ALGEBRAIC TORI AS NISNEVICH SHEAVES WITH TRANSFERS

BRUNO KAHN

Abstract. We relate $R$-equivalence on tori with Voevodsky’s theory of homotopy invariant Nisnevich sheaves with transfers and effective motivic complexes.

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1. Main results

Let $k$ be a field and let $T$ be a $k$-torus. The $R$-equivalence classes on $T$ have been extensively studied by several authors, notably by Colliot-Thélène and Sansuc in a series of papers including [4] and [5]: they play a central rôle in many rationality issues. In this note, we show that Voevodsky’s triangulated category of motives sheds a new light on this question: see Corollaries 1, 3 and 4 below.

More generally, let $G$ be a semi-abelian variety over $k$, which is an extension of an abelian variety $A$ by a torus $T$. Denote by $HI$ the category of homotopy invariant Nisnevich sheaves with transfers over $k$ in the sense of Voevodsky [20]. Then $G$ has a natural structure of an object of $HI$ ([18, proof of Lemma 3.2], [1, Lemma 1.3.2]). Let $L$ be the group of cocharacters of $T$.

Proposition 1. There is a natural isomorphism $G_{-1} \sim L$ in $HI$. 

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Here \(-1\) is the contraction operation of [19, p. 96], whose definition is recalled in the proof below.

**Proof.** Recall that if \(\mathcal{F}\) is a presheaf [with transfers] on smooth \(k\)-schemes, the presheaf [with transfers] \(\mathcal{F}^{-1}\) is defined by

\[
U \mapsto \operatorname{Coker}(\mathcal{F}(U \times \mathbf{A}^1) \to \mathcal{F}(U \times \mathbb{G}_m)).
\]

If \(\mathcal{F}\) is homotopy invariant, we may replace \(U \times \mathbf{A}^1\) by \(U\) and the rational point \(1 \in \mathbb{G}_m\) realises \(\mathcal{F}^{-1}(U)\) as a functorial direct summand of \(\mathcal{F}(U \times \mathbb{G}_m)\).

If \(\mathcal{F}\) is a Nisnevich sheaf [with transfers], \(\mathcal{F}^{-1}\) is defined as the sheaf associated to \(\mathcal{F}^{-1}\).

Now \(\mathcal{A}(U \times \mathbf{A}^1) \xrightarrow{\sim} \mathcal{A}(U \times \mathbb{G}_m)\) since \(\mathcal{A}\) is an abelian variety, hence \(\mathcal{A}^{-1} = 0\). We therefore have an isomorphism of presheaves \(T_p^{-1} \xrightarrow{\sim} G_p^{-1}\), and *a fortiori* an isomorphism of Nisnevich sheaves \(T_{-1} \xrightarrow{\sim} G_{-1}\).

Let \(p : \mathbb{G}_m \to \text{Spec } k\) be the structural map. One easily checks that the étale sheaf \(\operatorname{Coker}(T \xrightarrow{i} p_*p^*T)\) is canonically isomorphic to \(L\). Since \(i\) is split, its cokernel is still \(L\) if we view it as a morphism of presheaves, hence of Nisnevich sheaves. \(\square\)

From now on, we assume \(k\) perfect. Let \(\text{DM}^\text{eff}\) be the triangulated category of effective motivic complexes introduced in [20]: it has a \(t\)-structure with heart \(\text{HI}\). It also has a tensor structure and a (partially defined) internal Hom. We then have an isomorphism

\[
L[0] = G_{-1}[0] \simeq \operatorname{Hom}_{\text{DM}^\text{eff}}(\mathbb{G}_m[0], G[0])
\]

[11, Rk. 4.4], hence by adjunction a morphism in \(\text{DM}^\text{eff}\)

\[
L[0] \otimes \mathbb{G}_m[0] \to G.
\]

Let \(\nu_{\leq 0} G[0]\) denote the cone of (1): by [12, Lemma 6.3] or [9, §2], \(\nu_{\leq 0} G[0]\) is the *birational motivic complex* associated to \(G\). We want to compute its homology sheaves.

For this, consider a coflasque resolution\(^1\)

\[
0 \to Q \to L_0 \to L \to 0
\]

of \(L\) in the sense of [4, p. 179]. Taking a coflasque resolution of \(Q\) and iterating, we get a resolution of \(L\) by invertible lattices:

\[
\cdots \to L_n \to \cdots \to L_0 \to L \to 0.
\]

We set

\[
Q_n = \begin{cases} 
Q & \text{for } n = 1 \\
\ker(L_{n-1} \to L_{n-2}) & \text{for } n > 1.
\end{cases}
\]

\(^1\)See Section 2 for this and further terminology.
Theorem 1. a) Let $T_n$ denote the torus with cocharacter group $L_n$. Then $\nu_{\leq 0}G[0]$ is isomorphic to the complex

$$\cdots \to T_n \to \cdots \to T_0 \to G \to 0.$$ 

b) Let $S_n$ be the torus with cocharacter group $Q_n$. For any connected smooth $k$-scheme $X$ with function field $K$, we have

$$H_n(\nu_{\leq 0}G[0])(X) = \begin{cases} 
0 & \text{if } n < 0 \\
G(K)/R & \text{if } n = 0 \\
S_n(K)/R & \text{if } n > 0.
\end{cases}$$

The proof is given in Section 3.

