

# Algebraic tori as Nisnevich sheaves with transfers

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**ABSTRACT.** — We relate  $R$ -equivalence on tori with Voevodsky’s theory of homotopy invariant Nisnevich sheaves with transfers and effective motivic complexes.

**RÉSUMÉ.** — On relie la  $R$ -équivalence sur les tores aux faisceaux Nisnevich avec transferts invariants par homotopie et aux complexes motiviques effectifs, étudiés par Voevodsky.

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## Contents

<b>1</b>	<b>Main results . . . . .</b>	<b>699</b>
<b>2</b>	<b>Review of terminology for tori . . . . .</b>	<b>703</b>
<b>3</b>	<b>Proofs of Theorem 1.2 and Corollary 1.4 . . . . .</b>	<b>704</b>
<b>4</b>	<b>Stable birationality . . . . .</b>	<b>706</b>
<b>5</b>	<b>Some open questions . . . . .</b>	<b>712</b>
	<b>References . . . . .</b>	<b>715</b>

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### 1. Main results

Let  $k$  be a field and let  $T$  be a  $k$ -torus. The  $R$ -equivalence classes on  $T$  have been extensively studied by several authors, notably by Colliot-Thélène and Sansuc in a series of papers including [4] and [5]: they play a central rôle in many rationality issues. In this note, we show that Voevodsky’s triangulated category of motives sheds a new light on this question: see Corollaries 1.3, 1.7 and 1.8 below.

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More generally, let  $G$  be a semi-abelian variety over  $k$ , which is an extension of an abelian variety  $A$  by a torus  $T$ . Denote by HI the category of homotopy invariant Nisnevich sheaves with transfers over  $k$  in the sense of Voevodsky [19]. Then  $G$  has a natural structure of an object of HI ([17, proof of Lemma 3.2], [1, Lemma 1.3.2]). Let  $L$  be the group of cocharacters of  $T$ .

PROPOSITION 1.1. — *There is a natural isomorphism  $G_{-1} \xrightarrow{\sim} L$  in HI.*

Here  $_{-1}$  is the contraction operation of [18, p. 96], whose definition is recalled in the proof below.

*Proof.* — Recall that if  $\mathcal{F}$  is a presheaf [with transfers] on smooth  $k$ -schemes, the presheaf [with transfers]  $\mathcal{F}_{-1}^p$  is defined by

$$U \mapsto \text{Coker}(\mathcal{F}(U \times \mathbf{A}^1) \rightarrow \mathcal{F}(U \times \mathbb{G}_m)).$$

If  $\mathcal{F}$  is homotopy invariant, we may replace  $U \times \mathbf{A}^1$  by  $U$  and the rational point  $1 \in \mathbb{G}_m$  realises  $\mathcal{F}_{-1}^p(U)$  as a functorial direct summand of  $\mathcal{F}(U \times \mathbb{G}_m)$ .

If  $\mathcal{F}$  is a Nisnevich sheaf [with transfers],  $\mathcal{F}_{-1}$  is defined as the sheaf associated to  $\mathcal{F}_{-1}^p$ .

Now  $A(U \times \mathbf{A}^1) \xrightarrow{\sim} A(U \times \mathbb{G}_m)$  since  $A$  is an abelian variety, hence  $A_{-1}^p = 0$ . We therefore have an isomorphism of presheaves  $T_{-1}^p \xrightarrow{\sim} G_{-1}^p$ , and *a fortiori* an isomorphism of Nisnevich sheaves  $T_{-1} \xrightarrow{\sim} G_{-1}$ .

Let  $p : \mathbb{G}_m \rightarrow \text{Spec } k$  be the structural map. One easily checks that the étale sheaf  $\text{Coker}(T \xrightarrow{i} p_* p^* T)$  is canonically isomorphic to  $L$ . Since  $i$  is split, its cokernel is still  $L$  if we view it as a morphism of presheaves, hence of Nisnevich sheaves.  $\square$

From now on, we assume  $k$  perfect. Let  $\text{DM}_{-}^{\text{eff}}$  be the triangulated category of effective motivic complexes introduced in [19]: it has a  $t$ -structure with heart HI. It also has a tensor structure and a (partially defined) internal Hom. We then have an isomorphism

$$L[0] = G_{-1}[0] \simeq \underline{\text{Hom}}_{\text{DM}_{-}^{\text{eff}}}(\mathbb{G}_m[0], G[0])$$

[10, Rk. 4.4], hence by adjunction a morphism in  $\text{DM}_{-}^{\text{eff}}$

$$L[0] \otimes \mathbb{G}_m[0] \rightarrow G. \tag{1.1}$$

Let  $\nu_{\leq 0} G[0]$  denote the cone of (1.1): by [11, Lemma 6.3] or [8, §2],  $\nu_{\leq 0} G[0]$  is the *birational motivic complex* associated to  $G$ . We want to compute its homology sheaves.

