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TOPOLOGIES ON SCHEMES AND MODULUS PAIRS

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Abstract. We study relationships between the Nisnevich topology on smooth schemes and certain Grothendieck topologies on proper and not necessarily proper modulus pairs, which were introduced in previous papers. Our results play an important role in the theory of sheaves with transfers on proper modulus pairs.

Introduction

In [3], a theory of sheaves on *nonproper* modulus pairs has been studied as the first step to establish the theory of motives with modulus, which is to be a non-$\mathbf{A}^1$-invariant version of Voevodsky's category of motives given in [13]. This repaired the first part of the mistake in [5] (the ancestor of the theory) mentioned in the introduction of [3].

In [4], a theory of sheaves on *proper* modulus pairs is developed as the second step, thus repairing the second part of the mistake. The main point of these repairs is to prove that the categories $\mathbf{\underline{M}NST}$ and $\mathbf{MNST}$ of [5], which had been defined in an ad hoc way, are really categories of sheaves (with transfers) for suitable Grothendieck topologies having good formal properties.

The aim of the present paper is to provide some foundational results, which will be the key building blocks of the theory in [4]. To explain our aim in more detail, we first recall basic notions of modulus pairs from [3]. We fix a base field k and write $\mathbf{Sch}$ (resp. $\mathbf{Sm}$) for the category of separated k-schemes of finite type (resp. its full subcategory of smooth k-schemes).

A *modulus pair* is a pair

$$M = (\overline{M}, M^\infty),$$

where $\overline{M} \in \mathbf{Sch}$ and M^∞ is an effective Cartier divisor on $\overline{M}$ such that the complement of the divisor

$$M^{o} := \overline{M} - M^\infty$$

belongs to $\mathbf{Sm}$. These conditions imply that $\overline{M}$ is reduced and M^{o} is dense [3, Remark 1.1.2(3)]. We call $\overline{M}$ (resp. M^{o}) *the ambient space of* M (resp. *the interior of* M).

A morphism $f : M \to N$ of modulus pairs is a morphism $f^{o} : M^{o} \to N^{o}$ in $\mathbf{Sm}$, which satisfies the following *admissibility condition*: let Γ be the graph of the rational map $\overline{M} \dashrightarrow \overline{N}$ defined by f^{o}, and let $\Gamma^N \to \Gamma$ be the normalization, whence a diagram $\overline{M} \xleftarrow{a} \Gamma^N \xrightarrow{b} \overline{N}$. Then a is proper and we have $a^* M^\infty \geqslant b^* N^\infty$, where $a^* M^\infty$ and $b^* N^\infty$ denote the pullbacks of effective Cartier divisors (see [3, Definitions 1.1.1, 1.3.2, 1.3.3]). The composition of

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morphisms of modulus pairs is given by that of morphisms in $\mathbf{Sm}$. Thus, we obtain a category $\underline{\mathbf{M}}\mathbf{Sm}$ of modulus pairs.

A modulus pair M is *proper* if $\overline{M}$ is proper over k. We write $\mathbf{MSm}$ for the full subcategory of $\underline{\mathbf{M}}\mathbf{Sm}$ which consists of proper modulus pairs. It is our main object of study here.

Recall that the Nisnevich topology on $\mathbf{Sm}$ may be understood by means of a certain *cd-structure* in the sense of Voevodsky [14], which is *complete* and *regular* (see [14]). In [3], it is shown that $\underline{\mathbf{M}}\mathbf{Sm}$ also admits a complete and regular cd-structure, parallel to the previous one and denoted by $P_{\underline{\mathrm{MV}}}$. In [9], a more subtle cd-structure P_{MV} is defined on the category $\mathbf{MSm}$, and shown to be complete and regular as well. We recall in Section 1 the definitions of all these cd-structures.

This paper studies the relationship between the cd-structures $P_{\underline{\mathrm{MV}}}$ and P_{MV}. Our main theorems are too technical to be stated in this introduction; here they are nevertheless:

(1) Theorem 1.5.6 (cofinality theorem);
(2) Theorem 2.1.4 (existence of partial compactifications).

Let us roughly explain the contents of Theorem 1.5.6. Given a complete and regular cd-structure, distinguished squares yield long exact "Mayer–Vietoris" sequences for sheaves in the associated topology. Take a distinguished square S in $P_{\underline{\mathrm{MV}}}$. By Proposition 1.5.4, it may be embedded into a commutative square T in $\mathbf{MSm}$ by a collection of "compactifications" (see Definition 1.5.1). But T has no reason to be in P_{MV}. Theorem 1.5.6 says that, under a mild normality condition on S, one can always lift the embedding $S \hookrightarrow T$ to an embedding $S \to T'$ with $T' \in P_{\mathrm{MV}}$.

Endow $\underline{\mathbf{M}}\mathbf{Sm}$ and $\mathbf{MSm}$ with the Grothendieck topologies associated to these cd-structures, and $\mathbf{Sm}$ with the Nisnevich topology. Then the following result is a corollary of Theorems 1.5.6 and 2.1.4.

THEOREM 1. *The natural forgetful functors*

$$\underline{\omega}_s : \underline{\mathbf{M}}\mathbf{Sm} \to \mathbf{Sm}; \quad M \mapsto M^{\mathrm{o}},$$
$$\omega_s : \mathbf{MSm} \to \mathbf{Sm}; \quad M \mapsto M^{\mathrm{o}}$$

and the left adjoint to $\underline{\omega}_s$

$$\lambda_s : \mathbf{Sm} \to \underline{\mathbf{M}}\mathbf{Sm}; \quad X \mapsto (X, \emptyset)$$

are continuous and cocontinuous in the sense of [SGA4, Exposé III]. Moreover, the inclusion functor

$$\tau_s : \mathbf{MSm} \to \underline{\mathbf{M}}\mathbf{Sm}; \quad M \mapsto M$$

is continuous.

(See Section A.1 for a review of continuity and cocontinuity.)

REMARK 1. On the other hand, τ_s is not cocontinuous; see Remark 5.2.1. Rather, the content of Theorem 1.5.6 is, morally, that its pro-left adjoint $\tau_s^!$ is continuous for a natural topology on pro-$\mathbf{MSm}$ extending that of $\mathbf{MSm}$. However, developing this viewpoint would force us to get into unpleasant categorical and set-theoretic issues, and we prefer to skip it here.

This paper is organized as follows. In Section 1, we recall the definitions of the cd-structure on $\underline{\mathbf{M}}\mathbf{Sm}$ from [3] and that on $\mathbf{MSm}$ from [9]. Moreover, we state Theorem 1.5.6. In Section 2, we state and prove Theorem 2.1.4. In Section 3, we prove Theorem 1.5.6 in a special case. In Section 4, we complete the proof of Theorem 1.5.6. In Section 5, we prove Theorem 1. The appendices provide technical facts needed in the text.

§1. Recollection on cd-structures; the cofinality theorem

In this section, we recall definitions of the cd-structures on $\underline{\mathbf{M}}\mathbf{Sm}$ and $\mathbf{MSm}$ from [3, 9]. We assume that the reader is familiar with [14], part of whose results is summarized in [3, A.8].

1.1 The cd-structure on Sm

First, recall the Nisnevich cd-structure on $\mathbf{Sm}$. The following notation is useful.

DEFINITION 1.1.1. Let $\mathbf{Sq}$ denote the product category $[1]^2 = \{0 \to 1\}^2$. For any category $\mathcal{C}$, define $\mathcal{C}^{\mathbf{Sq}}$ as the category of functors $\mathbf{Sq} \to \mathcal{C}$. An object of $\mathcal{C}^{\mathbf{Sq}}$ is a commutative square in $\mathcal{C}$, and a morphism of $\mathcal{C}^{\mathbf{Sq}}$ is a morphism of commutative squares.

An object $S \in \mathcal{C}^{\mathbf{Sq}}$ will often be depicted as

$$\begin{array}{ccc} S(00) & \xrightarrow{v_S} & S(01) \\ {\scriptstyle q_S}\downarrow & & \downarrow{\scriptstyle p_S} \\ S(10) & \xrightarrow{u_S} & S(11). \end{array} \tag{1.1.1}$$

DEFINITION 1.1.2. An *elementary Nisnevich square* is an object of $\mathbf{Sch}^{\mathbf{Sq}}$ of the form

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & X, \end{array}$$

which satisfies the following properties:

(1) The square is Cartesian.
(2) The horizontal morphisms are open immersions.
(3) The vertical morphisms are étale.
(4) The morphism $(V - W)_{\mathrm{red}} \to (X - U)_{\mathrm{red}}$ is an isomorphism.

Elementary Nisnevich squares whose vertices are in $\mathbf{Sm}$ define a complete and regular cd-structure on $\mathbf{Sm}$ [15]. Moreover, the Grothendieck topology associated to the cd-structure is the Nisnevich topology.

1.2 The cd-structure on $\underline{\mathbf{M}}\mathbf{Sm}$

Next, we recall the definition of $\underline{\mathrm{MV}}$-squares from [3]. We start from the following definition.

DEFINITION 1.2.1.

(1) A morphism $f : M \to N$ is *ambient* if $f^{\mathrm{o}} : M^{\mathrm{o}} \to N^{\mathrm{o}}$ extends (uniquely) to a morphism $\overline{f} : \overline{M} \to \overline{N}$.

(2) An ambient morphism $f : M \to N$ is *minimal* if $M^\infty = \overline{f}^* N^\infty$.
(3) Let $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ (resp. $\mathbf{MSm}^{\mathrm{fin}}$) be the (nonfull) subcategory of $\underline{\mathbf{M}}\mathbf{Sm}$ (resp. $\mathbf{MSm}$) whose objects are the same as $\underline{\mathbf{M}}\mathbf{Sm}$ (resp. $\mathbf{MSm}$) and morphisms are ambient morphisms.

Definition 1.2.2. An $\underline{\mathrm{MV}}^{\mathrm{fin}}$-square is an object $S \in (\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}})^{\mathbf{Sq}}$ of the form (1.1.1) such that we have the following:

(1) All morphisms that appear in S are minimal.
(2) The square in $\mathbf{Sch}$

$$\begin{tikzcd} \overline{S}(00) \arrow[r, "\overline{v}_S"] \arrow[d, "\overline{q}_S"'] & \overline{S}(01) \arrow[d, "\overline{p}_S"] \\ \overline{S}(10) \arrow[r, "\overline{u}_S"] & \overline{S}(11) \end{tikzcd}$$

is an elementary Nisnevich square.

By [3, Proposition 3.2.2], the $\underline{\mathrm{MV}}^{\mathrm{fin}}$-squares form a complete and regular cd-structure on $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$.

Definition 1.2.3. An $\underline{\mathrm{MV}}$-square is an object $S \in \underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}$, which belongs to the essential image of the natural (nonfull) functor

$$(\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}})^{\mathbf{Sq}} \to \underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}.$$

By [3, Theorem 4.1.2], the $\underline{\mathrm{MV}}$-squares form a complete and regular cd-structure on $\underline{\mathbf{M}}\mathbf{Sm}$, denoted by $P_{\underline{\mathrm{MV}}}$.

1.3 The cd-structure on MSm

Finally, we recall the definition of MV-squares from [9, Section 4]. Recall that for any diagram $M_1 \to N \leftarrow M_2$ in $\underline{\mathbf{M}}\mathbf{Sm}$ (resp. in $\mathbf{MSm}$) such that $M_1^{\mathrm{o}} \times_{N^{\mathrm{o}}} M_2^{\mathrm{o}} \in \mathbf{Sm}$, the fiber product $M_1 \times_N M_2$ is representable in $\underline{\mathbf{M}}\mathbf{Sm}$ (resp. in $\mathbf{MSm}$) (see [3, Section 1.10] or [9, Section 2.2]) and coproducts exist (see [9, Definition 3.1.2]).

Theorem 1.3.1. (Off-diagonals; see [9, Theorem 3.1.3]) *For any morphism $f : M \to N$ in $\underline{\mathbf{M}}\mathbf{Sm}$ such that $f^{\mathrm{o}} : M^{\mathrm{o}} \to N^{\mathrm{o}}$ is étale, there exists a canonical decomposition in $\underline{\mathbf{M}}\mathbf{Sm}$*

$$M \times_N M \cong M \sqcup \mathrm{OD}(f),$$

where $\sqcup$ denotes the coproduct in $\underline{\mathbf{M}}\mathbf{Sm}$, and the morphism $M \to M \times_N M$ is the diagonal. If M and N are proper modulus pairs, so is $\mathrm{OD}(f)$.

Moreover, if S is a commutative square in $\underline{\mathbf{M}}\mathbf{Sm}$ of the form (1.1.1) such that

(1) u_S^{o} and v_S^{o} are open immersions and
(2) p_S^{o} and q_S^{o} are étale,

then the morphism $v_S \times v_S : S(00) \times_{S(10)} S(00) \to S(01) \times_{S(11)} S(01)$ induces a morphism

$$\mathrm{OD}(q_T) \to \mathrm{OD}(p_T)$$

in $\underline{\mathbf{M}}\mathbf{Sm}$.

DEFINITION 1.3.2. (See [9, Definition 4.2.1]) Let T be an object of $\mathbf{MSm^{Sq}}$ of the form

$$\begin{tikzcd} T(00) \arrow[r, "v_T"] \arrow[d, "q_T"'] & T(01) \arrow[d, "p_T"] \\ T(10) \arrow[r, "u_T"] & T(11). \end{tikzcd} \tag{1.3.1}$$

Then T is called an MV-*square* if the following conditions hold:

(1) T is a pullback square in $\mathbf{MSm}$.
(2) There exist an $\underline{\mathrm{M}}\mathrm{V}$-square S such that $S(11)\in\mathbf{MSm}$ and a morphism $S\to T$ in $\underline{\mathbf{M}}\mathbf{Sm^{Sq}}$ such that the induced morphism $S^{\mathrm{o}}\to T^{\mathrm{o}}$ is an isomorphism in $\mathbf{Sm^{Sq}}$ and $S(11)\to T(11)$ is an isomorphism in $\mathbf{MSm}$. In particular, T^{o} is an elementary Nisnevich square.
(3) $\mathrm{OD}(q_T)\to\mathrm{OD}(p_T)$ is an isomorphism in $\mathbf{MSm}$.

The MV-squares form a complete and regular cd-structure on $\mathbf{MSm}$, denoted by P_{MV} (see [9, Theorems 4.3.1, 4.4.1]).

1.4 A few lemmas

In this subsection, we collect some lemmas that were proven in previous works and will be used repeatedly in the sequel.