Corollary 1. The assignment $Sm(k) \ni X \mapsto \bigoplus_{x \in X(0)} G(k(x))/R$ provides $G/R$ with the structure of a homotopy invariant Nisnevich sheaf with transfers. In particular, any morphism $\varphi : Y \to X$ of smooth connected $k$-schemes induces a morphism $\varphi^* : G(k(X))/R \to G(k(Y))/R$. □

This functoriality is essential to formulate Theorem 2 below. For $\varphi$ a closed immersion of codimension 1, it recovers a specialisation map on $R$-equivalence classes with respect to a discrete valuation of rank 1 which was obtained (for tori) by completely different methods, e.g. [5, Th. 3.1 and Cor. 4.2] or [8]. (I am indebted to Colliot-Thélène for pointing out these references.)

Corollary 2. a) If $k$ is finitely generated, the $n$-th homology sheaf of $\nu_{\leq 0}G[0]$ takes values in finitely generated abelian groups, and even in finite groups if $n > 0$ or $G$ is a torus.

b) If $G$ is a torus, then $\nu_{\leq 0}G[0] = 0$ if $G$ is split by a Galois extension $E/k$ whose Galois group has cyclic Sylow subgroups. This condition is automatic if $k$ is (quasi-)finite.

The proof is also given in Section 3.

Given two semi-abelian varieties $G,G'$, we would now like to understand the maps

$$\text{Hom}_k(G,G') \to \text{Hom}_{\text{DM}}(\nu_{\leq 0}G[0],\nu_{\leq 0}G'[0]) \to \text{Hom}_{\text{HI}}(G/R,G'/R).$$

In Section 4, we succeed in elucidating the nature of their composition to a large extent, at least if $G$ is a torus. Our main result, in the spirit of Yoneda’s lemma, is

Theorem 2. Let $G,G'$ be two semi-abelian varieties, with $G$ a torus. Suppose given, for every function field $K/k$, a homomorphism $f_K : G(K)/R \to G'(K)/R$ such that $f_K$ is natural with respect to the functoriality of Corollary 1. Then
a) There exists an extension \( \tilde{G} \) of \( G \) by a permutation torus, and a homomorphism \( f : \tilde{G} \to G' \) inducing \((f_K)\).

b) \( f_K \) is surjective for all \( K \) if and only if there exist extensions \( \tilde{G}, \tilde{G}' \) of \( G \) and \( G' \) by permutation tori such that \( f_K \) is induced by a split surjective homomorphism \( \tilde{G} \to \tilde{G}' \).

The proof is given in §4.3. See Proposition 3, Corollary 5, Remark 4 and Proposition 4 for complements.

This relates to questions of stable birationality studied by Colliot-Thélène and Sansuc in [4] and [5], providing alternate proofs and strengthening of some of their results (at least over a perfect field).

More precisely, let us introduce the following terminology:

**Definition 1.**

a) A torus is *quasi-invertible* if it is a quotient of a invertible torus by a permutation torus.

b) An extension \( 0 \to T' \to T \to T'' \to 0 \) of tori is *Nisnevich-exact* if \( T(K) \to T''(K) \) is surjective for any function field \( K/k \).

(a) was suggested by Xun Jiang; see also [2]. See §2 for “permutation torus” and “invertible torus”.

Thanks to [19, Cor. 4.18], Nisnevich-exact sequences of tori are exact in the Nisnevich topology and even in the Zariski topology. It is easy to see that an extension as in b) is Nisnevich-exact if \( T' \) is invertible, but not necessarily if \( T' \) is only quasi-invertible. Using [4, Th. 2], one sees that quasi-invertible tori are universally \( R \)-trivial. Conversely:

**Corollary 3.**

a) Let \( G' \) be a semi-abelian \( k \)-variety such that \( G'(K)/R = 0 \) for any function field \( K/k \). Then \( G' \) is quasi-invertible.

b) In Theorem 2 b), assume that \( f_K \) is bijective for all \( K/k \). Then there exists an extension \( \tilde{G} \) of \( G \) by a permutation torus and a Nisnevich-exact extension \( \tilde{G}' \) of \( G' \) by a quasi-invertible torus such that \( f_K \) is induced by an isomorphism \( \tilde{G} \cong \tilde{G}' \).

**Proof.** a) This is the special case \( G = 0 \) of Theorem 2 b).

b) By Theorem 2 b), we may replace \( G \) and \( G' \) by extensions by permutation tori such that \( f_K \) is induced by a split surjection \( f : G \to G' \). Let \( T = \text{Ker} \ f \). Then \( T/R = 0 \) universally. By a), \( T \) is quasi-invertible. Replacing \( G' \) by \( G' \times T \), we get the desired statement.