For this, consider a coflasque resolution<sup>1</sup>

$$0 \rightarrow Q \rightarrow L_0 \rightarrow L \rightarrow 0 \tag{1.2}$$

of  $L$  in the sense of [4, p. 179]. Taking a coflasque resolution of  $Q$  and iterating, we get a resolution of  $L$  by invertible lattices:

$$\dots \rightarrow L_n \rightarrow \dots \rightarrow L_0 \rightarrow L \rightarrow 0. \tag{1.3}$$

We set

$$Q_n = \begin{cases} Q & \text{for } n = 1 \\ \text{Ker}(L_{n-1} \rightarrow L_{n-2}) & \text{for } n > 1. \end{cases}$$

**THEOREM 1.2.** — *a) Let  $T_n$  denote the torus with cocharacter group  $L_n$ . Then  $\nu_{\leq 0}G[0]$  is isomorphic to the complex*

$$\dots \rightarrow T_n \rightarrow \dots \rightarrow T_0 \rightarrow G \rightarrow 0.$$

*b) Let  $S_n$  be the torus with cocharacter group  $Q_n$ . For any connected smooth  $k$ -scheme  $X$  with function field  $K$ , we have*

$$H_n(\nu_{\leq 0}G[0])(X) = \begin{cases} 0 & \text{if } n < 0 \\ G(K)/R & \text{if } n = 0 \\ S_n(K)/R & \text{if } n > 0. \end{cases}$$

The proof is given in Section 3.

**COROLLARY 1.3.** — *The assignment  $Sm(k) \ni X \mapsto \bigoplus_{x \in X^{(0)}} G(k(x))/R$  provides  $G/R$  with the structure of a homotopy invariant Nisnevich sheaf with transfers. In particular, any morphism  $\varphi : Y \rightarrow X$  of smooth connected  $k$ -schemes induces a morphism  $\varphi^* : G(k(X))/R \rightarrow G(k(Y))/R$ .*

This functoriality is essential to formulate Theorem 1.5 below. For  $\varphi$  a closed immersion of codimension 1, it recovers a specialisation map on  $R$ -equivalence classes with respect to a discrete valuation of rank 1 which was obtained (for tori) by completely different methods, e.g. [5, Th. 3.1 and Cor. 4.2] or [7]. (I am indebted to Colliot-Thélène for pointing out these references.)

**COROLLARY 1.4.** — *a) If  $k$  is finitely generated, the  $n$ -th homology sheaf of  $\nu_{\leq 0}G[0]$  takes values in finitely generated abelian groups, and even in finite groups if  $n > 0$  or  $G$  is a torus.*

*b) If  $G$  is a torus, then  $\nu_{\leq 0}G[0] = 0$  if  $G$  is split by a Galois extension  $E/k$  whose Galois group has cyclic Sylow subgroups. This condition is automatic if  $k$  is (quasi-)finite.*

<sup>(1)</sup> See Section 2 for this and further terminology.

The proof is also given in Section 3.

Given two semi-abelian varieties  $G, G'$ , we would now like to understand the maps

$$\mathrm{Hom}_k(G, G') \rightarrow \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}}_{\nu_{\leq 0}}(\nu_{\leq 0}G[0], \nu_{\leq 0}G'[0]) \rightarrow \mathrm{Hom}_{\mathrm{HI}}(G/R, G'/R).$$

In Section 4, we succeed in elucidating the nature of their composition to a large extent, at least if  $G$  is a torus. Our main result, in the spirit of Yoneda's lemma, is

**THEOREM 1.5.** — *Let  $G, G'$  be two semi-abelian varieties, with  $G$  a torus. Suppose given, for every function field  $K/k$ , a homomorphism  $f_K : G(K)/R \rightarrow G'(K)/R$  such that  $f_K$  is natural with respect to the functoriality of Corollary 1.3. Then*

a) *There exists an extension  $\tilde{G}$  of  $G$  by a permutation torus, and a homomorphism  $f : \tilde{G} \rightarrow G'$  inducing  $(f_K)$ .*

b)  *$f_K$  is surjective for all  $K$  if and only if there exist extensions  $\tilde{G}, \tilde{G}'$  of  $G$  and  $G'$  by permutation tori such that  $f_K$  is induced by a split surjective homomorphism  $\tilde{G} \rightarrow \tilde{G}'$ .*

The proof is given in §4.3. See Proposition 4.7, Corollary 4.9, Remark 4.10 and Proposition 4.11 for complements.

This relates to questions of stable birationality studied by Colliot-Thélène and Sansuc in [4] and [5], providing alternate proofs and strengthening of some of their results (at least over a perfect field). More precisely, let us introduce the following terminology:

*Definition 1.6.* — a) *A torus is quasi-invertible if it is a quotient of a invertible torus by an permutation torus.*

b) *An extension  $0 \rightarrow T' \rightarrow T \rightarrow T'' \rightarrow 0$  of tori is Nisnevich-exact if  $T(K) \rightarrow T''(K)$  is surjective for any function field  $K/k$ .*

(a) was suggested by Xun Jiang; see also [2]. See §2 for “permutation torus” and “invertible torus”).

Thanks to [18, Cor. 4.18], Nisnevich-exact sequences of tori are exact in the Nisnevich topology and even in the Zariski topology. It is easy to see that an extension as in b) is Nisnevich-exact if  $T'$  is invertible, but not necessarily if  $T'$  is only quasi-invertible. Using [4, Th. 2], one sees that quasi-invertible tori are universally  $R$ -trivial. Conversely:

COROLLARY 1.7. — *a) Let  $G'$  be a semi-abelian  $k$ -variety such that  $G'(K)/R = 0$  for any function field  $K/k$ . Then  $G'$  is a quasi-invertible torus.*

*b) In Theorem 1.5 b), assume that  $f_K$  is bijective for all  $K/k$ . Then there exists an extension  $\tilde{G}$  of  $G$  by a permutation torus and a Nisnevich-exact extension  $\tilde{G}'$  of  $G'$  by a quasi-invertible torus such that  $f_K$  is induced by an isomorphism  $\tilde{G} \xrightarrow{\sim} \tilde{G}'$ .*

*Proof.* — a) This is the special case  $G = 0$  of Theorem 1.5 b).

b) By Theorem 1.5 b), we may replace  $G$  and  $G'$  by extensions by permutation tori such that  $f_K$  is induced by a split surjection  $f : G \rightarrow G'$ . Let  $T = \text{Ker} f$ . Then  $T/R = 0$  universally. By a),  $T$  is quasi-invertible. Replacing  $G'$  by  $G' \times T$ , we get the desired statement.  $\square$

Corollary 1.7 a) is a version of [5, Prop. 7.4] (taking [4, p. 199, Th. 2] into account). Theorem 1.5 was inspired by the desire to understand this result from a different viewpoint. Another characterisation of quasi-invertible tori in loc. cit. is that they are the retract-rational tori.