LEMMA 1.4.1. [6, Lemma 2.2] *Let $f:X\to Y$ be a surjective morphism of normal integral schemes, and let D,D' be two Cartier divisors on Y. If $f^*D'\leqslant f^*D$, then $D'\leqslant D$.*

LEMMA 1.4.2. [8, Lemma 3.14] *Let X be a quasicompact scheme and let D,E be Cartier divisors on X with $E\geqslant 0$. Assume that the restriction of D to the open subset $X\setminus E\subset X$ is effective. Then, there exists a natural number $n_0\geqslant 1$ such that $D+n\cdot E$ is effective for any $n\geqslant n_0$.*

LEMMA 1.4.3. [3, Lemma 1.3.7] *Let $f:X\to Y$ be a separated morphism of schemes, and let $U\subset X$ be an open dense subset. Assume that the image $f(U)$ of U is open in Y, and the induced morphism $U\to f(U)$ is proper (e.g., an isomorphism). Then, we have $f^{-1}(f(U))=U$.*

LEMMA 1.4.4. [3, Lemma 1.6.3] *Let $f:U\to X$ be an étale morphism of quasicompact and quasiseparated integral schemes. Let $g:V\to U$ be a proper birational morphism, $T\subset U$ be a closed subset such that $g_{|U-T}$ is an isomorphism, and S be the closure of $f(T)$ in X. Then there exists a closed subscheme $Z\subset X$ supported in S such that $U\times_X\mathbf{Bl}_Z(X)\to U$ factors through V.*

1.5 The cofinality theorem

Recall the following notion from [3, Definition 1.8.1].

DEFINITION 1.5.1. For $M\in\underline{\mathbf{M}}\mathbf{Sm}$, let $\mathbf{Comp}(M)$ be the category whose objects are morphisms $j:M\to N$ in $\underline{\mathbf{M}}\mathbf{Sm}$ such that

(1) $N\in\mathbf{MSm}$,
(2) j is ambient and minimal,
(3) the morphism $\overline{j}:\overline{M}\to\overline{N}$ is a dense open immersion, and

(4) there are effective Cartier divisors M_N^∞ and C on $\overline{N}$ such that $N^\infty = M_N^\infty + C$ and $|C| = \overline{N} - \overline{j}(\overline{M})$,

and morphisms $(j_1 : M \to N_1) \to (j_2 : M \to N_2)$ are morphisms $f : N_1 \to N_2$ in **MSm** such that $f \circ j_1 = j_2$. Note that for any $(j : M \to N) \in \mathbf{Comp}(M)$, the morphism $j^o : M^o \to N^o$ is an isomorphism in **Sm**. By [3, Lemma 1.8.2], the category $\mathbf{Comp}(M)$ is a cofiltered ordered set.

Example 1.5.2. Take $M = (\mathbf{A}^2, 0 \times \mathbf{A}^1)$, $N_1 = (\mathbf{P}^1 \times \mathbf{P}^1, 0 \times \mathbf{P}^1 + \infty \times \mathbf{P}^1 + \mathbf{P}^1 \times \infty)$, and $N_2 =$ (blowup of $\mathbf{P}^1 \times \mathbf{P}^1$ at $\infty \times \infty$, pullback of N_1^∞). Then N_1 and N_2 are both in $\mathbf{Comp}(M)$, and N_2 dominates N_1.

We give a similar definition for squares.

Definition 1.5.3. Let S be an object in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}$. Define $\mathbf{Comp}(S)$ as the category whose objects are morphisms $j : S \to T$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}$ such that for each $(ij) \in \mathbf{Sq}$, the morphism $j(ij) : S(ij) \to T(ij)$ belongs to $\mathbf{Comp}(S(ij))$, and whose morphisms $(j_1 : S \to T_1) \to (j_2 : S \to T_2)$ are morphisms $f : T_1 \to T_2$ in $\mathbf{MSm}^{\mathbf{Sq}}$ such that $f \circ j_1 = j_2$.

Proposition 1.5.4. *For any $S \in \underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}$, the category $\mathbf{Comp}(S)$ is cofiltered and ordered.*

Proof. Let $\tau_s : \mathbf{MSm} \to \underline{\mathbf{M}}\mathbf{Sm}$ be the inclusion functor. Then τ_s admits a pro-left adjoint [SGA4, I.8.11] $\tau_s^! : \underline{\mathbf{M}}\mathbf{Sm} \to \text{pro-}\mathbf{MSm}$, which is represented by **Comp**, that is, we have

$$\tau_s^!(M) = \text{“}\varprojlim_{(M \to N) \in \mathbf{Comp}(M)}\text{”} N$$

(see [3, Remark 1.8.5]). Thus, the assertion follows from Lemma C.1.1, applied to $\mathcal{C} = \mathbf{MSm}$, $\mathcal{C}' = \underline{\mathbf{M}}\mathbf{Sm}$, $u = \tau_s$, $v = \tau_s^!$, $I = \mathbf{Comp}$, and $\Delta = \mathbf{Sq}$. ∎

Definition 1.5.5. For any $S \in \underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}$, define $\mathbf{Comp}^{\mathrm{MV}}(S)$ as the full subcategory $\mathbf{Comp}(S)$ consisting of objects $S \to T$ such that T is an MV-square.

The main result of this paper is the following.

Theorem 1.5.6. *Let S be an $\underline{\mathrm{MV}}^{\mathrm{fin}}$-square with $\overline{S}(11)$ normal. Then, for any $(S \to T) \in \mathbf{Comp}(S)$, there exists $(S \to T') \in \mathbf{Comp}^{\mathrm{MV}}(S)$, which dominates $(S \to T)$ in $\mathbf{Comp}(S)$, and such that $T'(11) \to T(11)$ is ambient and minimal (hence an isomorphism in* **MSm***).*

In particular, $\mathbf{Comp}^{\mathrm{MV}}(S)$ is cofinal in $\mathbf{Comp}(S)$.

The proof of Theorem 1.5.6 will be given in Sections 3 and 4.

§2. Partial compactifications

2.1 Definition and statement

Definition 2.1.1. Let S be an $\underline{\mathrm{MV}}^{\mathrm{fin}}$-square.

(1) We say that S is *normal* if $\overline{S}(11)$ is normal. (Note that this implies that $\overline{S}(ij)$ is normal for all $i, j \in \{0, 1\}$).
(2) An $\underline{\mathrm{MV}}^{\mathrm{fin}}$-square is called *partially compact* if $S(11) \in \mathbf{MSm}$.
(3) A *partial compactification* of S is a morphism $S \to S'$ in $(\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}})^{\mathbf{Sq}}$ such that

 (a) S' is a partially compact $\underline{\mathrm{MV}}^{\mathrm{fin}}$-square,

(b) the morphism $S(11) \to S'(11)$ belongs to $\mathbf{Comp}(S(11))$,
(c) $\overline{S}(ij) \to \overline{S}'(ij)$ are open immersions, and
(d) $\overline{S}(ij) \xrightarrow{\sim} \overline{S}'(ij) \times_{\overline{S}'(11)} \overline{S}(11)$.

EXAMPLE 2.1.2. The simplest case is when S is given by a Nisnevich square of schemes with empty divisors (and this is the essential case).

Take (in characteristic $\neq 2$)

$$S(11) = (\mathbf{A}^1 - \{0\}, \emptyset)$$
$$S(10) = (\mathbf{A}^1 - \{0, 1\}, \emptyset)$$
$$S(01) = (\mathbf{A}^1 - \{0, -1\}, \emptyset)$$
$$S(00) = (\mathbf{A}^1 - \{0, 1, -1\}, \emptyset)$$

with $\overline{S}(01) \to \overline{S}(11)$ the square map $t \mapsto t^2$ (and the horizontal maps the inclusions). Then S is an $\underline{\mathrm{MV}}^{\mathrm{fin}}$-square, as it is a distinguished Nisnevich square if we forget the empty divisor. A partial compactification $S \to S'$ is given by

$$S'(11) = (\mathbf{P}^1, 0 + \infty)$$
$$S'(10) = (\mathbf{P}^1 - \{1\}, 0 + \infty)$$
$$S'(01) = (\mathbf{A}^1 - \{0, -1\}, \emptyset)$$
$$S'(00) = (\mathbf{A}^1 - \{0, 1, -1\}, \emptyset).$$

Remarks 2.1.3. (1) If $S \to S'$ is a partial compactification, then the isomorphism in Condition (d) induces an isomorphism $S(ij) \xrightarrow{\sim} S'(ij) \times_{S'(11)} S(11)$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$, where the right-hand side denotes the fiber product in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$, which exists by the minimality of the projection maps [3, Corollary 1.10.7].

(2) If $S \to S'$ is a partial compactification, then the induced morphism $S(ij)^{\mathrm{o}} \to S'(ij)^{\mathrm{o}}$ is an isomorphism for all $i, j \in \{0, 1\}$. This is true for $i = j = 1$ by Condition (2)(b). For other i, j, we need to prove $\overline{S}'(ij) - \overline{S}(ij) \subset |S'(ij)^{\infty}|$. The left-hand side equals the pullback of $\overline{S}'(11) - \overline{S}(11)$ by the map $\overline{S}'(ij) \to \overline{S}'(11)$ by Condition (2)(d). Since $\overline{S}'(11) - \overline{S}(11) \subset |S'(11)^{\infty}|$ by the previous case, we obtain $\overline{S}'(ij) - \overline{S}(ij) \subset |S'(11)^{\infty} \times_{\overline{S}'(11)} \overline{S}'(ij)| = |S'(ij)^{\infty}|$, where the last equality follows from the minimality of $S'(ij) \to S(11)$.

The main result of this section is the following theorem.

THEOREM 2.1.4. *For any $\underline{\mathrm{MV}}^{\mathrm{fin}}$-square S and for any compactification $T \in \mathbf{Comp}(S(11))$, there exists a partial compactification $S \to S'$ such that $S'(11) \in \mathbf{Comp}(S(11))$ dominates T, and the morphism $S'(11) \to T$ is minimal.*

The proof of Theorem 2.1.4 will be given in the following subsections.

2.2 The Zariski case

Before going into the proof of the general case, we will describe the proof in the case that S is a Zariski square, that is, that $\overline{p}_S : \overline{S}(01) \to \overline{S}(11)$ is an open immersion. This subsection is only used in the sequel as a guide for the reader.

Take any object $(S(11) \to T) \in \mathbf{Comp}(S(11))$, and set $Z_1 := \overline{S}(11) - \overline{S}(10)$ and $Z_2 := \overline{S}(01) - \overline{S}(00)$. Let $\overline{Z}_i$ be the closure of Z_i in $\overline{T}$ for $i = 1, 2$.

2.2.1 Special case

If $\overline{Z}_1 \cap \overline{Z}_2$ is empty, we set $\overline{S}'(10) := \overline{T} - \overline{Z}_1$, $\overline{S}'(01) := \overline{T} - \overline{Z}_2$, and $\overline{S}'(00) := \overline{S}'(10) \cap \overline{S}'(01)$. Moreover, set $S'(11) := T$, $S'(ij)^\infty := S'(11)^\infty \cap \overline{S}'(ij)$, and $S'(ij) := (\overline{S}'(11), S'(11)^\infty)$ for $(ij) \neq (11)$. Then we obtain a partial compactification $S \to S'$, where the maps $S(ij) \to S'(ij)$ are induced by natural open immersions.

2.2.2 General case

In general, let $\pi : \overline{T}_1 \to \overline{T}$ be the blowup of $\overline{T}$ along $\overline{Z}_1 \times_{\overline{T}} \overline{Z}_2$. Then the closure of Z_1 in $\overline{T}_1$ and the closure of Z_2 in $\overline{T}_1$ do not intersect. Therefore, by applying the above construction by replacing T with $T_1 := (\overline{T}_1, \pi^* T^\infty)$, we obtain a partial compactification of S.

The general case of Theorem 2.1.4 follows this strategy, with rather substantial complications.

2.3 A general construction

In this subsection, we make a preliminary construction for the proof of the general case. Set $Z_1 := \overline{S}(11) - \overline{S}(10)$ and $Z_1' := \overline{S}(01) - \overline{S}(00)$. Since $\overline{S}$ is an elementary Nisnevich square, the natural morphism $Z_1' \to Z_1$ is an isomorphism, and we have $Z_1' \cong Z_1 \times_{\overline{S}(11)} \overline{S}(01)$.

Contrary to the Zariski case, we cannot regard $\overline{S}(01)$ and $\overline{S}(00)$ as open subsets of $\overline{T}$. So, we take a compactification $\overline{S}(01) \to \overline{R}$ such that $\overline{p}_S : \overline{S}(01) \to \overline{S}(11)$ extends to a morphism $\overline{p} : \overline{R} \to \overline{T}$ of schemes over k,[1] and set $R := (\overline{R}, R^\infty) := (\overline{R}, \overline{p}^* T^\infty)$. Thus we obtain a minimal morphism $p : R \to T$.

In the Zariski case, we considered the closures of Z_1 and Z_2 in $\overline{T}$ and studied their intersection. In the general case, we will consider closures in $\overline{R}$.

We need the following elementary observation. Consider the open subscheme $U := \overline{p}^{-1}(\overline{S}(11))$ of $\overline{R}$. Then we have the commutative diagram

$$\begin{tikzcd}
Z_1' \arrow[r] \arrow[d, "\wr"'] \arrow[dr, phantom, "\square"] & \overline{S}(01) \arrow[r] \arrow[d, "\overline{p}_S"] & U \arrow[r] \arrow[dl] & \overline{R} \arrow[d, "\overline{p}"] \\
Z_1 \arrow[r] & \overline{S}(11) \arrow[rr] & & \overline{T},
\end{tikzcd}$$

where we regard Z_1' and Z_1 as reduced closed subschemes.

Lemma 2.3.1.

(1) *The inclusion $Z_1' \subset U$ is a closed immersion.*
(2) $Z_1' = \overline{p}^{-1}(Z_1) \cap \overline{S}(01)$.
(3) *Regard*

$$\overline{p}^{-1}(Z_1) := Z_1 \times_{\overline{T}} \overline{R},$$

as a closed subscheme of U. Then there exists an open and closed subscheme Z_3 of U such that $\overline{p}^{-1}(Z_1) = Z_1' \sqcup Z_3$.

[1]For example, take a compactification $\overline{S}(01) \to \overline{R}_0$ and define $\overline{R}$ as the graph of the rational map $\overline{R}_0 \dashrightarrow \overline{T}$.

(4) *Set* $Z_2 := U - (Z_3 \sqcup \overline{S}(01))$. *Then* Z_2 *is a closed subset of* U. *We endow* Z_2 *with the reduced scheme structure.*
(5) *The closed subschemes* Z_1', Z_2, Z_3 *of* U *are disjoint from each other.*

Proof. We prove (1). The composite $Z_1' \to Z_1 \to \overline{S}(11)$ is a proper morphism since $Z_1' \to Z_1$ is an isomorphism. Since it factors through $\overline{U}$ and since $\overline{U} \to \overline{T}$ is separated, we conclude that $Z_1' \to \overline{U}$ is proper, hence a closed immersion.

(2) follows from the isomorphism $Z_1' \cong Z_1 \times_{\overline{S}(11)} \overline{S}(01)$.

We prove (3). (1) implies that Z_1' is a closed subscheme of $\overline{p}^{-1}(Z_1)$. On the other hand, Z_1' is open also in $\overline{p}^{-1}(Z_1)$ by (2). Therefore, taking $Z_3 := \overline{p}^{-1}(Z_1) - Z_1'$, we finish the proof.