Corollary 3 a) is a version of [5, Prop. 7.4] (taking [4, p. 199, Th. 2] into account). Theorem 2 was inspired by the desire to understand this result from a different viewpoint. Another characterisation of quasi-invertible tori in loc. cit. is that they are the retract-rational tori.

**Corollary 4.** Let \( f : G \to G' \) be a rational map of semi-abelian varieties, with \( G \) a torus. Then the following conditions are equivalent:
(i) \( f_* : \nu_{\leq 0} G[0] \to \nu_{\leq 0} G'[0] \) is an isomorphism (see Proposition 3).

(ii) \( f_* : G(K)/R \to G'(K)/R \) is bijective for any function field \( K/k \).

(iii) \( f \) is an isomorphism, up to Nisnevich-exact extensions of \( G \) and \( G' \) by quasi-invertible tori and up to a translation. (See Lemma 6.) \( \square \)

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2. Review of terminology for tori

We take this terminology from [4] and [5].

Definition 2. Let \( G \) be a profinite group.

a) A lattice is a \( G \)-module which is finitely generated and free over \( \mathbb{Z} \).

b) A lattice \( L \) is
   - permutation if it affords a \( G \)-invariant \( \mathbb{Z} \)-basis.
   - invertible if it is isomorphic to a direct summand of a permutation lattice.
   - coflasque if \( H^1(H, L) = 0 \) for any open (hence closed) subgroup \( H \subseteq G \).
   - flasque if the dual lattice \( L^* \) is coflasque.

c) A coflasque resolution of a lattice \( L \) is a short exact sequence of lattices
   \[ 0 \to Q \to P \to L \to 0 \]
   where \( P \) is permutation and \( Q \) is coflasque. Dually, we have flasque [co]resolutions
   \[ 0 \to L \to P \to F \to 0 \]
   with \( P \) permutation and \( F \) flasque.

Proposition 2 ([4, p. 181, lemme 3]). Any lattice has a flasque and a coflasque resolution. \( \square \)

In [5, Lemma 0.6], the first statement of c) is extended to \( G \)-modules which are finitely generated over \( \mathbb{Z} \) but not necessarily free.

Let \( k_s \) be a separable closure of the field \( k \) and take \( G = \text{Gal}(k_s/k) \). Let \( T \) be a \( k \)-torus: we shall say that it is permutation, invertible,
flasque, coflasque, if its character group is (Colliot-Thélène and Sansuc use quasi-trivial for “permutation”). Any permutation torus is of the form $R_{E/k}\mathbb{G}_m$ (Weil restriction of scalars) for some étale $k$-algebra $E$.

3. Proofs of Theorem 1 and Corollary 2

Lemma 1. The exact sequence

$$0 \rightarrow T(k) \rightarrow G(k) \rightarrow A(k)$$

induces an exact sequence

$$0 \rightarrow T(k)/R \rightarrow G(k)/R \rightarrow A(k).$$

Proof. Let $f : \mathbb{P}^1 \dashrightarrow G$ be a $k$-rational map defined at 0 and 1. Its composition with the projection $G \rightarrow A$ is constant: thus the image of $f$ lies in a $T$-coset of $G$ defined by a rational point. This implies the injectivity of $i$, and the rest is clear. □

Let NST denote the category of Nisnevich sheaves with transfers. Recall that $\mathrm{DM}_{\text{eff}}^-$ may be viewed as a localisation of $D^-(\text{NST})$, and that its tensor structure is a descent of the tensor structure on the latter category [20, Prop. 3.2.3].

Lemma 2. If $G$ is an invertible torus, there is a canonical isomorphism in $D^-(\text{NST})$

$$L[0] \otimes \mathbb{G}_m \sim \rightarrow G[0].$$

In particular, $\nu_{\leq 0} G[0] = 0$.

Proof. We reduce to the case $T = R_{E/k}\mathbb{G}_m$, where $E$ is a finite extension of $k$. Let us write more precisely $\text{NST}(k)$ and $\text{NST}(E)$. There is a pair of adjoint functors

$$\text{NST}(k) \xrightarrow{f^*} \text{NST}(E), \quad \text{NST}(E) \xrightarrow{f_*} H\text{I}(k)$$

where $f : \text{Spec } E \rightarrow \text{Spec } k$ is the projection. Clearly,

$$f_*\mathbb{Z} = \mathbb{Z}_{\text{tr}}(\text{Spec } E), \quad f_*\mathbb{G}_m = T$$

where $\mathbb{Z}_{\text{tr}}(\text{Spec } E)$ is the Nisnevich sheaf with transfers represented by $\text{Spec } E$. Since $\mathbb{Z}_{\text{tr}}(\text{Spec } E) = L$, this proves the claim. □

Proof of Theorem 1. a) Recall that $L_0$ is an invertible lattice chosen so that $L_0(E) \rightarrow L(E)$ is surjective for any extension $E/k$. In particular, (2) and (3) are exact as sequences of Nisnevich sheaves; hence $L[0]$ is isomorphic in $D^-(\text{NST})$ to the complex

$$L_0 \rightarrow \cdots \rightarrow L_n \rightarrow \cdots \rightarrow L_0 \rightarrow 0.$$
(We may view (3) as a version of Voevodsky’s “canonical resolutions” as in [20, §3.2 p. 206].