COROLLARY 1.8. — *Let  $f : G \dashrightarrow G'$  be a rational map of semi-abelian varieties, with  $G$  a torus. Then the following conditions are equivalent:*

- (i)  $f_* : \nu_{\leq 0} G[0] \rightarrow \nu_{\leq 0} G'[0]$  is an isomorphism (see Proposition 4.7).
- (ii)  $f_* : G(K)/R \rightarrow G'(K)/R$  is bijective for any function field  $K/k$ .
- (iii)  $f$  is an isomorphism, up to Nisnevich-exact extensions of  $G$  and  $G'$  by quasi-invertible tori and up to a translation. (See Lemma 4.4.)

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## 2. Review of terminology for tori

We take this terminology from [4] and [5].

*Definition 2.1* Let  $G$  be a profinite group.

- a) A lattice is a  $G$ -module which is finitely generated and free over  $\mathbf{Z}$ .

b) A lattice  $L$  is

- permutation if it affords a  $G$ -invariant  $\mathbf{Z}$ -basis.
- invertible if it is isomorphic to a direct summand of a permutation lattice.
- coflasque if  $H^1(H, L) = 0$  for any open (hence closed) subgroup  $H \subseteq G$ .
- flasque if the dual lattice  $L^*$  is coflasque.

c) A coflasque resolution of a lattice  $L$  is a short exact sequence of lattices

$$0 \rightarrow Q \rightarrow P \rightarrow L \rightarrow 0$$

where  $P$  is permutation and  $Q$  is coflasque. Dually, we have flasque [co]resolutions

$$0 \rightarrow L \rightarrow P \rightarrow F \rightarrow 0$$

with  $P$  permutation and  $F$  flasque.

PROPOSITION 2.2 ([4, P. 181, LEMME 3]). — Any lattice has a flasque and a coflasque resolution.

In [5, Lemma 0.6], the first statement of c) is extended to  $G$ -modules which are finitely generated over  $\mathbf{Z}$  but not necessarily free.

Let  $k_s$  be a separable closure of the field  $k$  and take  $G = \text{Gal}(k_s/k)$ . Let  $T$  be a  $k$ -torus: we shall say that it is *permutation*, *invertible*, *flasque*, *coflasque*, if its character group is (Colliot-Thélène and Sansuc use *quasi-trivial* for “permutation”). Any permutation torus is of the form  $R_{E/k}\mathbb{G}_m$  (Weil restriction of scalars) for some étale  $k$ -algebra  $E$ .

### 3. Proofs of Theorem 1.2 and Corollary 1.4

LEMMA 3.1. — *The exact sequence*

$$0 \rightarrow T(k) \rightarrow G(k) \rightarrow A(k)$$

*induces an exact sequence*

$$0 \rightarrow T(k)/R \xrightarrow{i} G(k)/R \rightarrow A(k).$$

*Proof.* — Let  $f : \mathbf{P}^1 \dashrightarrow G$  be a  $k$ -rational map defined at 0 and 1. Its composition with the projection  $G \rightarrow A$  is constant: thus the image of  $f$  lies in a  $T$ -coset of  $G$  defined by a rational point. This implies the injectivity of  $i$ , and the rest is clear.  $\square$

Let NST denote the category of Nisnevich sheaves with transfers. Recall that  $\mathrm{DM}_-^{\mathrm{eff}}$  may be viewed as a localisation of  $D^-(\mathrm{NST})$ , and that its tensor structure is a descent of the tensor structure on the latter category [19, Prop. 3.2.3].

LEMMA 3.2. — *If  $G$  is an invertible torus, there is a canonical isomorphism in  $D^-(\mathrm{NST})$*

$$L[0] \otimes \mathbb{G}_m \xrightarrow{\sim} G[0].$$

In particular,  $\nu_{\leq 0}G[0] = 0$ .

*Proof.* — We reduce to the case  $T = R_{E/k}\mathbb{G}_m$ , where  $E$  is a finite extension of  $k$ . Let us write more precisely  $\mathrm{NST}(k)$  and  $\mathrm{NST}(E)$ . There is a pair of adjoint functors

$$\mathrm{NST}(k) \xrightarrow{f^*} \mathrm{NST}(E), \quad \mathrm{NST}(E) \xrightarrow{f_*} \mathrm{HI}(k)$$

where  $f : \mathrm{Spec}E \rightarrow \mathrm{Spec}k$  is the projection. Clearly,

$$f_*\mathbf{Z} = \mathbf{Z}_{\mathrm{tr}}(\mathrm{Spec}E), \quad f_*\mathbb{G}_m = T$$

where  $\mathbf{Z}_{\mathrm{tr}}(\mathrm{Spec}E)$  is the Nisnevich sheaf with transfers represented by  $\mathrm{Spec}E$ . Since  $\mathbf{Z}_{\mathrm{tr}}(\mathrm{Spec}E) = L$ , this proves the claim.  $\square$

*Proof of Theorem 1.2.* — a) Recall that  $L_0$  is an invertible lattice chosen so that  $L_0(E) \rightarrow L(E)$  is surjective for any extension  $E/k$ . In particular, (1.2) and (1.3) are exact as sequences of Nisnevich sheaves; hence  $L[0]$  is isomorphic in  $D^-(\mathrm{NST})$  to the complex

$$L = \dots \rightarrow L_n \rightarrow \dots \rightarrow L_0 \rightarrow 0.$$

(We may view (1.3) as a version of Voevodsky’s “canonical resolutions” as in [19, §3.2 p. 206].)