(4) immediately follows from (3).

(5) By construction, we have $U - \overline{S}(01) = Z_2 \sqcup Z_3$ and $Z_1' \subset \overline{S}(01)$. This finishes the proof. ∎

REMARK 2.3.2. In the Zariski case, we have $Z_3 = \emptyset$.

Let $\overline{Z}_1$ be the closure of Z_1 in $\overline{T}$. Moreover, let $\overline{Z}_1', \overline{Z}_2, \overline{Z}_3$ be the closures of Z_1', Z_2, Z_3 in $\overline{R}$, respectively, endowed with their reduced scheme structures.

LEMMA 2.3.3. *Set* $V := \overline{R} - (\overline{Z}_2 \cup \overline{Z}_3)$. *Then we have*

(1) $U \cap V = \overline{S}(01)$;
(2) $q^{-1}(\overline{S}(11)) = \overline{S}(01)$, *where* q *is the composite* $V \to \overline{R} \xrightarrow{\overline{p}} \overline{T}$.

Proof. We have $Z_2 \sqcup Z_3 = U - \overline{S}(01) \subset \overline{R} - \overline{S}(01)$ and $\overline{R} - \overline{S}(01)$ is closed in $\overline{R}$; hence $\overline{Z}_2 \cup \overline{Z}_3 \subset \overline{R} - \overline{S}(01)$, so $\overline{S}(01) \subset V$ and $U \cap V \supseteq \overline{S}(01)$. But $U - \overline{S}(01) = Z_2 \sqcup Z_3$ and $V \cap (Z_2 \sqcup Z_3) = \emptyset$; hence we have equality in (1). Finally, $q^{-1}(\overline{S}(11)) = U \cap V$, so (1) $\Longleftrightarrow$ (2). ∎

2.4 Proof of Theorem 2.1.4 in a special case

In the Zariski case, this subsection reduces to Subsubsection 2.2.1 (see Remark 2.4.1).

Let S, $(S(11) \to T) \in \mathbf{Comp}(S(11))$, and $\overline{S}(01) \to \overline{R}$ be as in the previous subsection. Moreover, define Z_1, Z_1', Z_2, Z_3 and $\overline{Z}_1, \overline{Z}_1', \overline{Z}_2, \overline{Z}_3$ in the same way as before (see Lemma 2.3.1).

In this subsection, we assume the following condition on T and $\overline{R}$.

$(*)_{T,\overline{R}}$ Let V, q be as in Lemma 2.3.3. Let $V_\flat \subset V$ be the flat locus of the composite $q : V \subset \overline{R} \xrightarrow{\overline{p}} \overline{T}$. Then $V_\flat$ contains $\overline{Z}_1'$.

REMARK 2.4.1. Assume that S is a Zariski square, that is, that $\overline{p}_S$ is an open immersion, and take $\overline{R}$ to be $\overline{T}$. Then the condition $(*)_{T,\overline{T}}$ is equivalent to $\overline{Z}_1 \cap \overline{Z}_2 = \emptyset$. Indeed, by Remark 2.3.2, we have $V_\flat = V = \overline{T} - \overline{Z}_2$. Moreover, we have $\overline{Z}_1 = \overline{Z}_1'$. Therefore $\overline{Z}_1' \subset V_\flat \Longleftrightarrow \overline{Z}_1 \cap \overline{Z}_2 = \emptyset$.

The general case will be treated in the next subsection.

Let $j : V_{\text{ét}} \subset V$ be the étale locus of q. Define

$$S'(11) := T,$$
$$S'(10) := (\overline{S}'(10), S'(10)^\infty) := (\overline{T} - \overline{Z}_1, T^\infty \cap (\overline{T} - \overline{Z}_1)),$$
$$S'(01) := (\overline{S}'(01), S'(01)^\infty) = (V_{\text{ét}}, j^* q^* T^\infty).$$

Then the open immersion $\overline{T}-\overline{Z}_1\to\overline{T}$ and the morphism $q\circ j:V_{\text{ét}}\to\overline{T}$ induce minimal morphisms $S'(10)\to S'(11)$ and $S'(01)\to S'(11)$.

Set $S'(00):=S'(10)\times_{S'(11)}S'(01)$ as the fiber product in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$, which exists by the minimality of (one of) the projections [3, Corollary 1.10.7]. In our situation, we have

$$S'(00)=(\overline{S}'(10)\times_{\overline{S}'(11)}\overline{S}'(01),\text{the pullback of }S'(11)^\infty).$$

Thus, we obtain a pullback diagram

$$S':\quad\begin{array}{ccc} S'(00) & \longrightarrow & S'(01)\\ \downarrow & & \downarrow\\ S'(10) & \longrightarrow & S'(11)\end{array}$$

in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$. By construction, for each $(ij)\in\mathbf{Sq}$, we have $\overline{S}(ij)\subset\overline{S}'(ij)$. Moreover, the open immersions induce minimal morphisms $S(ij)\to S'(ij)$. Therefore, we obtain a morphism $S\to S'$ in $(\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}})^{\mathbf{Sq}}$.

PROPOSITION 2.4.2. *The morphism $S\to S'$ is a partial compactification of S.*

We need the following two lemmas for the proof.

LEMMA 2.4.3. *In the factorization*

$$Z_1'\to q_\flat^{-1}(Z_1)\to Z_1,\tag{2.4.1}$$

both morphisms are isomorphisms.

Proof. We have the following commutative diagram:

$$\begin{array}{ccccc} Z_1' & \longrightarrow & \overline{S}(01) & \longrightarrow & V\\ \wr\downarrow & \square & \downarrow & \square & \downarrow q\\ Z_1 & \longrightarrow & \overline{S}(11) & \longrightarrow & \overline{T},\end{array}$$

where the left square is Cartesian since S is an $\underline{\mathrm{MV}}^{\mathrm{fin}}$-square, and the right square is also Cartesian thanks to Lemma 2.3.3. Since $\overline{S}(01)\subset V_\flat$, this implies that the commutative square

$$\begin{array}{ccc} Z_1' & \longrightarrow & V_\flat\\ \wr\downarrow & & \downarrow q_\flat\\ Z_1 & \longrightarrow & \overline{T}\end{array}$$

is also Cartesian. So the first morphism of (2.4.1) is an isomorphism, and hence so is the second one. This concludes the proof. ∎

For the next lemma, recall that we have $\overline{Z}_1'\subset V_\flat$ by assumption. From now on, we regard $\overline{Z}_1'$ and $\overline{Z}_1$ as reduced closed subschemes.

LEMMA 2.4.4. *The morphism $\overline{Z}_1'\to\overline{Z}_1$ is an isomorphism. Moreover, the induced morphism $\overline{Z}_1'\to q_\flat^{-1}(\overline{Z}_1):=\overline{Z}_1\times_{\overline{T}}V_\flat$ is an isomorphism.*

Proof. Let $q_{\flat,Z} : q_\flat^{-1}(\overline{Z}_1) \to \overline{Z}_1$ be the base change of $q : V_\flat \to \overline{T}$ by the closed immersion $\overline{Z}_1 \subset \overline{T}$. Then we obtain the following commutative diagram:

$$\begin{tikzcd} Z_1' \arrow[r] \arrow[d] & \overline{Z}_1' \arrow[r] \arrow[d] & q_\flat^{-1}(\overline{Z}_1) \arrow[r] \arrow[dl, "q_{\flat,Z}"] & V_\flat \arrow[r] \arrow[dr, "q_\flat"'] & V \arrow[d, "q"] \\ Z_1 \arrow[r] & \overline{Z}_1 \arrow[rrr] & & & \overline{T}. \end{tikzcd}$$

Claim 2.4.5. The morphism $q_{\flat,Z}$ is an isomorphism.

Proof. Since $q_\flat : V_\flat \to \overline{T}$ is flat, so is $q_{\flat,Z}$ by base change. Moreover, $q_{\flat,Z}$ is an isomorphism over the dense open subset $Z_1 \subset \overline{Z}_1'$ by Lemma 2.4.3. Therefore, $q_{\flat,Z}$ is an open immersion by Theorem B.1.1.

On the other hand, note that the morphism $\overline{Z}_1' \to \overline{Z}_1$ decomposes as $\overline{Z}_1' \subset q_{\flat,Z}^{-1}(\overline{Z}_1) \xrightarrow{q_{\flat,Z}} \overline{Z}_1$. Since $\overline{Z}_1' \to \overline{Z}_1$ is dominant and proper, it is surjective. Therefore, the open immersion $q_{\flat,Z}$ is indeed an isomorphism. This finishes the proof of Claim 2.4.5. ∎

Note that $\overline{Z}_1' \to \overline{Z}_1$ is surjective. By Claim 2.4.5, this implies that the closed immersion $\overline{Z}_1' \to q_\flat^{-1}(\overline{Z}_1)$ is also surjective. Since $\overline{Z}_1'$ is reduced by construction, and since $q_\flat^{-1}(\overline{Z}_1) \cong \overline{Z}_1$ is also reduced as $\overline{Z}_1$ is reduced by construction, the surjection $\overline{Z}_1' \to q_\flat^{-1}(\overline{Z}_1)$ must be an isomorphism of schemes. This finishes the proof of Lemma 2.4.4. ∎

Proof of Proposition 2.4.2. We will check Conditions (a)–(d) in Definition 2.1.1. Conditions (b) and (c) are satisfied by construction.

We check (d). The case $(ij) = (11)$ is obvious. The case $(ij) = (10)$ can be checked by

$$\overline{S}'(10) \cap \overline{S}(11) = (\overline{S}'(11) - \overline{Z}_1) \cap \overline{S}(11) = \overline{S}(11) - Z_1 = \overline{S}(10).$$

The case $(ij) = (01)$ follows from Lemma 2.3.3. The case $(ij) = (00)$ follows from $\overline{S}(00) \cong \overline{S}(10) \times_{\overline{S}(11)} \overline{S}(01)$.

Finally, we check Condition (a), that is, that S' is an $\underline{\mathrm{MV}}^{\mathrm{fin}}$-square. Since all edges of S' are minimal, it suffices to show that the square $\overline{S}'$ of schemes is an elementary Nisnevich square. The horizontal maps of $\overline{S}'$ are open immersions, and the vertical maps of $\overline{S}'$ are étale by construction. In view of Lemma 2.4.4, noting that $\overline{S}'(11) - \overline{S}(10) = \overline{Z}_1$, it suffices to prove the following claim.

Claim 2.4.6. $\overline{Z}_1' \subset \overline{S}'(01) = V_{\text{ét}}$.

Proof. Since $\overline{Z}_1' \to \overline{Z}_1$ is an isomorphism by Lemma 2.4.4, the flat morphism $q_\flat : V_\flat \to \overline{T}$ is unramified at each point of $\overline{Z}_1'$. This shows that $q_\flat$ is étale at each point of $\overline{Z}_1'$ by [EGA4-4, Theorem 17.6.1]. This finishes the proof of the claim. ∎

Thus, we have finished the proof of Proposition 2.4.2. ∎

2.5 A refinement of the general construction

PROPOSITION 2.5.1. *In Section 2.3, we may choose $\overline{S}(01) \to \overline{R}$ satisfying the following conditions.*

(1) $\overline{p}_S : \overline{S}(01) \to \overline{S}(11)$ *extends to a morphism* $\overline{p} : \overline{R} \to \overline{T}$.

(2) $\overline{Z}_1' \cap \overline{Z}_2 = \emptyset$ *and* $\overline{Z}_1' \cap \overline{Z}_3 = \emptyset$, *where* $\overline{Z}_1'$ *and* $\overline{Z}_2$ *are the closures of* Z_1' *and* Z_2 *in* $\overline{R}$.

Before going into the proof, we prepare a definition and a lemma, which will be used several times.

DEFINITION 2.5.2. Let $M \in \underline{\mathbf{M}}\mathbf{Sm}$, and let F be a closed subscheme of $\overline{M}$ such that $F \cap M^{\mathrm{o}} = \emptyset$. Let

$$\overline{\pi} : \mathbf{Bl}_F(\overline{M}) \to \overline{M}$$

be the blowup of $\overline{M}$ along F, and let

$$\overline{\nu} : \mathbf{Bl}_F(\overline{M})^N \to \mathbf{Bl}_F(\overline{M})$$

be the normalization of $\mathbf{Bl}_F(\overline{M})$. Set

$$\mathbf{Bl}_F(M) := (\mathbf{Bl}_F(\overline{M}), \overline{\pi}^* M^\infty),$$
$$\mathbf{Bl}_F(M)^N := (\mathbf{Bl}_F(\overline{M})^N, \overline{\nu}^* \overline{\pi}^* M^\infty).$$

By construction, $\mathbf{Bl}_F(M)^{\mathrm{o}} = (\mathbf{Bl}_F(M)^N)^{\mathrm{o}} = M^{\mathrm{o}}$. Moreover, the maps $\overline{\pi}$ and $\overline{\nu}$ induce minimal morphisms $\pi : \mathbf{Bl}_F(M) \to M$ and $\nu : \mathbf{Bl}_F(M)^N \to \mathbf{Bl}_F(M)$.

We call $\pi : \mathbf{Bl}_F(M) \to M$ (resp. $\pi\nu : \mathbf{Bl}_F(M)^N \to M$) *the blowup of* M *along* F (resp. *the normalized blowup of* M *along* F).

LEMMA 2.5.3. *Let* $M_0 \in \underline{\mathbf{M}}\mathbf{Sm}$ *and* $(M_0 \to M) \in \mathbf{Comp}(M_0)$. *Let* F *be a closed subscheme of* $\overline{M}$ *with* $F \cap M^{\mathrm{o}} = \emptyset$. *Assume that* $F \cap \overline{M}_0$ *is an effective Cartier divisor on* $\overline{M}_0$. *Then the following assertions hold.*

(1) *The open immersion* $\overline{M}_0 \to \overline{M}$ *lifts uniquely to an open immersion* $\overline{M}_0 \to \mathbf{Bl}_F(\overline{M})$. *This defines an object* $(M_0 \to \mathbf{Bl}_F(M)) \in \mathbf{Comp}(M_0)$, *which dominates* $M_0 \to M$.
(2) *If* $\overline{M}_0$ *is normal, the open immersion* $\overline{M}_0 \to \overline{M}$ *lifts uniquely to an open immersion* $\overline{M}_0 \to \mathbf{Bl}_F(\overline{M})^N$. *This defines an object* $(M_0 \to \mathbf{Bl}_F(M)^N) \in \mathbf{Comp}(M_0)$, *which dominates* $M_0 \to M$.

Proof. Since $(M_0 \to M) \in \mathbf{Comp}(M_0)$, there exist effective Cartier divisors $M_{0,M}^\infty, C_M$ on $\overline{M}$ such that $M^\infty = M_{0,M}^\infty + C_M$, $|C_M| = \overline{M} - \overline{M}_0$, and $M_{0,M}^\infty \cap \overline{M}_0 = M_0^\infty$.