By Lemma 2, \( L_n[0] \otimes G_m[0] \simeq T_n[0] \) is homologically concentrated in degree 0 for all \( n \). It follows that the complex

\[
T_n = \cdots \rightarrow T_n \rightarrow \cdots \rightarrow T_0 \rightarrow 0
\]

is isomorphic to \( L[0] \otimes G_m[0] \) in \( D^- (NST) \), hence a fortiori in \( D^{\text{eff}}(\mathcal{S}) \).

b) For any nonempty open subscheme \( U \subseteq X \) we have isomorphisms

\[
H_n(\nu_{\leq 0} G[0])(X) \xrightarrow{\sim} H_n(\nu_{\leq 0} G[0])(U) \xrightarrow{\sim} H_n(\nu_{\leq 0} G[0])(K)
\]

(e.g. [9, p. 912]). By a), the right hand term is the \( n \)-th homology group of the complex

\[
\cdots \rightarrow T_n(K) \rightarrow \cdots \rightarrow T_0(K) \rightarrow G(K) \rightarrow 0
\]

with \( G(K) \) in degree 0. By [4, p. 199, Th. 2], the sequences

\[
0 \rightarrow S_1(K) \rightarrow T_0(K) \rightarrow T(K) \rightarrow T(K)/R \rightarrow 0
\]

\[
0 \rightarrow S_{n+1}(K) \rightarrow T_n(K) \rightarrow S_n(K) \rightarrow S_n(K)/R \rightarrow 0
\]

are all exact. Using Lemma 1 for \( H_0 \), the conclusion follows from an easy diagram chase.

\[\square\]

Remark 1. As a corollary to Theorem 1, \( S_n(K)/R \) only depends on \( G \). This can be seen without mentioning \( D^{\text{eff}}(\mathcal{S}) \): in view of the reasoning just above, it suffices to construct a homotopy equivalence between two resolutions of the form (3), which easily follows from the definition of coflasque modules.

Proof of Corollary 2. a) This follows via Theorem 1 and Lemma 1 from [4, p. 200, Cor. 2] and the Mordell-Weil-Néron theorem. b) We may choose the \( L_n \), hence the \( S_n \) split by \( E/k \). The conclusion now follows from Theorem 1 and [4, p. 200, Cor. 3]. The last claim is clear.

\[\square\]

Remark 2. In characteristic \( p > 0 \), all finitely generated perfect fields are finite. To give some contents to Corollary 2 a) in this characteristic, one may pass to the perfect [one should say radicial] closure \( k \) of a finitely generated field \( k_0 \). If \( G \) is a semi-abelian \( k \)-variety, it is defined over some finite extension \( k_1 \) of \( k_0 \). If \( k_2/k_1 \) is a finite (purely inseparable) subextension of \( k/k_1 \), then the composition

\[
G(k_2) \xrightarrow{N_{k_2/k_1}} G(k_1) \rightarrow G(k_2)
\]

equals multiplication by \([k_2 : k_1]\). Hence Corollary 2 a) remains true at least after inverting \( p \).
4. Stable birationality

If $X$ is a smooth variety over a field $k$, we write $\text{Alb}(X)$ for its generalised Albanese variety in the sense of Serre [17]: it is a semi-abelian variety, and a rational point $x_0 \in X$ determines a morphism $X \to \text{Alb}(X)$ which is universal for morphisms from $X$ to semi-abelian varieties sending $x_0$ to 0.

We also write $\text{NS}(X)$ for the group of cycles of codimension 1 on $X$ modulo algebraic equivalence. This group is finitely generated if $k$ is algebraically closed [10, Th. 3].

4.1. Well-known lemmas. I include proofs for lack of reference.

**Lemma 3.** a) Let $G, G'$ be two semi-abelian $k$-varieties. Then any $k$-morphism $f : G \to G'$ can be written uniquely $f = f(0) + f'$, where $f'$ is a homomorphism.

b) For any semi-abelian $k$-variety $G$, the canonical map $G \to \text{Alb}(G)$ sending 0 to 0 is an isomorphism.

**Proof.** a) amounts to showing that if $f(0) = 0$, then $f$ is a homomorphism. By an adjunction game, this is equivalent to b). Let us give two proofs: one of a) and one of b).

**Proof of a.)** We may assume $k$ to be a universal domain. The statement is classical for abelian varieties [16, p. 41, Cor. 1] and an easy computation for tori. In the general case, let $T, T'$ be the toric parts of $G$ and $G'$ and $A, A'$ be their abelian parts. Let $g \in G(k)$. As any morphism from $T$ to $A'$ is constant, the $k$-morphism

$$\varphi_g : T \ni t \mapsto f(g + t) - f(g) \in G'$$

(which sends 0 to 0) lands in $T'$, hence is a homomorphism. Therefore it only depends on the image of $g$ in $A(k)$. This defines a morphism $\varphi : A \to \text{Hom}(T, T')$, which must be constant with value $\varphi_0 = f$. It follows that

$$(g, h) \mapsto f(g + h) - f(g) - f(h)$$

induces a morphism $A \times A \to T'$. Such a morphism is constant, of value 0.