By Lemma 3.2,  $L_n[0] \otimes \mathbb{G}_m[0] \simeq T_n[0]$  is homologically concentrated in degree 0 for all  $n$ . It follows that the complex

$$T = \dots \rightarrow T_n \rightarrow \dots \rightarrow T_0 \rightarrow 0$$

is isomorphic to  $L[0] \otimes \mathbb{G}_m[0]$  in  $D^-(\mathrm{NST})$ , hence *a fortiori* in  $\mathrm{DM}_-^{\mathrm{eff}}$ .

b) For any nonempty open subscheme  $U \subseteq X$  we have isomorphisms

$$H_n(\nu_{\leq 0}G[0])(X) \xrightarrow{\sim} H_n(\nu_{\leq 0}G[0])(U) \xrightarrow{\sim} H_n(\nu_{\leq 0}G[0])(K) \quad (3.1)$$

(e.g. [8, p. 912]). By a), the right hand term is the  $n$ -th homology group of the complex

$$\dots \rightarrow T_n(K) \rightarrow \dots \rightarrow T_0(K) \rightarrow G(K) \rightarrow 0$$

with  $G(K)$  in degree 0. By [4, p. 199, Th. 2], the sequences

$$\begin{aligned} 0 &\rightarrow S_1(K) \rightarrow T_0(K) \rightarrow T(K) \rightarrow T(K)/R \rightarrow 0 \\ 0 &\rightarrow S_{n+1}(K) \rightarrow T_n(K) \rightarrow S_n(K) \rightarrow S_n(K)/R \rightarrow 0 \end{aligned}$$

are all exact. Using Lemma 3.1 for  $H_0$ , the conclusion follows from an easy diagram chase.  $\square$

*Remark 3.3.* — As a corollary to Theorem 1.2,  $S_n(K)/R$  only depends on  $G$ . This can be seen without mentioning  $\mathrm{DM}_-^{\mathrm{eff}}$ : in view of the reasoning just above, it suffices to construct a homotopy equivalence between two resolutions of the form (1.3), which easily follows from the definition of coflasque modules.

*Proof of Corollary 1.4.* — a) This follows via Theorem 1.2 and Lemma 3.1 from [4, p. 200, Cor. 2] and the Mordell-Weil-Néron theorem. b) We may choose the  $L_n$ , hence the  $S_n$  split by  $E/k$ . The conclusion now follows from Theorem 1.2 and [4, p. 200, Cor. 3]. The last claim is clear.  $\square$

*Remark 3.4.* — In characteristic  $p > 0$ , all finitely generated perfect fields are finite. To give some contents to Corollary 1.4 a) in this characteristic, one may pass to the perfect [one should say radical] closure  $k$  of a finitely generated field  $k_0$ . If  $G$  is a semi-abelian  $k$ -variety, it is defined over some finite extension  $k_1$  of  $k_0$ . If  $k_2/k_1$  is a finite (purely inseparable) subextension of  $k/k_1$ , then the composition

$$G(k_2) \xrightarrow{N_{k_2/k_1}} G(k_1) \rightarrow G(k_2)$$

equals multiplication by  $[k_2 : k_1]$ . Hence Corollary 1.4 a) remains true at least after inverting  $p$ .

#### 4. Stable birationality

If  $X$  is a smooth variety over a field  $k$ , we write  $\mathrm{Alb}(X)$  for its generalised Albanese variety in the sense of Serre [16]: it is a semi-abelian variety, and a rational point  $x_0 \in X$  determines a morphism  $X \rightarrow \mathrm{Alb}(X)$  which is universal for morphisms from  $X$  to semi-abelian varieties sending  $x_0$  to 0.

We also write  $\mathrm{NS}(X)$  for the group of cycles of codimension 1 on  $X$  modulo algebraic equivalence. This group is finitely generated if  $k$  is algebraically closed [9, Th. 3].



#### 4.1. Well-known lemmas

I include proofs for lack of reference.

LEMMA 4.1. — *a) Let  $G, G'$  be two semi-abelian  $k$ -varieties. Then any  $k$ -morphism  $f : G \rightarrow G'$  can be written uniquely  $f = f(0) + f'$ , where  $f'$  is a homomorphism.*

*b) For any semi-abelian  $k$ -variety  $G$ , the canonical map  $G \rightarrow \text{Alb}(G)$  sending 0 to 0 is an isomorphism.*

*Proof.* — a) amounts to showing that if  $f(0) = 0$ , then  $f$  is a homomorphism. By an adjunction game, this is equivalent to b). Let us give two proofs: one of a) and one of b).