(1): The assumption shows that $\pi : \mathbf{Bl}_F(\overline{M}) \to \overline{M}$ is an isomorphism over $\overline{M}_0$. Therefore, the open immersion $j : \overline{M}_0 \to \overline{M}$ lifts uniquely to an open immersion $j_1 : \overline{M}_0 \to \mathbf{Bl}_F(\overline{M})$:

$$\begin{tikzcd} & \mathbf{Bl}_F(\overline{M}) \arrow[d, "\pi"] \\ \overline{M}_0 \arrow[ur, "j_1"] \arrow[r, "j"] & \overline{M}. \end{tikzcd}$$

By construction, we have

$$\mathbf{Bl}_F(M)^\infty = \pi^* M^\infty = \pi^* M_{0,M}^\infty + \pi^* C_M.$$

Moreover, we have

$$j_1^* \pi^* M_{0,M}^\infty = j^* M_{0,M}^\infty = M_0^\infty,$$

and

$$\begin{aligned}|\pi^* C_M| &= \pi^{-1}(|C_M|) = \pi^{-1}(\overline{M} - j(\overline{M}_0)) = \mathbf{Bl}_F(\overline{M}) - \pi^{-1}(j(\overline{M}_0)) \\ &= \mathbf{Bl}_F(\overline{M}) - \pi^{-1}(\pi j_1(\overline{M}_0)) \\ &= \mathbf{Bl}_F(\overline{M}) - j_1(\overline{M}_0),\end{aligned}$$

where the last equality follows from Lemma 1.4.3. This proves that j_1 defines an object $(M_0 \to \mathbf{Bl}_F(M)) \in \mathbf{Comp}(M_0)$, which dominates $M_0 \to M$.

(2): The assumption shows that $\pi\nu : \mathbf{Bl}_F(\overline{M})^N \to \overline{M}$ is an isomorphism over $\overline{M}_0$. Therefore, the open immersion $j : \overline{M}_0 \to \overline{M}$ lifts uniquely to an open immersion $j_2 : \overline{M}_0 \to \mathbf{Bl}_F(\overline{M})^N$. The rest of the argument is the same as above. This finishes the proof. ∎

Proof of Proposition 2.5.1. We start from a construction as in Subsection 2.3; for clarity, we write $\overline{R}_0$ instead of $\overline{R}$ but keep the other notation $(Z_1', \overline{Z}_1', \dots)$. Let

$$\pi_1 : \overline{R}_1 := \mathbf{Bl}_{\overline{Z}_1' \times_{\overline{R}_0} \overline{Z}_2}(\overline{R}_0) \to \overline{R}_0$$

be the blowup of $\overline{R}_0$ along $\overline{Z}_1' \times_{\overline{R}} \overline{Z}_2$. Then π_1 is an isomorphism over the open subscheme $\overline{S}(01) \subset \overline{R}_0$ since $\overline{Z}_1' \times_{\overline{R}} \overline{Z}_2 \cap \overline{S}(01) \subset Z_1' \cap Z_2 = \emptyset$ by Lemma 2.3.1(5). Therefore, the open immersion $\overline{S}(01) \to \overline{R}_0$ lifts uniquely to an open immersion $\overline{S}(01) \to \overline{R}_1$. Moreover, the strict transforms $\pi_1^\# \overline{Z}_1'$ and $\pi_1^\# \overline{Z}_2$, that is, the closures of Z_1' and Z_2 in $\overline{R}_1$, do not intersect.

Similarly, let

$$\pi_2 : \overline{R}_2 := \mathbf{Bl}_{\pi_1^\# \overline{Z}_1' \times_{\overline{R}_1} \pi_1^\# \overline{Z}_3}(\overline{R}_1) \to \overline{R}_1$$

be the blowup of $\overline{R}_1$ along $\pi_1^\# \overline{Z}_1' \times_{\overline{R}_1} \pi_1^\# \overline{Z}_3$. Then π_2 is an isomorphism over the open subscheme $\overline{S}(01) \to \overline{R}_1$ since $\pi_1^\# \overline{Z}_1' \times_{\overline{R}_1} \pi_1^\# \overline{Z}_3 \cap \overline{S}(01) \cong \overline{Z}_1' \times_{\overline{R}} \overline{Z}_3 \cap \overline{S}(01) \subset Z_1' \cap Z_3 = \emptyset$ by Lemma 2.3.1. Therefore, the open immersion $\overline{S}(01) \to \overline{R}_1$ lifts uniquely to an open immersion $\overline{S}(01) \to \overline{R}_2$. Moreover, the strict transforms $\pi_2^\# \pi_1^\# \overline{Z}_1'$ and $\pi_2^\# \pi_1^\# \overline{Z}_3'$, that is, the closures of Z_1' and Z_3 in $\overline{R}_2$, do not intersect. Recalling that $\pi_1^\# \overline{Z}_1' \cap \pi_1^\# \overline{Z}_2' = \emptyset$, we also see that the strict transforms $\pi_2^\# \pi_1^\# \overline{Z}_1'$ and $\pi_2^\# \pi_1^\# \overline{Z}_2'$, that is, the closures of $\overline{Z}_1'$ and $\overline{Z}_2$ in $\overline{R}_2$, do not intersect. Thus, setting $\overline{R} := \overline{R}_3$, we get the desired compactification $\overline{S}(01) \to \overline{R}$. ∎

2.6 End of proof of Theorem 2.1.4

It suffices to show the following.

PROPOSITION 2.6.1. *There exist $(S(11) \to T') \in \mathbf{Comp}(S(11))$ and a compactification $\overline{S}(01) \to \overline{R}_1$, which satisfy Condition $(*)_{T', \overline{R}_1}$ from Subsection 2.4, such that $S(11) \to T'$ dominates $S(11) \to T$ and the morphism $T' \to T$ is minimal.*

Proof. Take a compactification $\overline{S}(01) \to \overline{R}$ as in Proposition 2.5.1.

Recall that $V = \overline{R} - (\overline{Z}_2 \cup \overline{Z}_3)$. By assumption, we have $\overline{Z}_1' \subset V$. Moreover, we have $q^{-1}(\overline{S}(11)) = \overline{S}(01)$ by Lemma 2.3.3. In particular, $q : V \to \overline{T}$ is étale over $\overline{S}(11) \subset \overline{T}$.

Therefore, by the theorem of "platification" [11, Theorem 5.2.2] of Raynaud–Gruson, there exists a closed subscheme $C \subset \overline{T} - \overline{S}(11)$, which satisfies the following condition: define $T_1 := \mathbf{Bl}_C(T)$ (see Definition 2.5.2). Let

$$\pi_1' : \overline{R}_1 := \pi_1^\# \overline{R} \to \overline{R}$$

be the strict transform, that is, the blowup of $\overline{R}$ along the closed subscheme $\overline{p}^{-1}(C) \subset \overline{R}$. Let $\overline{p}_1 := \pi_1^{\#}\overline{p} : \overline{R}_1 \to \overline{T}_1$ be the lift of $\overline{p}$. Then the strict transform

$$V_1 := \pi_1^{\#}(V) = V \times_{\overline{T}} \overline{T}_1 \subset \overline{R}_1$$

of V is flat over $\overline{T}_1$.

Note that π_1 is an isomorphism over $\overline{S}(11)$ since $C \cap \overline{S}(11) = \emptyset$. Therefore, Lemma 2.5.3 shows that the open immersion $\overline{S}(11) \to \overline{T}$ lifts uniquely to an open immersion $\overline{S}(11) \to \overline{T}_1$, and it defines an object $(S(11) \to T_1) \in \mathbf{Comp}(S(11))$.

Moreover, $\pi_1' : \overline{R}_1 := \pi_1^{\#}\overline{R} \to \overline{R}$ is an isomorphism over $\overline{S}(01) \subset \overline{R}$ since $\overline{S}(01)$ lies over $\overline{S}(11)$. Therefore, the open immersion $\overline{S}(01) \to \overline{R}$ lifts uniquely to an open immersion $\overline{S}(01) \to \overline{R}_1$.

Since $\overline{Z}_1' \cap \overline{Z}_2 = \overline{Z}_1' \cap \overline{Z}_3 = \emptyset$, we have the corresponding equality for their strict transforms: $\pi_1^{\#}\overline{Z}_1' \cap \pi_1^{\#}\overline{Z}_2 = \pi_1^{\#}\overline{Z}_1' \cap \pi_1^{\#}\overline{Z}_3 = \emptyset$. Since $\pi_1^{\#}\overline{Z}_1', \pi_1^{\#}\overline{Z}_2, \pi_1^{\#}\overline{Z}_3$ are the closures of Z_1', Z_2, Z_3 in $\overline{R}_1$, respectively, the pair $(S(11) \to T_1) \in \mathbf{Comp}(S(11))$ and $\overline{S}(01) \to \overline{R}_1$ satisfies $(*)_{T_1,\overline{R}_1}$. Moreover, the blowup $\pi_1 : \overline{T}_1 \to \overline{T}$ induces a minimal morphism $T_1 \to T$ by construction. This finishes the proof. ∎

Corollary 2.6.2. *If S is a normal $\underline{\mathrm{MV}}^{\mathrm{fin}}$-square and if $S \xrightarrow{j} S'$ is a partial compactification, then there exists a partial compactification $S \xrightarrow{j_1} S_1'$ such that S_1' is normal and a minimal morphism $S_1' \xrightarrow{p} S$ such that $pj_1 = j$.*

Proof. Let $S_1'(ij) = \mathbf{Bl}_\emptyset(S'(ij))^N$ be the normalized blowup along the empty subscheme for $(ij) \in \mathbf{Sq}$. More explicitly, we have

$$S_1'(ij) = (\overline{S}'(ij)^N, S'(ij)^\infty \times_{\overline{S}'(ij)} \overline{S}'(ij)^N).$$

By Lemma 2.5.3, the morphisms $S \to S'(ij)$ uniquely lift to $(S(ij) \to S_1'(ij)) \in \mathbf{Comp}(S(ij))$ for all $(ij) \in \mathbf{Sq}$.

Since $\overline{S}'(ij) \to \overline{S}'(11)$ is étale for each $(ij) \in \mathbf{Sq}$, we have

$$\overline{S}'(ij)^N \cong \overline{S}'(ij) \times_{\overline{S}'(11)} \overline{S}'(11)^N,$$

and hence

$$S_1'(ij) \cong S'(ij) \times_{S'(11)} S'(11)^N,$$

where the right-hand side denotes the fiber product in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$, which exists by the minimality of (one of) the projection maps [3, Corollary 1.10.7]. Therefore, $S_1'(ij)$'s form an $\underline{\mathrm{MV}}^{\mathrm{fin}}$-square S_1'. This finishes the proof. ∎

§3. Cofinality of MV-squares: the partially compact case

3.1 Main result

In this section, we prove the following special case of Theorem 1.5.6.

Theorem 3.1.1. *Theorem 1.5.6 is true if S is partially compact in the sense of Definition 2.1.1(2).*

This is the technical heart of the paper. The strategy is simple: we achieve successively conditions (1) and (3) of Definition 1.3.2, Condition (2) being automatic.

In the sequel, we fix a normal $\underline{\mathrm{MV}}^{\mathrm{fin}}$-square S; it will be assumed partially compact only in Subsection 3.6.

3.2 Another general construction

Here, we prepare a general setting, which will be used in the proof of Theorem 3.1.1.

Let $(S \to T) \in \mathbf{Comp}(S)$. Since $(S(ij) \to T(ij)) \in \mathbf{Comp}(S(ij))$ for each $(ij) \in \mathbf{Sq}$, we can find an effective Cartier divisor D_{ij} on $\overline{T}(ij)$ such that $|D_{ij}| = \overline{T}(ij) - \overline{S}(ij)$.

Write

$$\begin{tikzcd} T(00) \arrow[r, "v_T"] \arrow[d, "q_T"'] \arrow[dr, "d_T"] & T(01) \arrow[d, "p_T"] \\ T(10) \arrow[r, "u_T"] & T(11), \end{tikzcd}$$

where $d_T := p_T v_T = u_T q_T$. All morphisms in the diagram are ambient by assumption.

We shall need the following Condition 3.2.3. We recall a definition and a lemma.

Definition 3.2.1. Two effective Cartier divisors D, E on a scheme X *have a universal supremum* if $D \times_X E$ is an effective Cartier divisor on X. If D, E have a universal supremum, we define an effective Cartier divisor $\sup(D, E)$ on X by $\sup(D, E) := D + E - D \times_X E$.

Lemma 3.2.2. *Let D, E be effective Cartier divisors on a scheme X, which have a universal supremum. Then, for any morphism $f : Y \to X$ in $\mathbf{Sch}$ such that Y is normal and such that $f(T) \not\subset |D| \cup |E|$ for any irreducible component T of Y, the effective Cartier divisors f^*D and f^*E have a universal supremum, and $f^* \sup(D, E) = \sup(f^*D, f^*E)$. Moreover, if we regard $\sup(f^*D, f^*E)$ as a Weil divisor on the normal scheme Y, it coincides with the smallest Weil divisor which is larger than f^*D and f^*E.*

Proof. See [3, Definition 1.10.2, Remark 1.10.3] or [9, Lemma 2.2.1]. □

Conditions 3.2.3.

(1) T is ambient, that is, $T \in (\mathbf{MSm}^{\mathrm{fin}})^{\mathbf{Sq}}$.
(2) $\overline{T}(ij)$ is normal for each $(ij) \in \mathbf{Sq}$.
(3) $\overline{q}_T^* D_{10}$ and $\overline{v}_T^* D_{01}$ have a universal supremum on $\overline{T}(00)$.

Lemma 3.2.4. *For any $(S \to T_0) \in \mathbf{Comp}(S)$, there exists $(S \to T) \in \mathbf{Comp}(S)$ which dominates $(S \to T_0)$ and such that T satisfies Conditions 3.2.3.*

Proof. By the graph trick, there exists $(S \to T_1) \in \mathbf{Comp}(S)$, which dominates $(S \to T_0) \in \mathbf{Comp}(S)$, such that $T_1 \in (\mathbf{MSm}^{\mathrm{fin}})^{\mathbf{Sq}}$.

Set $F := \overline{q}_{T_1}^* D_{10} \times_{\overline{T}(00)} \overline{v}_{T_1}^* D_{01}$, and let

$$T(00) := \mathbf{Bl}_F(T_1(00))^N \to T_1(00)$$

be the normalized blowup of $T_1(00)$ along F (see Definition 2.5.2). Since $F \cap \overline{S}(00) = \emptyset$ by construction, and since $\overline{S}(00)$ is normal by assumption, Lemma 2.5.3(2) shows that the open immersion $\overline{S}(00) \to \overline{T}_1(00)$ lifts uniquely to an open immersion $\overline{S}(00) \to \overline{T}(00)$, which defines an object $(S(00) \to T(00)) \in \mathbf{Comp}(S(11))$.