**Proof of b.)** This is true if $G$ is abelian, by rigidity and the equivalence between a) and b). In general, any morphism from $G$ to an abelian variety is trivial on $T$. This shows that the abelian part of $\text{Alb}(G)$ is $A$. Let $T' = \text{Ker}(\text{Alb}(G) \to A)$. We also have the counit morphism $\text{Alb}(G) \to G$, and the composition $G \to \text{Alb}(G) \to G$ is the identity. Thus $T$ is a direct summand of $T'$. It suffices to show that $\dim T' = \dim T$. Going to the algebraic closure, we may reduce to $T = \mathbb{G}_m$. 
Then consider the line bundle completion \( \tilde{G} \rightarrow A \) of the \( \mathbb{G}_m \)-bundle \( G \rightarrow A \). It is sufficient to show that the kernel of

\[
\text{Alb}(G) \rightarrow \text{Alb}(\tilde{G}) = A
\]

is 1-dimensional. This follows for example from [1, Cor. 10.5.1].

**Lemma 4.** Suppose \( k \) algebraically closed, and let \( G \) be a semi-abelian \( k \)-variety. Let \( A \) be the abelian quotient of \( G \). Then the map

\[
(5) \quad \text{NS}(A) \rightarrow \text{NS}(G)
\]

is an isomorphism.

**Proof.** Let \( T = \text{Ker}(G \rightarrow A) \) and \( X(T) \) be its character group. Choosing a basis \( (e_i) \) of \( X(T) \), we may complete the \( \mathbb{G}_m \)-torsor \( G \) into a product of line bundles \( \tilde{G} \rightarrow A \). The surjection

\[
\text{Pic}(A) \rightarrow \text{Pic}(\tilde{G}) \rightarrow \text{Pic}(G)
\]

show the surjectivity of (5). Its kernel is generated by the classes of the irreducible components \( D_i \) of the divisor with normal crossings \( \tilde{G} - G \). These components correspond to the basis elements \( e_i \). Since the corresponding \( \mathbb{G}_m \)-bundle is a group extension of \( A \) by \( \mathbb{G}_m \), the class of the 0 section of its line bundle completion lies in \( \text{Pic}^0(A) \), hence goes to 0 in \( \text{NS}(\tilde{G}) \). \qed

**Lemma 5.** Let \( X \) be a smooth \( k \)-variety, and let \( U \subseteq X \) be a dense open subset. Then there is an exact sequence of semi-abelian varieties

\[
0 \rightarrow T \rightarrow \text{Alb}(U) \rightarrow \text{Alb}(X) \rightarrow 0
\]

with \( T \) a torus. If \( \text{NS}(\bar{U}) = 0 \) (this happens if \( U \) is small enough), there is an exact sequence of character groups

\[
0 \rightarrow X(T) \rightarrow \bigoplus_{x \in X^{(1)}-U^{(1)}} \mathbb{Z} \rightarrow \text{NS}(\bar{X}) \rightarrow 0.
\]

**Proof.** This follows for example from [1, Cor. 10.5.1]. \qed

**Lemma 6.** Let \( f : G \rightarrow G' \) be a rational map between semi-abelian \( k \)-varieties, with \( G \) a torus. Then there exists an extension \( \tilde{G} \) of \( G \) by a permutation torus and a homomorphism \( \tilde{f} : \tilde{G} \rightarrow G' \) which extends \( f \) up to translation in the following sense: there exists a rational section \( s : G \rightarrow \tilde{G} \) of the projection \( \pi : \tilde{G} \rightarrow G \) and a rational point \( g' \in G'(k) \) such that \( f = \tilde{f}s + g' \). If \( f \) is defined at 0\(_G\) and sends it to 0\(_{G'}\), then \( g' = 0 \).
Proof. Let $U$ be an open subset of $G$ where $f$ is defined. We define $\tilde{G} = \text{Alb}(U)$. Applying Lemmas 5 and 3 b) and using $\text{NS} (\tilde{G}) = 0$, we get an extension

$$0 \to P \to \tilde{G} \to G \to 0$$

where $P$ is a permutation torus, as well as a morphism $\tilde{f} = \text{Alb}(f) : \tilde{G} \to G'$. Let us first assume $k$ infinite. Then $U(k) \neq \emptyset$ because $G$ is unirational. A rational point $g \in U$ defines an Albanese map $s : U \to \tilde{G}$ sending $g$ to $0_{\tilde{G}}$. Since $P$ is a permutation torus, $g \in G(k)$ lifts to $\tilde{g} \in \tilde{G}(k)$ (Hilbert 90) and we may replace $s$ by a morphism sending $g$ to $\tilde{g}$. Then $s$ is a rational section of $\pi$. Moreover, $f = \tilde{f}s + g'$ with $g' = f(g) - \tilde{f}(\tilde{g})$. The last assertion follows.