*Proof of a).* — We may assume  $k$  to be a universal domain. The statement is classical for abelian varieties [15, p. 41, Cor. 1] and an easy computation for tori. In the general case, let  $T, T'$  be the toric parts of  $G$  and  $G'$  and  $A, A'$  be their abelian parts. Let  $g \in G(k)$ . As any morphism from  $T$  to  $A'$  is constant, the  $k$ -morphism

$$\varphi_g : T \ni t \mapsto f(g + t) - f(g) \in G'$$

(which sends 0 to 0) lands in  $T'$ , hence is a homomorphism. Therefore it only depends on the image of  $g$  in  $A(k)$ . This defines a morphism  $\varphi : A \rightarrow \underline{\text{Hom}}(T, T')$ , which must be constant with value  $\varphi_0 = f$ . It follows that

$$(g, h) \mapsto f(g + h) - f(g) - f(h)$$

induces a morphism  $A \times A \rightarrow T'$ . Such a morphism is constant, of value 0.

*Proof of b).* — This is true if  $G$  is abelian, by rigidity and the equivalence between a) and b). In general, any morphism from  $G$  to an abelian variety is trivial on  $T$ . This shows that the abelian part of  $\text{Alb}(G)$  is  $A$ . Let  $T' = \text{Ker}(\text{Alb}(G) \rightarrow A)$ . We also have the counit morphism  $\text{Alb}(G) \rightarrow G$ , and the composition  $G \rightarrow \text{Alb}(G) \rightarrow G$  is the identity. Thus  $T$  is a direct summand of  $T'$ . It suffices to show that  $\dim T' = \dim T$ . Going to the algebraic closure, we may reduce to  $T = \mathbb{G}_m$ .

Then consider the line bundle completion  $\bar{G} \rightarrow A$  of the  $\mathbb{G}_m$ -bundle  $G \rightarrow A$ . It is sufficient to show that the kernel of

$$\text{Alb}(G) \rightarrow \text{Alb}(\bar{G}) = A$$

is 1-dimensional. This follows for example from [1, Cor. 10.5.1]. □

LEMMA 4.2. — *Suppose  $k$  algebraically closed, and let  $G$  be a semi-abelian  $k$ -variety. Let  $A$  be the abelian quotient of  $G$ . Then the map*

$$\mathrm{NS}(A) \rightarrow \mathrm{NS}(G) \tag{4.2}$$

*is an isomorphism.*

*Proof.* — Let  $T = \mathrm{Ker}(G \rightarrow A)$  and  $X(T)$  be its character group. Choosing a basis  $(e_i)$  of  $X(T)$ , we may complete the  $\mathbb{G}_m^n$ -torsor  $G$  into a product of line bundles  $\tilde{G} \rightarrow A$ . The surjection

$$\mathrm{Pic}(A) \xrightarrow{\sim} \mathrm{Pic}(\tilde{G}) \twoheadrightarrow \mathrm{Pic}(G)$$

show the surjectivity of (4.2). Its kernel is generated by the classes of the irreducible components  $D_i$  of the divisor with normal crossings  $\tilde{G} - G$ . These components correspond to the basis elements  $e_i$ . Since the corresponding  $\mathbb{G}_m$ -bundle is a group extension of  $A$  by  $\mathbb{G}_m$ , the class of the 0 section of its line bundle completion lies in  $\mathrm{Pic}^0(A)$ , hence goes to 0 in  $\mathrm{NS}(A)$ .  $\square$

LEMMA 4.3. — *Let  $X$  be a smooth  $k$ -variety, and let  $U \subseteq X$  be a dense open subset. Then there is an exact sequence of semi-abelian varieties*

$$0 \rightarrow T \rightarrow \mathrm{Alb}(U) \rightarrow \mathrm{Alb}(X) \rightarrow 0$$

*with  $T$  a torus. If  $\mathrm{NS}(U \otimes_k \bar{k}) = 0$  (this happens if  $U$  is small enough), there is an exact sequence of character groups*

$$0 \rightarrow X(T) \rightarrow \bigoplus_{x \in X^{(1)} - U^{(1)}} \mathbf{Z} \rightarrow \mathrm{NS}(\bar{X}) \rightarrow 0.$$

*Proof.* — This follows for example from [1, Cor. 10.5.1].  $\square$

LEMMA 4.4. — *Let  $f : G \dashrightarrow G'$  be a rational map between semi-abelian  $k$ -varieties, with  $G$  a torus. Then there exists an extension  $\tilde{G}$  of  $G$  by a permutation torus and a homomorphism  $\tilde{f} : \tilde{G} \rightarrow G'$  which extends  $f$  up to translation in the following sense: there exists a rational section  $s : G \dashrightarrow \tilde{G}$  of the projection  $\pi : \tilde{G} \rightarrow G$  and a rational point  $g' \in G'(k)$  such that  $f = \tilde{f}s + g'$ . If  $f$  is defined at  $0_G$  and sends it to  $0_{G'}$ , then  $g' = 0$ .*

*Proof.* — Let  $U$  be an open subset of  $G$  where  $f$  is defined. We define  $\tilde{G} = \mathrm{Alb}(U)$ . Applying Lemmas 4.3 and 4.1 b) and using  $\mathrm{NS}(G \otimes_k \bar{k}) = 0$ , we get an extension

$$0 \rightarrow P \rightarrow \tilde{G} \rightarrow G \rightarrow 0$$

where  $P$  is a permutation torus, as well as a morphism  $\tilde{f} = \mathrm{Alb}(f) : \tilde{G} \rightarrow G'$ .

Let us first assume  $k$  infinite. Then  $U(k) \neq \emptyset$  because  $G$  is unirational. A rational point  $g \in U$  defines an Albanese map  $s : U \rightarrow \tilde{G}$  sending  $g$  to  $0_{\tilde{G}}$ . Since  $P$  is a permutation torus,  $g \in G(k)$  lifts to  $\tilde{g} \in \tilde{G}(k)$  (Hilbert 90) and we may replace  $s$  by a morphism sending  $g$  to  $\tilde{g}$ . Then  $s$  is a rational section of  $\pi$ . Moreover,  $f = \tilde{f}s + g'$  with  $g' = f(g) - \tilde{f}(\tilde{g})$ . The last assertion follows.