For $(ij) \in \mathbf{Sq} - \{(00)\}$, define

$$T(ij) := (\overline{T}_1(ij)^N, T(ij)^\infty \times_{\overline{T}(ij)} \overline{T}(ij)^N).$$

Then, for each $(ij) \in \mathbf{Sq} - \{(00)\}$, the morphism $S(ij) \to T_1(ij)$ lifts uniquely to an object $S(ij) \to T(ij)$ by Lemma 2.5.3(2), noting that the normalization is the normalized blowup along the closed subset $\emptyset$. Moreover, for each $(ij) \to (kl)$ in $\mathbf{Sq}$, the morphism $\overline{T}_1(ij) \to \overline{T}_1(kl)$ lifts to a morphism $\overline{T}(ij) \to \overline{T}(kl)$ by the universal property of normalization. Therefore, T is ambient.

The other conditions in Conditions 3.2.3 hold by construction. ∎

In the rest of this subsection, we fix $(S \to T) \in \mathbf{Comp}(S)$ verifying Conditions 3.2.3. Set

$$D := \sup(\overline{q}_T^* D_{10}, \overline{v}_T^* D_{01}),$$

where the notation on the right-hand side is from Definition 3.2.1. Then D is an effective Cartier divisor on $\overline{T}(00)$ by assumption.

LEMMA 3.2.5. $|D| = |D_{00}| = \overline{T}(00) - \overline{S}(00)$.

Proof. The second equality is by definition. To show the first one, taking the complements of both sides, we are reduced to proving $\overline{S}(00) = \overline{q}_T^{-1}(\overline{S}(10)) \cap \overline{v}_T^{-1}(\overline{S}(01))$. The inclusion $\overline{S}(00) \subset \overline{q}_T^{-1}(\overline{S}(10)) \cap \overline{v}_T^{-1}(\overline{S}(01))$ is obvious. By the universal property of fiber product, there exists a unique morphism $\overline{q}_T^{-1}(\overline{S}(10)) \cap \overline{v}_T^{-1}(\overline{S}(01)) \to \overline{S}(10) \times_{\overline{S}(11)} \overline{S}(01) = \overline{S}(00)$, which is compatible with $\overline{q}_S$ and $\overline{v}_S$. We can check that this map is inverse to the inclusion map, by restricting them to the dense open subset $S^{\mathrm{o}}(00) = T^{\mathrm{o}}(00)$. This finishes the proof. ∎

LEMMA 3.2.6. *There exists a positive integer n_T such that for any $n \geqslant n_T$, we have*

$$T(00)^\infty \leqslant \overline{d}^* T(11)^\infty + nD.$$

Proof. We have $T(00)^\infty|_{\overline{S}(00)} = \overline{d}^* T(11)^\infty|_{\overline{S}(00)} = S(00)^\infty$ by the minimality of $S(00) \to T(00)$ and $S(00) \to S(11) \cong T(11)$. Therefore, in view of Lemmas 3.2.5 and 1.4.2, we finish the proof. ∎

For any nonnegative integers m and n, define $T_{m,n} \in (\mathbf{MSm}^{\mathrm{fin}})^{\mathbf{Sq}}$ by

$$\begin{gathered} T_{m,n}(11) := T(11) \\ T_{m,n}(10) := (\overline{T}(10), T(10)^\infty + mD_{10}) \\ T_{m,n}(01) := (\overline{T}(01), T(01)^\infty + nD_{01}) \\ T_{m,n}(00) := T_{m,n}(10) \times^{\mathrm{c}}_{T_{m,n}(11)} T_{m,n}(01), \end{gathered}$$

where $\times^{\mathrm{c}}$ denotes the canonical model of fiber product introduced in [9, Definition 2.2.2]. Note that for each $(ij) \in \mathbf{Sq} - \{(00)\}$, the open immersion $\overline{S}(ij) \to \overline{T}(ij)$ induces an object $(S(ij) \to T_{m,n}(ij)) \in \mathbf{Comp}(S(ij))$. Moreover, it is easy to see by the construction of the canonical model of fiber product that the morphism $S(00) \to T_{m,n}(00)$ in $\underline{\mathbf{M}}\mathbf{Sm}$, which is induced by the universal property of fiber product, is ambient and minimal, and defines an object $(S(00) \to T_{m,n}(00)) \in \mathbf{Comp}(S(00))$.

Therefore, we obtain $(S \to T_{m,n}) \in \mathbf{Comp}(S)$ for any m, n. Write

$$\begin{tikzcd} T_{m,n}(00) \arrow[r, "v_{T_{m,n}}"] \arrow[d, "q_{T_{m,n}}"'] \arrow[dr, "d_{T_{m,n}}"] & T_{m,n}(01) \arrow[d, "p_{T_{m,n}}"] \\ T_{m,n}(10) \arrow[r, "u_{T_{m,n}}"] & T_{m,n}(11). \end{tikzcd}$$

Note that $T_{m,n}$ is Cartesian in $\mathbf{MSm}$ for any m, n by construction.

3.3 Cofinality of Cartesian squares

PROPOSITION 3.3.1. *Let $(S \to T) \in \mathbf{Comp}(S)$ verifying Conditions 3.2.3. Let n_T be as in Lemma 3.2.6, and $T_{m,n}$ be as constructed above. Then for any integers $m, n \geqslant n_T$, there exists a morphism $(S \to T_{m,n}) \to (S \to T)$ in $\mathbf{Comp}(S)$.*

Proof. For $(ij) \in \mathbf{Sq} - \{(00)\}$ and for any m, n, there exists a natural morphism $T_{m,n}(ij) \to T(ij)$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ by construction.

Our task is to show that the isomorphism $T_{m,n}(00)^{\mathrm{o}} \to T(00)^{\mathrm{o}}$ in $\mathbf{Sm}$ defines a morphism $T_{m,n}(00) \to T(00)$ in $\mathbf{MSm}$ for $m, n \geqslant n_T$.

Let Γ be the graph of the rational map $\overline{T}_{m,n}(00) \dashrightarrow \overline{T}(00)$, and let $\Gamma^N \to \Gamma$ be the normalization of Γ. Then we obtain the following commutative diagrams of schemes:

$$\begin{tikzcd} \overline{T}_{m,n}(00) \arrow[d, "\overline{q}_{T_{m,n}}"'] & \Gamma^N \arrow[l, "a"'] \arrow[r, "b"] & \overline{T}(00) \arrow[d, "\overline{q}_T"] \\ \overline{T}_{m,n}(10) \arrow[rr, "="] & & \overline{T}(10), \end{tikzcd} \qquad \begin{tikzcd} \overline{T}_{m,n}(00) \arrow[d, "\overline{v}_{T_{m,n}}"'] & \Gamma^N \arrow[l, "a"'] \arrow[r, "b"] & \overline{T}(00) \arrow[d, "\overline{v}_T"] \\ \overline{T}_{m,n}(01) \arrow[rr, "="] & & \overline{T}(01). \end{tikzcd}$$

Claim 3.3.2. $\sup(a^*\overline{q}^*_{T_{m,n}} D_{10}, a^*\overline{v}^*_{T_{m,n}} D_{01}) = b^*D$.

Proof. We have $\overline{q}_T b = \overline{q}_{T_{m,n}} a$ and $\overline{v}_T b = \overline{v}_{T_{m,n}} a$ by the commutativity of the above diagrams. Therefore,

$$\begin{aligned} \sup(a^*\overline{q}^*_{T_{m,n}} D_{10}, a^*\overline{v}^*_{T_{m,n}} D_{01}) &= \sup(b^*\overline{q}^*_T D_{10}, b^*\overline{v}^*_T D_{01}) \\ &= b^* \sup(\overline{q}^*_T D_{10}, \overline{v}^*_T D_{01}) \\ &= b^* D. \end{aligned}$$

This finishes the proof. ∎

By the construction of $T_{m,n}(00)$, we have

$$T_{m,n}(00)^\infty = \sup(\overline{q}^*_{T_{m,n}} T_{m,n}(10)^\infty, \overline{v}^*_{T_{m,n}} T_{m,n}(01)^\infty),$$

and hence

$$a^*T_{m,n}(00)^\infty = \sup(a^*\overline{q}^*_{T_{m,n}} T_{m,n}(10)^\infty, a^*\overline{v}^*_{T_{m,n}} T_{m,n}(01)^\infty). \tag{3.3.1}$$

By the construction of $T_{m,n}$ and by the choice of m, n, we have

$$\begin{aligned} \overline{u}^*_{T_{m,n}} T(11)^\infty + n_T D_{10} \leqslant \overline{u}^*_{T_{m,n}} T(11)^\infty + m D_{10} \leqslant T_{m,n}(10)^\infty, \\ \overline{p}^*_{T_{m,n}} T(11)^\infty + n_T D_{01} \leqslant \overline{p}^*_{T_{m,n}} T(11)^\infty + n D_{01} \leqslant T_{m,n}(01)^\infty. \end{aligned}$$

Combining these with (3.3.1), we have

$$\begin{aligned} a^*T_{m,n}(00)^\infty &\geqslant a^*\overline{d}^*_{T_{m,n}} T(11)^\infty + n_T \sup(a^*\overline{q}^*_{T_{m,n}} D_{10}, a^*\overline{v}^*_{T_{m,n}} D_{01}) \\ &= b^*(\overline{d}^*_T T(11)^\infty + n_T D), \end{aligned}$$

where the last equality follows from Claim 3.3.2 and $\overline{d}_{T_{m,n}} a = \overline{d}_T b$. Therefore, in view of Lemma 3.2.6, we have

$$a^*T_{m,n}(00)^\infty \geqslant b^*T(00)^\infty,$$

which shows that the isomorphism $T_{m,n}(00)^{\mathrm{o}} \to T(00)^{\mathrm{o}}$ in $\mathbf{Sm}$ defines a morphism $T_{m,n}(00) \to T(00)$ in $\mathbf{MSm}$. This finishes the proof of Proposition 3.3.1. ∎

COROLLARY 3.3.3. *The subset*

$$\{(S \to T) \in \mathbf{Comp}(S) \mid T \in (\mathbf{MSm}^{\mathrm{fin}})^{\mathbf{Sq}}, T \textit{ is Cartesian in } \mathbf{MSm}\}$$

is cofinal in $\mathbf{Comp}(S)$. ∎

3.4 Topological study of a certain diagram

By Corollary 3.3.3, we may and do assume in Theorem 3.1.1 that $T \in (\mathbf{MSm}^{\mathrm{fin}})^{\mathbf{Sq}}$ and that T is Cartesian in $\mathbf{MSm}$. Since T^{o} is an elementary Nisnevich square, the morphism $\mathrm{OD}(q_T)^{\mathrm{o}} \to \mathrm{OD}(p_T)^{\mathrm{o}}$ is an isomorphism in $\mathbf{Sm}$, and it induces an admissible morphism $\mathrm{OD}(q_T) \to \mathrm{OD}(p_T)$. Our task is to modify T in order to make this morphism invertible in $\mathbf{MSm}$. In this subsection, we prepare the ground to show that we only need to increase multiplicities of divisors, which will be done in the next subsection.

We take the notation of Section 3.2. Let Γ be the graph of the birational map $\overline{\mathrm{OD}(q_T)} \dashrightarrow \overline{\mathrm{OD}(p_T)}$. Let $\nu : \Gamma \to \overline{\mathrm{OD}(p_T)}$ be the natural map, and let $s_i : \overline{\mathrm{OD}(p_T)} \to \overline{T}(01)$ be the natural ith projection for $i = 1, 2$.

Set $H := \nu^* s_1^* D_{01} \times_\Gamma \nu^* s_2^* D_{01}$, let

$$\pi : \Gamma_1 := \mathbf{Bl}_H(\Gamma)^N \to \Gamma \tag{3.4.1}$$

be the normalized blowup of Γ along H, and set $b := \nu\pi$. Then $b^* s_1^* D_{01}$ and $b^* s_2^* D_{01}$ have a universal supremum (see Definition 3.2.1) by construction. Set

$$D' := \sup(b^* s_1^* D_{01}, b^* s_2^* D_{01}). \tag{3.4.2}$$

By [9, Proposition 2.2.7(3), Lemma 4.1.4], there are natural open immersions $j_q : \overline{\mathrm{OD}(q_S)} \to \overline{\mathrm{OD}(q_T)}$ and $j_p : \overline{\mathrm{OD}(p_S)} \to \overline{\mathrm{OD}(p_T)}$. They lift uniquely to open immersions $j'_q : \overline{\mathrm{OD}(q_S)} \to \Gamma_1$ and $j'_p : \overline{\mathrm{OD}(p_S)} \to \Gamma_1$ since $\overline{\mathrm{OD}(q_S)}$ and $\overline{\mathrm{OD}(p_S)}$ are normal by construction and since $H \cap j_p(\overline{\mathrm{OD}(p_S)}) = H \cap j_q(\overline{\mathrm{OD}(q_S)}) = \emptyset$.

Thus we obtain commutative diagrams of schemes $(i = 1, 2)$

$$\begin{tikzcd}
\overline{\mathrm{OD}(q_S)} \arrow[rr, "\sim"] \arrow[d, "j_q"] \arrow[dr, "j'_q"] & & \overline{\mathrm{OD}(p_S)} \arrow[d, "j_p"] \arrow[dl, "j'_p"'] \\
\overline{\mathrm{OD}(q_T)} \arrow[d, "t_i"] & \Gamma_1 \arrow[l, "a"'] \arrow[r, "b"] & \overline{\mathrm{OD}(p_T)} \arrow[d, "s_i"] \\
\overline{T}(00) \arrow[rr, "\overline{v}_T"] & & \overline{T}(01),
\end{tikzcd} \tag{3.4.3}$$

where t_i is the natural ith projection and a is the composite $\Gamma_1 \to \Gamma \to \overline{\mathrm{OD}(q_T)}$.

Set

$$U := j'_q(\overline{\mathrm{OD}(q_S)}) = j'_p(\overline{\mathrm{OD}(p_S)}). \tag{3.4.4}$$

Consider the composite

$$c : \overline{\mathrm{OD}(p_T)} \to \overline{T(01) \times^{\mathrm{c}}_{T(11)} T(01)} \to \overline{T}(11),$$

where the first morphism is the natural inclusion and the second is the structural morphism.

LEMMA 3.4.1. $a^*\mathrm{OD}(q_T)^\infty|_U = b^*\mathrm{OD}(p_T)^\infty|_U = b^*c^*T(11)^\infty|_U$.

Proof. The natural morphism $\mathrm{OD}(q_S) \to \mathrm{OD}(p_S)$ is an isomorphism in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ by [9, Proposition 4.1.5]. Moreover, the morphisms $\mathrm{OD}(q_S) \to \mathrm{OD}(q_T)$ and $\mathrm{OD}(p_S) \to \mathrm{OD}(p_T)$ are minimal. Therefore, the first equality follows from the commutativity of (3.4.3). The second assertion follows from the minimality of the natural morphism $\mathrm{OD}(p_S) \to S(10) \times^{\mathrm{c}}_{S(11)} S(01) \to S(11) \to T(11)$. This finishes the proof. ∎

LEMMA 3.4.2. $\overline{\mathrm{OD}(q_T)} - j_q(\overline{\mathrm{OD}(q_S)}) = t_1^{-1}\overline{v}_T^{-1}(|D_{01}|) \cup t_2^{-1}\overline{v}_T^{-1}(|D_{01}|)$.