If $k$ is finite, then $U$ has at least a zero-cycle $g$ of degree 1, which is enough to define the Albanese map $s$. We then proceed as above (lift every closed point involved in $g$ to a closed point of $\tilde{G}$ with the same residue field).

$\blacksquare$

Lemma 7. Let $G$ be a finite group, and let $A$ be a finitely generated $G$-module. Then

a) There exists a short exact sequence of $G$-modules $0 \to P \to F \to A \to 0$, with $F$ torsion-free and flasque, and $P$ permutation.

b) Let $B$ be another finitely generated $G$-module, and let $0 \to P' \to E \to B \to 0$ be an exact sequence with $P'$ an invertible module. Then any $G$-morphism $f : A \to B$ lifts to $\tilde{f} : F \to E$.

Proof. a) is the contents of [5, Lemma 0.6, (0.6.2)]. b) The obstruction to lifting $f$ lies in $\text{Ext}^1_G(F, P') = 0$ [4, p. 182, Lemme 9].

$\blacksquare$

4.2. Functoriality of $\nu_{\leq 0}G$. We now assume $k$ perfect.

Lemma 8. Let

$$0 \to P \to G \to H \to 0$$

be an exact sequence of semi-abelian varieties, with $P$ an invertible torus. Then $\nu_{\leq 0}G[0] \stackrel{\sim}{\to} \nu_{\leq 0}H[0]$.

Proof. As $P$ is invertible, (6) is exact in NST hence defines an exact triangle

$$P[0] \to G[0] \to H[0] \rightrightarrows$$

in $\text{DM}_{\text{eff}}$. The conclusion then follows from Lemma 2. $\blacksquare$

Proposition 3. Let $G, G'$ be two semi-abelian $k$-varieties, with $G$ a torus. Then a rational map $f : G \dasharrow G'$ induces a morphism $f_* : \nu_{\leq 0}G[0] \to \nu_{\leq 0}G'[0]$, hence a homomorphism $f_* : G(K)/R \to G'(K)/R$. 


for any extension $K/k$. If $K$ is infinite, $f_*$ agrees up to translation with the morphism induced by $f$ via the isomorphism $U(K)/R \cong G(K)/R$ from [4, p. 196 Prop. 11], where $U$ is an open subset of definition of $f$.

Proof. By Lemma 6, $f$ induces a homomorphism $\tilde{G} \to G'$ where $\tilde{G}$ is an extension of $G$ by a permutation torus. By Lemma 8, the induced morphism

$$\nu_{\leq 0}\tilde{G}[0] \to \nu_{\leq 0}G'[0]$$

factors through a morphism $f_* : \nu_{\leq 0}G[0] \to \nu_{\leq 0}G'[0]$.

The claims about $R$-equivalence classes follow from Theorem 1 b) and Lemma 6. □

Remark 3. The proof shows that $f' = f_*$ if $f'$ differs from $f$ by a translation by an element of $G(k)$ or $G'(k)$.

Corollary 5. If $T$ and $T'$ are birationally equivalent $k$-tori, then $\nu_{\leq 0}T[0] \simeq \nu_{\leq 0}T'[0]$. In particular, the groups $T(k)/R$ and $T'(k)/R$ are isomorphic.

Proof. The proof of Proposition 3 shows that $f \mapsto f_*$ is functorial for composable rational maps between tori. Let $f : T \dashrightarrow T'$ be a birational isomorphism, and let $g : T' \dashrightarrow T$ be the inverse birational isomorphism. Then we have $g_* f_* = 1_{\nu_{\leq 0}T[0]}$ and $f_* g_* = 1_{\nu_{\leq 0}T'[0]}$. The last claim follows from Theorem 1. □

Remark 4. It is proven in [4] that a birational isomorphism of tori $f : T \dashrightarrow T'$ induces a set-theoretic bijection $f_* : T(k)/R \cong \to T'(k)/R$ (p. 197, Cor. to Prop. 11) and that the group $T(k)/R$ is abstractly a birational invariant of $T$ (p. 200, Cor. 4). The proof above shows that $f_*$ is an isomorphism of groups if $f$ respects the origins of $T$ and $T'$. This solves the question raised in [4, mid. p. 397]. The proofs of Lemma 6 and Proposition 3 may be seen as dual to the proof of [4, p. 189, Prop. 5], and are directly inspired from it.

4.3. Faithfulness and fullness.

Proposition 4. Let $f : G \dashrightarrow G'$ be a rational map between semi-abelian varieties, with $G$ a torus. Assume that the map $f_* : G(K)/R \to G'(K)/R$ from Proposition 3 is identically 0 when $K$ runs through the finitely generated extensions of $k$. Then there exists a permutation torus $P$ and a factorisation of $f$ as

$$G \overset{\tilde{f}}{\longrightarrow} P \overset{g}{\longrightarrow} G'$$

where $\tilde{f}$ is a rational map and $g$ is a homomorphism. If $f$ is a morphism, we may choose $\tilde{f}$ as a homomorphism.
Conversely, if there is such a factorisation, then \( f_* : \nu_{\leq 0}G[0] \to \nu_{\leq 0}G'[0] \) is the 0 morphism.

