$, i.e. the trivial bundle on L. Its Chern class is zero. We have that i_∞ has the equation $t = 0$. Hence $i_\infty^* \mathcal{O}(1)$ is given by $(s, l, 0) \mapsto s$, i.e. the bundle L on S. $\square$

Putting the two lemmas together we get the proposition. Note that there are many choices of signs involved. We did not enter into this issue because it is unimportant for our final result.

REFERENCES

Abh69 S. S. Abhyankar, *Resolution of singularities of algebraic surfaces*, in *Algebraic geometry, Internat. Colloq.*, Tata Institute of Fundamental Research, Bombay, 1968 (Oxford University Press, London, 1969), 1–11.

BBD82 A. A. Beilinson, J. Bernstein and P. Deligne, *Faisceaux pervers*, Astérisque **100** (1982).

Big04 S. Biglari, *Motives of reductive groups*, Thesis, University of Leipzig (2004).

DG70 M. Demazure and A. Grothendieck, *Schémas en groupes, II*, Lecture Notes in Mathematics, vol. 152 (Springer, Berlin, 1970).

DP80 A. Dold and D. Puppe, *Duality, trace and transfers*, in *Proc. int. conf. on geometric topology*, Varsovie, 1978 (PWN, Varsovie, 1980), 81–102.

EKLV98 H. Esnault, B. Kahn, M. Levine and E. Viehweg, *The Arason invariant and mod 2 algebraic cycles*, J. Amer. Math. Soc. **11** (1998), 73–118.

Hir64 H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero, I and II*, Ann. of Math. (2) **79** (1964), 109–203, 205–326.

Hub00 A. Huber, *Realization of Voevodsky's motives*, J. Algebraic Geom. **9** (2000), 755–799.

Hub03 A. Huber, *Slice filtration on motives and the Hodge conjecture*, Math. Nachrichten, to appear.

dJo96 P. A. de Jong, *Smoothness, semi-stability and alterations*, Publ. Math. Inst. Hautes Études Sci. **83** (1996), 51–93.

Kah99 B. Kahn, *Motivic cohomology of smooth geometrically cellular varieties*, in *Algebraic K-theory*, Seattle, 1997, Proceedings of Symposia in Pure Mathematics, vol. 67 (American Mathematical Society, Providence, RI, 1999), 149–174.

Kah00 B. Kahn, *Weight filtration and mixed Tate motives*, in *Proc. Kinosaki algebraic geometry conf.*, ed. K. Ohno (2000), 112–116.

Kah02 B. Kahn, *K-theory of semi-local rings with finite coefficients and étale cohomology*, K-theory **25** (2002), 99–138.

KS06 B. Kahn and R. Sujatha, *Birational motives, I*, in preparation.

Köc91 B. Köck, *Chow motif and higher Chow theory of G/P*, Manuscripta Math. **70** (1991), 363–372.

Lev93a M. Levine, *Tate motives and the vanishing conjectures for algebraic K-theory*, in *Algebraic K-theory and algebraic topology*, Lake Louise, AB, 1991, NATO Advanced Study Institute Ser. C Math. Phys. Sci., vol. 407 (Kluwer Academic, Dordrecht, 1993), 167–188.

Lev93b M. Levine, *The algebraic K-theory of the classical groups and some twisted forms*, Duke Math. J. **70** (1993), 405–443.

Lev98 M. Levine, *Mixed motives*, Mathematical Surveys and Monographs, vol. 57 (American Mathematical Society, Providence, RI, 1998).

Lip78 J. Lipman, *Desingularization of two-dimensional schemes*, Ann. of Math. (2) **107** (1978), 151–207.

SV00 A. Suslin and V. Voevodsky, *Bloch–Kato conjecture and motivic cohomology with finite coefficients*, in *The arithmetic and geometry of algebraic cycles*, Banff, AB, 1998, NATO Advanced Study Institute Ser. C Math. Phys. Sci., vol. 548 (Kluwer Academic, Dordrecht, 2000), 117–189.

Tot06 B. Totaro, *Chow groups, Chow cohomology, and linear varieties*, J. Algebraic Geom., to appear.

Voe00a V. Voevodsky, *Cohomological theory of presheaves with transfers*, in *Cycles, transfers and motivic cohomology theories*, Annals of Mathematics Studies, vol. 143 (Princeton University Press, Princeton, NJ, 2000), 87–137.

Voe00b V. Voevodsky, *Triangulated categories of motives over a field*, in *Cycles, transfers and motivic cohomology theories*, Annals of Mathematics Studies, vol. 143 (Princeton University Press, Princeton, NJ, 2000), 188–238.

Voe02a V. Voevodsky, *Open problems in the motivic stable homotopy theory, I*, in *Motives, polylogarithms and Hodge theory, Part I*, Irvine, CA, 1998, International Press Lecture Series, vol. 3, I (International Press, Somerville, MA, 2002), 3–34.

Voe02b V. Voevodsky, *Motivic cohomology groups are isomorphic to higher Chow groups in any characteristic*, Internat. Math. Res. Not. **2002**, no. 7 (2002), 351–355.

Voe02c V. Voevodsky, *Cancellation theorem*, Preprint (2002); available on the K-theory preprint archive, no. 541.

Voe03a V. Voevodsky, *Motives over simplicial schemes*, Preliminary version (2003); available on the K-theory preprint archive, no. 638.

Voe03b V. Voevodsky, *Motivic cohomology with $\mathbf{Z}/2$-coefficients*, Publ. Math. Inst. Hautes Études Sci. **98** (2003), 59–104.

Wei04 C. Weibel, *A road map of motivic homotopy and homology theory*, in *Axiomatic, enriched and motivic homotopy theory*, NATO Sci. Ser. II Math. Phys. Chem., vol. 131 (Kluwer Academic, Dordrecht, 2004), 385–392.

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