Proof. Set

$$A := t_1^{-1}\overline{v}_T^{-1}(\overline{S}(01)) \cap t_2^{-1}\overline{v}_T^{-1}(\overline{S}(01)).$$

Then the assertion is equivalent to the equality

$$j_q(\overline{\mathrm{OD}(q_S)}) = A.$$

The inclusion $\subset$ follows from the commutativity of (3.4.3). By the universal property of the fiber product, we have a natural morphism

$$\gamma : A \to \overline{S}(01) \times_{\overline{S}(11)} \overline{S}(01).$$

Claim 3.4.3. $\gamma(A) \cap \Delta(\overline{S}(01)) = \emptyset$.

Proof. First, note that $\overline{\mathrm{OD}(q_T)}$ is the closure of

$$\mathrm{OD}(q_T^{\mathrm{o}}) = \mathrm{OD}(q_S^{\mathrm{o}}) = S(00)^{\mathrm{o}} \times_{S(10)^{\mathrm{o}}} S(00)^{\mathrm{o}} - \Delta(S(00)^{\mathrm{o}})$$

by construction (see the proof of [9, Theorem 3.1.3]). Therefore, since $A \subset \overline{\mathrm{OD}(q_T)}$ by construction, the open immersion

$$\mathrm{OD}(q_T^{\mathrm{o}}) \to A$$

is also dense. In particular, $\mathrm{OD}(q_T^{\mathrm{o}})$ is dense in $\gamma(A)$.

Since $\mathrm{OD}(q_T^{\mathrm{o}}) \cap \Delta(S(01)^{\mathrm{o}}) = \emptyset$ and since $\Delta(S(01)^{\mathrm{o}})$ is dense in $\Delta(\overline{S}(01))$, we have $\mathrm{OD}(q_T^{\mathrm{o}}) \cap \Delta(\overline{S}(01)) = \emptyset$. Therefore, since $\Delta(\overline{S}(01))$ is closed in $\overline{S}(01) \times_{\overline{S}(11)} \overline{S}(01)$, we have

$$\gamma(A) \cap \Delta(\overline{S}(01)) \subset \overline{\mathrm{OD}(q_T^{\mathrm{o}})} \cap \Delta(\overline{S}(01)) = \emptyset.$$

This finishes the proof of Claim 3.4.3. ∎

By Claim 3.4.3, the morphism γ induces

$$A \to j_q(\overline{\mathrm{OD}(q_S)}).$$

This map is inverse to the inclusion morphism since their restrictions to the dense interior $S^{\mathrm{o}}(01) \times_{S^{\mathrm{o}}(11)} S^{\mathrm{o}}(01)$ are the identity. Therefore, $j_q(\overline{\mathrm{OD}(q_S)}) = A$. This finishes the proof of Lemma 3.4.2. ∎

LEMMA 3.4.4. *We have $\Gamma_1 - U = |D'|$ (see (3.4.2) and (3.4.4) for the definitions of D' and U).*

Proof. By applying a^{-1} to both sides of the equality in Lemma 3.4.2, by using the commutativity of the above diagram and by Lemma 1.4.3, we have

$$\Gamma_1 - U = b^{-1}s_1^{-1}(|D_{01}|) \cup b^{-1}s_2^{-1}(|D_{01}|) = |D'|,$$

where the last equality follows from the construction of D'. This finishes the proof. ∎

3.5 Increasing multiplicities

By Lemmas 1.4.2, 3.4.1, and 3.4.4, there exists a positive integer n such that

$$a^*\mathrm{OD}(q_T)^\infty \leqslant b^*c^*T(11)^\infty + nD'. \tag{3.5.1}$$

Set

$$T' := T_{0,n},$$

where the right-hand side is defined as in Subsection 3.2.

Then there exists a natural morphism $T' \to T$ in $(\mathbf{MSm}^{\mathrm{fin}})^{\mathbf{Sq}}$, and we obtain the commutative diagram

$$\begin{tikzcd} \mathrm{OD}(q_{T'}) \arrow[r] \arrow[d] & \mathrm{OD}(q_T) \arrow[d] \\ \mathrm{OD}(p_{T'}) \arrow[r] & \mathrm{OD}(p_T) \end{tikzcd} \tag{3.5.2}$$

in $\mathbf{MSm}$ by the functorial property of OD (see [9, Theorem 3.1.3]).

Lemma 3.5.1. *Diagram (3.5.2) is Cartesian in* $\mathbf{MSm}$.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd} \mathrm{OD}(q_{T'}) \arrow[r] \arrow[d] & \mathrm{OD}(q_T) \arrow[r, "\pi_q"] \arrow[d] & T(10) \arrow[d, "u_T"] \\ \mathrm{OD}(p_{T'}) \arrow[r] & \mathrm{OD}(p_T) \arrow[r, "\pi_p"] & T(11), \end{tikzcd}$$

in $\mathbf{MSm}$, where π_p and π_q are the natural projections to the bases of fiber products. Since T is Cartesian, the right square is Cartesian by [9, Proposition 3.1.4]. Since T' is also Cartesian and since $T'(11) = T(11)$ and $T'(10) = T(10)$ by construction, the large square is Cartesian by [9]. Therefore, a general argument shows that the left square is Cartesian. This finishes the proof. ∎

The main point of this subsection is the following.

Proposition 3.5.2. *The natural morphism* $\mathrm{OD}(q_{T'}) \to \mathrm{OD}(p_{T'})$ *is an isomorphism in* $\mathbf{MSm}$.

Proof. We will construct an inverse morphism. By Lemma 3.5.1, it suffices to show that $\mathrm{OD}(p_{T'}) \to \mathrm{OD}(p_T)$ lifts to a morphism $\mathrm{OD}(p_{T'}) \to \mathrm{OD}(q_T)$.

Let Γ_2 be the graph of the rational map $\overline{\mathrm{OD}(p_{T'})} \dashrightarrow \Gamma_1$. Then we obtain the following commutative diagrams ($i = 1, 2$),

$$\begin{tikzcd} \overline{\mathrm{OD}(p_{T'})} \arrow[d, "s'_i"] \arrow[dd, bend right, "c'"'] & \Gamma_2^N \arrow[l, "a'"'] \arrow[r, "b'"] & \Gamma_1 \arrow[r, "b"] & \overline{\mathrm{OD}(p_T)} \arrow[d, "s_i"] \arrow[dd, bend left, "c"] \\ \overline{T}'(01) \arrow[rrr, "="] \arrow[d, "\overline{p}_{T'}"] & & & \overline{T}(01) \arrow[d, "\overline{p}_T"] \\ \overline{T}'(11) \arrow[rrr, "="] & & & \overline{T}(11), \end{tikzcd}$$

where s'_i are natural projections and $c' := \overline{p}_{T'} s'_1 = \overline{p}_{T'} s'_2$.

By (3.5.1),

$$(b')^* a^* \mathrm{OD}(q_T)^\infty \leqslant (b')^* (b^* c^* T(11)^\infty + nD').$$

The commutativity of the above diagram shows

$$(b')^* b^* c^* T(11)^\infty = (a')^* (c')^* T(11)^\infty.$$

By (3.4.2),

$$\begin{aligned}(b')^* D' &= \sup((b')^* b^* s_1^* D_{01}, (b')^* b^* s_2^* D_{01}) \\ &= \sup((a')^* (s'_1)^* D_{01}, (a')^* (s'_2)^* D_{01}).\end{aligned}$$

Combining these, we get

$$(b')^* a^* \mathrm{OD}(q_T)^\infty \leqslant \sup_{i=1,2} ((a')^* ((c')^* T(11)^\infty + n(s'_i)^* D_{01})).$$

By the admissibility of $p_T : T(01) \to T(11)$, and noting that $p_T = p_{T'}$ by construction, we have

$$(c')^* T(11)^\infty = (s'_i)^* \overline{p}_{T'}^* T(11)^\infty \leqslant (s'_i)^* T(01)^\infty$$

for each $i = 1, 2$.

Therefore, we have

$$(b')^* a^* \mathrm{OD}(q_T)^\infty \leqslant \sup_{i=1,2} ((a')^* (s'_i)^* (T(01)^\infty + nD_{01})). \tag{3.5.3}$$

Recall that $T'(01)^\infty = T(01)^\infty + nD_{01}$ by the definition of $T' = T_{0,n}$. Moreover, by the construction of OD, we have $\mathrm{OD}(p_{T'})^\infty = \sup_{i=1,2}((s'_i)^* T'(01)^\infty)$ (see the proof of [9, Theorem 3.1.3]). Therefore, we have

$$\begin{aligned}\sup_{i=1,2} ((a')^* (s'_i)^* (T(01)^\infty + nD_{01})) &= \sup_{i=1,2} ((a')^* (s'_i)^* T'(01)^\infty) \\ &= (a')^* \sup_{i=1,2} ((s'_i)^* T'(01)^\infty) \\ &= (a')^* \mathrm{OD}(p_{T'})^\infty.\end{aligned}$$

Combined with (3.5.3), this implies

$$(b')^* a^* \mathrm{OD}(q_T)^\infty \leqslant (a')^* \mathrm{OD}(p_{T'})^\infty. \tag{3.5.4}$$

Let Γ_3 be the graph of the rational map $\overline{\mathrm{OD}(p_{T'})} \dashrightarrow \overline{\mathrm{OD}(q_T)}$, and let $\Gamma_3^N \to \Gamma_3$ be the normalization of Γ_3. Recalling that Γ_2 is the graph of the rational map $\overline{\mathrm{OD}(p_{T'})} \dashrightarrow \Gamma_1$, we have a natural birational morphism $\Gamma_2 \to \Gamma_3$. Thus, we have the following commutative diagram:

$$\begin{tikzcd} \overline{\mathrm{OD}(p_{T'})} \arrow[d, "\mathrm{Id}"'] & \Gamma_2^N \arrow[l, "a'"'] \arrow[r, "b'"] \arrow[d, "\xi"] & \Gamma_1 \arrow[d, "a"] \\ \overline{\mathrm{OD}(p_{T'})} & \Gamma_3^N \arrow[l, "l"'] \arrow[r, "r"] & \overline{\mathrm{OD}(q_T)}, \end{tikzcd}$$

where ξ is induced from the morphism $\Gamma_2 \to \Gamma_3$ by the universal property of normalization.

By (3.5.4), and by the commutativity of the above diagram, we have

$$\xi^* r^* \mathrm{OD}(q_T)^\infty \leqslant \xi^* l^* \mathrm{OD}(p_{T'})^\infty.$$

Since ξ is proper birational by construction, Lemma 1.4.1 and (3.5.4) imply

$$r^* \mathrm{OD}(q_T)^\infty \leqslant l^* \mathrm{OD}(p_{T'})^\infty,$$

which shows that $\mathrm{OD}(p_{T'}) \to \mathrm{OD}(q_T)$ is defined in $\mathbf{MSm}$. This concludes the proof of Proposition 3.5.2. ∎

3.6 End of proof of Theorem 3.1.1

We now assume that S is partially compact. Then T' satisfies Condition (2) in Definition 1.3.2. Since T' is Cartesian in $\mathbf{MSm}$, it also satisfies Condition (1). Moreover, it satisfies Condition (3) by Proposition 3.5.2. Therefore, T' is an MV-square, and hence $T' \in \mathbf{Comp}^{\mathrm{MV}}(S)$. This finishes the proof.

§4. Cofinality of MV-squares: the general case

In this section, we complete the proof of Theorem 1.5.6.

4.1 Preparatory lemmas

LEMMA 4.1.1. *Let S be a $\underline{\mathrm{MV}}^{\mathrm{fin}}$-square, and let $S \to S'$ be a partial compactification. Let F be a closed subscheme of $\overline{S}'(11)$ such that $F \cap S'(11)^{\mathrm{o}} = \emptyset$, and set $F_{ij} := F \times_{\overline{S}'(11)} \overline{S}'(ij) \subset \overline{S}'(ij)$ for $(ij) \in \mathbf{Sq}$. Note that $F_{11} = F$ by definition.*

Let

$$S'_F(ij) := \mathbf{Bl}_{F_{ij}} S'(ij)$$

be the blowup of $S'(ij)$ along F_{ij} (see Definition 2.5.2). Note that for any morphism $(ij) \to (kl)$ in $\mathbf{Sq}$, the structure morphism $S'(ij) \to S'(kl)$ lifts to a morphism $S'_F(ij) \to S'_F(kl)$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ by the universal property of blowing up.

Then the resulting square S'_F is an $\underline{\mathrm{MV}}^{\mathrm{fin}}$-square. Moreover, the morphism $S \to S'$ lifts uniquely to a morphism $S \to S'_F$ in $(\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}})^{\mathbf{Sq}}$, and it is a partial compactification of S.

Proof. Since $\overline{S}(ij) \to \overline{S}(11)$ are étale (hence flat), we have

$$\overline{S}'_F(ij) \cong \overline{S}'(ij) \times_{\overline{S}'(11)} \overline{S}'_F(11). \tag{4.1.1}$$

Therefore, the resulting square $\overline{S}'_F$ is an elementary Nisnevich square as base change. Since $S'_F(ij)^\infty$ are the pullback of $S'_F(11)^\infty$ by construction, we obtain an $\underline{\mathrm{MV}}^{\mathrm{fin}}$-square S'_F.

By Lemma 2.5.3, the object $(S(11) \to S'(11)) \in \mathbf{Comp}(S(11))$ lifts uniquely to $(S(11) \to S'_F(11)) \in \mathbf{Comp}(S(11))$. Similarly, for $(ij) \in \mathbf{Sq} - \{(11)\}$, the open immersion $\overline{S}(ij) \to \overline{S}'(ij)$ lifts uniquely to an open immersion $\overline{S}(ij) \to \overline{S}'_F(ij)$ since $F_{ij} \cap \overline{S}(ij)$ is an effective Cartier divisor by assumption. Therefore, we obtain a morphism $S \to S'_F$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$.