**Proof.** By Lemma 6, we may reduce to the case where \( f \) is a homomorphism. Let \( K = k(G) \). By hypothesis, the image of the generic point \( \eta_G \in G(K) \) is \( R \)-equivalent to 0 on \( G'(K) \). By a lemma of Gille [7, Lemme II.1.1 b)], it is directly \( R \)-equivalent to 0: in other words, there exists a rational map \( h : G \times A^1 \to G' \), defined in the neighbourhood of 0 and 1, such that \( h|_{G \times \{0\}} = 0 \) and \( h|_{G \times \{1\}} = f \).

Let \( U \subseteq G \times A^1 \) be an open set of definition of \( h \). The 0 and 1-sections of \( G \times A^1 \) induce sections
\[
s_0, s_1 : G \to \text{Alb}(U)
\]
of the projection \( \pi : \text{Alb}(U) \to \text{Alb}(G \times A^1) = G \) such that \( \text{Alb}(h) \circ s_0 = 0 \) and \( \text{Alb}(h) \circ s_1 = f \). If \( P = \ker \pi \), then \( s_1 - s_0 \) induces a homomorphism \( \tilde{f} : G \to P \) such that the composition
\[
G \xrightarrow{\tilde{f}} P \to \text{Alb}(U) \xrightarrow{\text{Alb}(h)} G'
\]
equals \( f \). Finally, \( P \) is a permutation torus by Lemma 5.

The last claim follows from Lemma 2. \( \square \)

**Proof of Theorem 2.** a) Take \( K = k(G) \). The image of the generic point \( \eta_G \) by \( f_K \) lifts to a (non unique) rational map \( f : G \to G' \). Using Lemma 6, we may extend \( f \) to a homomorphism
\[
\tilde{f} : \tilde{G} \to G'
\]
where \( \tilde{G} \) is an extension of \( G \) by a permutation torus \( P \). Since \( \tilde{G}(K)/R \sim \to G(K)/R \), we reduce to \( \tilde{G} = G \) and \( \tilde{f} = f \).

Let \( L/k \) be a function field, and let \( g \in G(L) \). Then \( g \) arises from a morphism \( g : X \to G \) for a suitable smooth model \( X \) of \( L \). By assumption on \( K \mapsto f_K \), the diagram
\[
\begin{array}{ccc}
G(K)/R & \xrightarrow{f_K} & G'(K)/R \\
\downarrow{g^*} & & \downarrow{g^*} \\
G(L)/R & \xrightarrow{f_L} & G'(L)/R
\end{array}
\]
commutes. Applying this to \( \eta_K \in G(K) \), we find that \( f_L([g]) = [g \circ f] \), which means that \( f_L \) is the map induced by \( f \).

b) The hypothesis implies that \( G'(E)/R = 0 \) for any algebraically closed extension \( E/k \), which in turn implies that \( G' \) is also a torus. Applying a), we may, and do, convert \( f \) into a true homomorphism by replacing \( G \) by a suitable extension by a permutation torus. Applying Lemma 7 a) to the cocharacter group of \( G \), we then get a resolution.
0 → \text{P} \rightarrow Q \rightarrow G \rightarrow 0 \text{ with } Q \text{ coflasque and } \text{P} \text{ permutation. Hence we may (and do) further assume } G \text{ coflasque.}

Let \( K = k(G') \) and choose some \( g \in G(K) \) mapping modulo \( R \)-equivalence to the generic point of \( G' \). Then \( g \) defines a rational map \( g : G' \rightarrow G \) such that \( fg \) is \( R \)-equivalent to \( 1_{G'} \). It follows that the induced map

\[
1 - fg : G'/R \rightarrow G'/R
\]

is identically 0.

Reapplying Lemma 6, we may find an extension \( \tilde{G}' \) of \( G' \) by a suitable permutation torus which converts \( g \) into a true homomorphism. Since \( G \) is coflasque, Lemma 7 b) shows that \( f : G \rightarrow G' \) lifts to \( \tilde{f} : G \rightarrow \tilde{G}' \). Then (7) is still identically 0 when replacing \((G', f)\) by \((\tilde{G}', \tilde{f})\).

Summarising: we have replaced the initial \( G \) and \( G' \) by suitable extensions by permutation tori, such that \( f \) lifts to these extensions and there is a homomorphism \( g : G' \rightarrow G \) such that (7) vanishes identically. Hence \( 1 − fg \) factors through a permutation torus \( \text{P} \). Let \( G_1 = G \times \text{P} \) and consider the maps

\[
f_1 = (f, v) : G_1 \rightarrow G', \quad g_1 = \begin{pmatrix} g \\ u \end{pmatrix} : G' \rightarrow G_1.
\]

Then \( f_1 g_1 = 1 \) and \( G' \) is a direct summand of \( G_1 \) as requested. \( \square \)

5. SOME OPEN QUESTIONS

\textbf{Question 1.} Are lemma 6 and Proposition 3 still true when \( G \) is not a torus?