If  $k$  is finite, then  $U$  has at least a zero-cycle  $g$  of degree 1, which is enough to define the Albanese map  $s$ . We then proceed as above (lift every closed point involved in  $g$  to a closed point of  $\tilde{G}$  with the same residue field).  $\square$

LEMMA 4.5. — *Let  $G$  be a finite group, and let  $A$  be a finitely generated  $G$ -module. Then*

a) *There exists a short exact sequence of  $G$ -modules  $0 \rightarrow P \rightarrow F \rightarrow A \rightarrow 0$ , with  $F$  torsion-free and flasque, and  $P$  permutation.*

b) *Let  $B$  be another finitely generated  $G$ -module, and let  $0 \rightarrow P' \rightarrow E \rightarrow B \rightarrow 0$  be an exact sequence with  $P'$  an invertible module. Then any  $G$ -morphism  $f : A \rightarrow B$  lifts to  $\tilde{f} : F \rightarrow E$ .*

*Proof.* — a) is the contents of [5, Lemma 0.6, (0.6.2)]. b) The obstruction to lifting  $f$  lies in  $\text{Ext}_G^1(F, P') = 0$  [4, p. 182, Lemme 9].  $\square$

## 4.2. Functoriality of $\nu_{\leq 0}G$

We now assume  $k$  perfect.

LEMMA 4.6. — *Let*

$$0 \rightarrow P \rightarrow G \rightarrow H \rightarrow 0 \tag{4.3}$$

*be an exact sequence of semi-abelian varieties, with  $P$  an invertible torus. Then  $\nu_{\leq 0}G[0] \xrightarrow{\sim} \nu_{\leq 0}H[0]$ .*

*Proof.* — As  $P$  is invertible, (4.3) is exact in NST hence defines an exact triangle

$$P[0] \rightarrow G[0] \rightarrow H[0] \xrightarrow{+1}$$

in  $\text{DM}_{-}^{\text{eff}}$ . The conclusion then follows from Lemma 3.2.  $\square$

PROPOSITION 4.7. — *Let  $G, G'$  be two semi-abelian  $k$ -varieties, with  $G$  a torus. Then a rational map  $f : G \dashrightarrow G'$  induces a morphism  $f_* : \nu_{\leq 0}G[0] \rightarrow \nu_{\leq 0}G'[0]$ , hence a homomorphism  $f_* : G(K)/R \rightarrow G'(K)/R$  for any extension  $K/k$ . If  $K$  is infinite,  $f_*$  agrees up to translation with the morphism induced by  $f$  via the isomorphism  $U(K)/R \xrightarrow{\sim} G(K)/R$  from [4, p. 196 Prop. 11], where  $U$  is an open subset of definition of  $f$ .*

*Proof.* — By Lemma 4.4,  $f$  induces a homomorphism  $\tilde{G} \rightarrow G'$  where  $\tilde{G}$  is an extension of  $G$  by a permutation torus. By Lemma 4.6, the induced morphism

$$\nu_{\leq 0}\tilde{G}[0] \rightarrow \nu_{\leq 0}G'[0]$$

factors through a morphism  $f_* : \nu_{\leq 0}G[0] \rightarrow \nu_{\leq 0}G'[0]$ .

The claims about  $R$ -equivalence classes follow from Theorem 1.2 b) and Lemma 4.4.  $\square$

*Remark 4.8.* — The proof shows that  $f'_* = f_*$  if  $f'$  differs from  $f$  by a translation by an element of  $G(k)$  or  $G'(k)$ .

**COROLLARY 4.9.** — *If  $T$  and  $T'$  are birationally equivalent  $k$ -tori, then  $\nu_{\leq 0}T[0] \simeq \nu_{\leq 0}T'[0]$ . In particular, the groups  $T(k)/R$  and  $T'(k)/R$  are isomorphic.*

*Proof.* — The proof of Proposition 4.7 shows that  $f \mapsto f_*$  is functorial for composable rational maps between tori. Let  $f : T \dashrightarrow T'$  be a birational isomorphism, and let  $g : T' \dashrightarrow T$  be the inverse birational isomorphism. Then we have  $g_*f_* = 1_{\nu_{\leq 0}T[0]}$  and  $f_*g_* = 1_{\nu_{\leq 0}T'[0]}$ . The last claim follows from Theorem 1.2.  $\square$

*Remark 4.10.* — It is proven in [4] that a birational isomorphism of tori  $f : T \dashrightarrow T'$  induces a set-theoretic bijection  $f_* : T(k)/R \xrightarrow{\sim} T'(k)/R$  (p. 197, Cor. to Prop. 11) and that the group  $T(k)/R$  is abstractly a birational invariant of  $T$  (p. 200, Cor. 4). The proof above shows that  $f_*$  is an isomorphism of groups if  $f$  respects the origins of  $T$  and  $T'$ . This solves the question raised in [4, mid. p. 397]. The proofs of Lemma 4.4 and Proposition 4.7 may be seen as dual to the proof of [4, p. 189, Prop. 5], and are directly inspired from it.