The last thing to check is Condition (c) of Definition 2.1.1. Since $S \to S'$ is a partial compactification, the natural morphism

$$\overline{S}(ij) \xrightarrow{\sim} \overline{S}'(ij) \times_{\overline{S}'(11)} \overline{S}(11)$$

is an isomorphism. Combining this with (4.1.1) as above, for each $(ij) \in \mathbf{Sq}$, we have

$$\begin{aligned}\overline{S}'_F(ij) \times_{\overline{S}'_F(11)} \overline{S}(11) &\cong \overline{S}'(ij) \times_{\overline{S}'(11)} \overline{S}'_F(11) \times_{\overline{S}'_F(11)} \overline{S}(11)\\ &\cong \overline{S}'(ij) \times_{\overline{S}'(11)} \overline{S}(11)\\ &\cong \overline{S}(ij).\end{aligned}$$

This finishes the proof of Lemma 4.1.1. ∎

LEMMA 4.1.2. *Let $f : M \to N$ be a morphism in $\underline{\mathbf{M}}\mathbf{Sm}$ with $N \in \mathbf{MSm}$. Assume that f is ambient and minimal, and that $f^o : M^o \to N^o$ is an isomorphism. Then there exists an isomorphism $N \xrightarrow{\sim} N'$ in $\mathbf{MSm}$ such that the composite $M \to N \xrightarrow{\sim} N'$ belongs to $\mathbf{Comp}(M)$.*

Proof. Take any compactification $j : \overline{M} \to X$, and let Γ_1 be the graph of the birational map $g : X \dashrightarrow \overline{N}$. Since the composite $gj = \overline{f}$ is a morphism of schemes, the open immersion j lifts uniquely to an open immersion $j_1 : \overline{M} \to \Gamma_1$. Let

$$\pi : \Gamma_2 := \mathbf{Bl}_{(\Gamma_1 - j_1(\overline{M}))_{\mathrm{red}}}(\Gamma_1) \to \Gamma_1$$

be the blowup of Γ_1 along the closed subscheme $(\Gamma_1 - j_1(\overline{M}))_{\mathrm{red}}$. Then j_1 lifts to an open immersion $j_2 : \overline{M} \to \Gamma_2$ since π is an isomorphism over $j_1(\overline{M})$. Thus we obtain the following commutative diagram:

$$\begin{tikzcd} \Gamma_2 \arrow[r, "\pi"] & \Gamma_1 \arrow[d] \arrow[dr, "p"] & \\ \overline{M} \arrow[u, "j_2"] \arrow[ur, "j_1"] \arrow[r, "j"] & X \arrow[r, dotted, "g"] & \overline{N}. \end{tikzcd}$$

Note that $\overline{f} = p\pi j_2$ by construction. Set

$$N' := (\overline{N}', (N')^\infty) := (\Gamma_2, \pi^* p^* N^\infty).$$

Then we have $j_2^*(N')^\infty = j_2^*\pi^*p^*N^\infty = \overline{f}^* N^\infty = M^\infty$, where the last equality follows from the minimality of f. Therefore, j_2 induces a minimal morphism $J : M \to N'$. Moreover, denoting by E the exceptional divisor of the blowup π, we have $E \leqslant (N')^\infty$ by construction. Thus, we obtain a decomposition $(N')^\infty = ((N')^\infty - E) + E$ as a sum of two effective Cartier divisors. Therefore, we have $(J : M \to N') \in \mathbf{Comp}(M)$.

By construction, $p\pi$ induces a minimal morphism $h : N' \to N$. Moreover, it induces an isomorphism $h^o : (N')^o \xrightarrow{\sim} N^o$. Therefore, h is an isomorphism in $\mathbf{MSm}$. The commutativity of the above diagram shows $hJ = f$; hence $h^{-1}f = J \in \mathbf{Comp}(M)$. This finishes the proof. ∎

4.2 A key proposition

PROPOSITION 4.2.1. *Let S be an $\underline{\mathrm{MV}}^{\mathrm{fin}}$-square, and let $(S \to T) \in \mathbf{Comp}(S)$ be any object. Then there exists a partial compactification $S \to S'$, an object $(S' \to T') \in \mathbf{Comp}(S')$, and a morphism $T' \to T$ in $\mathbf{MSm}^{\mathrm{fin}}$ such that the diagram*

$$\begin{tikzcd} S' \arrow[r] & T' \arrow[d] \\ S \arrow[u] \arrow[r] & T \end{tikzcd}$$

commutes. If S is normal, we can choose S' to be normal.

Proof. The first step of the proof is to find a partial compactification $S \to S'$ such that the morphism $S \to T$ extends to a morphism $S' \to T$ in $(\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}})^{\mathbf{Sq}}$.

By Theorem 2.1.4, take a partial compactification $S \to S'_1$ such that $(S(11) \to S'_1(11))$ dominates $(S(11) \to T(11))$ in $\mathbf{Comp}(S(11))$ and $S'_1(11) \to T(11)$ is minimal. For all $(ij) \in \mathbf{Sq}$, let Γ_{ij} denote the graph of the rational map $\overline{S}'_1(ij) \dashrightarrow \overline{T}(ij)$. Note that the projection $\Gamma_{ij} \to \overline{S}'_1(ij)$ is a proper birational morphism, which is an isomorphism over $\overline{S}(ij) = \overline{S}'_1(ij) \times_{\overline{S}'_1(11)} \overline{S}(11)$. Consider the étale morphism

$$f : \bigsqcup_{(ij)\in\mathbf{Sq}} \overline{S}'_1(ij) \to \overline{S}'_1(11).$$

By Lemma 1.4.4, we can find a closed subscheme F of $\overline{S}'_1(11)$, which is supported on $|S'_1(11)^\infty|$ and such that the base change of f along the blowup $\mathbf{Bl}_{F_1}\overline{S}'_1(11) \to \overline{S}'_1(11)$ factors through $\sqcup_{i,j}\Gamma_{ij}$.

Set $S'_2 := (S'_1)_F$, where the right-hand side is defined as in Lemma 4.1.1. Then S'_2 is an $\underline{\mathrm{MV}}^{\mathrm{fin}}$-square, and the morphism $S \to S'_1$ lifts uniquely to a partial compactification $S \to S'_2$. Moreover, we have a morphism $\overline{S}'_2 \to \overline{T}$ in $\mathbf{Sch}^{\mathbf{Sq}}$ by construction.

Since $(S(11) \to S'_2(11)) \in \mathbf{Comp}(S(11))$, there exists an effective Cartier divisor D_{11} on $\overline{S}'_2(11)$ such that $|D_{11}| = \overline{S}'_2(11) - \overline{S}(11)$. Since $\overline{S}(ij) = \overline{S}'_2(ij) \times_{\overline{S}'_2(11)} \overline{S}(11)$ by the definition of a partial compactification, if we set $D_{ij} := D_{11} \times_{\overline{S}'_2(11)} \overline{S}'_2(ij)$, we have $|D_{ij}| = \overline{S}'_2(ij) - \overline{S}(ij)$. By Lemma 1.4.2, there exists a positive integer n such that the morphism $\overline{S}'_2(ij) \to \overline{T}(ij)$ induces an admissible morphism

$$S'_3(ij) := (\overline{S}'_2(ij), S'_2(ij)^\infty + nD_{ij}) \to T(ij)$$

for each $(ij) \in \mathbf{Sq}$, and they induce minimal morphisms $S(ij) \to S'_3(ij)$. Since $\overline{S}'_3(ij) = \overline{S}'_2(ij)$ and since $S'_3(ij)^\infty \cap \overline{S}(ij) = S(ij)^\infty$ for each $(ij) \in \mathbf{Sq}$ by construction, the morphism $S \to S'_2$ lifts uniquely to a partial compactification $S \to S'_3$.

We set $S' := S'_3$. Then $(S \to S') \in \mathbf{Comp}(S)$ and the morphism $S \to T$ lifts to a morphism $S' \to T$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ by construction, as required. If S is normal, we replace S' by its normalization as in Corollary 2.6.2.

Take now any $(S' \to T'_1) \in \mathbf{Comp}(S')$. Then the graph trick shows that there exists $(S \to T'_2) \in \mathbf{Comp}(S')$, which dominates $S \to T'_1$, and the composite maps

$$\overline{h}_{ij} : \overline{T}'_2(ij) \to \overline{T}'_1(ij) \dashrightarrow \overline{T}(ij)$$

are morphisms of schemes.

For each $(ij) \in \mathbf{Sq}$, let F_{ij} be an effective Cartier divisor on $\overline{T}'(ij)$ such that $|F_{ij}| = \overline{T}'(ij) - \overline{S}'(ij)$, which exists by the definition of $\mathbf{Comp}$. By Lemma 1.4.2, noting that $\overline{h}^*_{ij}T^\infty(ij) \cap \overline{S}'(ij) \leqslant S'^\infty(ij) = T'^\infty(ij) \cap \overline{S}'(ij)$, there exist positive integers m_{ij} such that we have

$$\overline{h}^*_{ij}T^\infty(ij) \leqslant T'^\infty(ij) + m_{ij}F_{ij}$$

for all $i, j \in \{0, 1\}$. Define

$$T'_{m_{ij}} := (\overline{T}'(ij), T'^\infty(ij) + m_{ij}F_{ij}).$$

Clearly, by choosing $\boldsymbol{m} = (m_{ij})_{i,j\in\{0,1\}}$ appropriately, we obtain admissible morphisms $T'_{m_{ij}} \to T'_{m_{i'j'}}$ for any morphism $(i,j) \to (i',j')$ in the diagram category $\mathbf{Sq}$. Therefore, we obtain a diagram

$$T' := T'_{\boldsymbol{m}} : \begin{tikzcd} T'_{m_{00}} \arrow[r] \arrow[d] & T'_{m_{01}} \arrow[d] \\ T'_{m_{10}} \arrow[r] & T'_{m_{11}}, \end{tikzcd}$$

which belongs to $\mathbf{Comp}(S')$, and its image in $\mathbf{Comp}(S)$ dominates T. This finishes the proof. ∎

4.3 End of proof of Theorem 1.5.6

Let S be a normal $\underline{\mathrm{MV}}^{\mathrm{fin}}$-square, and let $(j : S \to T) \in \mathbf{Comp}(S)$ be a compactification. Apply Proposition 4.2.1. By Theorem 3.1.1, there exists $(j'_1 : S' \to T'_1) \in \mathbf{Comp}^{\mathrm{MV}}(S')$, which dominates $S' \to T'$. Thus we have the following commutative diagram in $\underline{\mathbf{M}}\mathbf{Sm}$:

$$\begin{tikzcd} & T'_1 \arrow[d] \\ S' \arrow[ur, "j'_1"] \arrow[r, "j'"] & T' \arrow[d] \\ S \arrow[u, "f"] \arrow[r, "j"] & T. \end{tikzcd}$$

Note that $j'_1 f$ is ambient and minimal, and it induces an isomorphism $S^{\mathrm{o}} \xrightarrow{\sim} (T'_1)^{\mathrm{o}}$ by assumption. Therefore, by Lemma 4.1.2, there exists an isomorphism $g : T'_1 \xrightarrow{\sim} T'_2$ in $\mathbf{MSm}$ such that $(g j'_1 f : S \to T'_2) \in \mathbf{Comp}(S)$. Since T'_1 is an MV-square by construction, so is T'_2. Moreover, $S \to T'_2$ dominates $S \to T$ in $\mathbf{Comp}(S)$ by construction. This finishes the proof.

§5. Continuity and cocontinuity

In this section, we prove Theorem 1.

5.1 Continuity

Note that τ_s, $\underline{\omega}_s$, and ω_s all preserve fiber products, as required in Proposition A.1.2(1). The continuity of $\underline{\omega}_s$ and λ_s is obvious by Lemma A.2.1(a) since they send distinguished squares to distinguished squares. This implies the cocontinuity of λ_s, by Proposition A.1.3(2). In the case of τ_s, by the same lemma, we must show that $\{\tau_s T(01), \tau_s T(10)\}$ is an $\underline{\mathrm{MV}}$-cover of $\tau_s T(11)$ for any MV-square $T = (T(ij))$. By [SGA4, Exposé II, Theorem 4.4], this is the case if and only if, for any sheaf of sets F on $\underline{\mathbf{M}}\mathbf{Sm}$, the map

$$F(\tau_s(T(11))) \to F(\tau_s(T(01))) \times F(\tau_s(T(10)))$$

is injective. By Condition (2) of Definition 1.3.2, there is an $\underline{\mathrm{MV}}$-square S mapping to $\tau_s T$ and such that $S(11) \xrightarrow{\sim} \tau_s T(11)$, and hence the conclusion. But Condition (2) of Definition 1.3.2 says that this morphism is dominated by a morphism $S(01) \sqcup S(10) \to \tau_s(T(11))$ for some $\underline{\mathrm{MV}}$-square S such that $S(11) \xrightarrow{\sim} \tau_s(T(11))$, hence the conclusion. Finally, ω_s is continuous as a composition of continuous functors.

5.2 Cocontinuity

It suffices to check the conditions of Lemma A.2.1 for $\underline{\omega}_s$ and ω_s. The Nisnevich cd-structure on **Sm** is complete [15, Theorem 2.2]. Therefore, Condition (1) holds for $\underline{\omega}_s$ and ω_s. We now check that they both satisfy Condition (2).

Take any $M \in \underline{\mathbf{M}}\mathbf{Sm}$ and take any elementary Nisnevich square S in **Sm** such that $\underline{\omega}_s(M) \cong S(11)$. Take any $M_c \in \mathbf{Comp}(M)$. Then, regarding S as a square in $\underline{\mathbf{M}}\mathbf{Sm}$, Corollary 2.6.2 implies that there exists a partial compactification $S \to S'$ such that S' is normal, $S'(11) \in \mathbf{Comp}(S(11)) = \mathbf{Comp}(M^o)$ dominates $M_c \in \mathbf{Comp}(M) \subset \mathbf{Comp}(M^o)$, and the morphism $S'(11) \to M_c$ is minimal. In particular, $S'(11) \to M_c$ is an isomorphism in **MSm**.

5.2.1 Case of $\underline{\omega}_s$

Define a square S_1 in $\underline{\mathbf{M}}\mathbf{Sm}$ as follows: for any $(ij) \in \mathbf{Sq}$, set

$$\overline{S}_1(ij) = \overline{S}'(ij) \times_{\overline{M}_c} \overline{M},$$
$$S_1^{\infty}(ij) = S'^{\infty}(ij) \times_{\overline{M}_c} \overline{M},$$
$$S_1(ij) = (\overline{S}_1(ij), S_1^{\infty}(ij)).$$

Then S_1 is an $\underline{\mathrm{MV}}^{\mathrm{fin}}$-square (hence an $\underline{\mathrm{MV}}$-square). Moreover, the natural morphism $S_1(11) \to M$ is an isomorphism in $\underline{\mathbf{M}}\mathbf{Sm}$ since it is minimal, $\overline{S}_1(11) \to \overline{M}$ is proper surjective, and $S_1^o(11) = S^o(11) = M^o$. Therefore, the square S_1' obtained by replacing $S_1(11)$ in S_1 with M is also an $\underline{\mathrm{MV}}$-square, and we have $\underline{\omega}_s(S_1') = S$. Thus, Condition (2) of Lemma A.2.1 holds for $\underline{\omega}_s$, and therefore $\underline{\omega}_s$ is cocontinuous.

5.2.2 Case of ω_s

Here we take $M \in \mathbf{MSm}$; hence $M = M_c$. By Theorem 1.5.6, the category $\mathbf{Comp}^{\mathrm{MV}}(S')$ is cofinal in $\mathbf{Comp}(S')$. In particular, it is nonempty. Take any object $T \in \mathbf{Comp}^{\mathrm{MV}}(S')$. Then T is by definition an MV-square such that $T(11) = S'(11) \cong M$. Therefore, the square T' obtained by replacing $T(11)$ in T with M is also an MV-square, and we have $\omega_s(T') = S$. Thus, ω_s satisfies Condition (2) in Lemma A.2.1.