This is far from clear in general, starting with the case where \( G \) is an abelian variety and \( G' \) a torus. Let me give a positive answer in the case of an elliptic curve.

\textbf{Proposition 5.} The answer to Question 1 is yes if the abelian part \( A \) of \( G \) is an elliptic curve.

\textit{Proof.} Arguing as in the proof of Proposition 3, we get for an open subset \( U \subseteq G \) of definition for \( f \) an exact sequence

\[
0 \rightarrow \mathbb{G}_m \rightarrow P \rightarrow \text{Alb}(U) \rightarrow G \rightarrow 0
\]

where \( P \) is a permutation torus. Here we used that \( \text{NS}(\tilde{G}) \cong \mathbb{Z} \), which follows from Lemma 4.

The character group \( X(P) \) has as a basis the geometric irreducible components of codimension 1 of \( G - U \). Up to shrinking \( U \), we may assume that \( G - U \) contains the inverse image \( D \) of 0 \( \in A \). As the
divisor class of 0 generates \( \text{NS}(A) \), \( D \) provides a Galois-equivariant splitting of the map \( \mathbb{G}_m \to P \). Thus its cokernel is still a permutation torus, and we conclude as before. \( \square \)

**Question 2.** Can one formulate a version of Theorem 2 and Corollary 3 providing a description of the groups \( \text{Hom}_{\text{DM}_{\text{eff}}}(\nu \leq 0 G[0], \nu \leq 0 G'[0]) \) and \( \text{Hom}_{\text{Hil}}(G/R, G'/R) \) (at least when \( G \) and \( G' \) are tori)?

The proof of Theorem 2 suggests the presence of a closed model structure on the category of tori (or lattices), which might provide an answer to this question.

For the last question, let \( G \) be a semi-abelian variety. Forgetting its group structure, it has a motive \( M(G) \in \text{DM}_{\text{eff}} \). Recall the canonical morphism

\[
M(G) \to G[0]
\]

induced by the “sum” maps

\[
(c(X, G) \xrightarrow{\sigma} G(X))
\]

for smooth varieties \( X \) ([18, (6), (7)], [1, §1.3]).

The morphism (8) has a canonical section

\[
G(X) \xrightarrow{\gamma} c(X, G)
\]

given by the graph of a morphism: this section is functorial in \( X \) but is not additive.

Consider now a smooth equivariant compactification \( \bar{G} \) of \( G \). It exists in all characteristics. For tori, this is written up in [3]. The general case reduces to this one by the following elegant argument I learned from M. Brion: if \( G \) is an extension of an abelian variety \( A \) by a torus \( T \), take a smooth projective equivariant compactification \( Y \) of \( T \). Then the bundle \( G \times_T Y \) associated to the \( T \)-torsor \( G \to A \) also exists: this is the desired compactification.

Then we have a diagram of birational motives

\[
\begin{array}{ccc}
\nu \leq 0 M(G) & \xrightarrow{\sim} & \nu \leq 0 M(\bar{G}) \\
\text{ } \downarrow^{\nu \leq 0 \sigma} & & \\
\nu \leq 0 G[0]. & & \\
\end{array}
\]

By [12], we have \( H_0(\nu \leq 0 M(\bar{G}))(X) = CH_0(\bar{G}_{k(X)}) \) for any smooth connected \( X \). Hence the above diagram induces a homomorphism

\[
CH_0(\bar{G}_{k(X)}) \to G(k(X))/R
\]

which is natural in \( X \) for the action of finite correspondences (compare Corollary 1). One can probably check that this is the homomorphism
of [13, (17) p. 78], reformulating [4, Proposition 12 p. 198]. Similarly, the set-theoretic map
\[ G(k(X))/R \to CH_0(\bar{G}_{k(X)}) \]
of [4, p. 197] can presumably be recovered as a birational version of (9), using perhaps the homotopy category of schemes of Morel and Voevodsky [15].

In [13], Merkurjev shows that (11) is an isomorphism for $G$ a torus of dimension at most 3. This suggests:

**Question 3.** Is the map $\nu_{\leq 0}\sigma$ of Diagram (10) an isomorphism when $G$ is a torus of dimension $\leq 3$?

In [14], Merkurjev gives examples of tori $G$ for which (12) is not a homomorphism; hence its (additive) left inverse (11) cannot be an isomorphism. Merkurjev’s examples are of the form $G = R^1_{K/k}\mathbb{G}_m \times R^1_{L/k}\mathbb{G}_m$, where $K$ and $L$ are distinct biquadratic extensions of $k$. This suggests:

**Question 4.** Can one study Merkurjev’s examples from the above viewpoint? More generally, what is the nature of the map $\nu_{\leq 0}\sigma$ of Diagram (10)?

We leave all these questions to the interested reader.

**References**


Institut de Mathématiques de Jussieu, UMR 7586, Case 247, 4 place Jussieu, 75252 Paris Cedex 05, France
E-mail address: kahn@math.jussieu.fr