### 4.3. Faithfulness and fullness

**PROPOSITION 4.11.** — *Let  $f : G \dashrightarrow G'$  be a rational map between semi-abelian varieties, with  $G$  a torus. Assume that the map  $f_* : G(K)/R \rightarrow G'(K)/R$  from Proposition 4.7 is identically 0 when  $K$  runs through the finitely generated extensions of  $k$ . Then there exists a permutation torus  $P$  and a factorisation of  $f$  as*

$$G \xrightarrow{\tilde{f}} P \xrightarrow{g} G'$$

where  $\tilde{f}$  is a rational map and  $g$  is a homomorphism. If  $f$  is a morphism, we may choose  $\tilde{f}$  to be a homomorphism.

Conversely, if there is such a factorisation, then  $f_* : \nu_{\leq 0}G[0] \rightarrow \nu_{\leq 0}G'[0]$  is the 0 morphism.

*Proof.* — By Lemma 4.4, we may reduce to the case where  $f$  is a homomorphism. Let  $K = k(G)$ . By hypothesis, the image of the generic point  $\eta_G \in G(K)$  is  $R$ -equivalent to 0 on  $G'(K)$ . By a lemma of Gille [6, Lemme II.1.1 b)], it is directly  $R$ -equivalent to 0: in other words, there exists a rational map  $h : G \times \mathbf{A}^1 \dashrightarrow G'$ , defined in the neighbourhood of 0 and 1, such that  $h|_{G \times \{0\}} = 0$  and  $h|_{G \times \{1\}} = f$ .

Let  $U \subseteq G \times \mathbf{A}^1$  be an open set of definition of  $h$ . The 0 and 1-sections of  $G \times \mathbf{A}^1 \rightarrow G$  induce sections

$$s_0, s_1 : G \rightarrow \text{Alb}(U)$$

of the projection  $\pi : \text{Alb}(U) \rightarrow \text{Alb}(G \times \mathbf{A}^1) = G$  such that  $\text{Alb}(h) \circ s_0 = 0$  and  $\text{Alb}(h) \circ s_1 = f$ . If  $P = \text{Ker}\pi$ , then  $s_1 - s_0$  induces a homomorphism  $\tilde{f} : G \rightarrow P$  such that the composition

$$G \xrightarrow{\tilde{f}} P \rightarrow \text{Alb}(U) \xrightarrow{\text{Alb}(h)} G'$$

equals  $f$ . Finally,  $P$  is a permutation torus by Lemma 4.3.

The last claim follows from Lemma 3.2. □

*Proof of Theorem 1.5.* — a) Take  $K = k(G)$ . The image of the generic point  $\eta_G$  by  $f_K$  lifts to a (non unique) rational map  $f : G \dashrightarrow G'$ . Using Lemma 4.4, we may extend  $f$  to a homomorphism

$$\tilde{f} : \tilde{G} \rightarrow G'$$

where  $\tilde{G}$  is an extension of  $G$  by a permutation torus  $P$ . Since  $\tilde{G}(K)/R \xrightarrow{\sim} G(K)/R$ , we reduce to  $\tilde{G} = G$  and  $\tilde{f} = f$ .

Let  $L/k$  be a function field, and let  $g \in G(L)$ . Then  $g$  arises from a morphism  $g : X \rightarrow G$  for a suitable smooth model  $X$  of  $L$ . By assumption on  $K \mapsto f_K$ , the diagram

$$\begin{array}{ccc} G(K)/R & \xrightarrow{f_K} & G'(K)/R \\ g^* \downarrow & & g^* \downarrow \\ G(L)/R & \xrightarrow{f_L} & G'(L)/R \end{array}$$

commutes. Applying this to  $\eta_K \in G(K)$ , we find that  $f_L([g]) = [g \circ f]$ , which means that  $f_L$  is the map induced by  $f$ .

b) The hypothesis implies that  $G'(E)/R = 0$  for any algebraically closed extension  $E/k$ , which in turn implies that  $G'$  is also a torus. Applying a), we may, and do, convert  $f$  into a true homomorphism by replacing  $G$  by a suitable extension by a permutation torus. Applying Lemma 4.5 a) to the cocharacter group of  $G$ , we then get a resolution  $0 \rightarrow P_1 \rightarrow Q \rightarrow G \rightarrow 0$  with  $Q$  coflasque and  $P_1$  permutation. Hence we may (and do) further assume  $G$  coflasque.

Let  $K = k(G')$  and choose some  $g \in G(K)$  mapping modulo  $R$ -equivalence to the generic point of  $G'$ . Then  $g$  defines a rational map  $g : G' \dashrightarrow G$  such that  $fg$  is  $R$ -equivalent to  $1_{G'}$ . It follows that the induced map

$$1 - fg : G'/R \rightarrow G'/R \tag{4.4}$$

is identically 0.

Reapplying Lemma 4.4, we may find an extension  $\tilde{G}'$  of  $G'$  by a suitable permutation torus which converts  $g$  into a true homomorphism. Since  $G$  is coflasque, Lemma 4.5 b) shows that  $f : G \rightarrow G'$  lifts to  $\tilde{f} : G \rightarrow \tilde{G}'$ . Then (4.4) is still identically 0 when replacing  $(G', f)$  by  $(\tilde{G}', \tilde{f})$ .

Summarising: we have replaced the initial  $G$  and  $G'$  by suitable extensions by permutation tori, such that  $f$  lifts to these extensions and there is a homomorphism  $g : G' \rightarrow G$  such that (4.4) vanishes identically. Hence  $1 - fg$  factors through a permutation torus  $P$  thanks to Proposition 4.11. Write  $u : G' \rightarrow P$  and  $v : P \rightarrow G'$  for homomorphisms such that  $1 - fg = vu$ . Let  $G_1 = G \times P$  and consider the maps

$$f_1 = (f, v) : G_1 \rightarrow G', \quad g_1 = (g)u : G' \rightarrow G_1.$$

Then  $f_1g_1 = 1$  and  $G'$  is a direct summand of  $G_1$  as requested. □

## 5. Some open questions

*Question 5.1.* — Are lemma 4.4 and Proposition 4.7 still true when  $G$  is not a torus?