This finishes the proof of Theorem 1.

Remark 5.2.1. The functor τ_s is not cocontinuous: take $M \in \mathbf{MSm}$ of dimension 1. Cover $\tau_s M$ by two affine opens $S(01), S(10)$ (with the minimal modulus structure). Let $f : N \to M$ be a morphism in **MSm**. If $\tau_s f$ factors through $S(01)$ or $S(10)$, then its image is finite since $\overline{N}$ is proper. But any term of a cover of N is surjective on the ambient spaces, which is a contradiction.

Appendix A. Continuous and cocontinuous functors

A.1 Review of the notions

We write $\hat{\mathcal{C}}$ for the category of presheaves of sets on a category $\mathcal{C}$. If $\mathcal{C}$ is a site, we write $\tilde{\mathcal{C}} \subset \hat{\mathcal{C}}$ for the full subcategory of sheaves of sets, $i_{\mathcal{C}}$ for this inclusion, and $a_{\mathcal{C}}$ for its left adjoint (sheafification).

In this subsection, we recall some facts from [SGA4, Exposé III], where all sites are assumed to be "U-sites" in the sense of [SGA4, II.3.0.2]. We implicitly make this assumption below; it is automatic for sites defined by a cd-structure.

DEFINITION A.1.1. [SGA4, III.1.1 and III.2.2] Let $\mathcal{C}$ and $\mathcal{D}$ be sites, and let $u:\mathcal{C}\to\mathcal{D}$ be a functor. We say that u is *continuous* (resp. *cocontinuous*) if the functor $u^*:\hat{\mathcal{D}}\to\hat{\mathcal{C}}$ (resp. $u_*:\hat{\mathcal{C}}\to\hat{\mathcal{D}}$) carries sheaves to sheaves.

PROPOSITION A.1.2.

(1) *[SGA4, III.1.6]. Suppose that u preserves the fiber products involved in base changes under morphisms coming from covering families of $\mathcal{C}$. Then u is continuous if and only if, for any cover $\{U_i\to X\}$ in $\mathcal{C}$, $\{u(U_i)\to u(X)\}$ is a cover in $\mathcal{D}$.*
(2) *The functor u is cocontinuous if and only if, for any $X\in\mathcal{C}$ and any cover $\{V_i\to u(X)\}$ in $\mathcal{D}$, there is a cover $\{U_j\to X\}$ in $\mathcal{C}$ such that $\{u(U_j)\to u(X)\}$ refines $\{V_i\to u(X)\}$ (i.e., for each j, $u(U_j)\to u(X)$ factors through V_i for some i).*

Proof of (2). By [SGA4, III, 2.1 and 2.2], u is cocontinuous if and only if, for any $Y\in\mathcal{C}$ and any covering sieve R of $u(Y)$, the sieve of Y generated by the arrows $Z\to Y$ such that $u(Z)\to u(Y)$ factors through R is a covering sieve. This condition is clearly equivalent to that stated in (2). ∎

PROPOSITION A.1.3.

(1) *[SGA4, III.1.3] If u is continuous, the functor $u^t:\tilde{\mathcal{D}}\to\tilde{\mathcal{C}}$ given by Definition A.1.1 has a left adjoint u_t, given by the formula $u_t=a_{\mathcal{C}}u_!i_{\mathcal{D}}$.*
(2) *[SGA4, III.2.5] Let v be left adjoint to u. Then u is continuous if and only if v is cocontinuous.*

A.2 The case of cd-structures

LEMMA A.2.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites, and let $u:\mathcal{C}\to\mathcal{D}$ be a functor. Assume that the topologies on $\mathcal{C}$ and $\mathcal{D}$ are generated by cd-structures $P_{\mathcal{C}}$ and $P_{\mathcal{D}}$, respectively.*

(a) *Assume that $P_{\mathcal{C}}$ is complete and that u verifies the condition of Proposition A.1.2. Then u is continuous if and only if it sends elementary covers to covers.*
(b) *Assume the following:*
    (1) *$P_{\mathcal{D}}$ is complete.*
    (2) *For any $X'\in\mathcal{C}$ and for any distinguished square $Q\in P_{\mathcal{D}}$ such that $Q(11)=u(X')$, there exists $Q'\in P_{\mathcal{C}}$ such that $u(Q')=Q$ and $Q'(11)=X'$.*

Then u is cocontinuous.

Proof. (a) Necessity is obvious. Sufficiency: since $P_{\mathcal{C}}$ is complete, any cover can be refined by a simple cover as in [14, Definition 2.2] (this is the definition of complete; see [14, Definition 2.3]). Therefore, it suffices to show that u sends simple covers to covers. For any simple cover $\mathcal{V}=\{V_l\to X\}_{l\in L}$ of an object $X\in\mathcal{C}$, define $n_{\mathcal{V}}:=|L|$ (note that $n_{\mathcal{V}}$ is finite). We prove the assertion by induction on $n_{\mathcal{V}}\geqslant 1$.

If $n_{\mathcal{V}}=1$, then $\mathcal{V}$ consists of an isomorphism since if a simple cover is obtained by composing an elementary cover at least once, then the cardinality of the indexing set must be >1. Therefore the statement is trivial.

Assume that $n_{\mathcal{V}} > 1$ and take a distinguished square $Q \in P_{\mathcal{C}}$ of the form

$$\begin{tikzcd} B \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ A \arrow[r, "e"] & X \end{tikzcd}$$

and simple covers $\mathcal{Y} = \{p_s : Y_s \to Y\}_{s\in S}$ and $\mathcal{A} = \{q_t : A_t \to A\}_{t\in T}$ such that $\mathcal{V} = \{p \circ p_s, e \circ q_t\}_{s\in S, t\in T}$. Then obviously, we have $n_{\mathcal{Y}} < n_{\mathcal{V}}$ and $n_{\mathcal{A}} < n_{\mathcal{V}}$, and we are done by induction.

(b) Take any $X' \in \mathcal{C}$. By the same argument as in (a), it suffices to show the following assertion: for any simple cover $\mathcal{V} = \{V_j \to X\}_{j\in J}$, there exists a cover $\mathcal{U} = \{U_i \to X'\}_{i\in I}$ such that $u(\mathcal{U})$ refines $\mathcal{V}$. We proceed by induction on $n_{\mathcal{V}} \geqslant 1$, the case $n_{\mathcal{V}} = 1$ being trivial as above.

Assume that $n_{\mathcal{V}} > 1$. With the same notation as in (a), but with $Q \in P_{\mathcal{D}}$, by (2), there exists $Q' \in P_{\mathcal{C}}$ of the form

$$\begin{tikzcd} B' \arrow[r] \arrow[d] & Y' \arrow[d, "p'"] \\ A' \arrow[r, "e'"] & X' \end{tikzcd}$$

such that $u(Q') = Q$. In particular, we have $u(Y') = Y$ and $u(A') = A$. By the induction hypothesis, there exist covers $\mathcal{Y}' = \{p'_{s'} : Y'_{s'} \to Y'\}_{s'\in S'}$ and $\mathcal{A}' = \{q'_{t'} : A'_{t'} \to A'\}_{t'\in T'}$ such that $u(\mathcal{Y}')$ and $u(\mathcal{A}')$ refine $\mathcal{Y}$ and $\mathcal{A}$, respectively. Then we obtain a cover $\mathcal{U} = \{p' \circ p'_{s'}, e' \circ q'_{t'}\}_{s'\in S', t'\in T'}$ of X' such that $u(\mathcal{U}) = \{p \circ u(p'_{s'}), e \circ u(q'_{t'})\}_{s'\in S', t'\in T'}$ refines $\mathcal{V}$ by construction. This finishes the proof. ∎

Appendix B. A criterion for an open immersion

In this appendix, we state an important result that we found in [12, Tag:081M][2] (see also [7, Corollary 2.2] and [10, Theorem 2.7] in the affine case). We include its proof for completeness.

THEOREM B.1.1. *Let $f : X \to S$ be a morphism of schemes with X Noetherian, and let $U \subset S$ be an open subset. Assume the following conditions hold:*

(i) *f is separated, locally of finite presentation, and flat.*
(ii) *$f^{-1}(U) := U \times_S X \to U$ is an isomorphism.*
(iii) *The inclusion $U \to S$ is quasicompact and scheme-theoretically dense.*

Then, f is an open immersion.

Proof. First, consider the case that f is finite. Then, the proof is easy. Indeed, since f is finite flat of finite presentation, it is finite locally free. Since f is an isomorphism over a dense open subset, its degree is equal to 1. Hence it is an isomorphism everywhere.

Next, we treat the general case. Since f is flat of finite type, the image $f(X) \subset S$ is open. Therefore, we may assume that f is surjective. We want to prove that f is an isomorphism. By the above argument, it suffices to prove that f is finite.

[2] https://stacks.math.columbia.edu/tag/081M.

Claim B.1.2. f is quasifinite.

Proof. Since the problem is local on S, we may assume that S is affine. Then, since X is quasicompact by assumption, the morphism f is quasicompact. Moreover, f is of relative dimension 0 since f is flat and birational. Therefore, f is quasicompact and locally quasifinite and hence quasifinite. ∎

We need the following propositions from [1, Section 2.3 Proposition 8(a), Section 2.5 Proposition 2].

PROPOSITION B.1.3. (Étale localization of quasifinite morphisms) *Let $f : X \to Y$ be locally of finite type. Let x be a point of X, and set $y := f(x)$.*

If f is quasifinite at x, then there exists an étale neighborhood $Y' \to Y$ of y such that the morphism $f' : X' \to Y'$, obtained from f by the base change $Y' \to Y$, induces a finite morphism $f'|_{U'} : U' \to Y'$, where U' is an open neighborhood of the fiber of $X' \to X$ above x. In addition, if f is separated, U' is a connected component of X'. ∎

PROPOSITION B.1.4. (Compatibility between schematic images and flat base changes) *Let $f : X \to Y$ be an S-morphism, which is quasicompact and quasiseparated. Let $g : S' \to S$ be a flat morphism, and denote by $f' : X' \to Y'$ the S'-morphism obtained from f by base change. Let Z (resp. Z') be the schematic image of f (resp. f'). Then, $Z \times_S S'$ is canonically isomorphic to Z'.* ∎

Since the finiteness of f is Zariski local on S, it suffices to check it over an open neighborhood of a fixed point $s \in S$. Take a point $x \in X$ above s. Take an étale neighborhood $g : S' \to S$ of s as in Proposition B.1.3, and set $X' := X \times_S S'$. Denote by f' the induced morphism $X' \to S'$. Since f is separated, quasifinite, and locally of finite type, there exists a connected component $V' \subset X'$ such that

$$f'|_{V'} : V' \to X' \to S'$$

is finite, and V' is an open neighborhood of the fiber of x. Since f is flat, so is f'; hence the image $f'(V') \subset S'$ is an open subset. By shrinking S', we may assume that $V' \to S'$ is surjective. Since f is an isomorphism over $U \subset S$, f' is an isomorphism over $g^{-1}(U) \subset S'$. Therefore, combining with the surjectivity of $f'|_{V'}$, we have

$$(f')^{-1}(g^{-1}(U)) \subset V'. \tag{B.1.1}$$

On the other hand, since the map $g \circ f'$ is a flat morphism, Proposition B.1.4 implies that the open subset

$$(f')^{-1}(g^{-1}(U)) \subset X' = V' \sqcup X'_1 \tag{B.1.2}$$

is schematically dense, where X'_1 is an open and closed subset of X'. Therefore, (1.1) shows that $X'_1 = \emptyset$ and $X' = V'$. Therefore, we have $f' = f'|_{V'}$; hence f' is finite.

By replacing S with the image of $S' \to S$, we may assume that $S' \to S$ is an fpqc-cover. Since finiteness is an fpqc-local property, we conclude that f is finite. This finishes the proof of Theorem B.1.1. ∎

Appendix C. Cofilteredness for diagrams

Let $u : \mathcal{C} \to \mathcal{D}$ have a pro-left adjoint $v : \mathcal{D} \to \text{pro-}\mathcal{C}$. We give ourselves a system of subcategories $(I(d) \subset d \downarrow u)_{d\in\mathcal{D}}$ representing v. Thus each $I(d)$ is ordered and cofiltered, and $v(d) = \text{“}\varprojlim\text{”}_{c\in I(d)} c$ for any d.

LEMMA C.1.1. *Let Δ be a finite category without loops: the collections of objects and morphisms of Δ are finite and the only endomorphisms of objects are the identities. For $d \in \mathcal{D}$, define a subcategory $I(\underline{d})$ of $\underline{d} \downarrow u^\Delta$ as follows: an object X (resp. morphism f) of $\underline{d} \downarrow u^\Delta$ is in $I(\underline{d})$ if and only if $X(\delta)$ (resp. $f(\delta)$) is in $I(d(\delta))$ for all $\delta \in \Delta$. Then the category $I(\underline{d})$ is ordered and cofiltered for all $\underline{d} \in \mathcal{D}^\Delta$.*

Proof. The assertion ‘ordered’ is obvious. For ‘cofiltered’, we proceed by induction on $\#Ob(\Delta)$. We may assume Δ nonempty. The finiteness and “no loop” hypotheses imply that Δ has an object δ_0 such that no arrow leads to δ_0. Let Δ' be the subcategory of Δ obtained by removing δ_0 and all the arrows leaving from δ_0. Let $X_1 : \underline{d} \to u^\Delta(\underline{c_1})$, $X_2 : \underline{d} \to u^\Delta(\underline{c_2})$ be two objects of $I(\underline{d})$. By induction, we may find $Y_3 : \underline{d} \mid \Delta' \to u^{\Delta'}(\underline{c_3}') \in I(\underline{d} \mid \Delta')$ sitting above $X_1 \mid \Delta'$ and $X_2 \mid \Delta'$. Let $f : \delta_0 \to \delta$ be an arrow, with $\delta \in \Delta'$: by the functoriality of v, there exists a commutative diagram in $\mathcal{D}$

$$\begin{tikzcd} d(\delta_0) \arrow[r, "\varphi(f)"] \arrow[d, "d(f)"'] & u(c(f)) \arrow[d, "u(\psi(f))"'] \\ d(\delta) \arrow[r, "Y_3(\delta)"] & u(c_3'(\delta)) \end{tikzcd}$$

with $\varphi(f) \in I(d(\delta_0))$. Since $I(d(\delta))$ is cofiltered, we may find an object $d(\delta_0) \xrightarrow{g} u(c) \in I(d(\delta))$ sitting above all $\varphi(f)$’s as well as $X_1(\delta_0)$ and $X_2(\delta_0)$. Then, together with Y_3, $X_3(\delta_0) =: g$ completes the construction of X_3 dominating X_1 and X_2. ∎

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[3]This paper is withdrawn. See https://arxiv.org/pdf/1511.07124v5.pdf.

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Acronyms

[EGA4-4] A. Grothendieck and J. Dieudonné, *Éléments de goémétrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, Quatrième partie*, Publ. Math. Inst Hautes Études Sci. **32** (1967), 5–361.
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