This is far from clear in general, starting with the case where  $G$  is an abelian variety and  $G'$  a torus. Let me give a positive answer in the case of an elliptic curve.

**PROPOSITION 5.2.** — *The answer to Question 5.1 is yes if the abelian part  $A$  of  $G$  is an elliptic curve.*

*Proof.* — Arguing as in the proof of Proposition 4.7, we get for an open subset  $U \subseteq G$  of definition for  $f$  an exact sequence

$$0 \rightarrow \mathbb{G}_m \rightarrow P \rightarrow \mathrm{Alb}(U) \rightarrow G \rightarrow 0$$

where  $P$  is a permutation torus. Here we used that  $\mathrm{NS}(\bar{G}) \simeq \mathbf{Z}$ , which follows from Lemma 4.2.

The character group  $X(P)$  has as a basis the geometric irreducible components of codimension 1 of  $G - U$ . Up to shrinking  $U$ , we may assume that  $G - U$  contains the inverse image  $D$  of  $0 \in A$ . As the divisor class of 0 generates  $\mathrm{NS}(\bar{A})$ ,  $D$  provides a Galois-equivariant splitting of the map  $\mathbb{G}_m \rightarrow P$ . Thus its cokernel is still a permutation torus, and we conclude as before.  $\square$

*Question 5.3.* — Can one formulate a version of Theorem 1.5 and Corollary 1.7 providing a description of the groups  $\mathrm{Hom}_{\mathrm{DM}_{-}^{\mathrm{eff}}}(\nu_{\leq 0}G[0], \nu_{\leq 0}G'[0])$  and  $\mathrm{Hom}_{\mathrm{HI}}(G/R, G'/R)$  (at least when  $G$  and  $G'$  are tori)?

The proof of Theorem 1.5 suggests the presence of a closed model structure on the category of tori (or lattices), which might provide an answer to this question.

For the last question, let  $G$  be a semi-abelian variety. Forgetting its group structure, it has a motive  $M(G) \in \mathrm{DM}_{-}^{\mathrm{eff}}$ . Recall the canonical morphism

$$M(G) \rightarrow G[0]$$

induced by the “sum” maps

$$c(X, G) \xrightarrow{\sigma} G(X) \tag{5.1}$$

for smooth varieties  $X$  ([17, (6), (7)], [1, §1.3]).

The morphism (5.1) has a canonical section

$$G(X) \xrightarrow{\gamma} c(X, G) \tag{5.2}$$

given by the graph of a morphism: this section is functorial in  $X$  but is not additive.

Consider now a smooth equivariant compactification  $\bar{G}$  of  $G$ . It exists in all characteristics. For tori, this is written up in [3]. The general case reduces to this one by the following elegant argument I learned from M. Brion: if  $G$  is an extension of an abelian variety  $A$  by a torus  $T$ , take a

smooth projective equivariant compactification  $Y$  of  $T$ . Then the bundle  $G \times^T Y$  associated to the  $T$ -torsor  $G \rightarrow A$  also exists: this is the desired compactification.

Then we have a diagram of birational motives

$$\begin{array}{ccc} \nu_{\leq 0}M(G) & \xrightarrow{\sim} & \nu_{\leq 0}M(\bar{G}) \\ \nu_{\leq 0}\sigma \downarrow & & \\ \nu_{\leq 0}G[0] & & \end{array} \quad (5.3)$$

By [11], we have  $H_0(\nu_{\leq 0}M(\bar{G}))(X) = CH_0(\bar{G}_{k(X)})$  for any smooth connected  $X$ . Hence the above diagram induces a homomorphism

$$CH_0(\bar{G}_{k(X)}) \rightarrow G(k(X))/R \quad (5.4)$$

which is natural in  $X$  for the action of finite correspondences (compare Corollary 1.3). One can probably check that this is the homomorphism of [12, (17) p. 78], reformulating [4, Proposition 12 p. 198]. Similarly, the set-theoretic map

$$G(k(X))/R \rightarrow CH_0(\bar{G}_{k(X)}) \quad (5.5)$$

of [4, p. 197] can presumably be recovered as a birational version of (5.2), using perhaps the homotopy category of schemes of Morel and Voevodsky [14].

In [12], Merkurjev shows that (5.4) is an isomorphism for  $G$  a torus of dimension at most 3. This suggests:

*Question 5.4.* — Is the map  $\nu_{\leq 0}\sigma$  of Diagram (5.3) an isomorphism when  $G$  is a torus of dimension  $\leq 3$ ?

In [13], Merkurjev gives examples of tori  $G$  for which (5.5) is not a homomorphism; hence its (additive) left inverse (5.4) cannot be an isomorphism. Merkurjev's examples are of the form  $G = R_{K/k}^1 \mathbb{G}_m \times R_{L/k}^1 \mathbb{G}_m$ , where  $K$  and  $L$  are distinct biquadratic extensions of  $k$ . This suggests:

*Question 5.5.* — Can one study Merkurjev's examples from the above viewpoint? More generally, what is the nature of the map  $\nu_{\leq 0}\sigma$  of Diagram (5.3)?

We leave all these questions to the interested reader.



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